Guarded Impredicative Polymorphism
Extended Version

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Abstract
The design space for type systems that support impredicative instantiation is extremely complicated. One needs to strike a balance between expressiveness, simplicity for both the end programmer and the type system implementor, and how easily the system can be integrated with other advanced type system concepts. In this paper, we propose a new point in the design space, which we call guarded impredicativity. Its key idea is that impredicative instantiation in an application is allowed for type variables that occur under a type constructor. The resulting type system has a clean declarative specification — making it easy for programmers to predict what will type and what will not —, allows for a smooth integration with GHC’s `OutsideIn(X)` constraint solving framework, while giving up very little in terms of expressiveness compared to systems like HMF, HML, FPH and MLF. We give a sound and complete inference algorithm, and prove a principal type property for our system.

CCS Concepts • Theory of computation → Type structures;

Keywords Type systems, impredicative polymorphism, constraint-based inference

Yet it is tantalising: what is wrong with the type `∀a. a → a`, a list of polymorphic functions? If we have `xs :: [∀a. a → a]`, why can’t we write `(head xs)`? Not every programmer wants such types but, when they do, it is very annoying that they are disallowed, apparently for obscure technical reasons. That is why we keep trying.

So what is the problem? The difficulty is that to accept `head xs` we must instantiate the type variable of `head`’s type with a polymorphic type. More precisely, since `head :: ∀p. [p] → p`, we must instantiate `p` with `∀a. a → a`. This instantiation seems deceptively simple, but in practice it is extremely hard to combine with type inference. We respond to this challenge by making the following contributions:

- Every attempt to combine type inference with impredicativity involves a design trade-off between complexity, expressiveness, and annotation burden. Our key contribution is a new trade-off, which we call guarded instantiation or GI (Section 2).
- GI is simple: simple for the programmer to understand (Section 2.1-2.3), simple in its declarative specification and metatheory (Section 3); and simple in its implementation (Section 4). We do not extend the syntax of (System F) types in order to provide a specification of the type system (unlike previous work [1, 9, 23]), nor do we introduce new forms of annotations [19] or side-conditions that require principal types [8].
- Despite GI’s relative simplicity, it accepts without annotation particularly celebrated and practically important examples, such as `runST $ e` (Figure 2).
- We give a declarative type system for GI for a small core language, highlighting the key ideas of our system (Section 3). Then we show simple extensions to handle a more full-fledged language, including type annotations (Section 3.4), `let` bindings (Section 3.5), and pattern matching (Appendix A). The system has a notion of `principal type` akin to Hindley-Milner type systems, that is, existence of a monomorphic substitution mediating between types. In particular, impredicativity is never guessed in GI (Section 3.6). The resulting system can express any System F program.
We begin with an informal introduction to GI, based on constraints. The type inference algorithm is a modest extension of the constraint-based algorithm already used by GHC. Type-correct programs can readily be elaborated into System FC, GHC’s intermediate language, without extensions. Our inference algorithm scales readily to handle GADTs, type classes, higher kinds, type-level functions and other type system features.

We provide a prototype implementation of the whole system, integrated with Haskell’s type classes. Type inference for imprecadativity is dense with related work, as we discuss in Section 6. A small but useful contribution to a dense field is Figure 2, which presents key examples from the literature and shows how each major system behaves.

2 The key idea: intuition and examples

We begin with an informal introduction to GI, which we make fully precise in Section 3. In this discussion we make use of functions defined in Figure 1.

2.1 Exploiting the easy case

What is hard about typing (head ids)? Nothing! In head’s type the variable p appears under a list type constructor. Given the type of ids, [∀a. a → a], it is plain as a pikestaff that we must instantiate p with ∀a. a → a. The difficulty comes when we have a “naked” or “un-guarded” type variable in one of the arguments, such as p in single :: ∀p. p → [p]. Now if we examine (single id), it is not clear whether we should instantiate p with ∀a. a → a, or with Int → Int, or some other monomorphic type.

In fact, (single id) does not have a most general type. It has both of these two incomparable types ∀a. [a → a] and [∀a. a → a]. To make things worse (single id) is a perfectly typeable Hindley-Milner program (with the former type) so we must allow this type. But to support imprecadativity, we must also allow the latter. But under which conditions?

Our approach is to exploit the common case. We focus on n-ary applications (f e₁...eₙ). It is more conventional to deal with binary applications, but in fact n-ary applications (unencumbered with intervening let or case constructs) are wildly dominant in practice, and we can get much better typing by treating the application all at once. Then we adopt this rule to type such n-ary applications:

The Instantiation Rule. Given a n-argument call f e₁...eₙ to a function f :: ∀a₁...aₙ. σ₁ → ... → σₙ → ϕ, where m ≤ n, a type variable aᵢ may be instantiated to:

1. A polymorphic type ϕ, if aᵢ appears under a type constructor in one of the σᵢ (note that we only take in consideration as many types as arguments given). In this case we say that aᵢ is guarded in the type of f.

2. A top-level monomorphic type μ (see Figure 3), if aᵢ appears in any of the σᵢ at all.

3. A fully monomorphic type τ, otherwise.

This rule is carefully crafted. To illustrate, consider these examples (consult Figure 1 for the types):

• (map poly) is OK because in the type of map :: ∀a. b. (a → b) → [a] → [b], both type variables appear under the type constructor (→) in the first argument. We can instantiate both to (∀a. a → a), by case (1) of the Rule, as required to match the type of poly.

• (single ids) is OK because we can instantiate single :: ∀a. a → [a] with the top-level-monomorphic type ∀b. [b], using case (2) of the Instantiation Rule.

• ((i) id) is a partial application of (id). So although a appears guarded in the second argument of (id) :: ∀a. a → [a], in this call we can only take advantage of the first argument (see m ≤ n in the rule.). Hence a can only be instantiated by a top-level monomorphic type, by clause (2) of the rule. If we add a second argument in the call, such as ((i) id ids), instantiation may be polymorphic since the second argument is now taken into consideration.

• (id poly (λx. x)) is a tricky one. Here id is applied to two arguments although its type, ∀a. a → a, apparently only has one; moreover the type of id’s second argument must be polymorphic, and the (λx. x) must be generalised. But the Instantiation Rule says that this application is OK: the type of poly is (∀a. a → a) → (Int, Bool), a top-level monomorphic type. Thus the instantiation of id to that type is allowed by clause (2) of the Instantiation Rule.

The Instantiation Rule is still informal (which we remedy in Section 3) but it is very helpful to have a rule of thumb to explain to a programmer what will and will not work.

2.2 Ignoring the context of a call

Notice that the Instantiation Rule takes no account of the context of the call. For example, consider ids + single id. We know that the result of (single id) must be [∀a. a → a], given that ids has the same type, and you might think that would be enough to fix the instantiation of single. But not in GI! The swamp beckons, and we stay on dry land.

Moreover, the Instantiation Rule allows the programmer to understand imprecadativity in a simple bottom-up way. For example, consider the expression (map head (single ids)) (Figure 2). In GI, the types for head and single ids are instantiated independently, so we never need to consider the interaction between the arguments. This modularity pays off in the metatheory too.

There is a price to pay, however. As a degenerate case, a function application without any arguments – that is, a variable – may only instantiate fully monomorphically – no polymorphism, even if it appears under a type constructor.
Thus, the empty list constructor [ ] :: ∀a. [a] cannot be assigned a type [∀a. a → a]. We describe how to loosen this restriction in Section 3.3.

2.3 Lambda

In common with many other approaches to impredicativity, we take a conservative position on lambda-bound variables. Consider g (λf. (f ‘x’. f True)), where g :: ((∀a. a → a) → (Char, Bool)) → Int. Since g can only be applied to a function whose argument is itself polymorphic, you could imagine that information being propagated to f and so the program could be accepted. In common with many other systems, we reject all programs that require a lambda-bound variable to be polymorphic, unless it is explicitly annotated:

The Lambda Rule. Every lambda abstraction whose argument is polymorphic must be annotated. Otherwise, the bound variable can only have a fully monomorphic type.

By a “fully monomorphic type” we mean “no foralls anywhere”. Nothing about guardedness here! In contrast, MLF requires an annotation only when the argument is used more than once in the body with different polymorphic types.

While the Lambda Rule deals with the arguments to lambdas, it says nothing about the return type. To get a polymorphic return type, an annotation needs to be provided. For example, for λ(x :: ∀a. a → a). x x, GI infers the type (∀a. a → a) → b → b, and not (∀a. a → a) → (∀a. a → a).

To get the latter type, we have to write λ(x :: ∀a. a → a). (x x :: ∀a. a → a) instead.

2.4 Expressiveness

By treating n-ary applications as a whole, and taking guardedness from both the function type and the argument types, we can infer impredicative instantiations in many practically-useful situations. We summarise a collection of examples culled from the literature in Figure 2. This table also compares our system with others, but we defer discussion of related work to Section 6.

A celebrated example is the function ($) :: (a → b) → a → b. Haskell users use this function all the time to remove parentheses in their code, as in (runST $ do {...}) (the type of runST is given in Figure 2). This call absolutely requires impredicative instantiation of the variable a in the type of ($).

It is so annoying to reject this program that GHC implements a special, built-in typing rule for f $ x. Of course, that is horribly non-modular: if the programmer re-defines another version of ($), even with the same type, some programs cease to type check. In GI both type variables appear under the (→) constructor, so impredicative instantiation is allowed.

The lack of support for impredicative types is painful. For example, consider the following from Haskell’s lens library:

\[
\text{type } \text{Lens } s t a b = \forall f. \text{Functor } f \Rightarrow (a \rightarrow f b) \rightarrow s \rightarrow f t
\]

Programmers think of a lens as a first-class value, and are perplexed when they cannot put a lens into a list or other data structure. With GI, many more lens-manipulating programs become well-typed.

One might worry about the order of quantifiers. Take:

\[
f :: (\forall a. a \rightarrow b \rightarrow b) \rightarrow \text{Int} \\
g :: [\forall a. a \rightarrow b \rightarrow b] \rightarrow \text{Int}
\]

The application (f x) is well-typed in GI, despite the differing quantifier ordering, because we compare \( f \)'s argument type and \( x \)'s actual type using substitution; effectively we instantiate and re-generalise. In contrast, GI does not accept the application (g xs), because under a list constructor we compare the types using equality. Happily, while top-level quantifiers (such as those for \( x \)) are invisibly inferred (with unpredictable ordering), nested quantifiers, such as those in \( g \) and \( xs \)'s type, are never inferred but rather declared through a type signature. This makes accidental incompatibility vanishingly rare in practice, as we verify in Section 5.
A. Serrano, J. Hage, D. Vytiniotis, and S. Peyton Jones

Figure 2. Comparison of type systems

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B INFERENCES OF POLYMORPHIC ARGUMENTS

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D APPLICATION FUNCTIONS

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E η-EXPANSION

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Figure 3. Syntax of the language

Lambda Rule (Section 2.3), we provide explicitly-annotated lambda abstractions, \(\lambda(x :: \sigma). e\), to support lambdas whose bound variable must have a polymorphic type. Annotations and lets are treated in their own sections, we first focus on the core language with variables, applications, and lambdas.

Types (Figure 3) are classified by three “sorts”, u, t, and m. *Polymorphic types* \(\sigma, \phi\), of sort u, have unrestricted polymorphism. Top-level monomorphic types, \(\mu, \eta\), of sort t, have no polymorphism at the top level, but permit arbitrary nested polymorphic types under a type constructor. Finally, fully-monomorphic types, r, of sort m, have no trace of polymorphism. Fully monomorphic types correspond to monotypes in the Hindley-Milner tradition. These are the only types which can be assigned to un-annotated lambda-bound variables. We extend this notion to substitutions, and sometimes speak of fully monomorphic substitution to mean that the image of the substitution contains only types of that sort.

For a substitution \(\theta\), the image of a type variable \(a\) is denoted by \(\theta(a)\), and similarly for sort assignments.

### 3.2 Typing rules

The typing judgment \(\Gamma \vdash e : \sigma\) is given in Figure 4, along with some auxiliary judgments.

Rules **AnnAbs** and **ANNAbs** concern lambda abstractions, and are straightforward. **ANNAbs** deals with a lambda \((\lambda(x :: \phi). e)\) where the user has supplied a type annotation \(\phi\): we...
Guarded Impredicative Polymorphism

<table>
<thead>
<tr>
<th>$\sigma$ respects $s$</th>
<th>$\theta$ respects $\Lambda$</th>
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<tr>
<td>$\sigma$ respects $u$</td>
<td>$\mu$ respects $t$</td>
</tr>
<tr>
<td>$\tau$ respects $m$</td>
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</tr>
<tr>
<td>$\forall a. \mu \triangleright^s \Delta \backslash a$</td>
<td>ARGPOLY</td>
</tr>
<tr>
<td>$\Gamma \phi \triangleright^s [\text{ftv}(\phi) \mapsto u]$</td>
<td>ARGGUARD</td>
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<tr>
<td>$\sigma \triangleright^s \Delta$</td>
<td>ARGRES</td>
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<tr>
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<td>INSTMONO</td>
</tr>
<tr>
<td>$\phi_1 \triangleright^s \sigma_2; \mu$</td>
<td>INSTARRANGE</td>
</tr>
<tr>
<td>$\mu \triangleright^s \sigma$</td>
<td>INSTPOLY</td>
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<tr>
<td>$\mu \triangleright^s e; \mu$</td>
<td>INSTMONO</td>
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<tr>
<td>$\phi_1 \triangleright^s \sigma_2; \mu$</td>
<td>INSTPOLY</td>
</tr>
<tr>
<td>$\Gamma \vdash \text{fun} e \cdot \sigma$</td>
<td>EXPRHEAD</td>
</tr>
<tr>
<td>$\Gamma \vdash \text{var} e : \sigma$</td>
<td>ARGGEN</td>
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</table>

Figure 4. Declarative type system

simply bring $x$ into scope with type $\phi$. As discussed in Section 2.3, where there is no annotation (rule Abs) we insist that $x$ has a fully-monomorphic type $\tau$.

All the action is in rule App for $n$-ary applications ($e_0, e_1, \ldots, e_n$). First, note that a lone variable is treated as a nullary application. Second, the typing rules allow us to break an $n$-ary application in different ways, because $e_0$ could itself be an application. In practice we always choose $e_0$ to not be an application, so we get as many arguments as possible, and thereby maximise the opportunities for guardedness.

The first step in App is typing the head of the application $e_0$. The corresponding judgment $\Gamma \vdash \text{fun} e$ either looks for a variable in the environment or uses the normal typing judgment if the head is another kind of term. After typing the head, rule App instantiates the type of the head with the instantiation judgment $\lesssim^s \sigma$, yielding a list of argument types $\sigma_1, \ldots, \sigma_n$. App uses $\vdash^{\text{args}}$ to check each argument $e_i$ against the corresponding $\sigma_i$. This argument-checking judgement $\vdash^{\text{args}}$ can generalise the inferred type of the argument to match the type $\sigma$ expected by the function, to support higher-rank polymorphism. Without such a rule, we would not be able
to type check poly (\(\lambda x. x\)), which requires the argument to be of type \(\forall a. a \rightarrow a\).

For example, an application of a function to three arguments would give rise to the following instantiation:

\[
\forall ab. a \rightarrow (\forall c. [a] \rightarrow b \rightarrow c \rightarrow [b]) \leq_m \sigma_1, [\sigma_a], \tau_b; (\tau_c \rightarrow [\tau_b])
\]

This example illustrates several points. First, the bit-vector \(\bar{m}\) corresponds 1-1 with the arguments in the application. For now, the bit-values are irrelevant, but we will use them in Section 3.3. Second, the instantiation judgement returns a list of argument types that matches the length of the vector \(\Delta\) (rule \textsc{InstArrow}); in this case, there are three arguments, and the instantiated argument types are \(\sigma_a\), \([\sigma_a]\), and \(\tau_b\). Third, the judgement can instantiate nested foralls.

Fourth, and core to our contribution, each type variable is instantiated, by rule \textsc{InstPoly}, with a type whose sort reflects the way the type variable appears in the function type. This analysis is performed by the judgement \(\sigma \vdash_{\forall a} \Delta\), which returns a classification \(\Delta\) of the free type variables of \(\sigma\). This classification directly implements the three cases of the Instantiation Rule (Section 2.1):

1. If \(a\) appears under a type constructor (i.e. guarded) in any of the first \(n\) arguments of \(\sigma\), then \(a\) may be instantiated by an unrestricted type \(\phi\) (rule \textsc{ArgGuard}).
2. If \(a\) appears in one of the first \(n\) arguments of \(\sigma\), then \(a\) may be instantiated by a top-level monomorphic type \(\mu\) (rule \textsc{ArgTyVar}).
3. If \(a\) appears only in the result type of \(\sigma\), after stripping off \(n\) arguments, then \(a\) may be instantiated by a type of sort \(s\) (rule \textsc{ArgGen}).

The invocation of \(\leq_m\) in rule \textsc{App} the sort \(s\) is always fully monomorphic \(m\), but we need the extra generality for annotations (Section 3.4).

Once we have \(\Delta\) to classify each type variable rule, \textsc{InstPoly} instantiates the function type with a \(\Delta\)-respecting substitution \(\theta\), and recurses. Figure 4 also defines what it means for a substitution \(\theta\) to "respect" a classification \(\Delta\).

Unlike some other systems (see [13] for a comprehensive account), all type constructors in \(\text{GI}\) are invariant, including functions. This means that neither \(\forall a. a \rightarrow a\) \(\nless_m\) \(\text{Int} \rightarrow \text{Int}\) nor \(\text{Int} \rightarrow (\forall a. a \rightarrow a) \nless_m\) \(\text{Int} \rightarrow \text{Int}\). However, because \(\nless\) handles foralls nested to the right of arrows (rule \textsc{InstPoly}), we can often work around the lack of covariance using \(\eta\)-expansion. For example, suppose we have functions \(f : \forall a. a \rightarrow (\forall b. b \rightarrow a)\), and \(g : (\text{Int} \rightarrow \text{Int} \rightarrow \text{Int}) \rightarrow \text{Bool}\). Then (\(g\) \(f\)) is ill-typed; but (\(g (\lambda xy. f \_ y)\)) is well typed.

### 3.3 Single variables

The type system described up to now propagates polymorphism between arguments and from the arguments to the return type of the function – this is the essence of guardedness. Alas, this creates a problem for single variables, which are treated as nullary applications: since there are no arguments, no type variable is considered guarded, and thus impredicativity is forbidden. We can see this by expanding the derivation of \(\Gamma \vdash_{\forall \sigma} x : \sigma\) for the case of a variable \(x\).

\[
\frac{x : \phi \in \Gamma}{\Gamma \vdash_{\forall \sigma} x : \phi} \quad \text{VARHEAD}
\]

\[
\frac{\tilde{b} \notin \Gamma}{\Gamma \vdash_{\forall \sigma} x : \phi} \quad \text{ARGEND}
\]

The highlighted premise, \(\phi \nless_m \epsilon ; \mu\), forces instantiation to use only fully monomorphic types.

That is embarrassing, because we cannot typecheck, say \texttt{choose} \{\} \texttt{ids} from Figure 2. From the type of \texttt{ids} it is obvious that \{\} should be given a type \([\forall a. a \rightarrow a]\). But we cannot do so, because such instantiation is not fully monomorphic. We get back into swampy waters.

For the case of single variables with a rank-1 polymorphic type (quantifiers appear only at top-level) we can get out of the swamp. The \textsc{VarGen} rule in Figure 5 formalizes this idea: if a single variable \(x\) appears as the argument of an application – hence the use of the judgement \(\vdash_{\forall \sigma}\) – and it has a rank-1 type \(\forall \sigma\), we may instantiate impredicatively before generalizing. This is enough to cover the case of \texttt{choose} \{\} \texttt{ids}, because the type of \{\} \texttt{ids} has the right shape. Note, however, that it is still the case that the justification for instantiating the type of the first argument \{\} with top-level monomorphic type has to come from the second argument \texttt{ids}. If such justification is not forthcoming, as it should be in the case of \texttt{choose} \{\} \{\}, then both arguments should be instantiated fully monomorphically.

We manage the bookkeeping to distinguish these situations by keeping track of a vector of bits \(\bar{m}\). Its elements correspond to the arguments in an application: a \(\ast\) means that the rule \textsc{VarGen} was applied to type the corresponding argument; application of \textsc{ArgGen} is represented by \(\bullet\). The rule \textsc{ArgEndStar} ensures that whenever the rule \textsc{VarGen} was used to type a given argument, we reset the sorts of the free variables to \(m\) so that however these type variables were instantiated, this information cannot be used to justify the impredicative instantiation of other arguments.

### 3.4 Annotations

The \textsc{VarGen} rule makes it possible to accept more programs, but it is restricted to a very special class of expressions. For the general case, we provide annotations as part of the syntax. Since annotations fully specify the types, we do not
need to impose guardedness restrictions on those variables appearing in the result. Take the expression single (λx. x). Due to the type of single being ∀p. p → [p], the type of the expression must be [τ → τ] for a monomorphic r. If we want instead to obtain [∀a. a → a], we can just annotate the result, thus single (λx. x) :: [∀a. a → a].

Rule ANNAPP is almost identical to APP, except for the choice of parameter to the instantiation judgment, which is n. This implies that in contrast to non-annotated applications, variables in the result type of the function might be substituted by any type, polymorphic or not. This is sensible, the annotation tells us exactly what the types are that those variables should be instantiated with.

Annotations also free us from having a different judgment for declarations and expressions. For every combination f :: σ :: e in the source code, we just need to pose the problem of checking if e :: σ for well-typedness.

### 3.5 let bindings

The simplest way to type let x = e₁ in e₂ is to see it as a shorthand for (λx. e₂) e₁. Alas, such a translation imposes an important restriction on the type of x: it must be fully monomorphic even though the type of e₁ might be more general. The reason is that we try to guess the type of e₁ by looking at the way it is used in e₂, instead of looking at e₁ itself. But there is no need to be so restrictive! The rule LET in Figure 4 works in the other direction: the type obtained from typing e₁ is put in the environment as the one for x, allowing the type of x to be fully polymorphic instead.

One difference between let bindings in GI and Hindley-Milner is that the latter always generalizes the type of a let-bound identifier before passing it to the body of the let. Vytiniotis et al. [21] argue however that let-generalisation is not so important in practice and that in complex type systems how to generalize is not completely clear. If desired, generalisation can be obtained by annotating the bound expression, let x = (e₁ :: φ) in e₂.

### 3.6 Metatheory

Impredicativity is a great tool, but we do not want to lose those programs which only require top-level polymorphism. The following theorem states this fact, except for the different take on let generalization we discussed in Section 3.5.

**Theorem 3.1** (Compatibility with rank-1 polymorphism). Let e be an expression in the syntax of the lambda-calculus with predicative rank-1 polymorphism. If Γ ⊢ e : τ in that type system, then Γ ⊢ e : τ in GI.

One key property of GI is that all impredicative instantiations are settled by the shape of the expression and the types in the environment, modulo some monomorphic substitution. For that reason, we say that impredicative polymorphism is **not guessed** in GI.

**Theorem 3.2** (Impredicative instantiation is not guessed). Let Γ be an environment and e an expression. For every pair of fully monomorphic substitutions θ₁ and θ₂, if θ₁ Γ ⊢ e : σ₁ and θ₂ Γ ⊢ e : σ₂, then there exists a polymorphic type σ* and fully monomorphic substitutions φ₁ and φ₂ such that σ₁ = φ₁ σ*.

**Corollary 3.3.** Let Γ be a closed environment (that is, no type in Γ contains a free variable). If Γ ⊢ e : σ₁ and Γ ⊢ e : σ₂, then there is a polymorphic type σ* and fully monomorphic substitutions φ₁ and φ₂ such that σ₁ = φ₁ σ*.

This property suggests a notion of **principal type** similar to the one found in Hindley-Milner. A principal type for an expression e is defined as a type σ* for which any other type assignment φ to e is equal to θσ* for a fully monomorphic substitution θ. The fact that we only need to consider fully monomorphic substitutions here is a direct consequence of Theorem 3.2. The proof of the principal types property, however, is a corollary of other properties of the inference process, which we describe in Section 4.4. Note also that this theorem only promises that if GI accepts an expression, there was no guessing involved. But there are expressions for which only one choice of polymorphism is possible, yet GI cannot find it and an annotation is required.

In the remainder of this section we look at some properties of GI concerning derivations and stability under transformations. We use the notation e₁[e₂] to refer to a context in which e₂ appears. Proofs are given in Appendix D.1.

**Theorem 3.4** (Substitution). If Γ ⊢ u : σ and Γ, x : σ ⊢ e[x] : ϕ, then Γ ⊢ e[u] : ϕ.

The converse result does not holds: we cannot in general abstract over part of an expression. In practice, that means that we cannot always introduce a let, as in changing e₀ e₁ e₂ to let x = e₀ e₁ in x e₂. Our APP rule is responsible for propagating information between arguments, if we introduce a let for part of an expression, this bound is lost.

**Theorem 3.5.** Let app ::= ∀a b. (a → b) → a → b and revapp ::= ∀a b. a → (a → b) → b be the application and reverse application functions, respectively. Given two expressions f and e such that Γ ⊢ f : σ₀, and σ₀ ≤_e ε; σ₁ → φ then:

Γ ⊢ f e : φ ⇔ Γ ⊢ app f e : ϕ ⇔ Γ ⊢ revapp e f : ϕ

The hypothesis σ₀ ≤_e ε; σ₁ → φ means that type variables in f may only be instantiated with fully monomorphic variables. Thus, this transformation only respects well-typedness for the predicative fragment, but not in general when impredicative instantiation is allowed. One example of this restriction is the application f ids, where f has type ∀a. [a] → [a], which cannot be turned into app f ids. The reason is that in the original application the type variable a in f has to be instantiated with a polymorphic type ∀b. b → b. However, this is not allowed in the form with app.

One property which does not hold in GI is **full subject reduction**. If Γ ⊢ e : σ and e β-reduces to e′, it may not hold...
that $\Gamma \vdash e' : \sigma$. For example, \textit{app} \textit{auto} is typeable, but not its reduced form $\lambda x. \text{auto} \ x$, since it requires an annotation on $x$. Subject reduction holds in a milder form: if $\Gamma \vdash e : \sigma$, $e$ $\beta$-reduces to $e'$ and $\Gamma \vdash e' : \phi$, then $\sigma$ and $\phi$ coincide.

4 Type inference using constraints

In the previous section we described GI from a declarative perspective and now we turn to describing an efficient type inference algorithm for it.

Following Pottier and Rémy [15], we first walk over the syntax tree of the source program and generate \textit{typing constraints}, a process that typically introduces many \textit{unification variables} that stand for as-yet-unknown types. Next, we solve those constraints producing a \textit{type substitution} for these unification variables. By separating type inference in two simpler problems, the implementation and conceptual overhead with new source language and type system features remains low. For example, earlier work dubbed $\text{OuterSourceIn}(X)$ applies these ideas to a language like Haskell, with type classes, type-level functions, GADTs, and the like [21]. Another advantage is more sophisticated type-error diagnosis [6, 20, 25].

4.1 Constraints

The main challenge of type inference for impredicativity concern instantiation and generalisation of terms with polymorphic type. Consider the call (\textit{head ids True}), when fully elaborated we want to generate this System F term:

$$\text{head } (\forall a. \ a \rightarrow a) \ ids \ \text{Bool True}$$

That is, we instantiate \textit{head} at type $(\forall a. \ a \rightarrow a)$, then apply it to $\text{ids}$, to produce a result of type $(\forall a. \ a \rightarrow a)$. Now we must in turn instantiate that type with $\text{Bool}$ to get a function of type $(\text{Bool} \rightarrow \text{Bool})$ which we can apply to $\text{True}$. This second instantiation is problematic because, at constraint generation time, we do not yet know what type we are going to instantiate \textit{head} at; all we know is that (\textit{head ids}) has type $\alpha$ for some as-yet-unknown type $\alpha$. So we want to defer the instantiation decision.

Sometimes we must defer generalisation decisions too. Consider the function application $(\cdot) : (\lambda x. \ x)$ \textit{ids}. In System F terms, we want to infer the following elaborated program:

$$(\cdot) : (\forall a. \ a \rightarrow a) \ (\lambda a. \ \lambda (x : a). \ x) \ ids$$

in which $(\cdot)$ is instantiated at type $(\forall a. \ a \rightarrow a)$, and $(\cdot)$’s first argument is generalised to have that polymorphic type. Now consider constraint generation for this expression. We may instantiate the type of $(\cdot)$ with a fresh unification variable, $\alpha$ say. Ultimately the type of $\text{ids}$ forces $\alpha$ to be $\forall a. a \rightarrow a$, but we don’t know that yet. Moreover, in the final program we will need to generalise the type of $(\lambda x. \ x)$, but again at constraint generation time we don’t know that type either.

When we don’t know something at constraint generation time, the solution is to \textit{defer the choice}, by \textit{generating a constraint that represents that choice}. This is the key idea of the constraint solving approach. The game is to develop a constraint language that neatly embodies the choices that we want to defer, and a solver that can subsequently make those choices. With that in mind, Figure 6 gives the syntax of our constraint language.

As mentioned earlier, constraint generation produces many \textit{unification variables}, each of which stands for an as-yet-unknown type. Looking at Figure 6, a key idea is that unification variables are drawn from three distinct “alphabets”: $\alpha^\ast$ for each of the threes sorts $s$. (Sorts were introduced in Figure 3.) The sort of a unification variable specifies the possible types that the unification variable can stand for; operationally, a unification variable of sort $s$ may only be unified with types belonging to that sort.

The syntax presents several kinds of constraints $C$. These constraints do not form part of the source language; they are internal to the solver. Equality constraints are self explanatory. Instantiation constraints arise from the occurrence of a polymorphic variable, whose type must be instantiated – but that decision must be deferred (embodied in a constraint). Quantification constraints arise from explicit user type signatures, and pattern matching on data types involving existentials and GADTs. Both are fairly conventional. However \textit{generalisation constraints} are new; they precisely embody the deferred decision about generalisation that we mention above.

4.2 Constraint generation

Constraint \textit{generation} is described in Figure 7 as a four-element judgment $\Gamma \vdash e : \sigma \leadsto C$. The first two elements are inputs: the environment $\Gamma$ and the expression $e$ for which to generate constraints. The output of the process is a type $\sigma$
assigned to the expression, possibly including some unification variables, and the set of extended constraints \( C \) that the types must satisfy.

Rules \( \text{Abs} \) and \( \text{AnnAbs} \) are not surprising; they just extend the environment with a new unification variable or a given polymorphic type, respectively, and then proceed to generate constraints for the body of the abstraction. The usage of a fully monomorphic variable in \( \text{Abs} \) mimics the restriction on the expected argument type that the fruits of constraint generation for each argument \( \sigma_i \) are wrapped up, along with the expected argument type \( \alpha_i \) from the function, into a generalisation constraint for the solver to deal with later. The reason for the additional argument \( \bar{\nu} \) to the judgement is that we need to forbid generalisation over variables which are visible in the environment or the whole application.

Rule \( \text{App} \) is where most of the work is done. Just like the declarative specification (Figure 4), the head of the application is typed using an ancillary judgment \( \Gamma \vdash \text{app} e : \phi \leadsto C \), which either looks up a variable in the environment or threads the information to the normal gathering process.

Another ancillary judgment \( \Gamma ; \bar{\nu} \vdash_{\text{arg}} e : \sigma \leadsto C \) is used to generate constraints on the arguments. In this judgement we use introduce a completely new constraint form, \( \eta \leq \sigma \), which we call generalisation constraint. These constraints allow us to defer the generalisation decisions to the solver, as sketched in Section 4.1. The constraint \( (\forall \bar{\nu}). C \Rightarrow \phi \leq \sigma \) should be read “a term of type \( \phi \) with constraints \( C \) and unification variables \( \bar{\nu} \) can be instantiated and/or generalised to have type \( \sigma \)”. Even looking at the syntax alone, you can see that the fruits of constraint generation for each argument \( e_i \) are wrapped up, along with the expected argument type \( \alpha_i \) from the function, into a generalisation constraint for the solver to deal with later.

4.3 Constraint solving

The solver takes the generated constraint \( C \) and its free unification variables \( \bar{\nu} = \text{fuv}(C) \), and repeatedly applies the solver rules in Figure 8, until no rule applies. The result is a residual constraint. If the residual constraint is in solved form, then the program is well typed; if not, the unsolved constraints (e.g. \( \text{Int} \sim \text{Bool} \)) represent type errors that can be reported to the user. We concentrate first on the solver rules that incrementally solve the constraint.

Each of the rules in Figure 8 rewrites a configuration \( C; \bar{\nu} \) to another configuration. The unification variables \( \bar{\nu} \) are existentially quantified, so you can think of a configuration as representing \( \exists \bar{\nu}. C \). Rule \( \text{conj} \) and \( \text{forall} \) are structural rules: the former allows a rule to be applied to one part of a conjunction, while the latter allows a rule to be applied under a quantification. To avoid clutter we implicitly assume that
the rules are read modulo commutativity and associativity of \( \land \); that is why \textbf{CONJ} only has to handle the left conjunct. The rules \textbf{TIDENT} and \textbf{TFORALL} remove \( \top \) constraints from the set, since they are identities for the conjunction.

### 4.3.1 Basic rules

Rule \textbf{EQREFL} removes trivial equality constraints \( \sigma \sim \sigma \). Rule \textbf{EQMONO} indicates that two types headed by constructors are equal if and only if their heads coincide and all the arguments are equal. \textbf{EQSUBST} is the only rule that involves the interaction of two constraints. It applies the substitution of a unification variable to any other constraints conjoined with \( \land \), provided sorts are respected and the substitution does not lead to an infinite type (hence the occurs check). Notice that the equality constraint is not discarded; it remains in case it is needed again; indeed, these equality constraints remain in a solved constraint.

Given the different behaviour embodied by the different sorts of variables, the solver has to propagate this information. \textbf{EQVAR} ensures that whenever we have two variables with different sorts, the least restrictive one is substituted by the most restrictive. For example, when we have an unrestricted \( \alpha^m \) and a top-level monomorphic \( \beta^i \), then \( \alpha^m \)

should be replaced by \( \beta^i \), and not the other way around. Full monomorphism goes deeper: \textbf{EQFULLY} ensures that if a type \( \sigma \) is equated with a fully monomorphic variable \( \alpha^m \), all the variables in \( \sigma \) become fully monomorphic too.

One difference between these rules and other presentations is that we do not rewrite an unsatisfiable constraint, such as \( \text{Int} \sim \text{Bool} \), to \( \bot \). Instead, that constraint is simply stuck, and we can report it at the end.

Note that two polymorphic types need to be syntactically equal (modulo \( \alpha\)-equality) to match under the \textbf{EQREFL} rule. This means that \( (\forall a \ b. \ a \rightarrow b \rightarrow b) \sim (\forall b \ a. \ a \rightarrow b \rightarrow b) \) does not hold in our system. As we discuss in Section 2.4, this is not problematic, since on application type variables are instantiated and regeneralized using the \( \preceq \) relation.

### 4.3.2 Instantiation and generalisation constraints

For instantiation constraints \( \preceq^\mu_{\forall \alpha} \), we follow closely the judgement with the same name in the declarative specification. Rule \textbf{INSTE} encodes the fact that once all arguments are processed, and thus \( \overline{\alpha} \) is an empty vector of bits, the remaining types must be equal. This is a consequence of the invariance of type constructors. On the other hand, if we have a top-level monomorphic type \( \mu \) and the vector of bits

\[ \textsf{freshen}^\mu_{\forall \alpha}(\sigma) \rightarrow (\overline{\upsilon}, \mu) \]

\[ \mu \preceq^i_{\forall \alpha} \top \text{ fresh} \quad \upsilon_i = \alpha^i(a_i) \]

\[ \textsf{freshen}^\mu_{\forall \alpha}(\forall \alpha, \mu) \rightarrow (\overline{\upsilon}, [\alpha \mapsto \overline{\upsilon}] \mu) \]
is not empty, the only possibility of for μ to be a function type. This is the goal of the \textsc{inst}→ rule.

\textsc{instvl} instantiates a polytype σ with fresh unification variables, much as in the usual Damas-Milner algorithm, except that we must use a sort-respecting instantiation. This is done by \textsc{freshen}_γ, which in turn uses the already-introduced classification judgment \( \triangleright \text{inst} \) (Figure 4). Finally, the new variables enter the set of existentially quantified variables.

Finally, we come to generalisation constraints, which (recall Section 4.1) express a deferred generalisation decision. Rule \textsc{instvl} is simple: if the right hand side has no top-level foralls (it is of the form \( \eta \)) then there is no generalisation to be done, so it suffices to release all the captured constraints \( C \) and existentials \( \nu \) into the current constraint.

\textsc{instvr} is where actual generalisation takes place. In order to forge some intuition, let us look at the constraint

\[(\forall \{ \alpha \beta \}. (\alpha \leq^m e; \beta) \Rightarrow \alpha \rightarrow \beta) \leq (\forall p. p \rightarrow p)\]

This generalisation constraint says that by performing some solving and possibly abstracting over some of the variables \( \alpha \) and \( \beta \), we should get the polymorphic type \( \forall p. p \rightarrow p \). Following standard practice, we skolemise the type on the right, introducing a fresh skolem or rigid variable \( p \), which should not be unified.

\[(\alpha \leq^m e; \beta) \land (\alpha \rightarrow \beta \tau p \rightarrow p)\]

We obtain a solution by making \( \alpha \rightarrow \beta \sim p \rightarrow p \). In order for this solution to remain valid, we must guarantee that the skolem \( p \) does not escape to the outer world. We recall this restriction by means of a fresh quantification constraint

\[\forall p. \exists \alpha . (\alpha \leq^m e; \beta) \land (\alpha \rightarrow \beta \sim a \rightarrow a)\]

Rule \textsc{instvr} achieves this rather neatly simply by doing skolemisation and pushing the \( \exists \) inside; then \textsc{forall} and \textsc{instvl} will do the rest.

The rules applicable to instantiation and generalisation constraints do not handle every case. In particular, whenever an unrestricted variable appears in one of the sides of the constraint, there are good reasons to wait:

1. If we have \( a^n \leq^m e; \mu \) we cannot turn it directly into \( a^{n} \sim \mu \), because \( a^n \) might be unified later to a polymorphic type and we need instantiation.
2. Similarly, if we have \( (\forall \{ \alpha \}. C \Rightarrow \mu) \leq a^n \), and \( a^n \) is later substituted by a polytype, we must skolemise.

The guardedness restrictions are carefully crafted to ensure that the solver is confluent and that it is never completely stuck, unless the constraint set as a whole is inconsistent. A single constraint can be stuck for some time, but if the whole set is consistent, by steps applied to other constraints, it will eventually become unstuck.

**Theorem 4.1.** Suppose \( \Gamma \vdash e : \sigma \rightarrow C \). Then \( C \) is either inconsistent, or can be rewritten to a new set \( C' \) without instantiation and generalisation constraints which fixes the value of all unrestricted and top-level monomorphic variables.

\[\frac{\overline{a}; \overline{a}; \emptyset \vdash \top \text{ solved}}{\text{SolvedT}}\]

\[\sigma \text{ respects } s \quad \overline{ftv}(s) \subseteq \overline{a} \cup \overline{a} \]

\[\frac{\overline{a}; \overline{a}; \{ \beta \} \vdash \beta \sim \sigma \text{ solved}}{\text{SolvedVar}}\]

\[\overline{a}; \overline{a}; \overline{p_1} \vdash C_1 \text{ solved} \quad \overline{a}; \overline{a}; \overline{p_2} \vdash C_2 \text{ solved} \]

\[\frac{\overline{a}; \overline{a}; \overline{p_1} \vdash C_1 \land C_2 \text{ solved}}{\text{SolvedConj}}\]

\[\overline{v} = \overline{p_1} \cup \overline{p_2} \quad \overline{a} \vdash b \quad \overline{a} \cup \overline{p_1} \vdash \overline{p_2} \vdash C \text{ solved} \]

\[\frac{\overline{a}; \overline{a}; \emptyset \vdash \forall \overline{b}. \forall \overline{C}. C \text{ solved}}{\text{SolvedQuant}}\]

**Figure 9.** Definition of solved set of constraints

Theorem 4.1 tells us that the process of solving can be divided into two phases. In the first phase all constraints, including instantiation and generalization constraints, are turned into a set of equalities, possibly with different quantification levels. This is an instance of the problem of first-order unification under a mixed prefix [3], for which a complete solving algorithm is described by Pottier and Rémy [15].

**4.4 Soundness, principality and completeness**

The inference algorithm presented here – gathering the constraints from an expression followed by solving them – satisfies the usual properties of soundness, principality, and completeness with respect to the declarative specification.

In order to state the theorems we need some ancillary notions. A constraint is in solved form if it consists only of quantification and equality constraints (\( \nu \sim \sigma \)); and the equalities constitute a well-sorted idempotent substitution of its unification variables. For example

\[\exists a^m. (a^m \sim \text{Int}) \land (\forall b. \exists a^n. b^n \sim (b \rightarrow \text{Int}))\]

is in solved form. Being in solved form is more than a simple syntactic property; here are two constraints that are not:

\[\exists a. (a \sim \text{Int}) \land (a \sim \text{Bool}) \quad \exists a. \ldots (\forall b.a \sim [b]) \ldots\]

In the first there are two equalities for \( a \) (we should apply \textsc{eqsubst} to make progress); in the second, there is a skolem-escape problem. However it is OK for a unification variable to have no equalities; it is simply unconstrained.

Figure 9 defines solved form precisely. We keep a set of variables \( \overline{p} \) for which we ensure that there is precisely one equality constraint, and another set \( \overline{\alpha} \) (the unconstrained variables) for which there are none. Rule \textsc{solvedvar} expects precisely one \( \beta \), checks well-sortedness, and also checks that \( \sigma \) does not mention any variables other than the skolens and unconstrained unification variables – the latter check ensures idempotence. Rule \textsc{solvedconj} partitions the \( \overline{\beta} \) between the two conjunctions. Rule \textsc{solvedquant} partitions the local existentials \( \overline{v} \) into the unconstrained sets, \( \gamma_1 \) and \( \gamma_2 \) resp. Rule \textsc{solvedt} simply states that \( \top \) is a solved form without consuming any variables.
A second auxiliary notion which we need to state the results is substitution induced by a solved form.

\[ C_s = E \land R, \text{ where } E \text{ are all the equalities in } C_s \]
\[ \bar{C}_s = \left[ \alpha \mapsto \sigma \mid \alpha \sim \sigma \in E \right] \]

**Theorem 4.2** (Soundness). Let \( \Gamma \) be a closed environment and \( e \) an expression. If \( \Gamma \vdash e : \sigma \leadsto C \) and \( C_s \) is a solution for \( C \) with an induced substitution \( \bar{C}_s \), then we have \( \Gamma \vdash e : \bar{C}_s(\sigma) \).

**Theorem 4.3** (Principality). Suppose \( \Gamma \vdash e : \sigma \). Then there exists a type \( \phi \) such that \( \Gamma \vdash e : \phi \), and for every other \( \Gamma \vdash e : \sigma' \), there is a fully monomorphic substitution \( \pi \) such that \( \sigma' = \pi \phi \).

**Theorem 4.4** (Completeness). Let \( \Gamma \) be a closed environment and \( e \) an expression. If \( \Gamma \vdash e : \sigma \) then \( \Gamma \vdash e : \phi \leadsto C \) and \( C \) can reach a solved form.

Proofs of these results are given in Appendix D.2.

### 4.5 Alternative solver for equalities

Unfortunately, once we extend the language of types, by introducing type classes and local assumptions, the approach by Pottier and Rémy [15] is no longer applicable. With that in mind, we introduce a different approach to solve the problem of unification under a mixed prefix, which does scale to handle these extensions. The approach is rather simple – a single rule \textit{float} in Figure 10 – and is directly inspired by how GHC handles constraints: by floating constraints out from inside a quantification constraint. When can we do that? Precisely when the constraint does not mention the skolem. But what about the existentials? Consider:

\[ \exists a \ldots (\forall a. \exists \beta. (\alpha \sim [\beta]) \land C) \ldots \]

We would like to float the constraint \( (\alpha \sim [\beta]) \) out of the quantification constraint, but then \( \beta \) would be out of scope. We can solve this by “promoting” \( \beta \): producing a fresh \( \beta' \) that lives in the outer scope, and making \( \beta \) equal to it, thus:

\[ \exists a, \beta' \ldots (\alpha \sim [\beta']) \land (\forall a. \exists \beta. (\beta \sim \beta' \land C)) \ldots \]

All this is expressed directly by rule \textit{float}. If we cannot float, we have a skolem escape error; for example, consider:

\[ \exists a \ldots (\forall a. \exists \beta. (\alpha \sim [a]) \land C) \ldots \]

Here we cannot float \( (\alpha \sim [a]) \) because it mentions the skolem \( a \), so an inner skolem has leaked into an outer scope \( (\alpha \) is bound further out). Floating makes manifest that skolem escape has not happened, and brings the constraint nearer to solved form.

**Conjecture 4.5.** The solver presented in Figure 8 is complete for unification problems under a mixed prefix.

### 5 Practical matters

We have implemented a prototype of the type inference process described in this paper, including support for Haskell’s type classes by extending GI as described in Appendix B. The expressions in Figure 2 are accepted or rejected as described by the table, in the GI column.

GI does not support any co- or contra-variance in function types. For example, the constraint \( \text{Int} \rightarrow (\forall a. a \rightarrow a) \leq \text{Int} \rightarrow \text{Int} \rightarrow \text{Int} \) does not hold. In contrast, GHC with the \textit{RankNTypes} extension supports some amount of variance. Libraries such as Scrap Your Boilerplate use this fact very often in definitions with a \( \forall \) to the right of an arrow:

\[
\begin{align*}
\forall a. a \rightarrow (\forall b. b \rightarrow b) \\
\forall x. y \rightarrow y
\end{align*}
\]

Other uses of variance can be worked around by \( \eta \)-expansion. Consider \( (\text{flip } f) \), where \( \text{flip} \)'s type is in Figure 1. This is ill-typed because \text{flip} requires an argument of type \( a \rightarrow b \rightarrow c \), but \( f \)'s type, after instantiation, looks like \( \tau \rightarrow \forall b. b \rightarrow b \). The fix is simple: just \( \eta \)-expand the argument, thus \((\text{flip } (\lambda x \rightarrow f x))\). GHC does this automatically at the moment, but in fact this \( \eta \)-expansion is unsound in general, since a change in the laziness behavior can be observed.

One might worry that if GI is integrated in GHC many existing Haskell libraries would need to be modified. To quantify this impact, we modified GHC to impose those restrictions and rebuilt all the packages in Stackage which require the \textit{RankNTypes} extension. In order to minimize the annotation burden, we added a simple special case to support function definition in the style of Scrap Your Boilerplate.

With this done, very few packages required modifications, and modifications were always \( \eta \)-expansions. In particular, of the 2,400 packages in Stackage, 609 use \textit{RankNTypes}; of these, only 75 required manual changes, all of which were simple \( \eta \)-expansions. One (\textit{singletons}) would require larger changes, because it uses Template Haskell to generate Haskell code; so it needs to generate \( \eta \)-expanded code. Two more failed for reasons we have yet to investigate. Our conclusion is that the impact of our proposed changes is extremely minor, especially since GHC’s current covert \( \eta \)-expansion strategy is unsound in the first place.

### 6 Related work

Full type inference for System F is undecidable [24] – partial type inference with known generalisation positions but unknown instantiations can be reduced to higher-order unification [14]. System F lacks principal types, making modular type inference and the addition of ML-style let-bindings impossible. Higher-rank type inference with \textit{predicative} instantiation has some successful solutions [4, 13, 17], exploiting a mix of annotation propagation and unification.

On the other hand, no solution for impredicativity with a good benefit-to-weight ratio has been presented to-date.
MLF \[1, 2, 18\] is an extension of System F based on quantification with instance and equality bounds. The resulting system is powerful, but also quite complex to implement; in return we get back principal types. There have been several attempts to simplify the user-facing part of MLF to System F types. FPH \[23\] exposes a “box” structure around inferred types (that would be hidden under a constraint in MLF). Flexible types \[9\], also known as HML, avoid quantification over equality constraints. Implementing these systems in a working compiler is a significant undertaking, and so is the implementation of MLF with only “flexible” type bounds in constraints – fails to type check almost the same programs as those systems. To determine why a program fails to type check (and how it should be fixed) it suffices to determine whether some function has been instantiated to a type with top-level polymorphism in its arguments. For example, \((\text{choose id} \; \text{ids})\) fails to type check because it requires the instantiation of \((\forall a. a \to a) \to (\forall a. a \to a) \to (\forall a. a \to a)\) and no argument has a type with a top-level constructor.

HMF is based on local decisions about polymorphic instantiations and – without an extension to \(n\)-ary applications – fails to type check programs where the local instantiation has to be delayed to take more arguments into account (e.g. fails to type check \(id : ids\), though it does generalize in argument positions (e.g. \(\text{snoc} \; id \; ids\) is accepted, for \(\text{snoc} : \forall a. [a] \to a \to [a]\)). Leijen \[8\] proposes an extension of the basic algorithm to \(n\)-ary applications that makes these examples type check: after the function type is instantiated enough to cover all the arguments, we proceed to type check the arguments in a computed order, instead of left-to-right.

The arguments that must be type checked against a naked type variable coming from the instantiated function type are postponed and checked last, in the hope that the rest of the arguments will by that time determine any impredicative instantiations. The procedure is iterated, possibly uncovering impredicative instantiations in each round. Under that extension \(id : ids\) is accepted. Alas, the new algorithm is not accompanied with a declarative specification. Our system achieves a similar effect (and we conjecture is equally expressive), thanks to the delaying we get from the use of constraints. In the end, the main important difference of HMF with GI is that GI provides a declarative specification and an algorithm that easily integrates in a pre-existing constraint-based type inference engine.

### 7 Further work

GI seeks a sweet spot that balances simplicity with expressiveness. We are also exploring some nearby variants. For example, extending \textsc{Vargen} to handle larger expressions would allow us to accept examples A9, C8, C9 in Figure 2; and by improving skolemisation we could accept E3. It remains to be seen whether the extra expressiveness justifies the extra complexity, but GI seems to be an encouragingly robust base camp.
References


Data constructor \[ \exists K \]

Expressions / terms \[ e ::= \ldots | \text{case } e_0 \text{ of } \{ K x \rightarrow e \} \]

\[\Gamma \vdash e : \sigma\]

for each branch \(K_i x_i \rightarrow e_i\) with \(K_i : \forall \alpha \exists b_i \rightarrow \sigma_i \rightarrow \alpha \in \Gamma\)

\[\Gamma, x_i : [a \mapsto \phi_0] \sigma_i \vdash e_i : \phi_*\]

\[\Gamma \vdash \text{case } e_0 \text{ of } \{ K x \rightarrow e \} : \phi_*\]

**Figure 11.** Decl. type system with pattern matching

### A Pattern matching

Introducing pattern matching in the language gives no surprises, as witnessed by the rule **Case** in Figure 11. We first need to check that the scrutinized expression \(e_0\) can be given a type compatible with the indicated data constructor. It is important to note that the mere presence of the data constructors is enough to know the type constructor \(T\), there is no inference at that point. Then we introduce new term variables in each branch, whose type is obtained by combining the type of the constructor with the inferred values for the type variables in \(T\), namely \(\phi_0\). The return type of every branch should be equal, \(\phi_*\) (note that we do not allow the types of the branches to be instantiated).

The corresponding **Case** rule for constraint gathering is given in Figure 12. In this case we need a quantification constraint to introduce new skolem constants, for the existentially quantified type variables \(b_i\) in the data constructor.

### B Integration with other language features

One of the main advantages of organising a type checker around the concept of constraints is that extensions to the type system can be accommodated for quite easily. Indeed, this is one of the main advantages of our system over existing work. In this section we describe how to deal with other forms of constraint beyond the standard equality constraints we now support.

For example, Haskell supports *type classes*, which restrict the scope of a polymorphic abstraction to a subset of types. The archetypal example is *Eq*, which describes the types with support for decidable equality. Such a type class constraint is visible in the type of the equality operator (≡) \(\forall a. \text{Eq } a \Rightarrow a \rightarrow a \rightarrow \text{Bool}\). Similarly, languages like OCaml and Pure-Script feature *row* (or *record*) types like \{ x :: Point, y :: Point \} in addition to usual ADTs.

The good news is that a constraint-based formulation of typing makes it easy to cope with new concepts if they can also be described in terms of constraints. Several examples in the literature, like HM(X) [15] and OutsideIn(X) [21], are actually frameworks which can be parametrized by different constraint systems, hence the \(X\) in their names.

The modifications needed to accommodate the new kinds of constraints in GI are given in Figure 13. First, we split up the syntax of constraints into so-called simple and extended constraints. The former consists of constraints that can be used by programmers in their programs, while the second category consists of additional constraints that are internal to the solver. Syntax for any new kind of constraints can be added to the syntax of simple constraints \(Q\); we have in fact done so for the specific example of type classes. We also modify the syntax of polymorphic types so that they may contain a number of simple constraints.

Now, suppose that we need to check that \(\forall a. \text{Ord } a \Rightarrow a \rightarrow \text{Bool}\) is an instance of \(\forall a. \text{Eq } a \Rightarrow a \rightarrow \text{Bool}\) in other words \(\forall(a). \text{Eq } a \Rightarrow a \rightarrow \text{Bool}\) \(\leq\) \(\forall(a). \text{Ord } a \Rightarrow a \rightarrow \text{Bool}\). During this process we are allowed to assume that \(\text{Ord } a\) holds in order to discharge \(\text{Eq } a\). To be able to store this information in our extended constraints, we modify the syntax of quantification constraints to include the assumed information as part of an implication, in this case

\[\forall a. (\text{Ord } a \Rightarrow \forall(a). \text{Eq } a \Rightarrow a \rightarrow \text{Bool} \leq a \rightarrow \text{Bool})\]

Implication constraints are also introduced by the updated rules for constraint gathering. An annotated application – rule **AnnApp** – may also mention constraints, which are assumed while checking the enclosed expression. Rule **Case** allows some constraints to be locally valid, which means that the system gains support for generalized algebraic data types (GADTs).

Figure 14 presents the updates needed for the solver to cope with this new form of constraints. The solver now relates two sets of constraints: the *assumptions* and the *wanted* constraints. Whenever we go under an implication, we introduce the constraints that are part of the antecedent as additional assumptions. The solver is allowed to rewrite in both sets, and more importantly, to use an assumed constraint to rewrite a wanted one. The modifications to the rules are very much in line with Vytiniotis et al. [21].

The rules that deal with polymorphic types, namely **InstVL** and **InstVR** need to be updated. In the former case, a constraint in the type becomes an obligation to prove that it holds. In the latter case the constraints become assumptions that we are allowed to use while solving; these are stored in an implication constraint. Note that the guardedness restrictions do not change by the introduction of constraints.

Floating of constraints and promotion of variables also require changes, as described in rule **float**. In particular, we are only allowed to float equality constraints, which in the vanilla system were the only kind of simple constraints. The reason is that other kinds of constraints may incorporate information of the assumptions while they are solved, and these assumptions only hold in their respective branches. Equality constraints do not pose this problem.
\[ \Gamma \vdash e : \sigma \rightsquigarrow C \]

for each branch \( K_i \vec{x}_i \rightarrow e_i \)

\[ \Gamma \vdash e_0 : \sigma_0 \rightsquigarrow C_0 \quad \bar{\alpha}, \beta \text{ fresh} \]

\[ \begin{align*}
K_i & : \forall \bar{a} \bar{b}_i, \bar{\sigma}_i \rightarrow \top \bar{a} \in \Gamma \\
\Gamma, x_i : [\bar{a} \mapsto a^\alpha] \bar{\sigma}_i \vdash e_i : \phi_i \rightsquigarrow C_i \\
\bar{v}_i & = \text{fuv}(\phi_i, C_i) - \text{fuv}(\Gamma) - \bar{\alpha}
\end{align*} \]

\[ \Gamma \vdash \text{case } e_0 \text{ of } \{ K \vec{x} \rightarrow e \} : \beta^\alpha \rightsquigarrow C_0 \land (\sigma_0 \ll^e \epsilon ; \top a^\alpha) \land \forall \bar{b}_i, \exists \bar{\eta}_i, (C_i \land \beta^\alpha \rightsquigarrow \phi_i) \]

**Figure 12.** Constraint generation for pattern matching

- **Polymorphic types**
  \( \sigma, \phi ::= \alpha^\nu \mid \forall \alpha. \ Q \Rightarrow \mu \)

- **Simple constraints**
  \( Q ::= \top \mid Q_1 \land Q_2 \mid \sigma \rightsquigarrow \phi \mid \cdots \)

- **Equality**
  \( \cdots \)

- **Open for extension**
  \( C \sigma_1 \ldots \sigma_n \)

- **Type classes, for example**
  \( \forall \alpha. \ Q \mid C_1 \land Q_2 \)

- **Instantiation**
  \( \sigma \preceq^\tau \phi ; \mu \)

- **Generalisation**
  \( \sigma \preceq \bar{\tau} \)

- **Quantification and implication**
  \( \therefore \forall \bar{\alpha}. \ \exists \bar{\eta}_i. \ (Q \supset C) \)

\[ \begin{align*}
\Gamma & \vdash e : \sigma \rightsquigarrow C \\
\vdots & \\
\vdots & \\
\text{ANNAPP} & \text{for each branch } K_i \vec{x}_i \rightarrow e_i \\
\Gamma & \vdash e_0 : \sigma_0 \rightsquigarrow C_0 \quad \bar{\alpha}, \beta \text{ fresh} \]

\[ \begin{align*}
\Gamma & : \forall \bar{a} \bar{b}_i, \bar{\sigma}_i \rightarrow \top \bar{a} \in \Gamma \\
\Gamma, x_i : [\bar{a} \mapsto a^\alpha] \bar{\sigma}_i \vdash e_i : \phi_i \rightsquigarrow C_i \\
\bar{v}_i & = \text{fuv}(\phi_i, C_i) - \text{fuv}(\Gamma) - \bar{\alpha}
\end{align*} \]

\[ \Gamma \vdash \text{case } e_0 \text{ of } \{ K \vec{x} \rightarrow e \} : \beta^\alpha \rightsquigarrow C_0 \land (\sigma_0 \ll^e \epsilon ; \top a^\alpha) \land \forall \bar{b}_i, \exists \bar{\eta}_i, (Q_i \supset C_i \land \beta^\alpha \rightsquigarrow \phi_i) \]

**Figure 13.** Extensions for integration with other constraints

The \( C ; \bar{v} \Rightarrow C' ; \bar{v}' \) relation only deals with one set of constraints. But as implications enter the game, we need to consider the interaction between two sets: the assumed and the wanted ones. For that matter we introduce a new rewriting judgment \( Q_a ; C ; \bar{v} \Rightarrow Q'_a ; C' ; \bar{v}' \), where assumed constraints \( Q_a \) are rewritten to \( Q'_a \) and wanted constraints \( C \) to \( C' \). Note that set of assumed constraints consists only of simple constraints.

Rules \textsc{assumed} and \textsc{wanted} allow us to apply any rule to any of the two sets involved in solving. Rule \textsc{interact} deals with the interaction of an assumption and a wanted constraint; the result is put on the wanted set. For example, if you have \( \alpha \sim \text{Int} ; \beta \sim [\alpha] ; \alpha, \beta \), the rule moves to \( \alpha \sim \text{Int} ; \beta \sim [\text{Int}] ; \alpha, \beta \). Rule \textsc{dupl} is a simple case of interaction, in which a constraint in the wanted set is “crossed out” if it is already in the assumed set. Note that information may only flow from assumptions to wanted constraints, and never the other way around.

The last rule, an updated version of \textsc{forall}, is responsible for dealing with implication constraints. The constraints \( C \) inside of the implication are rewritten in an environment where the set of assumed constraints is enlarged with the antecedent \( Q \) from the implication.

### C From and to System F

Our system is as expressive as System F, the gold standard for a fully-expressive impredicative type system. We show so by describing a translation from every System F expression to GI in Figure 15. We also give the converse translation from GI terms to System F terms in Figure 16 (as usual, \( e \) denotes an empty list). In particular, this proves that GI is sound as a type system. For the sake of conciseness we do not include the translations of pattern matching.
Theorem C.1 (Embedding of System F). Let $e$ be a System F expression. If $\Gamma \vdash^F e : \sigma \leadsto e'$, as defined in Figure 15, then $\Gamma \vdash e' : \sigma$ in GI.

The main difference between GI and System F is that in the former impredicative instantiation is restricted by guardedness. The presented translation relies on annotations to work around them.

- Following GI we present $n$-ary application – which in the case of System F also includes type application. The given application rule only works if the guardedness restrictions are satisfied, otherwise an annotation on $e_0$ needs to be added before applying the rule.
- In GI the result of a non-annotated application always gets a top-level monomorphic type. This is not the case in System F, and thus an annotation may be required to further generalize.
- Completely unrestricted instantiation is only available via annotations. In order to apply this translation, we need to split applications so that guardedness guarantees are always met. Every time we split, we introduce a new annotation guiding the type checking process.

**Proof.** By induction on the typing derivation in System F, $\Gamma \vdash^F e : \sigma$.

\[
\frac{C \; \overrightarrow{v} \implies C' \; \overrightarrow{v}'}\]

\[
\begin{array}{c}
\text{[INST\forall]} \quad (\forall \overrightarrow{\alpha}. \, Q \Rightarrow \mu) \leq^m_{\overrightarrow{\alpha}} \overrightarrow{\eta} ; \overrightarrow{v} \quad \implies \quad [a \mapsto \overrightarrow{v}][Q] \land (\mu' \leq^m_{\overrightarrow{\eta}} \overrightarrow{\eta}) : \overrightarrow{v}, \overrightarrow{v}'
\end{array}
\]

where freshen$^m_{\overrightarrow{\alpha}}(\forall \overrightarrow{\alpha}. \mu) \implies (\overrightarrow{v}, \mu')$

\[
\begin{array}{c}
\text{[INST\forall]} \quad e \leq (\forall \overrightarrow{\alpha}. \, Q \Rightarrow \mu) ; \overrightarrow{v} \quad \implies \quad \forall \overrightarrow{\alpha}. (\forall \overrightarrow{\eta} [\overrightarrow{v} \leq \mu] ; \overrightarrow{v})
\end{array}
\]

$F$ is an equality

\[
\begin{array}{c}
\text{[FLOAT]} \quad \overrightarrow{\alpha} \in \text{fuv}(F) \cap \overrightarrow{v}_{\text{in}} \quad \overrightarrow{\gamma} \text{ fresh} \quad \overrightarrow{E} = \bigwedge_{\alpha' \in F} \alpha' \sim \gamma^p
\end{array}
\]

\[
\forall \overrightarrow{\alpha}. \exists \overrightarrow{\alpha}_{\text{in}}. (Q \supset C \land F) ; \overrightarrow{v} \quad \implies \quad [\overrightarrow{\alpha} \mapsto \gamma^p]F \land \forall \overrightarrow{\alpha}. \exists \overrightarrow{\alpha}_{\text{in}}. (Q \supset C \land F) ; \overrightarrow{v}, \overrightarrow{\gamma}^p
\]

\[
\begin{array}{c}
\frac{Q_a ; C ; \overrightarrow{v} \implies Q'_a ; \overrightarrow{v}}{[\text{Assumed}]} \frac{Q_a ; \overrightarrow{v} \implies Q'_a ; \overrightarrow{v}}{[\text{Wanted}]} \frac{\frac{Q_a ; C ; \overrightarrow{v} \implies Q'_a ; C ; \overrightarrow{v}'}{[\text{Interact}]}}{\frac{Q_a \land Q ; C \land C' ; \overrightarrow{v} \implies Q_a \land Q ; C \land C' ; \overrightarrow{v}}{[\text{DUP}]}} \frac{Q_a \land Q ; C ; \overrightarrow{v}_{\text{in}} \implies Q' ; C' ; \overrightarrow{v}_{\text{in}}}{[\text{Forall}]}
\end{array}
\]

Figure 14. Solving rules for implication constraints

**Variable.** We need to derive $\Gamma \vdash x : \sigma$, for a general $\sigma = \forall \alpha. \mu$. We recall that single variables are treated like 0-ary application, so some amount of instantiation must take place.

\[
\begin{array}{c}
\frac{x : \sigma \in \Gamma}{\frac{\Gamma \vdash x : \mu}{\text{VarHead} \quad \sigma \leq^m_{\epsilon} \epsilon ; \mu}}
\end{array}
\]

If the set of quantified variables $\overrightarrow{a}$ is empty, then this derivation is all we need. If it is not, we need an annotation to re-generalise those.

\[
\begin{array}{c}
\frac{x : \sigma \in \Gamma}{\frac{\Gamma \vdash (x : \forall \overrightarrow{\alpha}. \mu) : \forall \overrightarrow{\alpha}. \mu}{\text{AnnApp} \quad \sigma \leq^m_{\epsilon} \epsilon ; \mu}}
\end{array}
\]

We could get a smaller translation in some cases by using the VarGen rule, but the simpler type system with annotations is enough for our goals.

**Abstraction.** By induction hypothesis we are able to type check the body of the abstraction. The distinction between fully monomorphic and unrestricted types in the translation ensures that we can use the right Abs or AnnAbs rule from the declarative specification.
Type abstraction. There are two cases to consider:

- \([e']\) is not an application. In this case the annotation is treated as an annotated application with zero arguments. By induction hypothesis, we know that \(\Gamma \vdash e' : \phi[a]\). We assume without loss of generality that \(\phi = \forall \overline{b}. \mu\), and thus \(\Gamma \vdash e : \forall \overline{b}. \mu[a]\). Now we can build the following derivation:

\[
\begin{align*}
\Gamma \vdash e' &\vdash \forall \overline{b}. \mu[a] \\
\Gamma \vdash \text{fun} e' &\vdash \forall \overline{b}. \mu[a] \quad \forall \overline{b}. \mu[a] \leq^i \epsilon; \mu[a] \\
\Gamma \vdash (e' : \forall \overline{b}. \mu) &\vdash \forall \overline{b}. \mu
\end{align*}
\]

- \([e']\) is an application. In this case \(e'\) results from the application of either \(\text{App}\) or \(\text{ANNApp}\) in the declarative specification. By inspection of the rule \(\text{ANNApp}\), we can see that we can always choose to quantify over more variables via an annotation.

(Type) application. The premises about guardedness ensure that we can apply the rule \(\text{App}\) or \(\text{ANNApp}\) in the declarative specification. As discussed in the main text, if at a certain application we cannot apply this rule, we can always split the application, annotate the head and then use the application rule again.

D Proofs

D.1 Declarative specification

Lemma D.1. Let \(\Gamma\) be an environment and \(e\) an expression. For every pair of fully monomorphic substitutions \(\theta_1\) and \(\theta_2\), if \(\theta_1 \Gamma \vdash \text{fun} e : \sigma_1\) and \(\theta_1 \Gamma \vdash \text{fun} e : \sigma_2\), then there exists a polymorphic type \(\sigma^*\) and fully monomorphic substitutions \(\phi_1\) and \(\phi_2\) such that \(\sigma_1 = \phi_1 \sigma^*\).

Theorem 3.2 (Impredicative instantiation is not guessed). Let \(\Gamma\) be an environment and \(e\) an expression. For every pair of fully monomorphic substitutions \(\theta_1\) and \(\theta_2\), if \(\theta_1 \Gamma \vdash e : \sigma_1\) and \(\theta_2 \Gamma \vdash e : \sigma_2\), then there exists a polymorphic type \(\sigma^*\) and fully monomorphic substitutions \(\phi_1\) and \(\phi_2\) such that \(\sigma_1 = \phi_1 \sigma^*\).

Proof. We prove Lemma D.1 and Theorem 3.2 by mutual induction over the typing derivation of \(e\).

Proof of Lemma D.1. We distinguish two cases:

- Case \text{VARHEAD}. The two derivations to consider are:

\[
\begin{align*}
\theta_1 \Gamma \vdash \text{fun} x &\vdash \theta_1 \sigma \\
\theta_2 \Gamma \vdash \text{fun} x &\vdash \theta_2 \sigma
\end{align*}
\]

Thus we can take \(\sigma^* = \sigma\) and as substitutions those applied to \(\Gamma\), which are fully monomorphic by our assumptions.

- Case \text{EXPRHEAD}. Follows by induction over the premise.
\[
\begin{align*}
\sigma & \leq^* \psi; \mu \leadsto \psi_1, \ldots, \psi_n, \psi_r \\
\mu & \leq^* e; \mu \leadsto e \quad \text{InstMono} \\
\phi_2 & \leq^* \sigma_2, \ldots, \sigma_n; \mu \leadsto \psi_2, \ldots, \psi_n, \psi_r \quad \text{InstArrow} \\
\phi_1 \rightarrow \phi_2 & \leq^* \sigma_1, \ldots, \sigma_n; \mu \leadsto e, \psi_2, \ldots, \psi_n, \psi_r \\
\forall a, \mu \triangleright^* \Delta & \theta \text{ respects } \Delta \quad \theta \mu \leq^* \sigma_1, \ldots, \sigma_n; \eta \leadsto e, \psi_2, \ldots, \psi_n, \psi_r \quad \text{InstPoly}
\end{align*}
\]

\[\Gamma \vdash \text{fun} \ e : \sigma \leadsto e_F\]

\[\begin{align*}
\Gamma \vdash x : \sigma & \in \Gamma \\
\Gamma \vdash \text{fun} \ x : \sigma & \leadsto x \\
\Gamma \vdash e : \sigma & \leadsto e_F \\
\Gamma \vdash e : \sigma & \leadsto e_F \\
\Gamma \vdash \text{arg} \ e & : \sigma \leadsto e_F \\
\Gamma \vdash e : \forall a, \mu \leadsto e_F & \triangleleft \bar{a} \notin \Gamma \\
\Gamma \vdash \text{arg} \ e : \forall b. \lbrack a \mapsto \tau \rbrack \mu \leadsto b, e_F \bar{a} & \triangleleft \bar{a} \notin \Gamma \\
\Gamma \vdash x : \forall b. \tau & \in \Gamma \\
\Gamma \vdash \text{arg} \ x : \forall b. \lbrack a \mapsto \sigma \rbrack \tau & \leadsto b, x \bar{a} \\
\end{align*}\]

\[\begin{align*}
\Gamma \vdash \lambda x. \ e : \sigma & \leadsto \lambda(x :: \tau). e_F \\
\end{align*}\]

\[\begin{align*}
\Gamma \vdash \lambda(x :: \phi). \ e : \phi & \leadsto \sigma = \lambda(x :: \sigma). e_F \\
\end{align*}\]

\[\begin{align*}
\Gamma \vdash \text{fun} \ e_0 : \phi & \leadsto e_0, F \\
\phi & \leq^* e_0, \ldots, e_0; \sigma_1, \ldots, \sigma_n; \mu \leadsto \psi_1, \ldots, \psi_n, \psi_r \\
\Gamma & \vdash \text{arg} \ e_1 : \sigma_1 \leadsto e_F, 1 \\
\Gamma & \vdash \text{arg} \ e_n : \sigma_n \leadsto e_F, n \\
\Gamma & \vdash e_0, e_1, \ldots, e_n : \mu \leadsto e_0, F \psi_1, e_F, 1 \ldots \psi_n, e_F, n, \psi_r \\
\Gamma & \vdash \text{fun} \ e_0 : \phi \leadsto e_0, F \\
\phi & \leq^* e_0, \ldots, e_0; \sigma_1, \ldots, \sigma_n; \eta \leadsto \psi_1, \ldots, \psi_n, \psi_r \\
\Gamma & \vdash \text{arg} \ e_1 : \sigma_1 \leadsto e_F, 1 \\
\Gamma & \vdash \text{arg} \ e_n : \sigma_n \leadsto e_F, n \\
\Gamma & \vdash (e_0, e_1, \ldots, e_n :: \forall b. \eta) : \forall b. \eta \leadsto b, e_0, F \psi_1, e_F, 1 \ldots \psi_n, e_F, n, \psi_r \\
\Gamma & \vdash e_1 : \phi \leadsto e_F, 1 \\
\Gamma, x : \phi & \vdash e_2 : \sigma \leadsto e_F, 2 \\
\Gamma & \vdash \text{let} \ x = e_1 \ \text{in} \ e_2 : \sigma \leadsto (\lambda(x :: \phi). e_F, 2) e_F, 1
\end{align*}\]

\textbf{Figure 16.} Translation to System F
Proof of Theorem 3.2. We distinguish seven cases:

- **Case Abs.** The derivations look like:

\[
\begin{align*}
\theta_1 \Gamma, x : r_1 &\vdash e : \sigma_1 \\
\theta_2 \Gamma &\vdash \lambda x. e : r_2 \rightarrow \sigma_2
\end{align*}
\]

We cannot apply the induction hypothesis yet, since the environments in the premises are not of the right shape. Consider instead the environment \(\Gamma' = \Gamma, x : \alpha\) for a fresh \(\alpha\), and the substitutions

\[
\theta'_1 = [\alpha \mapsto r_1] \circ \theta_1 \quad \theta'_2 = [\alpha \mapsto r_2] \circ \theta_2
\]

These substitutions are fully monomorphic, since all of \(\theta_i\) and \(\tau_i\) \((1 \leq i \leq 2)\) are fully monomorphic by hypothesis. We have then the following equalities over the environments

\[
\theta_1 \Gamma, x : r_1 = \theta'_1 \Gamma' \quad \theta_2 \Gamma, x : r_2 = \theta'_2 \Gamma'
\]

and thus we can apply the induction hypothesis to \(e\) to obtain \(\sigma^*, \phi_1\) and \(\phi_2\) such that \(\phi_1 \vdash \sigma^*\). Consider now the extended substitutions:

\[
\phi'_1 = [\alpha \mapsto r_1] \circ \phi_1 \quad \phi'_2 = [\alpha \mapsto r_2] \circ \phi_2
\]

Since \(\tau_1\) and \(\tau_2\) are fully monomorphic types, \(\phi'_i\) are fully monomorphic substitutions. Take \(\sigma' = \alpha \rightarrow \sigma^*\). We have, for each derivation, that

\[
\phi'_1 \sigma' = \phi'_1(\alpha \rightarrow \sigma^*) = \phi'_1 \alpha \rightarrow \phi'_2 \sigma' = \tau_1 \rightarrow \sigma_i
\]

And we are done: \(\sigma', \phi'_1\) and \(\phi'_2\) are the desired outputs of the theorem.

- **Case AnnAbs.** Similar to Abs.

- **Case App.** In this case the derivations have the following shape for each \(i \in \{1, 2\}\):

\[
\begin{align*}
\theta_1 \Gamma \vdash \text{fun } e_0 : \theta_1 \phi \\
\phi &\vdash^{m} \sigma_{i,1}, \ldots, \sigma_{i,n}; \mu_i \\
\theta_2 \Gamma \vdash \text{arg } e_j : \sigma_{i,j}
\end{align*}
\]

We have two choices: either \(\phi\) is a top-level monomorphic type \(\eta\), or \(\theta_1 \phi = \forall \eta. \theta_2 \eta\). This shall remain true during the derivation of \(\vdash^m\). Intuitively, the substitutions do not alter the “polymorphism structure” of the type. In particular, as result type of the instantiation, we get \(\mu_1\) and \(\mu_2\) for which we know that there exists a common \(\mu^*\) such that:

\[
\begin{align*}
\mu_1 &= \phi_1^m \xi_1 \theta_1 \mu^* \\
\mu_2 &= \phi_2^m \xi_2 \theta_2 \mu^*
\end{align*}
\]

and all of \(\phi_1^m, \xi_1\) and \(\theta_2\) are fully monomorphic. We can use \(\text{VARGEN}\) we need to work a bit more. If for each type variable there is one \(e_a\) where \(\text{VARGEN}\) has been applied, the proof still works. On the other hand, if for a type variable \(a\) all expressions where it appears make use of \(\text{VARGEN}\), we cannot guarantee the existence of the \(\sigma^*_j\) and fully monomorphic substitutions \(\xi_{i,a}\) by induction hypothesis. However, in that case the definition of \(\vdash^m\) ensures that the substitution for the type variable \(a\) is fully monomorphic, giving us the desired result.

- **Case AnnApp.** This case is trivial, since the annotation ensures that both derivations yield the same type.

- **Case Let.** In this case the derivations are:

\[
\begin{align*}
\theta_i \Gamma &\vdash e_1 : \phi_i \\
\theta_i \Gamma, x : \phi_i &\vdash e_2 : \sigma_i \\
\theta_i \Gamma &\vdash \text{let } x = e_1 \text{ in } e_2 : \sigma_i
\end{align*}
\]

for \(i \in \{1, 2\}\). We cannot readily apply the induction hypothesis to the second premise – which would give us the desired conclusion –, since the environment may not be in the right shape. By the induction hypothesis on the first premise, there exists a type \(\phi^*\) and fully monomorphic substitutions \(\phi_1\) and \(\phi_2\) such that \(\phi_1 = \phi_2 \phi^*\). We can build a new version of \(\phi^*, \phi'_i\), where all the free variables are fresh – and thus disjoint from those in \(\Gamma\) – and corresponding \(\phi'_i\) where the domain is replaced.
by these free variables. Consider \( \theta'_i = \theta_i \circ \varphi'_i \):

\[
\begin{align*}
\theta'_i(\Gamma, x : \phi') &= \theta'_i(\Gamma), x : \theta'_i \phi'
\end{align*}
\]

- (substitutions have disjoint domains)

- (by definition of \( \varphi'_i \))

\[\theta'_i(\Gamma), x : \phi_i\]

This means that we have found a fully monomorphic substitution \( \theta'_i \) for the environment, which allows us to apply induction on the second premise and reach the desired conclusion.

- **Case Case.** By induction on the first premise of Case, we get \( \sigma_0 \) and two substitutions \( \varphi_1 \) and \( \varphi_2 \) such that \( \sigma_{0,i} = \varphi_i \sigma_0 \) for each of the two derivations. Without loss of generality, we can assume that \( \sigma_0 = \forall \bar{a}. \mu_0 \).

Then \( \sigma_{0,i} = \forall \bar{a}. \varphi_i \mu_0 \).

The next step is to notice that the instantiation judgment is nullary. As a result, all the type variables in \( \bar{a} \) must be substituted by fully monomorphic types. Let \( \pi_i \) be the substitution for each of the derivations. Then we know that \( T_{\varphi_0,i} = \pi_i \varphi_i \mu_0 = T_{\pi_i \varphi_i \mu_0} \).

Playing the same game we did for \( \text{LET} \), we can build substitutions \( \theta'_i \) such that:

\[
\theta'_i(\Gamma, x_i : [a \mapsto \phi_0,i] \sigma_j) = \theta_i(\Gamma, x_i : [a \mapsto \pi_i \varphi_i \phi_0] \sigma_j)
\]

The last step is applying induction hypothesis to one of the branches and obtain the desired common polymorphic type and fully monomorphic instantiations. The branch we choose does not matter, since they all give the same type as result.

\( \square \)

**Theorem 3.4** (Substitution). If \( \Gamma \vdash u : \sigma \) and \( \Gamma, x : \sigma \vdash e[x] : \phi \), then \( \Gamma \vdash e[u] : \phi \).

**Proof.** By induction over the expression \( e[x] \). The most interesting case is when \( x \) is the head of an application, that is, \( e = x \cdot e_1 \ldots e_n \), and \( u \) is also an application, \( u = u_0 \cdot u_1 \ldots u_m \).

Note that in that case the assigned types are always top-level monomorphic, so \( \sigma = \mu \) and \( \phi = \eta \).

The derivations involved look like:

\[
\begin{align*}
\Gamma \vdash u_0 : \sigma_u \\
\sigma_u \leq^m \pi_{\sigma_0, \ldots, \sigma_m} \sigma_1, \ldots, \sigma_m ; \mu \\
\Gamma \vdash u_0 u_1 \ldots u_m : \mu \\
\Gamma \vdash u_0 : \sigma_u \\
\Gamma \vdash \cdot u_i : \sigma_i \\
\Gamma \vdash e : \phi \\
\mu \leq^m \phi_{i_1}, \ldots, \phi_{i_n} ; \eta \\
\Gamma \vdash e_{i_j} : \phi_{i_j} \\
\Gamma \vdash e_j : \phi_j \\
\Gamma \vdash e : \phi
\end{align*}
\]

and the question is whether we can derive:

\[
\Gamma \vdash u_0 : \sigma_u \\
\Gamma \vdash u_0 : \sigma_u \\
\Gamma \vdash \cdot u_i : \sigma_i \\
\Gamma \vdash e : \phi \\
\mu \leq^m \phi_{i_1}, \ldots, \phi_{i_n} ; \eta \\
\Gamma \vdash e_{i_j} : \phi_{i_j} \\
\Gamma \vdash e_j : \phi_j \\
\Gamma \vdash e : \phi
\]

All the premises on this last rule come directly from those in the hypotheses, except for the instantiation

\[
\sigma_u \leq^m \pi_{\sigma_1, \ldots, \sigma_m, \phi_1, \ldots, \phi_n} ; \eta \\
\Gamma \vdash e : \phi
\]

For the \( n \) last components we can reuse the derivation in the second hypothesis. For the first \( m \) components, inspection on the rules of guardedness show that any instantiation with \( m \) arguments is admissible when \( m + n \) are considered. \( \Box \)

**Theorem 3.5.** Let \( \text{app} : \forall a. b. (a \rightarrow b) \rightarrow a \rightarrow b \) and \( \text{revapp} : \forall a. b. (a \rightarrow b) \rightarrow b \) be the application and reverse application functions, respectively. Given two expressions \( f \) and \( e \) such that \( \Gamma \vdash^\text{fun} f : \sigma_0 \) and \( \sigma_0 \leq^m \epsilon ; \sigma_1 \rightarrow \phi \) then:

\[
\Gamma \vdash f : \phi \iff \Gamma \vdash \text{app} f : e : \phi \iff \Gamma \vdash \text{revapp} e : f : \phi
\]

**Proof.** Let us compare the derivations of the first two.

\[
\begin{align*}
\Gamma \vdash f : \phi \\
\Gamma \vdash \text{app} f : \sigma_0 \\
\sigma_0 \leq^m \epsilon ; \sigma_1 \rightarrow \phi \\
\Gamma \vdash \text{revapp} e : f : \phi
\end{align*}
\]

The application of \( \text{app} \) can be instantiated with any type, given that both variables \( a \) and \( b \) are guarded. We need to consider the two different ways in which \( \Gamma \vdash^\text{arg} f : \sigma_1 \rightarrow \mu \) can be derived. If it is derived using \( \text{ARGGEN} \), then we have:

\[
\Gamma \vdash f : \sigma_0 \\
\epsilon \leq^m \sigma_1 \rightarrow \phi \\
\Gamma \vdash \cdot f : \sigma_1 \rightarrow \phi
\]

Then the premises for the whole derivation are the same, except for the highlighted one. We know that if \( \sigma_0 \leq^m \epsilon ; \sigma_1 \rightarrow \phi \), then \( \sigma_0 \leq^m \epsilon ; \sigma_1 ; \phi \), by inspection of the rule relating instantiation and function types.

The other possibility is that \( f \) is a variable and the rule for single variables, \( \text{VARGEN} \), applies:

\[
\begin{align*}
f : \forall \bar{\rho}. \tau \leq^m \epsilon ; \sigma_1 \rightarrow \phi \\
\Gamma \vdash^\text{arg} f : \sigma_1 \rightarrow \phi
\end{align*}
\]

In this case \( \text{VARGEN} \) does not rule our polymorphic instantiation. However, the \( \leq^m \) judgment applied to \( \text{app} \) will ignore this argument for guardedness purposes. That means that we can only instantiate impredicative those type variables coming for \( e \), exactly the same as we could with \( f \).

The proof for \( \text{revapp} \) is similar to the one for \( \text{app} \). \( \square \)
D.2 Constraint-based formulation

Theorem 4.1. Suppose $\Gamma \vdash e : \sigma \leadsto C$. Then $C$ is either inconsistent, or can be rewritten to a new set $C'$ without instantiation and generalisation constraints which fixes the value of all unrestricted and top-level monomorphic variables.

Lemma D.2. Suppose $\Gamma \vdash e : \sigma \leadsto C$. Then $C$ is either inconsistent, or can be rewritten to a new set $C'$ without instantiation or generalisation constraints which fixes the value of all unrestricted and top-level monomorphic variables.

Proof. We prove both theorems by mutual induction. Note that Lemma D.2 follows simply from Theorem 4.1, as the head typing judgment either produces no constraints with the rule VarHead or refers to the normal typing judgment as a premise with the rule ExprHead.

For Theorem 4.1, we proceed by induction on the constraint generation judgment $\Gamma \vdash e : \sigma \leadsto C$.

Case Abs and AnnAbs. Follows directly from induction hypothesis, since the constraints from the body are copied as the output.

Case App. We need to consider three sets of constraints: the set $C$ coming from generating constraints for the head, the set of instantiation constraints $C_{\leq}$ which extract the expected types of arguments from the function types, and the set of generalisation constraints $C_{<}$ which check that the actual arguments fit into those expected types. The third set is obtained from the ancillary judgement $\vdash^{\arg}$.

The desired properties for the first set of constraints, $C$, follow by induction. Since $C$ fixes all the unrestricted and top-level monomorphic variables, this implies that $\phi$ is fixed up to fully monomorphic variables. As a result, we can rewrite all the constraints in $C_{\leq}$ into equalities – but note that instantiations may introduce new variables to be fixed.

First consider the case in which all applications of $\vdash^{\arg}$ use the AnnGen rule. In that case, the newly-introduced variables can be divided between unrestricted $\overline{u}$, top-level monomorphic $\overline{t}$, and fully monomorphic $\overline{m}$. Only the first two interact with arguments, by definition. For the first set we know that for each variable $\alpha$ there is at least one argument of the form $\forall \overline{u}. T \overline{\phi}$, where $\alpha \in ftv(\overline{\phi})$. For that argument there is a corresponding expression $e_j$ and a generalisation constraint, $\forall \{\overline{y}\} \cdot C_j \Rightarrow \sigma_j \leq (\forall \overline{u}. T \overline{\phi})$.

Rule inst\forall and inst\forallL apply, rewriting this constraint to $C_j \land \sigma_j \leq^m e ; T \overline{\phi}$, possibly under an universal quantifier $\forall \overline{\nu}$. Now we can apply the induction hypothesis to $C_j$, which means that we can rewrite these constraints to a set of type equalities which fixes the unrestricted and top-level monomorphic variables.

We still have a remaining constraint $\sigma_j \leq^m e ; T \overline{\phi}$ and we have not proven yet that the variable $\alpha \in ftv(\overline{\phi})$ is fixed. But since $\sigma_j$ is fixed up to fully monomorphic components, we can decide which instantiation rule to apply; in either case a type equality is produced. In turn, this equality fixes the variable $a$. As a result, the generalisation constraint can be completely turned into equalities, as desired. Furthermore, this value of $\beta$ is floated out of the universal quantification – if it was ever introduced – up to the point where the variable was introduced.

Once the value of $\alpha$ is fixed — because of its appearance under a type constructor —, we can deal with those cases in which the variable appears alone as an argument,

$$(\forall \{\overline{y}\} \cdot C_k \Rightarrow \sigma_k) \leq a$$

But now we already know the type for $\alpha$ up to its fully monomorphic components! Thus, we are able to decide which rule in the solver to apply, and then apply the induction hypothesis. The end result is again a set of equalities, maybe under a universal quantifier.

Arguments which feature a top-level monomorphic variable $\beta \in \overline{t}$ are dealt with in the same way; the fact that the variable is top-level monomorphic is enough to unwrap the generalisation constraint. Then, the solving proceeds as with $\alpha$ unrestricted.

Finally, no constraints are generated over the set of fully monomorphic types $\overline{m}$. This is OK, since these variables are already fixed up to fully monomorphic components, by definition. This case also applies when all the expressions featuring a variable $\alpha$ result from the application of VarGen.

Case AnnApp. This case is almost identical to App. The only difference is that now there is a set of variables $\overline{u}$ which would have been classified as either top-level or fully monomorphic which are now classified as unrestricted.

As in the case of App, we apply the induction hypothesis over the head to guarantee that the set of constraints $C$ can be turned into a set of equalities under a mixed prefix. Since these equalities fix $\sigma$ up to fully monomorphic components, we can still apply the same reasoning to the set of instantiation constraints $C_{<}$. In particular, the constraint $\beta_n \leq^v e ; \eta$ is turned into an equality, too, and fixes the value of all those $\overline{u}$ variables – those are the ones appearing in the assignment to $\beta_n$ and $\eta$ is completely known since it is explicitly given by the programmer.

For the remaining variables we apply the same reasoning as before, distinguishing between unrestricted and top-level monomorphic variables and going through each of the arguments.

D.2.1 Solved form and solutions

In Section 4.4 we introduced the notion of a solved form. In the following results we also need the notion of when a solved form $C_s$ is a solution for a set of constraints $C$. We give the corresponding judgment $C_s \models C$ in Figure 17.
Given two solutions $C_s$ and $D_s$, which range over the same set of variables, we say that $C_s$ is more general than $D_s$ if there exists a fully monomorphic substitution $\pi$ such that for every pair of assignments $\alpha \sim \sigma$ in $C_s$, and $\alpha \sim \phi$ in $D_s$, $\phi = \pi \sigma$.

The solver described in [15] returns most general solutions for a given set of equalities under a mixed prefix. This is a consequence of the fact that they reuse the first-order solver in [11], which always returns most general substitutions.

D.2.2 Soundness and principality

In the proofs of the following results we focus on the core language without let, do and pattern matching. Nevertheless, the results still hold when the language is extended: no surprises there.

**Lemma D.3.** Suppose that $C_s$ is a solution for the constraint:

$$\sigma \preceq_{\leq_m} \alpha_1^n, \ldots, \alpha_n^n, \delta^1$$

Then we have a derivation for:

$$\exists \sigma \subseteq_{\leq_m} C_s(\alpha_1^n), \ldots, C_s(\alpha_n^n); C_s(\delta)$$

(as defined in the declarative specification)

**Proof:** The definition for $\models$ correspond one to one to the rules for $\preceq$ in the declarative specification. $\square$

**Theorem D.4** (Soundness, solution version). Let $\Gamma$ be an environment and $e$ an expression. Then, for every set of constraints $C'$,

1. If $\Gamma \vdash_{\text{fun}} e : \sigma \rightarrow C$, and $C_s \models C \land C'$, then we can build a derivation for $\exists \sigma \subseteq C_s(\Gamma) \vdash_{\text{fun}} e : C_s(\sigma)$.
2. If $\Gamma \vdash e : \sigma \rightarrow C$ and $C_s \models C \land C'$, then we can build a derivation for $\exists \sigma \subseteq C_s(\Gamma) \vdash e : C_s(\sigma)$.
3. If $\Gamma ; \bar{v} \vdash_{\text{app}} e : \sigma \rightarrow C$ and $C_s \models C \land C'$, then we have a derivation for $\exists \sigma \subseteq C_s(\Gamma) \vdash_{\text{app}} e : C_s(\phi)$.

**Proof:** We prove this theorem by mutual induction over the expression $e$.

**Proof of (1).** We distinguish two cases by inversion:

- **Case VarHead.** We have in this case that $\Gamma \vdash_{\text{fun}} x : \sigma \rightarrow \bar{s}$ if $x : \sigma \in \Gamma$. In this case $C_s$ is a solution of $C'$.
  
  By applying the substitution $\exists \sigma \subseteq C_s(\Gamma)$ to the environment we get that $x : C_s(\sigma) \in C_s(\Gamma)$, which allows us to conclude that $C_s(\Gamma) \vdash_{\text{fun}} x : C_s(\sigma)$.

- **Case ExprHead.** Follows directly from the induction hypothesis and (2).

**Proof of (2).** We distinguish four cases by inversion:

- **Case Abs.** In this case the constraint gathering is:
  
  $$\Gamma \vdash \lambda x. e : \alpha^m \rightarrow \mu \rightarrow C$$

  By induction hypothesis we know that for every $C'$, if $C \land C'$ is solved to $C_s$, then we have $C_s(\Gamma), x : C_s(\alpha^m) \vdash e : C_s(\sigma)$. From this we can apply the corresponding rule from the declarative specification.

  $$\exists \sigma \subseteq C_s(\Gamma) \vdash \lambda x. e : C_s(\alpha^m) \rightarrow C_s(\sigma) \rightarrow_{\text{App}} C_s(\Gamma)$$

  Now we just need to notice that $C_s(\alpha) \hookrightarrow C_s(\sigma) \rightarrow C_s(\alpha \rightarrow \sigma)$ and we are done.

  - **Case AnnAbs.** Similar to the Abs case.

  - **Case App.** Suppose that the shape of the expression is $e_0 \ldots e_n$. By applying (1) on $e_0$ we know that a solution $C_s$ determines a derivation $C_s(\Gamma) \vdash_{\text{fun}} \tilde{e}_0 : C_s(\sigma)$. By Lemma D.3, a solution for the $\preceq$ constraint determines a derivation for $C_s(\sigma) \preceq_{\leq_m} \tilde{C}_s(\alpha_1^n), \ldots, \tilde{C}_s(\alpha_n^n); C_s(\delta^1)$. Finally, by applying (3) to each argument, we get the corresponding derivations for $C_s(\Gamma) \vdash_{\text{app}} \tilde{e}_i : C_s(\alpha_i^n)$.

  As a result, we can apply the App rule from the declarative specification.

  - **Case AnnApp.** Similar to the App case.
Proof of (3). We distinguish two cases by inversion:

- **Case ArgGen.** We can assume, without loss of generality, that the generalisation constraint has the form $(\forall \{\eta\}. C \Rightarrow \sigma) \leq (\forall \eta'. \eta)$ for a possibly empty set of variables $\eta$.

Let us first consider the simple case in which the set of quantified variables $\eta$ is empty. By the definition of solution, this implies that we have

$$C_s \vdash C \quad \text{and} \quad C_s \vdash \sigma \leq^m e; \eta$$

Now we can apply (2) to $C_s \vdash C$ and derive that $\widehat{C}_s(\Gamma) \vdash \widehat{C}_s(\sigma)$. By inspection of the rules, we know that a solution of the second constraint, $\sigma \leq^m e; \eta$, defines a fully monomorphic substitution in the declarative specification. As a result, we can apply rule ArgGen and obtain the desired result. The case for a not empty set of constraints is similar, one just has to realize that the rule for $|\cdot|$ for generalization constraints introduce the Skolem variables required by ArgGen.

- **Case VarGen.** In this case the generated constraint is $[\rho \mapsto \alpha^\theta] \tau \leq (\vec{\rho}. \eta)$. As in the last case of ArgGen, the definition of solution tells us that there is a $\forall \rho. C_s \vdash C_s([\rho \mapsto \alpha^\theta] \tau)$ such that $C_s \vdash C_s(C_s) \quad \text{and} \quad C_s \vdash C_s([\rho \mapsto \alpha^\theta] \tau) \leq^m e; C_s(\eta)$.

The second application of $|\cdot|$ is trivially equal to $C_s \vdash C_s([\rho \mapsto \alpha^\theta] \tau) \sim C_s(\eta)$, since the type in the left-hand side is by construction a top-level monomorphic one. This is enough to apply rule VarGen from the declarative specification.

\[\square\]

**Theorem 4.2 (Soundness).** Let $\Gamma$ be a closed environment and $e$ an expression. If $\Gamma \vdash e : \sigma \Rightarrow C \land C_s$ is a solution for $C$ with an induced substitution $\widehat{C}_s$, then we have $\Gamma \vdash e : \widehat{C}_s(\sigma)$.

**Proof.** By Theorem D.4 we know that the existence of such a solution $C_s$ implies that $\widehat{C}_s(\Gamma) \vdash e : \widehat{C}_s(\sigma)$. Since $\Gamma$ is closed, $\widehat{C}_s(\Gamma) = \Gamma$, as desired.

\[\square\]

**Theorem D.5 (Derivations provide solutions).** Let $\Gamma$ be an environment and $e$ an expression.

1. If $\theta \Gamma \vdash^\tau e : \sigma$ and $\Gamma \vdash^\tau e : \phi \sim C$, then there exists a solution $C_s$ for $C$ such that $\widehat{C}_s$ is fully monomorphic and $\widehat{C}_s(\theta \phi) \sim \sigma$.

2. If $\theta \Gamma \vdash e : \sigma$ and $\Gamma \vdash e : \phi \sim C$, then there exists a solution $C_s$ for $C$ such that $\widehat{C}_s$ is fully monomorphic and $\widehat{C}_s(\theta \phi) \sim \sigma$.

**Proof.** We prove this theorem by mutual induction over the expression $e$.

Proof of (1). We distinguish two cases by inversion:

- **Case VarHead.** In this case the generated set of constraints $C$ it empty, so we can just take an empty solved form as solution $C_s$. The corresponding substitution $\widehat{C}_s$ is the identity function, which is trivially fully monomorphic.

Now, we just need to prove that $\widehat{C}_s(\theta \phi) \sim \theta \phi \sim \sigma$. By inversion of the rule VarHead in the constraint-based formulation we know that $\phi$ must come from an element $x : \phi \in \Gamma$. On the other hand, we know that in the declarative specification $\sigma$ must come from $x : \sigma \in \theta \Gamma$. Since $\Gamma$ is the same in both derivations, it must be the case that $\sigma \sim \theta \phi$.

- **Case ExprHead.** Follows by induction on the premise.

Proof of (2). We distinguish four cases:

- **Case Abs.** The derivation in the declarative specification looks like:

$$\frac{\theta \Gamma, x : \tau \vdash e : \sigma }{\theta \Gamma + \lambda x.e : \tau \rightarrow \sigma}$$

Let us first rewrite the premise as $\theta' \Gamma, x : \alpha \vdash e : \sigma$, where $\theta' = [a \mapsto \tau] \circ \theta$. Now we can apply the induction hypothesis to obtain a solution $C_s$ such that $\widehat{C}_s'(\theta' \phi) \sim \sigma$, where $\phi$ is the type assigned to the abstraction body during constraint gathering.

Now consider the problem $C_s' \land \alpha \sim \tau$. We know that a solution $C_s$ exists, or otherwise the expression would be ill-typed. Furthermore, $C_s' \circ \theta' = C_s \circ \theta$ – we are just moving the $\alpha$ assignment from one place to the other. In conclusion, we have that:

$$C_s(\theta(\alpha \rightarrow \phi)) \sim C_s'(\theta'(\alpha \rightarrow \phi)) \sim \tau \rightarrow \sigma$$

- **Case AnnAbs.** Similar to the Abs case.

- **Case App.** The derivation in the specification is:

$$\frac{\theta \Gamma \vdash^\tau e_0 : \sigma_0 \quad \sigma_0 \leq^m_\mu \sigma_1, \ldots, \sigma_n; \mu \quad \theta \Gamma \vdash^\tau e_i : \sigma_i}{\theta \Gamma \vdash e_0 . . . e_n : \phi}$$

App

The constraints generated by this sequence of App rules are:

$$C_0 \land (\phi_0 \leq^m_\mu \alpha_1, \ldots, \alpha_n; \delta') \land C_1 \land \cdots \land C_n$$

By (1) applied to $e_0$ there exists a solution $C_s$ such that $\widehat{C}_s^0 \theta \phi_0 \sim \sigma_0$. From the derivation of $\sigma_0 \leq^m_\mu \sigma_1, \ldots, \sigma_n; \mu$ in the declarative specification we obtain a list of equalities $C_s^\tau$, in particular $\alpha_i^\tau \sim \sigma_i$ and $\delta_i^\tau \sim \mu$.

Let us call $C_s' = \widehat{C}_s^\tau \land C_s^\tau$ and consider for each argument the derivation $C_s'(\theta \Gamma) \vdash e_i : \widehat{C}_s^\tau(\sigma_i)$, which holds by substitutivity in the typing judgment. In the case $\sigma_i$ is polymorphic, the rules for $\tau^{\text{args}}$ perform the job required to obtain a solution, namely introducing rigid variables. We are thus left with $\widehat{C}_s'(\theta \Gamma) \vdash e_i : \widehat{C}_s^\tau(\eta_i)$, where we can apply induction hypothesis to
obtain $\mathit{Cs}^{\mathit{APP}}$. Define $C_i$ to be exactly $\mathit{Cs}^{\mathit{APP}}$ if no rigid variables were introduced and $\forall b, C_i \neq \mathit{Cs}^{\mathit{APP}}$ otherwise. The final solution is the conjunction:

$$C_h \land C_{\mathit{inst}} \land C_1 \land \ldots \land C_n$$

• Case ANNAPP. Similar to the APP case.

Corollary D.6. Suppose $\Gamma \vdash e : \sigma$ and let $\Gamma \vdash e : \phi \leadsto C$. Then there exists a solution $\mathit{Cs}$ for $C$ such that $\mathit{Cs}$ is monomorphic and $\mathit{Cs}(\phi) \sim \sigma$.

Corollary D.7. Suppose $\Gamma \vdash e : \sigma \leadsto C$. Then $C$ is either inconsistent, or can be rewritten to a solved form.

Proof: By Theorem 4.1 every consistent set of constraints can be turned into an instance of the problem of first-order unification under a mixed prefix. The solver in Pottier and Rémy [15] describes a complete algorithm for this problem, from which we obtain the desired solved form.

Corollary D.8 (Principality, solution version). Let $\Gamma \vdash e : \sigma$ and let $\Gamma \vdash e : \phi \leadsto C$. If $C$ is solved to $\mathit{Cs}$, then $\Gamma \vdash e : \mathit{Cs}(\phi)$ and there exists a monomorphic substitution $\pi$ such that $\pi \mathit{Cs}(\phi) \sim \sigma$.

Proof: By soundness we know that if $C$ is solved to $\mathit{Cs}$, then $\Gamma \vdash e : \mathit{Cs}(\phi)$. From the derivation of $\Gamma \vdash e : \sigma$ we obtain another solution $\mathit{Cs}'$ such that $\mathit{Cs}'(\phi) \sim \sigma$. Since the solver produces most general solutions we know in particular that there exists a fully monomorphic substitution $\pi$ such that $\pi \mathit{Cs}(\phi) = \mathit{Cs}'$. This gives the desired result.

Theorem 4.3 (Principality). Suppose $\Gamma \vdash e : \sigma$. Then there exists a type $\phi$ such that $\Gamma \vdash e : \phi$, and for every other $\Gamma \vdash e : \sigma'$, there is a fully monomorphic substitution $\pi$ such that $\sigma' = \pi \phi$.

Proof. Take $\Gamma \vdash e : \phi \leadsto C$. Given that the derivation $\Gamma \vdash e : \sigma$ exists, by Corollary D.7 we are guaranteed to obtain a solution $\mathit{Cs}$ for $C$. Then by Corollary D.8 we know that $\Gamma \vdash e : \mathit{Cs}(\phi)$ and $\sigma \sim \pi \mathit{Cs}(\phi)$ for where $\pi$ is a fully monomorphic substitution. Take $\sigma^* = \mathit{Cs}(\phi)$ and we are done.

Theorem 4.4 (Completeness). Let $\Gamma$ be a closed environment and $e$ an expression. If $\Gamma \vdash e : \sigma$ then $\Gamma \vdash e : \phi \leadsto C$ and $C$ can reach a solved form.

Proof. Consider a closed environment $\Gamma$ and an expression $e$. By Corollary D.6 we know that if we generate constraints $\Gamma \vdash e : \phi \leadsto C$, then there exists a solution for $C$. In particular, since there exists a solution the constraint set $C$ is not inconsistent.

Corollary D.7 states that every constraint set is either inconsistent or can be rewritten to a solved form. Given that our constraint set $C$ is not inconsistent, this implies that it can be rewritten to a solved form, as desired.