The Isolation Game: A Game of Distances

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Abstract

We introduce a new multi-player geometric game, which we will refer to as the *isolation game*, and study its Nash equilibria and best or better response dynamics. The isolation game is inspired by the Voronoi game, competitive facility location, and geometric sampling. In the Voronoi game studied by Dürr and Thang, each player's objective is to maximize the area of her Voronoi region. In contrast, in the isolation game, each player's objective is to position herself as far away from other players as possible in a bounded space.

Even though this game has a simple definition, we show that its game-theoretic behaviors are quite rich and complex. We consider various measures of farness from one player to a group of players and analyze their impacts to the existence of Nash equilibria and to the convergence of the best or better response dynamics: We prove that it is NP-hard to decide whether a Nash equilibrium exists, using either a very simple farness measure in an asymmetric space or a slightly more sophisticated farness measure in a symmetric space. Complementing to these hardness results, we establish existence theorems for several special families of farness measures in symmetric spaces: We prove that for isolation games where each player wants to maximize her distance to her m^{th} nearest neighbor, for any m, equilibria always exist. Moreover, there is always a better response sequence starting from any configuration that leads to a Nash equilibrium. We show that when $m = 1$ the game is a potential game — no better response sequence has a cycle, but when $m > 1$ the games are not potential. More generally, we study farness functions that give different weights to a player's distances to others based on the distance rankings, and obtain both existence and hardness results when the weights are monotonically increasing or decreasing. Finally, we present results on the hardness of computing best responses when the space has a compact representation as a hypercube.

Key words: Algorithmic game theory, Nash equilibrium, Computational complexity.

1 Introduction

In competitive facility location [4,5,7], data clustering [8], and geometric sampling [10], a fundamental geometric problem is to place a set of objects (such as facilities and cluster centers) in a space so that they are mutually far away from one another. Inspired by the study of Dürr and Thang [3] on the Voronoi game, we introduce a new multi-player geometric game called *isolation game*.

In an isolation game, there are k players that will locate themselves in a space (Ω, Δ) where $\Delta(x, y)$ defines the pairwise distance among points in Ω. If $\Delta(x, y) = \Delta(y, x)$, for all $x, y \in \Omega$, we say (Ω, Δ) is symmetric. The i^{th} player has a $(k-1)$-place function $f_i(\ldots, \Delta(p_i, p_{i-1}), \Delta(p_i, p_{i+1}), \ldots)$ from the $k-1$ distances to all other players to a real value, measuring the farness from her location p_i to the locations of other players. The objective of player i is to maximize $f_i(\ldots, \Delta(p_i, p_{i-1}), \Delta(p_i, p_{i+1}), \ldots)$, once the positions of other players $(\ldots, p_{i-1}, p_{i+1}, \ldots)$ are given.

Depending on applications, there could be different ways to measure the farness from a point to a set of points. The simplest farness function $f_i()$ could be the one that measures the distance from p_i to her nearest player. Games based on this measure are called *nearest-neighbor games*. Another simple measure is the total distance from p_i to other players. Games based on this measure are called *total distance games*. Other farness measures include the distance of p_i to her m^{th} nearest player, or a weighted combination of the distances from player i to other players.

In some cases, the isolation games with simple farness measures have similar behaviors as the multi-player Voronoi game [1,2,6] in discrete spaces. Recall that in the Voronoi game, the objective of each player is to maximize the area of her Voronoi cell in Ω induced by $\{p_1, ..., p_k\}$ — the set of points in Ω that are closer to p_i than to any other player. The Voronoi game has applications in competitive facility location, where merchants try to place their facilities to maximize their customer bases, and customers are assumed to go to the facility closest to them. Each player needs to calculate the area of her Voronoi cell to play the game, which could be expensive. In the nearest-neighbor isolation game, each player chooses to maximize her nearest-neighbor distance

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to other players. In some discrete spaces, like the discrete cycle graph, the longer nearest-neighbor distance a player has, the larger Voronoi area this player gains. This gives rise to the isolation game with these special farness measures.

The generalized isolation games may have applications in product design in a competitive market, where companies' profit may depend on the dissimilarity of their products to those of their competitors, which could be measured by the multi-dimensional features of products. Companies differentiate their products from those of their competitors by playing some kind of isolation games in the multi-dimensional feature space. The isolation game may also have some connection with political campaigns such as in a multi-candidate election, in which candidates, constrained by their histories of public service records, try to position themselves in the multi-dimensional space of policies and political views in order to differentiate themselves from other candidates.

We study the Nash equilibria [9] and best or better response dynamics of the isolation games. We consider various measures of farness from one player to a group of players and analyze their impact to the existence of Nash equilibria and to the convergence of best or better response dynamics in an isolation game. For simple measures such as the nearest-neighbor and the total-distance, it is quite straightforward to show that these isolation games are potential games when the underlying space is symmetric. Hence, the game has at least one Nash equilibrium and all better response dynamics converge. Surprisingly, we show that when the underlying space is asymmetric, Nash equilibria may not exist, and it is NP-hard to determine whether Nash equilibria exist in an isolation game.

The general isolation game is far more complex even for symmetric spaces, even if we restrict our attention only to uniform anonymous isolation games. We say an isolation game is *anonymous* if for all i, $f_i()$ is invariant under the permutation of its parameters. We say an anonymous isolation game is *uniform* if $f_i() = f_j()$ for all i, j. For instance, the two potential isolation games with the nearest-neighbor or total-distance measure mentioned above are uniform anonymous games. Even these classes of games exhibit different behaviors: some subclass of games always have Nash equilibria, some can always find better response sequences that converge to a Nash equilibrium, but some may not have Nash equilibria and determining the existence of a Nash equilibrium is NP-complete. We summarize our findings below.

First, We prove that for isolation games where each player wants to maximize her distance to her m^{th} nearest neighbor, Nash equilibria always exist. In addition, there is always a better response sequence starting from any configuration that leads to a Nash equilibrium. We show, however, that this isolation game is not a potential game — there are better response sequences that lead to cycles. Second, as a general framework, we model the farness function of a uniform anonymous game by a vector $\vec{w} = (w_1, w_2, \ldots, w_{k-1})$. Let $\vec{d_j} = (d_{j,1}, d_{j,2}, \ldots, d_{j,k-1})$ be the *distance vector* of player j in a configuration, which are distances from player j to the other $k-1$ players sorted in nondecreasing order, i.e., $d_{j,1} \le d_{j,2} \le \ldots \le d_{j,k-1}$. Then the utility of player j in the configuration is $\vec{w} \cdot \vec{d} = \sum_{i=1}^{k-1}(w_i \cdot d_{j,i})$. We show that a Nash equilibrium exists for increasing or decreasing weight vectors $\vec{w}$, when the underlying space (Ω, Δ) satisfies certain conditions, which are different for increasing and decreasing weight vectors. For a particular version of the decreasing weight vectors, namely $(1, 1, 0, \ldots, 0)$, we show that: (a) it is not potential even on a continuous one dimensional circular space; (b) starting from any configuration there is a best-response sequence that converges to a Nash equilibrium in a continuous one dimensional circular space; (c) in general symmetric spaces a Nash equilibrium may not exist, and (d) it is NP-complete to decide if a Nash equilibrium exists in general symmetric spaces. Combining with the previous NP-completeness result, we see that either a complicated space (asymmetric space) or a slightly complicated farness measure ($(1, 1, 0, \ldots, 0)$ instead of $(1, 0 \ldots, 0)$ or $(0, 1, 0, \ldots, 0)$) would make the determination of the existence of Nash equilibria difficult.

Table 1 summarizes the rich behaviors we found for various versions of the isolation games.

Table 1
Summary of results for various isolation games. "NE" stands for Nash Equilibrium; "NPC" means NP-complete to determine if a Nash equilibrium exists.

farness function	complexity of space (Ω, Δ)				
	1-d continuous circular	polarized	finite	symmetric	asymmetric
1, 1, 1, ..., 1	potential	potential	potential	potential [Thm 2]	NPC [Thm 4][Cor 5]
1, 0, 0, ..., 0	potential	potential	potential	potential [Thm 1]	NPC [Thm 4][Cor 5]
0, 1, 0, ..., 0	not potential [Lem 7]		∃ best-response sequence to NE [Thm 9]	∃ NE [Thm 6]	
1, 1, 0, ..., 0	not potential [Lem 13], ∃ best-response sequence to NE [Thm 14]		NPC [Thm 16]		
single selection			∃ better-response sequence to NE [Thm 8]	∃ NE [Thm 6]	
monotonic increasing	∃ NE	∃ NE [Thm 10]			
monotonic decreasing	∃ NE [Cor 12] not potential [Lem 13]		NPC [Thm 16]		

We also examine the hardness of computing best responses in spaces with compact representations such as a hypercube. We show that for one class of isolation games including the nearest-neighbor game as the special case it is NP-complete to compute best responses, while for another class of isolation games, the computation can be done in polynomial time.

The rest of the paper is organized as follows. Section 2 covers the basic definitions and notation. Section 3 presents the results for nearest-neighbor and total-distance isolation games. Section 4 presents results for other general classes of isolation games. Section 5 examines the hardness of computing best responses in isolation games. We conclude the paper in Section 6.

2 Notation

We use (Ω, Δ) to denote the underlying space, where we assume $\Delta(x, x) = 0$ for all $x \in \Omega$, $\Delta(x, y) > 0$ for all $x, y \in \Omega$ and $x \neq y$, and that (Ω, Δ) is bounded — there exists a real value B such that $\Delta(x, y) \leq B$ for every $x, y \in \Omega$. In general, (Ω, Δ) may not be symmetric or satisfy the triangle inequality. We always assume that there are k players in an isolation game and each player's strategy set is the entire Ω. A *configuration* of an isolation game is a vector $(p_1, p_2, \ldots, p_k)$, where $p_i \in \Omega$ specifies the position of player i. The utility function of player i is a $(k-1)$-place function $f_i(\ldots, \Delta(p_i, p_{i-1}), \Delta(p_i, p_{i+1}), \ldots)$. For convenience, we use $ut_i(c)$ to denote the utility of player i in configuration c.

We consider several classes of weight vectors in the uniform, anonymous isolation game. In particular, the *nearest-neighbor* and *total-distance* isolation games have the weight vectors $(1, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$, respectively; the *single-selection* games have vectors that have exactly one nonzero entry; the *monotonically-increasing* (or *decreasing*) games have vectors whose entries are monotonically increasing (or decreasing).

A *better response* of player i in configuration $c = (p_1, \ldots, p_k)$ is a new position $p'_i \neq p_i$ such that the utility of player i in configuration c' by replacing p_i with p'_i in c is larger than her utility in c. In this case, we say that c' is the *result of a better-response move* of player i in configuration c. A *best response* of player i in configuration $c = (p_1, \ldots, p_k)$ is a new position $p'_i \neq p_i$ that maximizes the utility of player i while player j remains at position p_j for all $j \neq i$. In this case, we say that c' is the *result of a best-response move* of player i in configuration c.

A (pure) *Nash equilibrium* of an isolation game is a configuration in which no player has any better response in the configuration. An isolation game is *better-*

response potential (or *best-response potential*) if there is a function F from the set of all configurations to a totally ordered set such that $F(c) < F(c')$ for any two configurations c and c' where c' is the result of a better-response move (or a best-response move) of some player at configuration c. We call F a *potential function.* It is easy to see that any better- (best-) response sequence in a better- (best-) response potential game is acyclic, and further more if Ω is finite, any better- (best-) response sequence eventually leads to a Nash equilibrium. Note that a better-response potential game is also a best-response potential game, but a best-response potential game may not be a better-response potential game. Henceforth, we use the shorthand "potential games" to refer to better-response potential games.

3 Nearest-neighbor and Total-distance Isolation Games

In this section, we focus on the isolation games with weight vectors $(1, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$. We show that both are potential games when Ω is symmetric, but when Ω is asymmetric and finite, it is NP-complete to decide whether those games have Nash equilibria.

Theorem 1 *The symmetric nearest-neighbor isolation game is a potential game.*

Proof: Consider a configuration $c = (p_1, p_2, \ldots, p_k)$. The utility of player i in configuration c in the nearest-neighbor isolation game is the distance between player i and her nearest neighbor. Let vector $\vec{u}(c) = (u_1, u_2, \ldots, u_k)$ be the vector of the utility values of all players in c sorted in increasing order, i.e., $u_1 \leq u_2 \leq \ldots \leq u_k$.

We claim that for any configuration c and c' such that c' is the result of a better-response move of a player i from position p_i to $p'_i \neq p_i$, we have $\vec{u}(c) < \vec{u}(c')$ in lexicographic order.

Let $\vec{u}(c) = (u_1, u_2, \ldots, u_k)$, and u_j be the utility of player i in c (i.e. $ut_i(c) = u_j$). Consider another player $s \neq i$ and her utility $u_t = ut_s(c)$. There are three cases. Case 1: $u_t < u_j$. Since Ω is symmetric, u_t cannot be the distance from s to i. Thus, after i's better-response move, the utility of s in c' is still u_t. Case 2: $u_t > u_j$. After i's better-response move, if the utility of s decreases, then $ut_s(c') = \Delta(p_s, p'_i) = \Delta(p'_i, p_s) \geq ut_i(c') > ut_i(c) = u_j$. That is, in any case the utility of s in c' is greater than u_j. Case 3: $u_t = u_j$. After i's better-response move, the utility of s either remains the same or increases.

Therefore, comparing vector $\vec{u}(c')$ with $\vec{u}(c)$, all elements in front of u_j remain the same or increase, and all elements after u_j either remain the same or the new value is greater than u_j, and u_j itself increases. This implies that $\vec{u}(c') > \vec{u}(c)$ in lexicographic order. Thus, the above claim holds.

We now can define a potential function $F(c) = \vec{u}(c)$ to show that the nearest-neighbor isolation game is potential. □

Theorem 2 *The symmetric total-distance isolation game is a potential game.*

Proof: We define a function F on configurations $c = (p_1, p_2, \ldots, p_k)$ to be $F(c) = \sum_{1 \le i,j \le k} \Delta(p_i, p_j)$, the sum of all pairwise distances of players in the configuration. If a player i has a better response $p'_i \neq p_i$ in c, then we know that $\sum_{j \neq i} \Delta(p'_i, p_j) > \sum_{j \neq i} \Delta(p_i, p_j)$. Since Ω is symmetric, we have $F(c') - F(c) = (\sum_{j,\ell \neq i} \Delta(p_\ell, p_j) + \sum_{j \neq i} \Delta(p'_i, p_j)) - (\sum_{j,\ell \neq i} \Delta(p_\ell, p_j) + \sum_{j \neq i} \Delta(p_i, p_j)) > 0$. Therefore, F is a potential function, and the total-distance isolation game is potential. □

The following lemma shows that the asymmetric isolation game may not have any Nash equilibrium for any nonzero weight vector. Thus, it also implies that asymmetric nearest-neighbor and total-distance isolation games may not have Nash equilibria.

Lemma 3 *Consider an asymmetric space $\Omega = \{v_1, v_2, \ldots, v_{\ell+1}\}$ with the distance function given by the following matrix with $t \geq \ell + 1$. Suppose that for every player i her weight vector $\vec{w}_i$ has at least one nonzero entry. Then, for any $2 \le k \le \ell$, there is no Nash equilibrium in the k-player isolation game.*

$$\left(\begin{array}{c|cccccc}
\Delta & v_1 & v_2 & v_3 & \ldots & v_\ell & v_{\ell+1} \\
\hline
v_1 & 0 & t-1 & t-2 & \ldots & t-\ell+1 & t-\ell \\
v_2 & t-\ell & 0 & t-1 & \ldots & t-\ell+2 & t-\ell+1 \\
\vdots & \vdots & & \ddots & & \vdots & \\
v_\ell & t-2 & t-3 & t-4 & \ldots & 0 & t-1 \\
v_{\ell+1} & t-1 & t-2 & t-3 & \ldots & t-\ell & 0
\end{array}\right)$$

Proof: Consider any configuration c. Because $2 \le k \le \ell$ and $|\Omega| = \ell + 1$, in configuration c there exists at least one free point not occupied by any player. Without loss of generality, we assume that one of the free points is v_i and the point v_{i-1} is already chosen by player t (if $v_i = v_1$, then $v_{i-1} = v_{\ell+1}$). Then we can see that for all $j \neq i$ and $j \neq i-1$, $\Delta(v_i, v_j) > \Delta(v_{i-1}, v_j)$. Since all weights are nonnegative and at least one weight is nonzero, player t could achieve better utility by moving from v_{i-1} to v_i. Therefore, c is not a Nash equilibrium. □

Theorem 4 *It is NP-complete to decide whether a finite, asymmetric nearest-*

neighbor or total-distance isolation game has a Nash equilibrium.

Proof: We first prove the case of the nearest-neighbor isolation game.

Suppose that the size of Ω is n. Then the distance function Δ has n^2 entries. The decision problem is clearly in *NP*. The NP-hardness can be proved by reduction from the Set Packing problem. An instance of the Set Packing problem includes a set $I = \{e_1, e_2, \ldots, e_m\}$ of m elements, a set $S = \{S_1, \ldots, S_n\}$ of n subsets of I, and a positive integer k. The decision problem is to decide whether there are k disjoint subsets in S. We now give the reduction.

The space Ω has $n + k + 1$ points, divided into a left set $L = \{v_1, v_2, \ldots, v_n\}$ and a right set $R = \{u_1, u_2, \ldots, u_{k+1}\}$. For any two different points $v_i, v_j \in L$, $\Delta(v_i, v_j) = 2n$ if $S_i \cap S_j = \emptyset$, and $\Delta(v_i, v_j) = 1/2$ otherwise. The distance function on R is given by the distance matrix in Lemma 3 with $\ell = k$ and $t = k + 1$. For any $v \in L$ and $u \in R$, $\Delta(v, u) = \Delta(u, v) = 2n$. Finally, the isolation game has $k + 1$ players.

We now show that there exists a Nash equilibrium for the nearest-neighbor isolation game on Ω if and only if there are k disjoint subsets in the Set Packing instance.

First, suppose that there is a solution to the Set Packing instance. Without loss of generality, assume that the k disjoint subsets are $S_1, S_2, \ldots, S_k$. Then we claim that configuration $c = (v_1, v_2, \ldots, v_k, u_1)$ is a Nash equilibrium. In this configuration, it is easy to verify that every player's utility is $2n$, the largest possible pairwise distance. Therefore, c is a Nash equilibrium.

Conversely, suppose that there is a Nash equilibrium in the nearest-neighbor isolation game. Consider the set R. If there is a Nash equilibrium c, then the number of players positioned in R is either $k + 1$ or at most 1 because of Lemma 3. If there are $k + 1$ players in R, then every player has utility 1, and thus each of them would want to move to points in L to obtain a utility of $2n$. Therefore, there cannot be $k + 1$ players positioned in R, which means that there are at least k players positioned in L.

Without loss of generality, assume that these k players occupy points $v_1, v_2, \ldots, v_k$ (which may have duplicates). We claim that subsets $S_1, S_2, \ldots, S_k$ form a solution to the Set Packing problem. Suppose, for a contradiction, that this is not true, which means there exist S_i and S_j among these k subsets that intersect with each other. By our construction, we have $\Delta(v_i, v_j) = 0$ or $1/2$. In this case, players at point v_i and v_j would want to move to some free points in R, since that will give them utilities of at least 1. This contradicts the assumption that c is a Nash equilibrium. Therefore, we find a solution for the Set Packing problem given a Nash equilibrium c of the nearest-neighbor isolation game.

The proof for the case of the total-distance isolation game is essentially the same, with only changes in players' utility values. □

If the space is infinite and asymmetric, we can also make reduction from the Set Packing problem. Given an instance of the Set Packing problem, we can construct a $(k+1)$-player isolation game. There are two types of points in the game space Ω: "particular" points and "normal" points.

There are $n+k+1$ "particular" points, divided into a left set L and a right set R. The distances between "particular" points are defined the same as in Theorem 4. There are infinitely many "normal" points. For these "normal" points, let the distance between any two remaining points be 0 and let the distance between any "particular" point and any "normal" point be 0.

With similar analysis, we can prove that the Set Packing instance has a solution if and only if there is a Nash equilibrium for the nearest-neighbor isolation game on Ω. The proof for the case of the total-distance isolation game in an infinite and asymmetric space is essentially the same with only changes in players' utility values. Hence we get the following corollary.

Corollary 5 *It is NP-complete to decide whether an infinite, asymmetric nearest-neighbor or total-distance isolation game has a Nash equilibrium.*

4 Isolation Games with Other Weight Vectors

In this section, we study several general classes of isolation games. We consider symmetric space (Ω, Δ) in this section.

4.1 Single-selection Isolation Games

First we consider the single-selection isolation games. Recall that the single-selection games have weight vectors that have exactly one nonzero entry.

Theorem 6 *A Nash equilibrium always exists in any single-selection symmetric game.*

Proof: We denote the k-player single-selection symmetric game as $\mathcal{G}$, in which the m^{th} entry of the weight vector is 1. We partition the k players into $t = \lceil \frac{k}{m} \rceil$ groups $G_1, G_2, \ldots, G_t$. If $m \mid k$, then let each group consist of m players; otherwise, let each group except G_t consist of m players, and let G_t consist of the remaining $k-(t-1)m$ players. We denote a t-player nearest-neighbor isolation game on Ω as $\mathcal{G}'$. By Theorem 1, we know that in game $\mathcal{G}'$

there exists a Nash equilibrium $c' = (p_1, \ldots, p_t)$. Without loss of generality, we assume that in $\mathcal{G}'$ player t at p_t has the minimum utility value among all the t players. Now we generate a configuration c for game $\mathcal{G}$ in which all the players in group G_i choose position p_i. We show that c is a Nash equilibrium for game $\mathcal{G}$.

If $m \mid k$, we consider an arbitrary player i of game $\mathcal{G}$ that belongs to group G_j and locates at p_j. It is easy to verify that the utility of player i of game $\mathcal{G}$ in c equals the utility of player j of game $\mathcal{G}'$ in c', because each group consists of m players. If player i of game $\mathcal{G}$ could increase her utility to $\Delta(p_a, p_b)$ by moving to p_a, then player j of game $\mathcal{G}'$ would also have a better response of p_a to increase her utility to $\Delta(p_a, p_b)$, which contradicts the assumption that c' is a Nash equilibrium of game $\mathcal{G}'$. Hence no player of game $\mathcal{G}$ in the configuration c has a better response, and thus c is a Nash equilibrium for game $\mathcal{G}$.

If $m \nmid k$, we consider an arbitrary player i of game $\mathcal{G}$ that belongs to group G_j and locates at p_j. Suppose that the position of the m-th nearest neighbor of player i in configuration c of game $\mathcal{G}$ is p_u. It is easy to verify that p_u is also the position of the nearest neighbor of player j in configuration c' of game $\mathcal{G}'$. Suppose, for a contradiction, that player i of game $\mathcal{G}$ has a better response p_a. We consider the following cases.

Case 1: $p_a \notin \{p_1, \ldots, p_t\}$. Then in game $\mathcal{G}'$, player j could move to position p_a and strictly increase her utility, which contradicts the assumption that c' is a Nash equilibrium for game $\mathcal{G}'$.

Case 2: $p_a \in \{p_1, \ldots, p_{t-1}\}$. This is impossible because in game $\mathcal{G}$, at each position in $\{p_1, \ldots, p_{t-1}\}$, there are m players, and if player i moves to $p_a \neq p_j$, her m nearest neighbors are all at position p_a and thus her utility would be zero.

Case 3: $p_a = p_t$. Since group G_t has less than m players, the m-th nearest neighbor of player i after her move is positioned at some point $p_s \neq p_t$. Since this is a better response of player i in game $\mathcal{G}$, we have $\Delta(p_t, p_s) > \Delta(p_j, p_u)$. It is easy to verify that p_s is the position of the nearest neighbor of player t in configuration c' of game $\mathcal{G}'$. Thus we have that in game $\mathcal{G}'$ player t's utility in configuration c' is larger than player j's utility in c', which contradicts our selection of p_t.

We reach contradictions in all three cases. Therefore, c must be a Nash equilibrium of game $\mathcal{G}$. □

Although Nash equilibria always exist in single-selection isolation games, the following lemma shows that they are not potential games.

Lemma 7 *Let $\Omega = \{A, B, C, D, E, F\}$ contain six points on a one-*

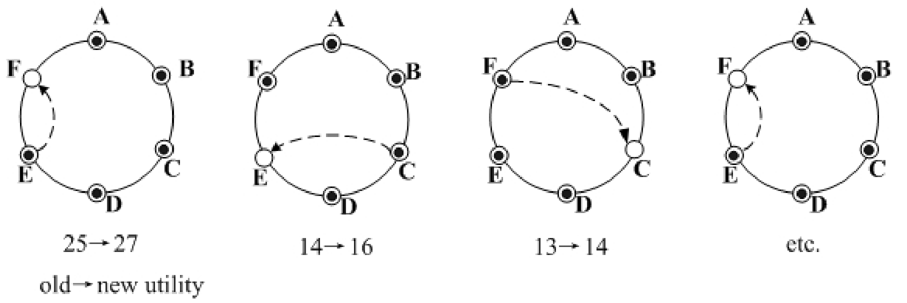

Fig. 1. An example of a better-response sequence that loops forever for a five-player isolation game with weight vector $(0, 1, 0, 0)$ on a one dimensional circular space with six points.

dimensional circular space with $\Delta(A, B) = 15$, $\Delta(B, C) = 11$, $\Delta(C, D) = 14$, $\Delta(D, E) = 16$, $\Delta(E, F) = 13$, *and* $\Delta(F, A) = 12$. *The five-player single-selection game with the weight vector* $(0, 1, 0, 0)$ *on* Ω *is not potential.*

Proof: Let the five players stand at A, B, C, D, and E respectively in the initial configuration. Their better response dynamics can iterate forever as shown in Figure 1. Hence this game is not a potential game. □

Surprisingly, the following theorem complements the previous lemma.

Theorem 8 *If* Ω *is finite, then for any single-selection game on* Ω *and any starting configuration* c*, there is a better-response sequence in the game that leads to a Nash equilibrium.*

Proof: Suppose that the nonzero weight entry is the m^{th} entry in the k-player single-selection isolation game with $m > 1$ (the case of $m = 1$ is already covered in the nearest-neighbor isolation game). For any configuration $c = (p_1, \ldots, p_k)$, the utility of player i is the distance between player i and her m^{th} nearest neighbor. Let vector $\vec{u}(c) = (u_1, u_2, \ldots, u_k)$ be the vector of the utility values of all players in c sorted in nondecreasing order, i.e., $u_1 \leq u_2 \leq \ldots \leq u_k$. We claim that for any configuration c, if c is not a Nash equilibrium, there must exist a finite sequence of configurations $c = c_0, c_1, c_2, \ldots, c_t = c'$, such that c_{i+1} is the result of a better-response move of some player in c_i for $i = 0, 1, \ldots, t-1$ and $\vec{u}(c) < \vec{u}(c')$ in lexicographic order.

We now prove this claim. Since the starting configuration $c_0 = c$ is not a Nash equilibrium, there exists a player i that can make a better response move to position p, resulting in configuration c_1. We have $ut_i(c_0) < ut_i(c_1)$. Let S_i be the set of player i's $m-1$ nearest neighbors in configuration c_1. We now repeat the following steps to find configurations $c_2, \ldots, c_t$. When in configuration c_j, we select a player a_j in S_i such that $ut_{a_j}(c_j) < ut_i(c_1)$ and move a_j to position p, the same position where player i locates. This gives configuration c_{j+1}. This is certainly a better-response move for a_j because

$ut_{a_j}(c_{j+1}) = ut_i(c_{j+1}) = ut_i(c_1) > ut_{a_j}(c_j)$, where the second equality holds because we only move the $m-1$ nearest neighbors of player i in c_1 to the same position as i, so it does not affect the distance from i to her m^{th} nearest neighbor. The repeating step ends when there is no more such player a_j in configuration c_j, in which case $c_j = c_t = c'$.

We now show that $\vec{u}(c) < \vec{u}(c')$ in lexicographic order. We first consider any player $j \notin S_i$. Either her utility does not change ($ut_j(c) = ut_j(c')$), or her utility change must be due to the changes of her distances to player i and players $a_1, a_2, \ldots, a_{t-1}$, who have moved to position p. Suppose that player j is at position q. Then $\Delta(p,q) \geq ut_i(c_1)$ because $j \notin S_i$. This means that if j's utility changes, her new utility $ut_j(c')$ must be at least $\Delta(p,q) \geq ut_i(c_1)$. For a player $j \in S_i$, if she is one of $\{a_1, \ldots, a_{t-1}\}$, then her new utility $ut_j(c') = ut_i(c') = ut_i(c_1)$; if she is not one of $\{a_1, \ldots, a_{t-1}\}$, then by definition $ut_j(c') \geq ut_i(c_1)$. Therefore, comparing the utilities of every player in c and c', we know that either her utility does not change, or her new utility is at least $ut_i(c') = ut_i(c_1) > ut_i(c)$, and at least player i herself strictly increases her utility from $ut_i(c)$ to $ut_i(c')$. With this result, it is straightforward to verify that $\vec{u}(c) < \vec{u}(c')$. Thus, our claim holds.

We may call the better-response sequence found in the above claim an epoch. We can apply the above claim to concatenate new epochs such that at the end of each epoch the vector $\vec{u}$ strictly increases in lexicographic order. Since the space Ω is finite, the vector $\vec{u}$ has an upper bound. Therefore, after a finite number of epochs, we must be able to find a Nash equilibrium, and all these epochs concatenated together form a better-response sequence that leads to the Nash equilibrium. This is clearly true when starting from any initial configuration. $\square$

We now consider a simple class of single-selection isolation games with weight vector $\vec{w} = (0, 1, 0, \ldots, 0)$, for which we are able to achieve the following stronger result than Theorem 8.

Theorem 9 *If Ω is finite, then for any isolation game with weight vector $\vec{w} = (0, 1, 0, \ldots, 0)$ on Ω and any starting configuration c, there is a best-response sequence in the game that leads to a Nash equilibrium.*

Proof: Since the weight vector is $\vec{w} = (0, 1, 0, \ldots, 0)$, for any configuration $c = (p_1, p_2, \ldots, p_k)$, the utility of player i is the distance between player i and her second nearest neighbor. Let vector $\vec{u}(c) = (u_1, u_2, \ldots, u_k)$ be the vector of the utility values of all players in c sorted in nondecreasing order, i.e., $u_1 \leq u_2 \leq \ldots \leq u_k$. We claim that for any configuration c, if c is not a Nash equilibrium, there must exist a finite sequence of configurations $c = c_0, c_1, c_2, \ldots, c_t = c'$, such that c_{i+1} is the result of a best-response move of some player in c_i for $i = 0, 1, \ldots, t-1$ and $\vec{u}(c) < \vec{u}(c')$ in lexicographic order.

We now prove this claim. Since the starting configuration $c_0 = c$ is not a Nash equilibrium, there exists a player i that can make a best response move to position p, resulting in configuration c_1. We have $ut_i(c_0) < ut_i(c_1)$. Suppose that the nearest neighbor of player i in configuration c_1 is player j_1 and her position is p_{j_1}.

First we consider any player t such that $t \neq j_1$, and suppose her position is p_t in both c_0 and c_1. We show that if $ut_t(c_1) < ut_t(c_0)$, then $ut_t(c_1) \geq ut_i(c_1)$. If this is not true, since only player i moves in configuration c_0 resulting in c_1, the decrease of player t's utility must be caused by the change in distance between player i and player t, i.e., $\Delta(p, p_t) \leq ut_t(c_1)$. Then we have $\Delta(p, p_t) \leq ut_t(c_1) < ut_i(c_1)$. Since player j_1 is the nearest neighbor of i in c_1, we have $\Delta(p, p_{j_1}) \leq \Delta(p, p_t) < ut_i(c_1)$. However, since the utility of player i is her distance to the second nearest neighbor, we have $ut_i(c_1) \leq \max(\Delta(p, p_t), \Delta(p, p_{j_1}))$, a contradiction.

Thus, we know from above that for any player $t \neq j_1$, her utility either does not decrease, or decreases but is still at least $ut_i(c_1)$. Now consider player j_1. If $ut_{j_1}(c_1) \geq ut_i(c_1)$, then it is easy to verify that $\vec{u}(c_0) < \vec{u}(c_1)$. In this case, let $c' = c_1$ and we are done.

Suppose now that $ut_{j_1}(c_1) < ut_i(c_1)$. We select player j_1 for the next best response move. Suppose player j_1 has a best response position q and the configuration changes to c_2. Note that player j_1 could move to position p and overlap with player i, in which case her new utility is $ut_{j_1}(c_2) = ut_i(c_1)$. Thus we know that $ut_{j_1}(c_2) \geq ut_i(c_1)$. If $ut_{j_1}(c_2) = ut_i(c_1)$, we can select q to be p, i.e., overlapping j_1 with i to achieve the best utility. In this case, it is again easy to verify that $\vec{u}(c_0) < \vec{u}(c_2)$, and we let $c' = c_2$ and we are done. If $ut_{j_1}(c_2) > ut_i(c_1)$, suppose j_2 is the nearest neighbor of j_1 in configuration c_2. We can repeat the above argument on j_2 in configuration c_2, and so on.

The above iteration either stops after a finite number of rounds, in which case we find a sequence of configurations $c = c_0, c_1, c_2, \ldots, c_t = c'$ such that c_{i+1} is the result of a best response move of some player in c_i for $i = 0, 1, \ldots, t-1$, and $\vec{u}(c) < \vec{u}(c')$ in lexicographic order, or it continues infinitely often, in which case we find an infinite sequence of configurations $c_0, c_1, c_2, \ldots$ as well as an infinite sequence of players $i, j_1, j_2, \ldots$ such that $ut_i(c_1) < ut_{j_1}(c_2) < ut_{j_2}(c_3) < \ldots$. The latter cannot be true, because the space is finite and bounded, the utility (or the distance) sequence cannot increase forever. Thus, our claim holds.

We may call the best-response sequence found in the above claim an epoch. We can apply the above claim to concatenate new epochs such that at the end of each epoch the vector $\vec{u}$ strictly increases in lexicographic order. Since the space Ω is finite, the vector $\vec{u}$ has an upper bound. Therefore, after a finite number of epochs, we must be able to find a Nash equilibrium, and all these epochs concatenated together form a best-response sequence that leads to the Nash equilibrium. This is clearly true when starting from any initial configuration. □

4.2 Monotonically-increasing Games

For monotonically-increasing games, we provide the following general condition that guarantees the existence of a Nash equilibrium. We say that a pair of points $u, v \in \Omega$ is a pair of *polar points* if for any point $w \in \Omega$, the inequality $\Delta(u, w) + \Delta(w, v) \leq \Delta(u, v)$ holds. We refer to spaces with a pair of polar points as *polarized spaces*, which include n-dimensional sphere (one-dimensional circular space being a special case), n-dimensional grid with L_1 norm as its distance function, etc.

Theorem 10 *If Ω has a pair of polar points, then any monotonically-increasing isolation game on Ω has a Nash equilibrium.*

Proof: Consider the k-player monotonically-increasing game with weight vector $\vec{w} = (w_1, \ldots, w_{k-1})$ on Ω, where $w_1 \leq w_2 \leq \ldots \leq w_{k-1}$. Let (u, v) be a pair of polar points of Ω. Construct the following configuration $c = (p_1, \ldots, p_k)$ such that $p_1 = \ldots = p_{\lceil \frac{k}{2} \rceil} = u$ and $p_{\lceil \frac{k}{2} \rceil + 1} = \ldots = p_k = v$. We show below that c is a Nash equilibrium.

First we prove that for any player j at u, she has no better response in c. By definition, her utility in configuration c is

$$ut_j(c) = \sum_{i=\lceil \frac{k}{2} \rceil}^{k-1} w_i \Delta(u, v)$$

If player j moves to position w with $\Delta(u, w) \leq \Delta(w, v)$ to obtain a new configuration c', then her new utility is

$$\begin{aligned}
ut_j(c') &= \sum_{i=1}^{\lceil \frac{k}{2} \rceil - 1} w_i \Delta(u, w) + \sum_{i=\lceil \frac{k}{2} \rceil}^{k-1} w_i \Delta(w, v) \\
&\leq \Delta(u, w) \sum_{i=\lceil \frac{k}{2} \rceil}^{k-1} w_i + \Delta(w, v) \sum_{i=\lceil \frac{k}{2} \rceil}^{k-1} w_i \\
&\qquad \text{(since the weight vector is monotonically increasing)} \\
&\leq \Delta(u, v) \sum_{i=\lceil \frac{k}{2} \rceil}^{k-1} w_i = ut_j(c) \quad \text{(since } (u, v) \text{ is a pair of polar points)}
\end{aligned}$$

If player j moves to w with $\Delta(u,w) > \Delta(w,v)$ to obtain a new configuration c'', then her new utility is

$$
\begin{aligned}
ut_j(c'') &= \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} w_i \Delta(w,v) + \sum_{i=\lfloor \frac{k}{2} \rfloor +1}^{k-1} w_i \Delta(u,w) \\
&\leq \Delta(w,v) \sum_{i=\lceil \frac{k}{2} \rceil}^{k-1} w_i + \Delta(u,w) \sum_{i=\lceil \frac{k}{2} \rceil}^{k-1} w_i \\
&\leq \Delta(u,v) \sum_{i=\lceil \frac{k}{2} \rceil}^{k-1} w_i = ut_j(c)
\end{aligned}
$$

Therefore, in either case, the utility of j does not increase after her move, and thus j has no better response in c.

In a similar manner, we can show that players at position v do not have better responses either. Therefore, configuration c is a Nash equilibrium. □

4.3 Monotonically-decreasing Games

Monotonically-decreasing games are more difficult to analyze than the previous variants of isolation games, and general results are not yet available. In this section, we first present a positive result for monotonically-decreasing games in a continuous one-dimensional circular space. We then present some hardness result for a simple type of weight vectors in general symmetric spaces.

The following theorem is a general result with monotonically-decreasing games as its special cases.

Theorem 11 *In a continuous one-dimensional circular space Ω, the isolation game on Ω with weight vector $\vec{w} = (w_1, w_2, \ldots, w_{k-1})$ always has a Nash equilibrium if $\sum_{t=1}^{k-1}(-1)^t w_t \leq 0$.*

Proof: Assume without loss of generality that the length of the full one-dimensional circle is k. We consider the configuration $c = (p_1, p_2, \ldots, p_k)$ where $\Delta(p_k, p_1) = \Delta(p_i, p_{i+1}) = 1$ for $1 \leq i \leq k-1$. We show below that c is a Nash equilibrium for the weight vector satisfying the condition in the theorem statement.

In configuration c, the distance vector of player i is $\vec{d_i} = (1,1,2,2,3,3,\ldots)$, and her utility is $\vec{w_i} \cdot \vec{d_i}$. Suppose that player i moves from p_i to $p_i{}'$ and $\Delta(p_i, p_i') = j + \epsilon$ where $j = 0, 1, \ldots, \lfloor (k-1)/2 \rfloor$, and $0 \leq \epsilon < 1$. Note that there are in general two points on the circle having the same distance to p_i, and since they are symmetric, the analysis of these two cases is the same. Let the new distance vector of player i after the move be $\vec{d}'_i$.

Case 1. $0 \le \epsilon \le 1/2$. In this case, we have

$$\vec{d}'_i = (\underbrace{\epsilon, 1-\epsilon, 1+\epsilon, 2-\epsilon, \ldots, j-1+\epsilon, j-\epsilon}_{2j}, \underbrace{j+1-\epsilon, j+1+\epsilon, j+2-\epsilon, \ldots}_{k-1-2j}).$$

And the difference of $\vec{d}_i$ and $\vec{d}'_i$ is

$$\vec{d}'_i - \vec{d}_i = (\underbrace{\epsilon-1, -\epsilon, \epsilon-1, -\epsilon, \ldots, \epsilon-1, -\epsilon}_{2j}, \underbrace{-\epsilon, \epsilon, -\epsilon, \epsilon, \ldots}_{k-1-2j}).$$

Therefore, we have

$$\begin{aligned}\vec{w} \cdot (\vec{d}'_i - \vec{d}_i) &\le -\epsilon \sum_{t=1}^{2j} w_t + \epsilon \sum_{t=2j+1}^{k-1} (-1)^t w_t \\ &\qquad \text{(since } \epsilon - 1 \le -\epsilon \text{ when } 0 \le \epsilon \le 1/2) \\ &\le \epsilon \sum_{t=1}^{k-1} (-1)^t w_t \le 0\end{aligned}$$

This means that in this case the move of player i does not improve her utility.

Case 2. $1/2 < \epsilon < 1$. Note that in this case, when k is odd, j is at most $(k-3)/2$, since, the largest distance to p_i is $(k-1)/2 + 1/2$, implying $\epsilon \le 1/2$ when $j = (k-1)/2$. Thus $k - 2j - 2 \ge 0$. In this case, we have

$$\vec{d}'_i = (\underbrace{1-\epsilon, \epsilon, 2-\epsilon, 1+\epsilon, \ldots, j-1+\epsilon, j+1-\epsilon}_{2j+1}, \underbrace{j+2-\epsilon, j+1+\epsilon, j+3-\epsilon, \ldots}_{k-2j-2}).$$

Furthermore, we have

$$\vec{d}'_i - \vec{d}_i = (\underbrace{-\epsilon, \epsilon-1, -\epsilon, \epsilon-1, \ldots, \epsilon-1, -\epsilon}_{2j+1}, \underbrace{1-\epsilon, \epsilon-1, 1-\epsilon, \epsilon-1, \ldots}_{k-2j-2})$$

Therefore, we have

$$\begin{aligned}\vec{w} \cdot (\vec{d}'_i - \vec{d}_i) &\le -(1-\epsilon) \sum_{t=1}^{2j+1} w_t + (1-\epsilon) \sum_{t=2j+2}^{k-1} (-1)^t w_t \\ &\qquad \text{(since } -\epsilon < \epsilon - 1 \text{ when } 1/2 < \epsilon < 1) \\ &\le (1-\epsilon) \sum_{t=1}^{k-1} (-1)^t w_t \le 0\end{aligned}$$

Thus the move of player i does not improve her utility either in this case.

Combining the two cases above, we conclude that configuration c is a Nash equilibrium. □

A monotonically-decreasing isolation game with weight vector $\vec{w} = (w_1, w_2, \ldots, w_{k-1})$ automatically satisfies the condition $\sum_{t=1}^{k-1}(-1)^t w_t \leq 0$. Hence we have the following corollary.

Corollary 12 *In a continuous one-dimensional circular space Ω, any monotonically-decreasing isolation game on Ω has a Nash equilibrium.*

We now consider a simple class of monotonically-decreasing games with weight vector $\vec{w} = (1, 1, 0, \ldots, 0)$ and characterize the Nash equilibria of the isolation game in a continuous one-dimensional circular space Ω. Although the game has a Nash equilibrium in a continuous one-dimensional circular space according to the above corollary, it is not potential, as shown by the following lemma.

Lemma 13 *Consider $\Omega = \{A, B, C, D, E, F\}$ that contains six points in a one-dimensional circular space with $\Delta(A, B) = 13$, $\Delta(B, C) = 5$, $\Delta(C, D) = 10$, $\Delta(D, E) = 10$, $\Delta(E, F) = 11$, and $\Delta(F, A) = 8$. The five-player monotonically-decreasing game on Ω with weight vector $\vec{w} = (1, 1, 0, 0)$ is not best-response potential (so not better-response potential either). This implies that the game on a continuous one-dimensional circular space is not better-response potential.*

Proof: Suppose that these five players stand at A, B, C, D, and E

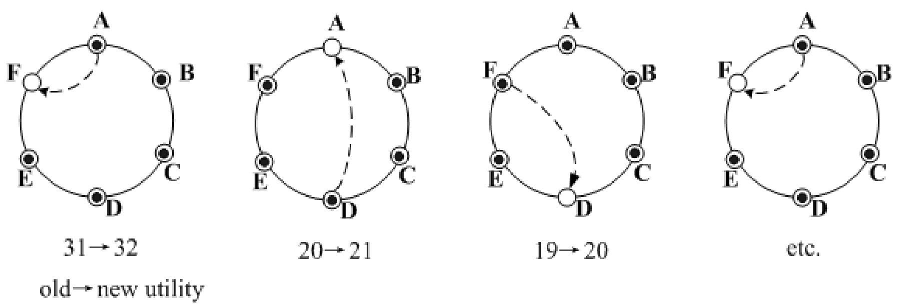

Fig. 2. An example of a best-response sequence that loops forever for a five-player isolation game with weight vector $(1, 1, 0, 0)$ on a one dimensional circular space with six points.

respectively in the initial configuration. Their best response dynamics can iterate forever as shown in Figure 2. Therefore, this special isolation game is not best-response potential on Ω. □

Even though the game is not potential, there exists a best-response sequence that converges to a Nash equilibrium starting from any configuration. This is similar to Theorem 8 for single-selection games, except that here the sequences are best-response ones, which are stronger than better-response ones.

Theorem 14 *For the monotonically-decreasing isolation game with weight vector $\vec{w} = (1, 1, 0, \ldots, 0)$ in a continuous one-dimensional circular space Ω, from any configuration c, there exists a sequence of best-response moves in which each player makes at most one best-response move and the sequence ends in a Nash equilibrium.*

Proof: Without loss of generality, we represent Ω as real points from 0 to 1 with 1 wrapped around to be the same as 0, and thus the position q of any player is a real value with $0 \leq q < 1$. We define clockwise distance α in Ω to be $\alpha(x, y) = y - x$ if $x \leq y$ and $\alpha(x, y) = 1 - x + y$ if $x > y$. Note that $\Delta(x, y) = \min(\alpha(x, y), \alpha(y, x))$, so the two distances are different. An arc from x to y, denoted as $\widehat{a}(x, y)$, is the portion of Ω from x to y in the clockwise direction, i.e., $\widehat{a}(x, y) = \{z \in \Omega \mid \alpha(x, z) \leq \alpha(x, y)\}$.

Given a configuration $c = (p_1, p_2, \ldots, p_k)$, we sort the positions of all k players in increasing order, breaking ties with players' identifiers, giving us a vector of positions $(q_1, q_2, \ldots, q_k)$ with $q_1 \leq q_2 \leq \ldots \leq q_k$, and a corresponding vector of players $(a_1, a_2, \ldots, a_k)$. For $i \in \{1, 2, \ldots, k\}$, define $i+ = i+1$, and $i- = i-1$, except that $k+ = 1$ and $1- = k$. For every q_i, we denote $\mathrm{succ}_c(q_i) = q_{i+}$ and $\mathrm{pred}_c(q_i) = q_{i-}$. We say that each player a_i is adjacent to two players a_{i-} and a_{i+}.

After the above preparation, we are now ready to proceed with the proof. Suppose that player i located at position p_i in configuration c has a best response p'_i, giving a new configuration c'. Let $p_a = \mathrm{pred}_c(p_i), p_b = \mathrm{succ}_c(p_i), p_c = \mathrm{pred}_{c'}(p'_i)$, and $p_d = \mathrm{succ}_{c'}(p'_i)$. Let p''_i be the middle point of the arc $\widehat{a}(p_c, p_d)$, i.e., $\alpha(p_c, p''_i) = \alpha(p_c, p_d)/2$. We claim that p''_i must also be a best response of player i in c, and $\alpha(p_a, p_b) \leq \alpha(p_c, p_d)$. This is because, in c', there are at least one player at p_c and one player at p_d, so $ut_i(c') \leq \Delta(p_c, p'_i) + \Delta(p'_i, p_d) \leq \alpha(p_c, p'_i) + \alpha(p'_i, p_d) = \alpha(p_c, p_d)$. On the other hand, if player i moves to p''_i to give configuration c'', we have $ut_i(c'') = \Delta(p_c, p''_i) + \Delta(p''_i, p_d) = \alpha(p_c, p''_i) + \alpha(p''_i, p_d) = \alpha(p_c, p_d)$. The first equality holds because with player i at the middle point in arc $\widehat{a}(p_c, p_d)$, the two players on p_c and p_d must be the two nearest players to i in c'', while the second equality holds because $\alpha(p_c, p''_i) = \alpha(p''_i, p_d) \leq 1/2$, so they are the same as the corresponding Δ distances. Therefore, we have $ut_i(c'') \geq ut_i(c')$. Since p'_i is already a best response of i in c, p''_i must also be a best response of i in c. Finally, if $\alpha(p_a, p_b) > \alpha(p_c, p_d)$, i would be able to move to the middle point of $\widehat{a}(p_a, p_b)$ to gain a utility $\alpha(p_a, p_b) > \alpha(p_c, p_d) \geq ut_i(c')$, contradicting the fact that c' is the result of i's best response move in c. Thus $\alpha(p_a, p_b) \leq \alpha(p_c, p_d)$.

With the above claim, we only need to focus on a best response that is at the middle point of two adjacent players. Henceforth, we refer to these as *middle-point* best responses. We denote $maxarc(c)$ as the length of the maximum arc of two adjacent players in configuration c, that is, for $c = (p_1, p_2, \ldots, p_k)$, $maxarc(c) = \max\{\alpha(p_i, \text{succ}_c(p_i)) \mid i = 1, 2, \ldots, k\}$. Next we claim that if configuration c' is the result of a middle-point best response move of some player in configuration c, then $maxarc(c') \leq maxarc(c)$.

Suppose, for a contradiction, that $maxarc(c') > maxarc(c)$. Let player i be the one that makes a best response move in c. Let p_i be the position of i in c, and let $p_a = \text{pred}_c(p_i)$ and $p_b = \text{succ}(p_i)$. Let p'_i be the position of i in c'. There are two cases. In the first case, p'_i is not in the arc $\widehat{a}(p_a, p_b)$. In this case, all arcs of two adjacent players in c remain the same or decrease in length, except $\widehat{a}(p_a, p_b)$, so by $maxarc(c') > maxarc(c)$ we have $\alpha(p_a, p_b) = maxarc(c')$, which is strictly greater than all other arcs in c. However, this would imply that player i's middle-point best response has to be the middle point of $\alpha(p_a, p_b)$, which means p'_i is still in the arc $\widehat{a}(p_a, p_b)$, a contradiction. In the second case, we consider that p'_i is still in the arc $\widehat{a}(p_a, p_b)$. The only arc length changes are from $\alpha(p_a, p_i)$ and $\alpha(p_i, p_b)$ to $\alpha(p_a, p'_i)$ and $\alpha(p'_i, p_b)$. However, we know that p'_i is the middle point of arc $\widehat{a}(p_a, p_b)$, so $\max(\alpha(p_a, p_i), \alpha(p_i, p_b)) \geq \alpha(p_a, p'_i) = \alpha(p'_i, p_b)$. Thus $maxarc(c') \leq maxarc(c)$. Therefore, the claim that $maxarc(c') \leq maxarc(c)$ holds.

We now consider a player i who takes a middle-point best response move in a configuration c, resulting in c'. We claim that for all subsequent middle-point best response sequence starting from c' resulting in c'', both of the two adjacent arcs of player i in c'' have lengths at least $maxarc(c'')/2$. We prove this claim by an induction on the number k of moves in the middle-point best response sequence starting in c'. In the base case $k = 0$, which means we consider c'. Since i just made a middle-point best response move, her two adjacent arcs in c' are of equal length, and the combined length must be $maxarc(c)$. From the previous claim, $maxarc(c) \geq maxarc(c')$, so each adjacent arc of i in c' is at least $maxarc(c')/2$. Hence the base case is true. We now show that the case of $k + 1$ is true if the case of k is true. If the $(k + 1)^{th}$ move is again by player i, then she moves to the middle point of the longest arc, and the case is the same as the base case. Let us now consider that a player $j \neq i$ makes the $(k + 1)^{th}$ move. Suppose the configuration after k middle-point best responses is c_k, and the configuration after one more middle-point best response is c_{k+1}. Let p_i be the position of i in both c_k and c_{k+1}. Let $p_a = \text{pred}_{c_k}(p_i)$ and $p_c = \text{pred}_{c_{k+1}}(p_i)$. If $\alpha(p_a, p_i) \leq \alpha(p_c, p_i)$, then by induction hypothesis $\alpha(p_c, p_i) \geq \alpha(p_a, p_i) \geq maxarc(c_k)/2$. By the previous claim we showed, $maxarc(c_k) \geq maxarc(c_{k+1})$, so we have $\alpha(p_c, p_i) \geq maxarc(c_{k+1})/2$. If $\alpha(p_a, p_i) > \alpha(p_c, p_i)$, it must be that player j moves to p_c as her best response in c_k. Then p_c is the middle point of $\widehat{a}(p_a, p_i)$, and because p_c is j's best response in c_k, $\alpha(p_c, p_i) = maxarc(c_k)/2 \geq maxarc(c_{k+1})/2$. Therefore, in c_{k+1} player i's adjacent arc $\widehat{a}(p_c, p_i)$ must be at least $maxarc(c_{k+1})/2$. The other adjacent arc of i can be argued symmetrically. Hence the claim is true.

We are now ready to construct the best response sequence leading to a Nash equilibrium from any configuration c. Initially let $S = \{1, 2, \dots, k\}$ be the set of all players, and let $c_0 = c$. In each round $i = 1, 2, \dots$, find a player $j_i \in S$ that can make a best response move in c_{i-1}. If no such player exists, then let $c_{i-1} = c'$ and we find the best response sequence $c_0, c_1, \dots, c_{i-1} = c'$. We will show that c' is a Nash equilibrium. If such a player exists, remove j_i from S, let j_i take a middle-point best response resulting in a configuration c_i, and then we go to round $i+1$. Suppose that the sequence ends at $c_t = c'$. We have $t \leq k$ since each player makes at most one move in the sequence.

We now show that c' is a Nash equilibrium. Suppose, for a contradiction, that this is not true, which means there exists a player l who could make a best response move in c'. Then this player l cannot be in S, since otherwise we will continue the iteration. Therefore player l makes one move in the sequence, and suppose that she ends at position p_l. Let $p_a = \mathrm{pred}_{c'}(p_l)$ and $p_b = \mathrm{succ}_{c'}(p_l)$. From the previous claim, we know that $\alpha(p_a, p_l) \geq maxarc(c')/2$ and $\alpha(p_l, p_b) \geq maxarc(c')/2$. This means that $\alpha(p_a, p_b) \geq maxarc(c')$, and if l has a best response in c', she always has a middle-point best response by moving to the middle point p'_l of arc $\widehat{a}(p_a, p_b)$. Since l's utility strictly increases, we know that $p_l \neq p'_l$, and in c' there must be another player j closer to l than one of the players at p_a and p_b. Without loss of generality, suppose that j is positioned at p_j in c' such that $p_j = \mathrm{pred}_{c'}(p_a)$ and $\alpha(p_j, p_l) < \alpha(p_l, p_b)$. Since $\alpha(p_j, p_l) = \alpha(p_j, p_a) + \alpha(p_a, p_l)$, $\alpha(p_a, p_l) \geq maxarc(c')/2$, and $\alpha(p_l, p_b) \leq maxarc(c')$, we have $\alpha(p_j, p_a) < maxarc(c')/2$. Suppose that player s is positioned at p_a in c'. From $\alpha(p_j, p_a) < maxarc(c')/2$, we know that player s does not make a move in the middle-point best response sequence from c_0 to $c_t = c'$, because otherwise by the previous claim we must have $\alpha(p_j, p_a) \geq maxarc(c')/2$. So s is still in S when we conclude the sequence at $c_t = c'$. However, in c', we have $ut_s(c') \leq \alpha(p_j, p_a) + \alpha(p_a, p_l) = \alpha(p_j, p_l) < \alpha(p_l, p_b)$, so player s may move to the middle of $\widehat{a}(p_l, p_b)$ and obtain a better utility $\alpha(p_l, p_b)$. This means that we can still find a player s in S that has a better (and thus best) response in c', contradicting the terminating condition of the sequence ending at c'. Therefore, c' must be a Nash equilibrium. It is clear that in the sequence every player makes at most one move. This concludes the proof of the theorem. □

If we extend from the one-dimensional circular space to a general symmetric space, there may be no Nash equilibrium for isolation games with weight vector $\vec{w} = (1, 1, 0, \dots, 0)$ at all, as shown in the following lemma.

Lemma 15 *There is no Nash equilibrium for the four-player isolation game with weight vector $\vec{w} = (1, 1, 0)$ in the space with five points $\{A, B, C, D, E\}$ and the following distance matrix, where $N > 21$ (note that this distance* *function also satisfies the triangle inequality).*

$$\left(\begin{array}{c|ccccc} \Delta & A & B & C & D & E \\ \hline A & 0 & N-6 & N-11 & N-1 & N-6 \\ B & N-6 & 0 & N-8 & N-10 & N-1 \\ C & N-11 & N-8 & 0 & N-1 & N-6 \\ D & N-1 & N-10 & N-1 & 0 & N-10 \\ E & N-6 & N-1 & N-6 & N-10 & 0 \end{array}\right)$$

Proof: Because the number of players is $k = 4$ and $|\Omega| = 5$, there exists at least one free point not occupied by any player in any configuration c. Now we will consider an arbitrary configuration c and show that it cannot be a Nash equilibrium.

If there exist two players i and j standing at the same point in c, then the utilities of player i and j are at most $N - 1$. However, player i could achieve utility at least $2N - 20$ by moving to a free point. Since $N > 21$, we have $2N - 20 > N - 1$. Hence a configuration with two players standing at the same point could not be a Nash equilibrium.

If the four players stand at different points in c, then we consider the remaining free point. There are five cases.

Case 1: the free point is A. The player standing at B has utility $2N - 18$, and she could achieve utility $2N - 17$ if she moves to A.

Case 2: the free point is B. The player standing at C has utility $2N - 17$, and she could achieve utility $2N - 16$ if she moves to B.

Case 3: the free point is C. The player standing at D has utility $2N - 20$, and she could achieve utility $2N - 19$ if she moves to C.

Case 4: the free point is D. The player standing at E has utility $2N - 12$, and she could achieve utility $2N - 11$ if she moves to D.

Case 5: the free point is E. The player standing at A has utility $2N - 17$, and she could achieve utility $2N - 16$ if she moves to E.

Combining the five cases above, a configuration with four players standing at different points is not a Nash equilibrium either. Therefore, there is no Nash equilibrium for the isolation game in this space. □

Using the above lemma as a basis, we further show that it is NP-complete to decide whether an isolation game with weight vector $(1,1,0,\ldots,0)$ on a general symmetric space has a Nash equilibrium. The proof is by reduction from the 3-Dimensional Matching problem.

Theorem 16 *In a finite symmetric space (Ω,Δ), it is NP-complete to decide the existence of a Nash equilibrium for the isolation game with weight vector $\vec{w}=(1,1,0,\ldots,0)$.*

Proof: It is straightforward to verify that the problem is in NP. We prove NP-hardness by reduction from the 3-Dimensional Matching problem.

An instance of the 3-Dimensional Matching problem includes three disjoint sets X, Y and Z (each of size n) and a set $T=\{T_1,\ldots,T_m\}\subseteq X\times Y\times Z$ of ordered triples. The decision problem is to decide whether there are n triples in T so that each element of $X\cup Y\cup Z$ is contained in exactly one of these triples. The construction of an instance of the isolation game from the above 3-Dimensional Matching instance is given below.

The finite symmetric space Ω has $m+5n$ points, divided into a left set $L=\{v_1,v_2,\ldots,v_m\}$ and n right sets $R_1,\ldots,R_n$ such that $R_i=\{u_{i1},u_{i2},\ldots,u_{i5}\}$ for $1\le i\le n$. Select a number $N>46$. For any two different points $v_i,v_j\in L$, $\Delta(v_i,v_j)=N$ if T_i and T_j have no common element, and $\Delta(v_i,v_j)=N/2+1$ otherwise. The distance within each R_i is given by the distance matrix in Lemma 15 with $u_{i1}=A,\ldots,u_{i5}=E$. For any $v\in L$ and $u\in R_i$, $\Delta(u,v)=\Delta(v,u)=N$ and for any $u\in R_i$ and $u'\in R_j$ with $i\neq j$, $\Delta(u,u')=N$. Finally, the isolation game on Ω has $4n$ players.

We claim that there exists a Nash equilibrium for this isolation game on Ω if and only if there are n disjoint triples in the 3-Dimensional Matching instance.

First, suppose that there is a solution to the 3-Dimensional Matching instance. Without loss of generality, assume that the n disjoint triples are $T_1,\ldots,T_n$. Then we claim that configuration $c=(v_1,\ldots,v_n,\ u_{11},\ldots,u_{n1},\ u_{12},\ldots,u_{n2},\ u_{15},\ldots,u_{n5})$ is a Nash equilibrium. In this configuration, every player in L has utility $2N$. Since it is the maximum possible utility, no player in L would want to change. Consider a player in R_i. Her utility is at least $2N-12$. If she moves to some point v_j in L, then because T_j at least intersects with one of $T_1,\ldots,T_n$, the maximum utility she may obtain is $3N/2<2N-12$ since $N>46$. So she will not move to L. If she moves to some other point in some R_j, using the distance matrix in Lemma 15, it is easy to verify that she will not gain any better utility either. Hence c is a Nash equilibrium.

Conversely, suppose that there is a Nash equilibrium in the isolation game. Because $k=4n$, $|\Omega|=5n+m$ and $|L|=m$, there exist at least n free points not occupied by any player in the right sets in any configuration.

Consider the number of players positioned in the right set R_i. If there is a Nash equilibrium c, then the number of players positioned in R_i cannot be 4 because of Lemma 15. We show that R_i has at most three players by arguing that the following two cases are impossible.

Case 1: There are more than five players positioned in R_i. Then there exist two players standing at the same point, and their utilities are at most $N-1$. Hence each of them would like to move to a free point to obtain a better utility of at least $2(N/2+1)$.

Case 2: There are five players positioned in R_i. If two or more players choose the same point, then it is the same as Case 1. If the five players are positioned in different points in R_i, then there must exist another right set R_j in which at most three players are positioned. Hence there are at least two free points in R_j. Select a free point $u_{js} \neq u_{j4}$. The player at u_{i4} could increase her utility value from $2N-20$ to at least $2N-19$ by moving to u_{js}.

Therefore, we know that every right set R_i contains at most three players in c, which means that there are at least n players positioned in L. Without loss of generality, we assume that there are n players who occupy points $v_1, \ldots, v_n$ (which may have duplicates). We claim that triples $T_1, \ldots, T_n$ form a solution to the 3-Dimensional Matching problem. Suppose, for a contradiction, that this is not true, which means that there exist T_i and T_j among these n triples that have some common element. By our construction, we have $\Delta(v_i, v_j) = 0$ or $N/2+1$. Then the utility of the player positioned at v_i or v_j is at most $3N/2+1$. In this case, both of them would want to move to a free point in a right set, since that will give them a utility of at least $2(N-11)$. When $N > 46$, we have $2(N-11) > 3N/2+1$. This contradicts the assumption that c is a Nash equilibrium. Therefore, we find a solution for the 3-Dimensional Matching problem given a Nash equilibrium c of the isolation game. □

5 Computation of Best Responses in High Dimensional Spaces

We now turn to the problem of computing the best response of a player in a configuration. A brute-force search on all points in the space can be done in $O(k \log k \sqrt{D})$, where D is the size of the distance matrix. This is fine if the distance matrix is explicitly given as input. However, it could become exponential if the space has a compact representation, such as an n-dimensional grid with the L_1 norm as the distance function. In this section, we present results in an n-dimensional hypercube $\{0,1\}^n$ with the Hamming distance, a special case of n-dimensional grids with the L_1 norm.

Theorem 17 *In a $2n$-dimensional hypercube $\{0,1\}^{2n}$, it is NP-complete to* *decide whether a player could move to a point so that her utility is at least $n-1$ in the k-player nearest-neighbor isolation game with k bounded by $poly(n)$.*

Proof: Computing the distance between a pair of vertices in $\{0,1\}^{2n}$ can be done in $O(n)$ time. Since k is bounded by $poly(n)$, verifying whether a player is at least $n-1$ away from all her neighbors can be done in polynomial time. Hence this problem is in NP.

We prove the NP-hardness by reduction from the NAE-3-SAT problem. NAE-3-SAT stands for "Not All Equal" 3SAT. An instance of NAE-3-SAT (which is a CNF formula) has m clauses $C_1, C_2, \ldots, C_m$, where each clause consists of three literals and there are n variables $x_1, \ldots, x_n$ in total. The decision problem is to decide whether there is a feasible assignment such that the CNF formula is satisfied and no clause of the formula has all literals evaluated to the same value. The construction of a configuration c of the nearest-neighbor isolation game from the above NAE-3-SAT instance is given below.

The space Ω is the $2n$-dimensional hypercube $\{0,1\}^{2n}$, so that each point in Ω is determined by $2n$ bits. The nearest-neighbor isolation game has $4m(n-2)+1$ players, divided into $m+1$ sets $S_1, \ldots, S_m, S_{m+1}$. The set S_{m+1} consists of only one player who is going to compute a better response, so her initial position does not matter. The set S_i ($i \neq m+1$) consists of $4(n-2)$ players. Clause C_i corresponds to S_i, with $4(n-2)$ players positioned at $p_{i,1}, \ldots, p_{i,4(n-2)}$ in Ω. Since each point $p_{i,j}$ is determined by $2n$ bits, we partition these $2n$ bits into n consecutive pairs such that the k^{th} pair stands for the $(2k-1)^{th}$ and $2k^{th}$ bits of $p_{i,j}$. We now show how to determine the $2n$ bits of $p_{i,j}$ given clause C_i.

Case 1: $1 \leq j \leq 2(n-2)$. We first fix the three pairs of $p_{i,j}$ corresponding to the three literals of C_i, in the following way:

- If C_i contains the literal x_k, then the k^{th} pair of $p_{i,j}$ is 11.
- If C_i contains the literal $\bar{x_k}$, then the k^{th} pair of $p_{i,j}$ is 00.

We then fix the remaining $n-3$ pairs. In the following description, the k^{th} remaining pair means the k^{th} pair after excluding the three pairs already determined above.

- If $j = 1$, then all the $n-3$ remaining pairs are 01.
- If $j = 2$, then all the $n-3$ remaining pairs are 10, i.e., flipping the bits of all $n-3$ remaining pairs in $p_{i,1}$.
- If $j = 2k+1$ for $1 \leq k \leq n-3$, the k^{th} remaining pair is 10 and all other $n-4$ remaining pairs are 01.
- If $j = 2k+2$ for $1 \leq k \leq n-3$, flipping the bits of all $n-3$ remaining pairs in $p_{i,j-1}$.

Case 2: $2(n-2)+1 \leq j \leq 4(n-2)$. In this case, $p_{i,j}$ is obtained by flipping all bits of $p_{i,j-2(n-2)}$.

Here we have fixed the $2n$ bits for all $p_{i,j}$ such that $1 \leq i \leq m$ and $1 \leq j \leq 4(n-2)$. We claim that there exists a p_0 in Ω such that $\min_{i,j} \Delta(p_0, p_{i,j}) \geq n-1$ if and only if there is a feasible assignment for the NAE-3-SAT instance.

First, suppose that there is a feasible assignment for the NAE-3-SAT instance. Then we fix the n pairs of p_0 in the following way, and claim that $\min_{i,j} \Delta(p_0, p_{i,j}) \geq n-1$.

- If x_i is 0 in this feasible assignment, then let the i^{th} pair of p_0 be 00.
- If x_i is 1 in this feasible assignment, then let the i^{th} pair of p_0 be 11.

Consider an arbitrary point $p_{i,j}$ whose corresponding clause C_i contains three variable x_a, x_b and x_c. Therefore, the a^{th}, b^{th} and c^{th} pairs of $p_{i,j}$ are either 00 or 11, and the remaining pairs are either 01 or 10. Since p_0 is fixed according to a solution of NAE-3-SAT, p_0 and $p_{i,j}$ have at least one and at most two common pairs among the a^{th}, b^{th}, and c^{th} pairs. Hence these three pairs could contribute two to $\Delta(p_0, p_{i,j})$. For each of the remaining $n-3$ pairs, p_0 is either 11 or 00, and $p_{i,j}$ is either 01 or 10. Thus the remaining $n-3$ pairs will contribute $n-3$ to $\Delta(p_0, p_{i,j})$. Therefore, $\Delta(p_0, p_{i,j}) \geq 2 + (n-3) = n-1$ for any $1 \leq i \leq m$ and $1 \leq j \leq 4(n-2)$.

Conversely, suppose that there is a point p_0 such that $\Delta(p_0, p_{i,j}) \geq n-1$ for any $1 \leq i \leq m$ and $1 \leq j \leq 4(n-2)$. We first show that each pair of p_0 is either 00 or 11. Suppose, for a contradiction, that at least one pair in p_0 is neither 00 nor 11.

Case 1: Only one pair in p_0 is neither 00 nor 11. Let this pair be the k^{th} pair, and its value is 01 (the argument for 10 is symmetric). There must exist a clause C_i that does not contain variable x_k, otherwise all clauses contain variable x_k and the instance is solvable in P. Suppose that C_i contains variables x_a, x_b, and x_c. We look at the remaining $n-3$ pairs of p_0 excluding the a^{th}, b^{th}, and c^{th} pairs. Consider position $p_{i,1}$, whose $n-3$ remaining pairs are all 01. The number of bits that are different between p_0 and $p_{i,1}$ in these $n-3$ remaining pairs is $n-4$, since the k^{th} pairs are the same, while each other pairs differ in exactly one bit. The number of bits that are different between p_0 and $p_{i,1}$ in the a^{th}, b^{th}, and c^{th} pairs could be 0, 2, 4 or 6, since all these pairs are either 00 or 11. If it is 0 or 2, then $\Delta(p_0, p_{i,1}) \leq n-4+2 < n-1$. If it is 4 or 6, then we instead use $p_{i,2(n-2)+2}$, whose a^{th}, b^{th}, and c^{th} pairs are flipped from $p_{i,1}$ while the remaining $n-3$ pairs are the same as p_{i-1}. In this case, $\Delta(p_0, p_{i,2(n-2)+2}) \leq n-4+2 < n-1$. Therefore, we can always find a position $p_{i,j}$ such that $\Delta(p_0, p_{i,j}) < n-1$, contradicting the choice of p_0.

Case 2: At least two pairs in p_0 are neither 00 nor 11. Let the k^{th} and the s^{th} pairs be such two pairs. There must exist a clause C_i that does not contain variable x_k and x_s, otherwise all clauses contain variable x_k or x_s and the instance is solvable in P. Suppose that C_i contains variables x_a, x_b, and x_c. We look at the remaining $n-3$ pairs of p_0 not containing the a^{th}, b^{th}, and c^{th} pairs. Suppose that these $n-3$ remaining pairs contain t_1 number of 00 or 11 pairs, t_2 number of 01 pairs, and t_3 number of 10 pairs, with $t_1+t_2+t_3 = n-3$. Without loss of generality we assume that $t_2 \geq t_3$ (the argument for $t_2 \leq t_3$ is symmetric). Moreover, since the k^{th} and s^{th} pairs are in the $n-3$ remaining pairs, we have $t_1 \leq n-5$. By the construction of the $4(n-2)$ positions of $p_{i,1}, \ldots, p_{i,4(n-2)}$, we first find a position $p_{i,j}$ with $1 \leq j \leq 2(n-2)$ with the following property: (a) if $t_3 = 0$, then all the $n-3$ remaining pairs of $p_{i,j}$ are 01 (actually $j = 1$ in this case); and (b) if $t_3 \geq 1$ with the ℓ^{th} pair being 10, then all the $n-3$ remaining pairs of $p_{i,j}$ are 01 except that the ℓ^{th} pair is 10. With this selection of $p_{i,j}$, the number of different bits between p_0 and $p_{i,j}$ in the $n-3$ remaining pairs is t_1 when $t_3 = 0$ and $t_1 + 2(t_3 - 1)$ when $t_3 \geq 1$. For the former, we already have $t_1 \leq n-5$, and for the latter, $t_1 + 2(t_3 - 1) \leq t_1 + t_2 + t_3 - 2 = n-5$. So they differ in at most $n-5$ bits among the $n-3$ remaining pairs. For the a^{th}, b^{th}, and c^{th} pairs, if they differ in at most 3 bits, then we have $\Delta(p_0, p_{i,j}) \leq n-5+3 < n-1$. If they differ in at least 4 bits, then let $j' = j+2(n-2)+(-1)^{j+1}$, and it is easy to verify that $p_{i,j'}$ flips all bits from $p_{i,j}$ in the a^{th}, b^{th}, and c^{th} pairs while has the same bits as $p_{i,j}$ in the remaining $n-3$ pairs. Then we have $\Delta(p_0, p_{i,j'}) \leq n-5+3 < n-1$. Thus, we can always find a $p_{i,j}$ whose distance to p_0 is less than $n-1$, a contradiction.

Therefore all pairs of p_0 are either 00 or 11. Then we can construct an assignment according to p_0 and prove that this assignment is a feasible solution for the NAE-3-SAT instance. The assignment is as follows.

- If the i^{th} pair in p_0 is 00, then we assign 0 to x_i.
- If the i^{th} pair in p_0 is 11, then we assign 1 to x_i.

To show that the above assignment is a solution, consider an arbitrary clause C_i. Comparing p_0 with $p_{i,1}$, we know that they differ in exactly $n-3$ bits in the $n-3$ remaining pairs excluding the three pairs corresponding to C_i's variables. For the three pairs corresponding to the three variables, they cannot differ in 0 pair or 3 pairs. If they differ in 0 pair, then $\Delta(p_0, p_{i,1}) = n-3 < n-1$; if they differ in 3 pairs, then taking $p_{i,1+2(n-2)}$, we have $\Delta(p_0, p_{i,1+2(n-2)}) = n-3 < n-1$, both contradicting the choice of p_0. So p_0 and $p_{i,1}$ differ in one or two pairs. This means exactly that the above assignment will make one or two literals true but not all of them or none of them. Since this is true for all clauses, it shows that the assignment is a solution to the NAE-3-SAT problem. □

The above theorem leads to the following hardness result in computing best responses for a general class of isolation games, with the nearest-neighbor game as a special case.

Corollary 18 *It is NP-hard to compute a best response for an isolation game in the space* $\{0,1\}^{2n}$ *with weight vector* $\vec{w} = (\underbrace{*,\ldots,*,1}_{c},0,\ldots,0)$ *where c is a constant and* $*$ *is either* 0 *or* 1.

Proof: If we want to compute a best response for a player x with weight vector $\vec{w'} = (1,0,\ldots,0)$, we can construct a configuration of the isolation game with weight vector $\vec{w} = (*,\ldots,*,1,0,\ldots,0)$. In the new game, we replace each player except x by c new players on the same point. And we can see that a best response for player x in the new game with weight vector $\vec{w}$ is just a best response in the original game with weight vector $\vec{w'}$. If we can compute a best response for the game with $\vec{w}$ efficiently, then we can compute a best response for the game with $\vec{w'}$ efficiently. However, Theorem 17 implies that it is NP-hard to compute a best response for $\vec{w'}$, so it is also NP-hard to compute a best response for $\vec{w}$. □

Contrasting to the above corollary, if the weight vector has only nonzero entries towards the end of the vector, it is easy to compute the best response, as shown in the following theorem.

Theorem 19 *A best response for a k-player isolation game in the space* $\{0,1\}^n$ *with* $\vec{w} = (0,\ldots,0,\underbrace{1,*,\ldots,*}_{c})$ *can be computed in polynomial time where c is a constant, k is bounded by poly*(n) *and* $*$ *is either* 0 *or* 1.

Proof: Consider a configuration $\kappa = (p_1,\ldots,p_k)$ in $\{0,1\}^n$ with player i positioned at p_i for all $1 \le i \le k$. And here we consider player k positioned at p_k. Let $\vec{w}_c$ be the vector of c entries that are the last c entries of $\vec{w}$. We give the following algorithm that computes a best response p'_k of player k in configuration κ, and we prove that its running time is bounded by *poly*(n).

Step 1: Repeat step 2 for all $\binom{k-1}{c}$ possible subsets S of c players among players $\{1,2,\ldots,k-1\}$.

Step 2: Given such a subset S, consider a reduced isolation game with $c+1$ players $S \cup \{k\}$ and weight vector $\vec{w}_c$. Consider the configuration κ_S in which every player $j \in S$ takes position p_j as in κ and player k takes position p_k. Compute the best response p_S of player k in this reduced game and its utility u_S.

Step 3: Return the best response p'_k as the p_S computed in step 2 with the largest utility u_S.

We prove the correctness of the algorithm by contradiction. Let player k's utility at position p'_k be u_1 in the original game. Suppose that p'_k is computed in step 2 in a reduced game with S as the subset and utility u_S. Since adding more players into the reduced game will only increase the c largest distances to player k at p'_k, we know that $u_1 \geq u_S$. Now suppose for a contradiction that there exists another position q that gives a better utility u_q to player k than position p'_k in the original game, i.e. $u_q > u_1$. Let S' be the subset of c players who have the c largest distances to q. Consider the reduced game with S' as the subset in step 2. Since in the original game, player k only cares about the c largest neighbors, in the reduced game k's utility at position q is also u_q. So the best response $p_{S'}$ computed in step 2 has utility $u_{S'} \geq u_q$. Thus, we have the sequence $u_{S'} \geq u_q > u_1 \geq u_S$. However, since p'_k is selected according to the largest utility in step 2, we have $u_S \geq u_{S'}$, a contradiction.

For time complexity, the number of iterations of step 2 is $\binom{k-1}{c}$, which is $poly(n)$ because k is bounded by $poly(n)$ and c is a constant. We now show how to compute each iteration of step 2 in polynomial time, and thus the overall algorithm is polynomial.

Consider a set S of c players. To compute the best response p_S in the reduced game with subset S, we argue that we only need to check a polynomial number of candidate positions. Without loss of generality, let $S = \{1, 2, \dots, c\}$. Construct matrix $M = (m_{ij})_{c \times n}$ where m_{ij} represents the j-th bit of point p_i.

Note that the number of unique column vectors in M is at most 2^c, which is a constant. Suppose that for the i^{th} unique column vector, there are t_i columns in M, corresponding to t_i bits in a candidate position. Given any point p in the hypercube, let n_i^p denotes the number of bits **1** in the t_i bits corresponding to the i^{th} unique column vector in M. We claim that, for any two points p and q, if $n_i^p = n_i^q$ for all i, then p and q give the same utility to player k in the reduced game. This is because the distance function Δ is the Hamming Distance function, which means the distance between two points only depends on the number of different bits between the two points. Therefore, only the number of bits **1** corresponding to each unique column vector affects the distances to the players in S. Hence, given matrix M, we only need to check $t_i + 1$ different bit assignments for each unique column vector. The total number of candidate positions we need to check is then $\prod_{i=1}^{2^c}(t_i + 1) = O(n^{2^c})$. Since c is a constant, this is polynomial to n. For each candidate position, computing the utility at this position certainly takes polynomial time. Therefore, each iteration of step 2 takes only polynomial time to complete. □

6 Final Remarks

The isolation game is very simple by its definition. However, as shown in this paper, the behaviors of its Nash equilibria and best response dynamics are quite rich and complex. This paper presents the first set of results on the isolation game and lays the ground work for the understanding of the impact of the farness measures and the underlying space to some basic game-theoretic questions about the isolation game. We summarize the results of this paper in Table 1. It remains an open question to fully characterize the isolation game. In particular, we would like to understand for what weight vectors the isolation game on simple spaces, such as d-dimensional grids, hypercubes, and torus grid graphs, has potential functions, has Nash equilibria, or has converging best (better) response sequences. What is the impact of distance functions, such as L1-norm or L2-norm to these questions? We would like to know whether it is NP-hard to determine if Nash equilibria exist in these special spaces when the input is the weight vector. What can we say about other continuous spaces such as squares, cubes, balls, and spheres? For example, is there a sequence of better response dynamics that converges to a Nash equilibrium in the isolation game on the sphere with $\vec{w} = (1, 1, 1, 0, \ldots, 0)$? What can we say about approximate Nash equilibria?

More concretely, in Lemma 7 we show an example in which a single-selection game with weight vector $(0, 1, 0, \ldots, 0)$ is not better-response potential in one dimensional circular space. However, we verify that the game is best-response potential. This phenomenon of being best-response potential but not better-response potential is rarely seen in other types of games. Moreover, our experiments lead us to conjecture that all games on the continuous one dimensional circular space with weight vector $(0, 1, 0, \ldots, 0)$ are best-response potential. If the conjecture is true, we will find a large class of games that are best-response potential but not better-response potential (the latter is implied by Lemma 7 for the continuous one dimensional space), an interesting phenomenon not known in other common games.

Another line of research is to understand the connection between the isolation game and the Voronoi game.

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