A renormalisation group analysis of the 4-dimensional continuous-time weakly self-avoiding walk

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Abstract

We prove $|x|^{-2}$ decay of the critical two-point function for the continuous-time weakly self-avoiding walk on $\mathbb{Z}^4$. The walk two-point function is identified as the two-point function of a supersymmetric field theory with quartic self-interaction, and the field theory is then analysed using renormalisation group methods.

This is joint work with David Brydges.

Papers at `http://www.math.ubc.ca/~slade`.

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Self-avoiding walk

Discrete-time model: Let $\mathcal{S}_n(x)$ be the set of $\omega : \{0, 1, \ldots, n\} \to \mathbb{Z}^d$ with: $\omega(0) = 0$, $\omega(n) = x$, $|\omega(i+1) - \omega(i)| = 1$, and $\omega(i) \neq \omega(j)$ for all $i \neq j$. Let $\mathcal{S}_n = \cup_{x \in \mathbb{Z}^d} \mathcal{S}_n(x)$.

Let $c_n(x) = |\mathcal{S}_n(x)|$. Let $c_n = \sum_x c_n(x) = |\mathcal{S}_n|$. Easy: $c_n^{1/n} \to \mu$. Declare all walks in $\mathcal{S}_n$ to be equally likely: each has probability c_n^{-1}.

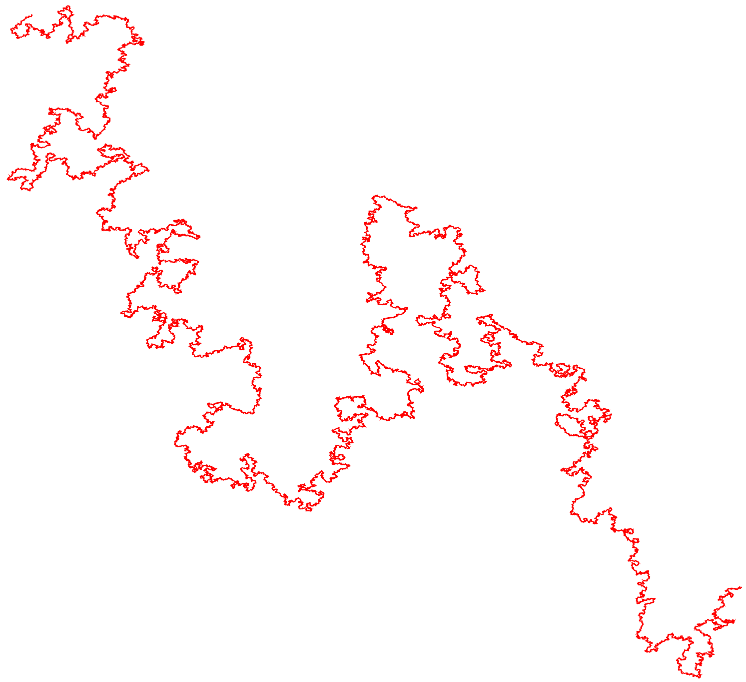
Two-point function: $G_z(x) = \sum_{n=0}^{\infty} c_n(x) z^n$, radius of convergence $z_c = \mu^{-1}$.

Predicted asymptotic behaviour:

$$c_n \sim A\mu^n n^{\gamma - 1}, \quad \mathbb{E}_n|\omega(n)|^2 \sim Dn^{2\nu}, \quad G_{z_c}(x) \sim c|x|^{-(d-2+\eta)},$$

with universal critical exponents γ, ν, η obeying $\gamma = (2 - \eta)\nu$.

A random SAW on $\mathbb{Z}^2$ with 10^6 steps



(Figure by T. Kennedy)

Dimensions $d \geq 4$

Theorem. (Brydges, Spencer (1985); Hara, Slade (1992); Hara (2008)...)
For $d \geq 5$,

$$c_n \sim A\mu^n, \quad \mathbb{E}_n|\omega(n)|^2 \sim Dn, \quad G_{z_c}(x) \sim c|x|^{-(d-2)}, \quad \frac{1}{\sqrt{Dn}}\omega(\lfloor nt \rfloor) \Rightarrow B_t.$$

Prediction is that upper critical dimension is 4, and asymptotic behaviour for $\mathbb{Z}^4$ has log corrections (Brézin, Le Guillou, Zinn-Justin 1973):

$$c_n \sim A\mu^n(\log n)^{1/4}, \quad \mathbb{E}_n|\omega(n)|^2 \sim Dn(\log n)^{1/4}, \quad G_{z_c}(x) \sim c|x|^{-2}.$$

Also, for susceptibility and correlation length, as $z \nearrow z_c$,

$$\chi(z) \sim \frac{A'|\log(1 - z/z_c)|^{1/4}}{1 - z/z_c}, \qquad \xi(z) \sim \frac{D'|\log(1 - z/z_c)|^{1/8}}{(1 - z/z_c)^{1/2}},$$

where

$$\chi(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad \frac{1}{\xi(z)} = -\lim_{n \to \infty} \frac{1}{n} \log G_z(ne_1).$$

Continuous-time weakly self-avoiding walk

This is a modification of the SAW model. We are interested in dimensions $d \geq 4$. Let E_0 denote the expectation for continuous-time nearest-neighbour simple random walk $X(t)$ on $\mathbb{Z}^d$ started from 0 (with $\mathrm{Exp}(1)$ holding times), and let

$$L_{u,T} = \int_0^T \delta_{u,X(s)} ds, \qquad I(0,T) = \sum_{u \in \mathbb{Z}^d} L_{u,T}^2.$$

Then

$$I(0,T) = \int_0^T \int_0^T \delta_{X(s),X(t)} ds\, dt.$$

Let $g \in (0,\infty)$. The two-point function is defined to be

$$G_{g,\nu}(x) = \int_0^\infty E_0 \left(e^{-gI(0,T)} \, \delta_{X(T),x} \right) e^{-\nu T} dT$$

(role of z now played by $e^{-\nu}$). A subadditivity argument shows that the susceptibility $\chi_g(\nu) = \sum_{x \in \mathbb{Z}^d} G_{g,\nu}(x)$ is finite if $\nu > \nu_c(g)$ and is infinite if $\nu < \nu_c(g)$.

Main result

Theorem (Brydges–Slade 2010). Let $d \geq 4$. There exists g_0 such that for $0 < g \leq g_0$,

$$G_{g,\nu_c}(x) = \frac{c}{|x|^{d-2}} + o\left(\frac{1}{|x|^{d-2}}\right).$$

Outlook: The method of proof (RG) has the potential to (but has not yet fully achieved):

- prove logarithmic corrections for susceptibility and correlation length for $d = 4$
- prove same result also with small nearest-neighbour attraction (Bauerschmidt)
- prove same result for a particular spread-out model of discrete-time strictly self-avoiding walk with exponentially decaying step weights
(explicitly the weight of a step is $(1 - a^{-1}\Delta)^{-1}(x, y)$ with $0 < a \ll 1$)

Related results:

- weakly SAW on 4-dimensional hierarchical lattice: Brydges, Evans, Imbrie (1992); Brydges, Imbrie (2003); and with different RG approach Ohno, Hara (2010+). The hierarchical lattice is a replacement of $\mathbb{Z}^4$ by a recursive structure which is well-suited to the RG.
- weakly self-avoiding Lévy walk on $\mathbb{Z}^3$ ($\alpha = \frac{3+\epsilon}{2}$, $d_c = 3 + \epsilon$): Mitter, Scoppola (2008).

Finite-volume approximation

Now we fix $g > 0$ and usually drop it from the notation.

Standard methods (Simon–Lieb inequality) show that

$$G_{\nu_c}(x) = \lim_{\nu \downarrow \nu_c} \lim_{\Lambda \uparrow \mathbb{Z}^d} G_{\Lambda,\nu}(x),$$

where $\Lambda = \mathbb{Z}^d / R\mathbb{Z}$ is a torus approximating $\mathbb{Z}^d$ and

$$G_{\Lambda,\nu}(x) = \int_0^\infty E_0^\Lambda \left(e^{-g I_\Lambda[0,T]} \delta_{X(T),x} \right) e^{-\nu T} dT,$$

with E_0^Λ the expectation for the continuous-time simple random walk on Λ,
and $I_\Lambda[0,T] = \sum_{v \in \Lambda} L_{v,T}^2$.

Thus we can work in finite volume, and slightly subcritical, as long as we maintain sufficient uniformity to take the limits.

Functional integral representation

Let $\varphi : \Lambda \to \mathbb{C}$. Let $\bar{\varphi}_x = u_x - iv_x$ denote the complex conjugate of $\varphi_x = u_x + iv_x$. Let Δ denote the discrete Laplacian on Λ, i.e., $\Delta\varphi_x = \sum_{y:|y-x|=1}(\varphi_y - \varphi_x)$. Let

$$\psi_x = \frac{1}{\sqrt{2\pi i}} d\varphi_x, \quad \bar{\psi}_x = \frac{1}{\sqrt{2\pi i}} d\bar{\varphi}_x,$$

$$\tau_x = \varphi_x\bar{\varphi}_x + \psi_x \wedge \bar{\psi}_x = u_x^2 + v_x^2 + \frac{1}{\pi} du_x \wedge dv_x,$$

$$\tau_{\Delta,x} = \frac{1}{2}\Big(\varphi_x(-\Delta\bar{\varphi})_x + (-\Delta\varphi)_x\bar{\varphi}_x + \psi_x \wedge (-\Delta\bar{\psi})_x + (-\Delta\psi)_x \wedge \bar{\psi}_x\Big),$$

where $\wedge$ is the standard anti-commutative wedge product. Then

$$G_{\Lambda,\nu}(x) = \int_{\mathbb{C}^\Lambda} e^{-\sum_{u\in\Lambda}(\tau_{\Delta,u} + g\tau_u^2 + \nu\tau_u)} \bar{\varphi}_0\varphi_x.$$

RHS is the two-point function of a supersymmetric field theory with boson field $(\varphi, \bar{\varphi})$ and fermion field $(\psi, \bar{\psi})$.

(Parisi, Sourlas 1980; McKane 1980; Luttinger 1983; Le Jan 1987;
Brydges, Evans, Imbrie 1992; Brydges, Imbrie 2003; Brydges, Imbrie, Slade 2009).

Meaning of the integral

The definition of an integral such as

$$G_{\Lambda,\nu}(x) = \int_{\mathbb{C}^\Lambda} e^{-\sum_{u\in\Lambda}(\tau_{\Delta,u}+g\tau_u^2+\nu\tau_u)}\bar{\varphi}_0\varphi_x$$

is as follows:

- expand entire integrand in power series about degree-zero part (*finite* sum), e.g.,

$$e^{\tau_v} = e^{\varphi_x\bar{\varphi}_x+\psi_x\bar{\psi}_x} = e^{\varphi_x\bar{\varphi}_x}\left(1+\psi_x\bar{\psi}_x\right),$$

- keep only terms with one factor $d\varphi_x$ and one $d\bar{\varphi}_x$ for each $x \in \Lambda$,
- write $\varphi_x = u_x + iv_x$, $\bar{\varphi}_x = u_x - iv_x$ and similarly for differentials,
- then use anti-commutativity to rearrange the differentials to $\prod_{x\in\Lambda} du_x dv_x$,
- and finally perform Lebesgue integral over $\mathbb{R}^{2|\Lambda|}$.

Such integrals have nice properties. Let $S(\Lambda) = \sum_{x\in\Lambda}(\tau_{\Delta,x} + m^2\tau_x)$. Then:

$$\int e^{-S(\Lambda)}F(\tau) = F(0), \qquad \int e^{-S(\Lambda)}\bar{\varphi}_0\varphi_x = (-\Delta+m^2)^{-1}(0,x).$$

Now we study the integral and forget about the walks.

Change of variables

The change of variable $\varphi_x \mapsto \sqrt{1+z_0}\varphi_x$, with $z_0 > -1$, gives

$$G_{\Lambda,\nu}(x) = (1+z_0) \int_{\mathbb{C}^\Lambda} e^{-\sum_u \left((1+z_0)\tau_{\Delta,u} + g(1+z_0)^2\tau_u^2 + \nu(1+z_0)\tau_u\right)} \bar{\varphi}_0 \varphi_x.$$

Introducing an *external field* $\sigma \in \mathbb{C}$, let

$$S(\Lambda) = \sum_{u\in\Lambda} \left(\tau_{\Delta,u} + m^2\tau_u\right),$$

$$V_0(\Lambda) = \sum_{u\in\Lambda} \left(g_0\tau_u^2 + \nu_0\tau_u + z_0\tau_{\Delta,u}\right) + \sigma\bar{\varphi}_0 + \bar{\sigma}\varphi_x,$$

$$g_0 = (1+z_0)^2 g, \quad \nu_0 = (1+z_0)\nu_c, \quad m^2 = (1+z_0)(\nu - \nu_c).$$

Then

$$G_{\Lambda,\nu}(x,y) = (1+z_0) \frac{\partial}{\partial\sigma}\frac{\partial}{\partial\bar{\sigma}}\Big|_0 \int e^{-S(\Lambda)-V_0(\Lambda)}.$$

Want to show that $\exists z_0$ such that first part of V_0 is a small perturbation and use

$$\lim_{m^2\downarrow 0}\lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{\partial}{\partial\sigma}\frac{\partial}{\partial\bar{\sigma}}\Big|_0 \int e^{-S(\Lambda)} e^{-\sigma\bar{\varphi}_0 - \bar{\sigma}\varphi_x} = (-\Delta)^{-1}(0,x) \sim \text{const}|x|^{-(d-2)}.$$

Gaussian "expectation"

For a positive definite $\Lambda \times \Lambda$ matrix C, and $A = C^{-1}$, let

$$S_A(\Lambda) = \sum_{x,y\in\Lambda} \Big(\varphi_x A_{xy} \bar{\varphi}_x + \psi_x A_{xy} \bar{\psi}_y\Big)$$

and, for a form F,

$$\mathbb{E}_C F = \int_{\mathbb{C}^\Lambda} e^{-S_A(\Lambda)} F.$$

Then $\mathbb{E}_C 1 = 1$. With $C = (-\Delta + m^2)^{-1}$, our goal is to compute

$$\lim_{m^2\downarrow 0} \lim_{\Lambda\uparrow\mathbb{Z}^4} G_{\Lambda,\nu}(x,y) = \lim_{m^2\downarrow 0} \lim_{\Lambda\uparrow\mathbb{Z}^4} (1+z_0) \left.\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\bar{\sigma}}\right|_0 \mathbb{E}_C e^{-V_0(\Lambda)}.$$

These integrals have much in common with standard Gaussian integrals. However, this is not ordinary probability theory and and in general $\mathbb{E}_C$ will be a Grassmann integral that take values in a space of differential forms.

Convolution integrals

Write $\phi = (\varphi, \bar{\varphi})$, $d\phi = (d\varphi, d\bar{\varphi})$.

Recall that $X \sim N(0, \sigma_1^2 + \sigma_2^2)$ has the same distribution as $X_1 + X_2$ where $X_1 \sim N(0, \sigma_1^2)$ and $X_2 \sim N(0, \sigma_2^2)$ are independent.

This finds expression for $\mathbb{E}_C$ via the following fact:

$$\mathbb{E}_{C_2+C_1} F = \mathbb{E}_{C_2} \circ \mathbb{E}_{C_1} \theta F,$$

where

$$(\theta F)(\phi, \xi, \psi, \eta) = F(\phi + \xi, \psi + \eta)$$

and $\mathbb{E}_{C_1}$ integrates out ξ and $\eta = \frac{1}{\sqrt{2\pi i}} d\xi$, leaving ϕ and $\psi = \frac{1}{\sqrt{2\pi i}} d\phi$ fixed. Then $\mathbb{E}_{C_2}$ integrates out ϕ and ψ.

Finite-range decomposition of covariance

Theorem (Brydges, Guadagni, Mitter 2004). Let $d > 2$. Fix a large L and suppose $|\Lambda| = L^{Nd}$. Let $C = (-\Delta + m^2)^{-1}$. It is possible to write:

$$C = \sum_{j=1}^{N} C_j$$

with C_j positive definite,

$$C_j(x,y) = 0 \quad \text{if} \quad |x-y| \geq \frac{1}{2} L^j$$

and, for $j = 1, \ldots, N-1$ and with $[\phi] = \frac{1}{2}(d-2)$ (so $[\phi] = 1$ for $d = 4$),

$$|C_j(x,x)| \leq O(L^{-2[\phi](j-1)}),$$

$$|\nabla_x^\alpha \nabla_y^\beta C_j(x,x)| \leq O(L^{-(2[\phi]+|\alpha|_1+|\beta|_1)(j-1)}).$$

The RG map

The covariance decomposition induces a field decomposition and allows the expectation to be done iteratively:

$$\phi = \sum_{j=1}^{N} \xi_j, \qquad d\phi = \sum_{j=1}^{N} d\xi_j, \qquad \mathbb{E}_C = \mathbb{E}_{C_N} \circ \cdots \circ \mathbb{E}_{C_2} \circ \mathbb{E}_{C_1}.$$

Write $\phi_j = \sum_{i=j+1}^{N} \xi_i$, with $\phi_0 = \phi$, $\phi_N = 0$. Then $\phi_j = \phi_{j+1} + \xi_{j+1}$. Let

$$Z_0 = Z_0(\phi, d\phi) = e^{-V_0(\Lambda)},$$

and

$$Z_j(\phi_j, d\phi_j) = \mathbb{E}_{C_j} \cdots \mathbb{E}_{C_1} Z_0.$$

In particular, our goal is to compute

$$Z_N = \mathbb{E}_C Z_0 = \mathbb{E}_C e^{-V_0(\Lambda)}$$

and we are led to study the RG map:

$$Z_{j+1} = \mathbb{E}_{C_{j+1}} Z_j.$$

Relevant, marginal, irrelevant directions

Let $d = 4$. The covariance estimates suggest that $\xi_{j+1,x} \approx L^{-j[\phi]} = L^{-j}$ and that this field is approximately constant over distance L^j. Thus, for a block B of side L^j,

$$\sum_{x \in B} |\xi_{j+1,x}|^p \approx |B| L^{-jp} = L^{j(4-p)},$$

which is *relevant* for $p < 4$, *marginal* for $p = 4$, *irrelevant* for $p > 4$.

Taking symmetries and derivatives into account, the relevant and marginal monomials are:

$$\tau, \qquad \tau_\Delta, \qquad \tau^2.$$

The role of $d = 4$: τ^2 is *relevant* for $d < 4$ and *irrelevant* for $d > 4$:

$$\sum_{x \in B} |\xi_{j+1,x}|^4 \approx |B| L^{-j4[\phi]} = L^{j(4-d)}.$$

The map $\mathbb{E}_{C_1} : Z_0 \mapsto Z_1$

This map takes a function of $\phi = \phi_1 + \xi_1$ to a function of ϕ_1 by integrating out ξ_1.

Write $Z_0(x) = I_0(x) = e^{-V_0(x)}$, and, for $X \subset \Lambda$, write

$$I_0(X) = \prod_{x \in X} I_0(x) = e^{-V_0(X)}.$$

This is a function of ϕ.

Let V_1 be a version of V_0 with modified coupling constants (g_1, ν_1, z_1) and regarded as a function of ϕ_1 (and $d\phi_1$). Let $I_1(x) = e^{-V_1(x)}$, this will be an approximation to Z_1. Let

$$\delta I_{1,x}(\phi_1, \xi_1) = I_{0,x}(\phi_1 + \xi_1) - I_{1,x}(\phi_1).$$

Then

$$\begin{aligned} Z_1(\Lambda) &= \mathbb{E}_{C_1} I_0(\Lambda) = \mathbb{E}_{C_1} \prod_{x \in \Lambda} \left(I_{1,x} + \delta I_{1,x} \right) \\ &= \mathbb{E}_{C_1} \sum_{X \subset \Lambda} I_1^{\Lambda \setminus X} \delta I_1^X = \sum_{X \subset \Lambda} I_1^{\Lambda \setminus X} \mathbb{E}_{C_1} \delta I_1^X . \end{aligned}$$

The $I \circ K$ representation

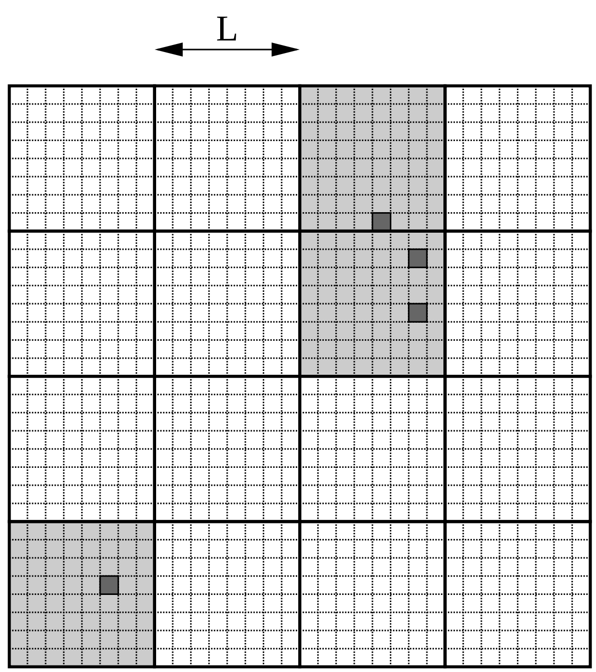
We write this as

$$Z_1(\Lambda) = \sum_{X \subset \Lambda} I_1^{\Lambda \setminus X} \mathbb{E}_{C_1} \delta I_1^X = \sum_{U \in \mathcal{P}_1} I_1^{\Lambda \setminus U} K_1(U),$$

where

$$K_1(U) = \sum_{X \in \overline{\mathcal{P}}_0(U)} I_1^{U \setminus X} \mathbb{E}_{C_1} \delta I_1^X$$

with factorisation property.



The $I \circ K$ representation

The formula

$$Z_1(\Lambda) = \sum_{X \subset \Lambda} I_1^{\Lambda \setminus X} \mathbb{E}_{C_1} \delta I_1^X = \sum_{U \in \mathcal{P}_1} I_1^{\Lambda \setminus U} K_1(U),$$

is an instance of the following "circle product."

Let $\mathcal{B}_j$ represent the blocks in a paving of Λ by blocks of side L^j, and let $\mathcal{P}_j$ denote the set of finite unions of such blocks. Given even forms F, G defined on $\mathcal{P}_j$, let

$$(F \circ G)(\Lambda) = \sum_{U \in \mathcal{P}_j} F(\Lambda \setminus U) G(U).$$

This defines an associative and commutative product. For $X \in \mathcal{P}_0$, let $K_0(X) = \delta_{X,\varnothing}$. Let K_1 be defined as above and let $I_1(U) = \prod_{x \in U} I_{1,x}$ for $U \in \mathcal{P}_1$. Then

$$Z_0(\Lambda) = I_0(\Lambda) = (I_0 \circ K_0)(\Lambda), \qquad Z_1(\Lambda) = (I_1 \circ K_1)(\Lambda).$$

Flow of coupling constants

Theorem. Let $d = 4$ ($d > 4$ is simpler). There is a choice of

$$V_{j,u} = g_j \tau_u^2 + \nu_j \tau_u + z_j \tau_{\Delta,u} + \lambda_j(\delta_{u,0}\sigma\bar{\varphi}_0 + \delta_{u,x}\bar{\sigma}\varphi_x) + q_j \frac{1}{2}(\delta_{u,0} + \delta_{u,x})\sigma\bar{\sigma}$$

which determines I_j, and of K_j, such that

$$Z_j(\Lambda) = (I_j \circ K_j)(\Lambda), \qquad Z_{j+1}(\Lambda) = \mathbb{E}_{C_{j+1}} Z_j(\Lambda) = (I_{j+1} \circ K_{j+1})(\Lambda),$$

and moreover

$$\begin{aligned}
g_{j+1} &= g_j - c_j g_j^2 + r_{g,j} \\
\nu_{j+1} &= \nu_j + 2 g_j C_{j+1}(0,0) + r_{\mu,j} \\
z_{j+1} &= z_j + r_{z,j} \\
K_{j+1} &= r_{K,j},
\end{aligned}$$

with additional equations for λ_j and q_j, such that the r's are error terms within an appropriately defined Banach space, and Lipschitz in (g_j, ν_j, z_j, K_j). K_j enters only in the error terms and these are independent of λ_j, q_j.

Fixed point theorem

We prove that there is a choice of initial conditions z_0 (which occurs in c of $c|x|^{-2}$) and ν_0 (which puts us at the critical point) such that the solution $(g_j, \nu_j, z_j, K_j)_{0 \le j \le N}$, in the limits $N \to \infty$ and $m^2 \to 0$, has limit

$$(g_j, \nu_j, z_j, K_j) \to (0,0,0,0) \quad \text{“infrared asymptotic freedom.”}$$

From this, estimates on K_N, and the specific form $q_j \approx \sum_{i=1}^{j} C_i(0,x) \to C_{\mathbb{Z}^4}(0,x)$ we obtain

$$\begin{aligned}
G_{\nu_c}(x) &= \lim_{\nu \downarrow \nu_c} (1+z_0) \lim_{N\to\infty} \frac{\partial}{\partial \sigma}\frac{\partial}{\partial \bar\sigma}\bigg|_0 Z_N(\Lambda) \\
&= \lim_{m^2 \downarrow 0} (1+z_0) \lim_{N\to\infty} \frac{\partial}{\partial \sigma}\frac{\partial}{\partial \bar\sigma}\bigg|_0 (I_N(\Lambda) + K_N(\Lambda)) \\
&= \lim_{m^2 \downarrow 0} (1+z_0) \lim_{N\to\infty} \frac{\partial}{\partial \sigma}\frac{\partial}{\partial \bar\sigma}\bigg|_0 (e^{-q_N \sigma\bar\sigma} + 0) \\
&= \lim_{m^2 \downarrow 0} (1+z_0) \lim_{N\to\infty} q_N \\
&= c'(-\Delta_{\mathbb{Z}^4})^{-1}(0,x) \sim c|x|^{-2}.
\end{aligned}$$