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The simple essence of automatic differentiation

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Target

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What's a derivative?

- Number
- Vector
- Covector
- Matrix
- Higher derivatives

Chain rule for each.

What's a derivative?

$$\mathcal{D} :: (a \to b) \to (a \to (a \multimap b))$$

where

$$\lim_{\varepsilon \to 0} \frac{\| f\ (a + \varepsilon) - (f\ a + \mathcal{D}\ f\ a\ \varepsilon) \|}{\| \varepsilon \|} = 0$$

See *Calculus on Manifolds* by Michael Spivak.

Composition

Sequential:

$$(\circ) :: (b \to c) \to (a \to b) \to (a \to c)$$
$$(g \circ f)\ a = g\ (f\ a)$$

$$\mathcal{D}\ (g \circ f)\ a = \mathcal{D}\ g\ (f\ a) \circ \mathcal{D}\ f\ a \quad \text{-- ``chain rule''}$$

What's a derivative?

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$$\mathcal{D}\ (g \circ f)\ a = \mathcal{D}\ g\ (f\ a) \circ \mathcal{D}\ f\ a \quad \text{-- ``chain rule''}$$

Parallel:

$$(\vartriangle) :: (a \to c) \to (a \to d) \to (a \to c \times d)$$
$$(f \vartriangle g)\ a = (f\ a, g\ a)$$

$$\mathcal{D}\ (f \vartriangle g)\ a = \mathcal{D}\ f\ a \vartriangle \mathcal{D}\ g\ a$$

Compositionality

Chain rule:

$$\mathcal{D}\ (g \circ f)\ a = \mathcal{D}\ g\ (f\ a) \circ \mathcal{D}\ f\ a \quad \text{-- non-compositional}$$

To fix, combine regular result with derivative:

$$\hat{\mathcal{D}} :: (a \to b) \to (a \to (b \times (a \multimap b)))$$
$$\hat{\mathcal{D}}\ f = f \vartriangle \mathcal{D}\ f \quad \text{-- specification}$$

Often much work in common to f and $\mathcal{D}\ f$.

Linear functions

Linear functions are their own perfect linear approximations.

$$
\begin{aligned}
\mathcal{D}\ \mathit{id}\quad a &= \mathit{id} \\
\mathcal{D}\ \mathit{fst}\ \ a &= \mathit{fst} \\
\mathcal{D}\ \mathit{snd}\ a &= \mathit{snd} \\
&\ldots
\end{aligned}
$$

For linear functions f,

$$\hat{\mathcal{D}}\ f\ a = (f\ a, f)$$

Abstract algebra for functions

$$
\begin{array}{l}
\textbf{class } \mathit{Category}\ (\leadsto)\ \textbf{where} \\
\quad \mathit{id}\ \ :: a \leadsto a \\
\quad (\circ) :: (b \leadsto c) \to (a \leadsto b) \to (a \leadsto c)
\end{array}
$$

$$
\begin{array}{l}
\textbf{class } \mathit{Category}\ (\leadsto) \Rightarrow \mathit{Cartesian}\ (\leadsto)\ \textbf{where} \\
\quad \mathit{exl} :: (a \times b) \leadsto a \\
\quad \mathit{exr} :: (a \times b) \leadsto b \\
\quad (\vartriangle) :: (a \leadsto c) \to (a \leadsto d) \to (a \leadsto (c \times d))
\end{array}
$$

Plus laws and classes for arithmetic etc.

Linear functions

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Abstract algebra for functions

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\end{array}
$$

Plus laws and classes for arithmetic etc.

Automatic differentiation

$$\textbf{newtype}\ D\ a\ b = D\ (a \to b \times (a \multimap b))$$

$$\hat{\mathcal{D}} :: (a \to b) \to D\ a\ b$$
$$\hat{\mathcal{D}}\ f = D\ (f \vartriangle \mathcal{D}\ f) \quad \text{-- not computable}$$

Require $\hat{\mathcal{D}}$ to preserve *Category* and *Cartesian* structure:

$$\hat{\mathcal{D}}\ id = id$$
$$\hat{\mathcal{D}}\ (g \circ f) = \hat{\mathcal{D}}\ g \circ \hat{\mathcal{D}}\ f$$

$$\hat{\mathcal{D}}\ exl = exl$$
$$\hat{\mathcal{D}}\ exr = exr$$
$$\hat{\mathcal{D}}\ (f \vartriangle g) = \hat{\mathcal{D}}\ f \vartriangle \hat{\mathcal{D}}\ g$$

The game: solve these equations for the RHS operations.

Solution: simple automatic differentiation

$$\textbf{newtype}\ D\ a\ b = D\ (a \to b \times (a \multimap b))$$

$$\mathit{linearD}\ f = D\ (\lambda a \to (f\ a, a))$$

$\textbf{instance}\ \mathit{Category}\ D\ \textbf{where}$

$\quad \mathit{id} = \mathit{linearD}\ \mathit{id}$

$\quad D\ g \circ D\ f = D\ (\lambda a \to \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ b\}\ \textbf{in}\ (c, g' \circ f'))$

$\textbf{instance}\ \mathit{Cartesian}\ D\ \textbf{where}$

$\quad \mathit{exl} = \mathit{linearD}\ \mathit{exl}$

$\quad \mathit{exr} = \mathit{linearD}\ \mathit{exr}$

$\quad D\ f \vartriangle D\ g = D\ (\lambda a \to \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ a\}\ \textbf{in}\ ((b, c), f' \vartriangle g'))$

$\textbf{instance}\ \mathit{NumCat}\ D\ \textbf{where}$

$\quad \mathit{negate} = \mathit{linearD}\ \mathit{negate}$

$\quad \mathit{add} = \mathit{linearD}\ \mathit{add}$

$\quad \mathit{mul} = D\ (\mathit{mul} \vartriangle (\lambda(a, b) \to \lambda(da, db) \to b * da + a * db))$

Automatic differentiation

$$\textbf{newtype}\ D\ a\ b = D\ (a \to b \times (a \multimap b))$$

$$\hat{\mathcal{D}} :: (a \to b) \to D\ a\ b$$
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The game: solve these equations for the RHS operations.

Solution: simple automatic differentiation

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Running examples

$$sqr :: Num\ a \Rightarrow a \rightarrow a$$
$$sqr\ a = a * a$$

$$magSqr :: Num\ a \Rightarrow a \times a \rightarrow a$$
$$magSqr\ (a, b) = sqr\ a + sqr\ b$$

$$cosSinProd :: Floating\ a \Rightarrow a \times a \rightarrow a \times a$$
$$cosSinProd\ (x, y) = (cos\ z, sin\ z)\ \mathbf{where}\ z = x * y$$

In categorical vocabulary:

$$sqr = mul \circ (id \vartriangle id)$$

$$magSqr = add \circ (mul \circ (exl \vartriangle exl) \vartriangle mul \circ (exr \vartriangle exr))$$

$$cosSinProd = (cos \vartriangle sin) \circ mul$$

Solution: simple automatic differentiation

$$\textbf{newtype}\ D\ a\ b = D\ (a \to b \times (a \multimap b))$$

$$linearD\ f = D\ (\lambda a \to (f\ a, a))$$

$\textbf{instance}\ Category\ D\ \textbf{where}$

$\quad id = linearD\ id$

$\quad D\ g \circ D\ f = D\ (\lambda a \to \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ b\}\ \textbf{in}\ (c, g' \circ f'))$

$\textbf{instance}\ Cartesian\ D\ \textbf{where}$

$\quad exl = linearD\ exl$

$\quad exr = linearD\ exr$

$\quad D\ f \vartriangle D\ g = D\ (\lambda a \to \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ a\}\ \textbf{in}\ ((b, c), f' \vartriangle g'))$

$\textbf{instance}\ NumCat\ D\ \textbf{where}$

$\quad negate = linearD\ negate$

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Running examples

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$$sqr = mul \circ (id \vartriangle id)$$

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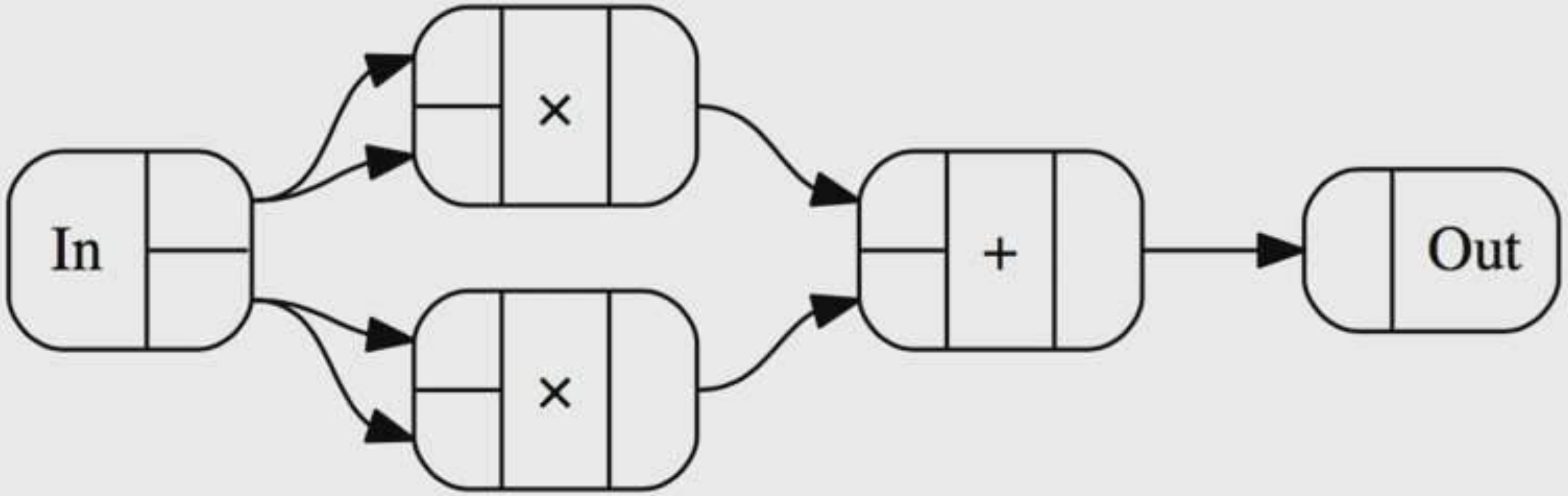
Visualizing computations

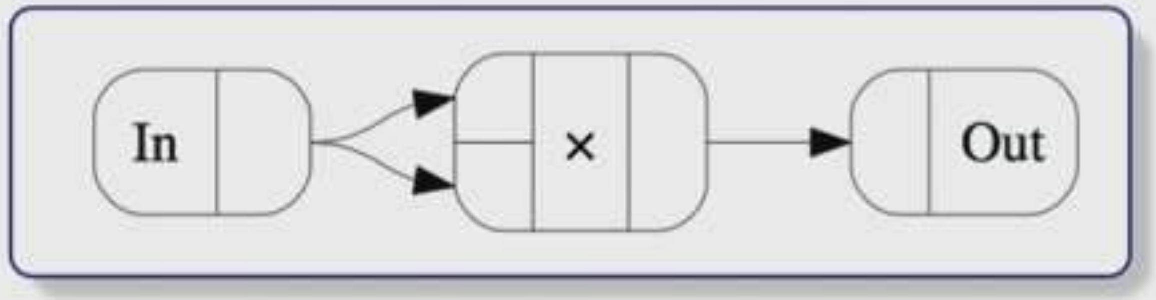
$$magSqr\ (a, b) = sqr\ a + sqr\ b$$

$$magSqr = add \circ (mul \circ (exl \vartriangle exl) \vartriangle mul \circ (exr \vartriangle exr))$$



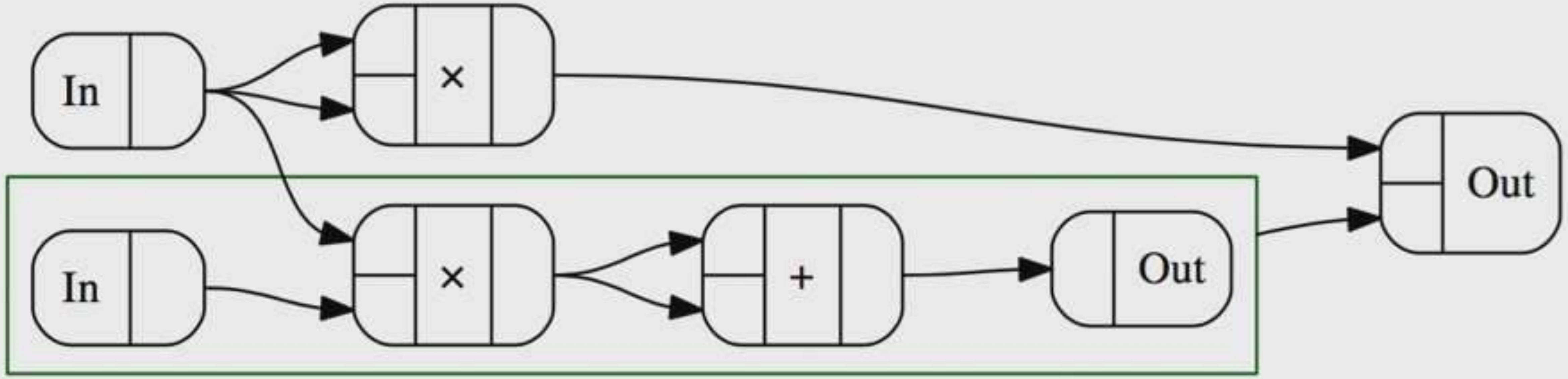
Auto-generated from Haskell code. See *Compiling to categories*.

AD example

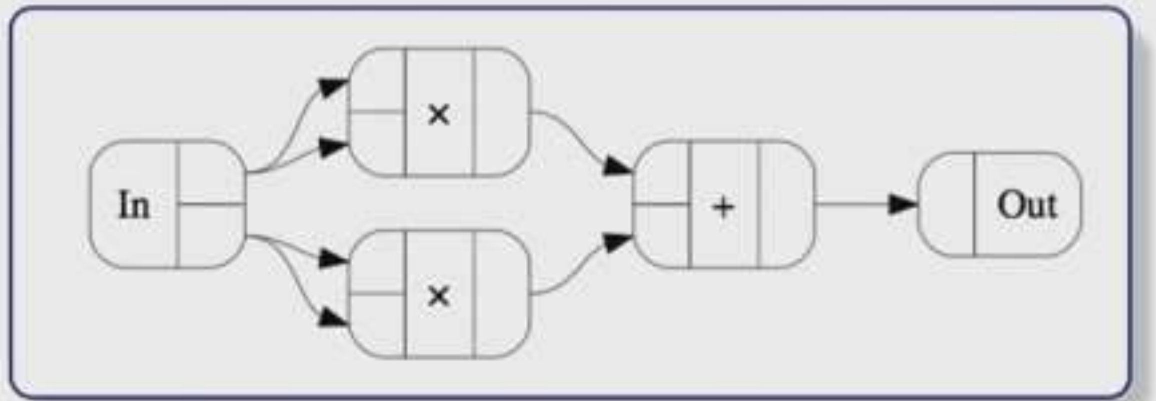


$$sqr\ a = a * a$$

$$sqr = mul \circ (id \vartriangle id)$$

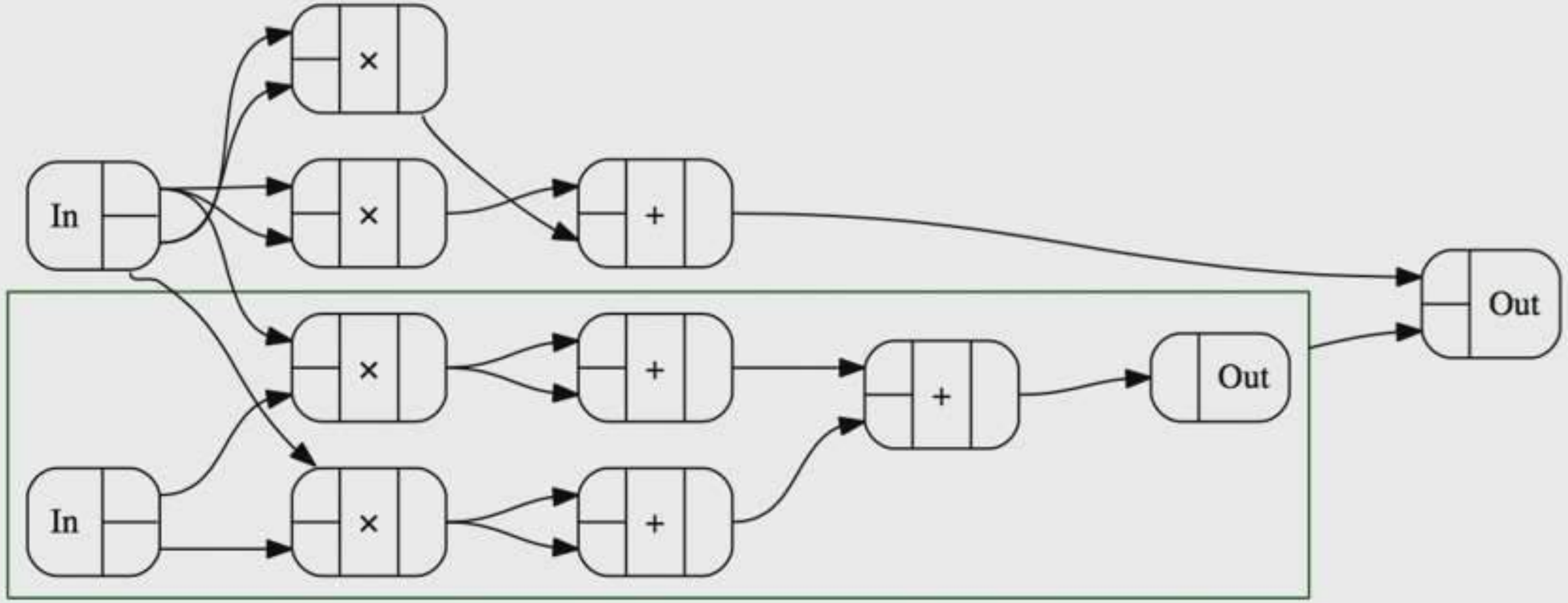


AD example

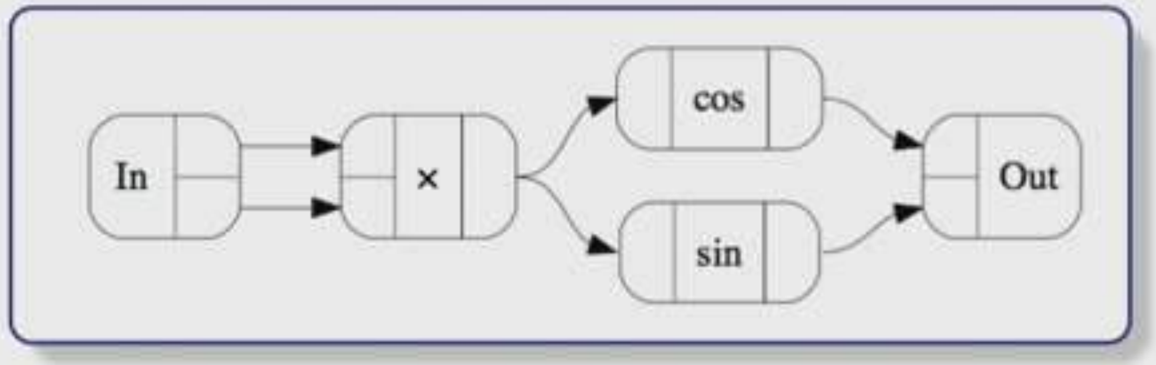


$$magSqr\ (a, b) = sqr\ a + sqr\ b$$

$$magSqr = add \circ (mul \circ (exl \vartriangle exl) \vartriangle mul \circ (exr \vartriangle exr))$$

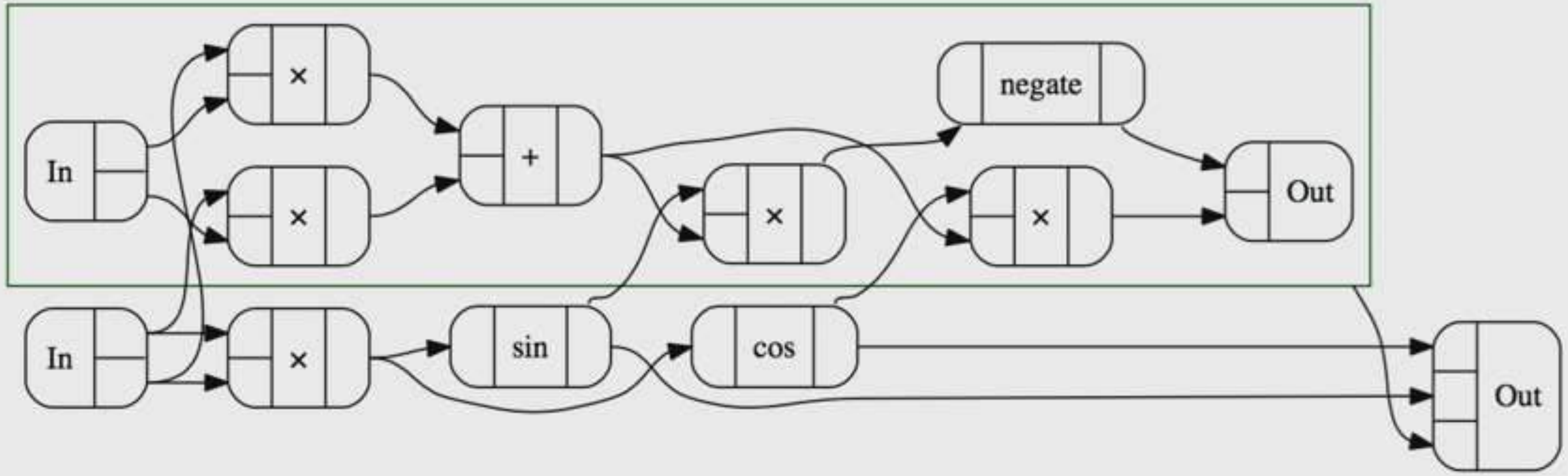


AD example



$$cosSinProd\ (x, y) = (cos\ z, sin\ z)\ \textbf{where}\ z = x * y$$

$$cosSinProd = (cos \vartriangle sin) \circ mul$$



Generalizing AD

$$\textbf{newtype}\ D\ a\ b = D\ (a \to b \times (a \multimap b))$$

$$linearD\ f = D\ (\lambda a \to (f\ a, f))$$

$$\begin{array}{l}\textbf{instance}\ Category\ D\ \textbf{where} \\ \quad id = linearD\ id \\ \quad D\ g \circ D\ f = D\ (\lambda a \to \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ b\}\ \textbf{in}\ (c, g' \circ f'))\end{array}$$

$$\begin{array}{l}\textbf{instance}\ Cartesian\ D\ \textbf{where} \\ \quad exl = linearD\ exl \\ \quad exr = linearD\ exr \\ \quad D\ f \vartriangle D\ g = D\ (\lambda a \to \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ a\}\ \textbf{in}\ ((b, c), f' \vartriangle g'))\end{array}$$

Each D operation just uses corresponding $(\multimap)$ operation.

Generalize from $(\multimap)$ to other cartesian categories.

Generalized AD

$\textbf{newtype}\ D_{(\rightsquigarrow)}\ a\ b = D\ (a \rightarrow b \times (a \rightsquigarrow b))$

$linearD\ f\ f' = D\ (\lambda a \rightarrow (f\ a, f'))$

$\textbf{instance}\ Category\ (\rightsquigarrow) \Rightarrow Category\ D_{(\rightsquigarrow)}\ \textbf{where}$
$\quad id = linearD\ id\ id$
$\quad D\ g \circ D\ f = D\ (\lambda a \rightarrow \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ b\}\ \textbf{in}\ (c, g' \circ f'))$

$\textbf{instance}\ Cartesian\ (\rightsquigarrow) \Rightarrow Cartesian\ D_{(\rightsquigarrow)}\ \textbf{where}$
$\quad exl = linearD\ exl\ exl$
$\quad exr = linearD\ exr\ exr$
$\quad D\ f \vartriangle D\ g = D\ (\lambda a \rightarrow \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ a\}\ \textbf{in}\ ((b, c), f' \vartriangle g'))$

$\textbf{instance} \ldots \Rightarrow NumCat\ D\ \textbf{where}$
$\quad negate = linearD\ negate\ negate$
$\quad add = linearD\ add\ add$
$\quad mul = ??$

Generalizing AD

$$\textbf{newtype}\ D\ a\ b = D\ (a \to b \times (a \multimap b))$$

$$\mathit{linearD}\ f = D\ (\lambda a \to (f\ a, f))$$

$$\begin{array}{l}\textbf{instance}\ \mathit{Category}\ D\ \textbf{where} \\ \quad \mathit{id} = \mathit{linearD}\ \mathit{id} \\ \quad D\ g \circ D\ f = D\ (\lambda a \to \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ b\}\ \textbf{in}\ (c, g' \circ f'))\end{array}$$

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Each D operation just uses corresponding $(\multimap)$ operation.

Generalize from $(\multimap)$ to other cartesian categories.

Generalized AD

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$\textbf{instance}\ Category\ (\leadsto) \Rightarrow Category\ D_{(\leadsto)}\ \textbf{where}$

$\quad id = linearD\ id\ id$

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$\textbf{instance}\ Cartesian\ (\leadsto) \Rightarrow Cartesian\ D_{(\leadsto)}\ \textbf{where}$

$\quad exl = linearD\ exl\ exl$

$\quad exr = linearD\ exr\ exr$

$\quad D\ f \vartriangle D\ g = D\ (\lambda a \to \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ a\}\ \textbf{in}\ ((b, c), f' \vartriangle g'))$

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$\quad negate = linearD\ negate\ negate$

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$\quad mul = ??$

Generalizing AD

$$\textbf{newtype}\ D\ a\ b = D\ (a \to b \times (a \multimap b))$$

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$\quad D\ g \circ D\ f = D\ (\lambda a \to \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ b\}\ \textbf{in}\ (c, g' \circ f'))$

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Each D operation just uses corresponding $(\multimap)$ operation.

Generalize from $(\multimap)$ to other cartesian categories.

Generalized AD

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$\textbf{instance}\ Cartesian\ (\leadsto) \Rightarrow Cartesian\ D_{(\leadsto)}\ \textbf{where}$
$\quad exl\ = linearD\ exl\ exl$
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$\textbf{instance}... \Rightarrow NumCat\ D\ \textbf{where}$
$\quad negate = linearD\ negate\ negate$
$\quad add\ = linearD\ add\ add$
$\quad mul = ??$

Numeric operations

Specific to (linear) *functions*:

$$mul = D\ (mul \vartriangle (\lambda(a, b) \rightarrow \lambda(da, db) \rightarrow b * da + a * db))$$

Rephrase:

$$\begin{aligned}&scale :: Multiplicative\ a \Rightarrow a \rightarrow (a \multimap a)\\&scale\ u = \lambda v \rightarrow u * v\end{aligned}$$

$$\begin{aligned}&(\triangledown) :: (a \multimap c) \rightarrow (b \multimap c) \rightarrow ((a \times b) \multimap c)\\&f \triangledown g = \lambda(a, b) \rightarrow f\ a + g\ b\end{aligned}$$

Now

$$mul = D\ (mul \vartriangle (\lambda(a, b) \rightarrow scale\ b \triangledown scale\ a))$$

Linear arrow vocabulary

$$
\begin{array}{l}
\textbf{class } \mathit{Category}\ (\leadsto) \textbf{ where} \\
\quad \mathit{id}\ \ :: a \leadsto a \\
\quad (\circ) :: (b \leadsto c) \to (a \leadsto b) \to (a \leadsto c)
\end{array}
$$

$$
\begin{array}{l}
\textbf{class } \mathit{Category}\ (\leadsto) \Rightarrow \mathit{Cartesian}\ (\leadsto) \textbf{ where} \\
\quad \mathit{exl}\ :: (a \times b) \leadsto a \\
\quad \mathit{exr} :: (a \times b) \leadsto b \\
\quad (\vartriangle) :: (a \leadsto c) \to (a \leadsto d) \to (a \leadsto (c \times d))
\end{array}
$$

$$
\begin{array}{l}
\textbf{class } \mathit{Category}\ (\leadsto) \Rightarrow \mathit{Cocartesian}\ (\leadsto) \textbf{ where} \\
\quad \mathit{inl}\ :: a \leadsto (a \times b) \\
\quad \mathit{inr} :: b \leadsto (a \times b) \\
\quad (\triangledown) :: (a \leadsto c) \to (b \leadsto c) \to ((a \times b) \leadsto c)
\end{array}
$$

$$
\begin{array}{l}
\textbf{class } \mathit{ScalarCat}\ (\leadsto)\ a \textbf{ where} \\
\quad \mathit{scale} :: a \to (a \leadsto a)
\end{array}
$$

Linear transformations as functions

$$\mathbf{newtype}\ a \rightarrow^{+} b = \mathit{AddFun}\ (a \rightarrow b)$$

$$\begin{array}{l}
\mathbf{instance}\ \mathit{Category}\ (\rightarrow^{+})\ \mathbf{where} \\
\quad \mathit{id}\ \ = \mathit{AddFun}\ \mathit{id} \\
\quad (\circ) = \mathit{inNew}_2\ (\circ)
\end{array}$$

$$\begin{array}{l}
\mathbf{instance}\ \mathit{Cartesian}\ (\rightarrow^{+})\ \mathbf{where} \\
\quad \mathit{exl}\ = \mathit{AddFun}\ \mathit{exl} \\
\quad \mathit{exr} = \mathit{AddFun}\ \mathit{exr} \\
\quad (\vartriangle) = \mathit{inNew}_2\ (\vartriangle)
\end{array}$$

$$\begin{array}{l}
\mathbf{instance}\ \mathit{Cocartesian}\ (\rightarrow^{+})\ \mathbf{where} \\
\quad \mathit{inl}\ = \mathit{AddFun}\ (,0) \\
\quad \mathit{inr} = \mathit{AddFun}\ (0,) \\
\quad (\triangledown) = \mathit{inNew}_2\ (\lambda f\ g\ (x, y) \rightarrow f\ x + g\ y)
\end{array}$$

$$\begin{array}{l}
\mathbf{instance}\ \mathit{Multiplicative}\ s \Rightarrow \mathit{ScalarCat}\ (\rightarrow^{+})\ s\ \mathbf{where} \\
\quad \mathit{scale}\ s = \mathit{AddFun}\ (s\ *)
\end{array}$$

Extracting a data representation

- How to extract a matrix or gradient vector?
- Sample over a domain *basis* (rows of identity matrix).
- For n-dimensional *domain*,
  - Make n passes.
  - Each pass works on n-D sparse ("one-hot") input.
  - Very inefficient.
- For gradient-based optimization,
  - High-dimensional domain.
  - Very low-dimensional (1-D) codomain.

Generalized matrices

$$\textbf{newtype}\ M_s\ a\ b = L\,(V_s\ b\,(V_s\ a\ s))$$

$$applyL :: M_s\ a\ b \to (a \to b)$$

Require *applyL* to preserve structure. Solve for methods.

Core vocabulary

Sufficient to build arbitrary "matrices":

$$
\begin{array}{lll}
\mathit{scale} :: a \to (a \leadsto a) & & \text{-- } 1 \times 1 \\
(\triangledown) & :: (a \leadsto c) \to (b \leadsto c) \to ((a \times b) \leadsto c) & \text{-- horizontal juxt} \\
(\vartriangle) & :: (a \leadsto c) \to (a \leadsto d) \to (a \leadsto (c \times d)) & \text{-- vertical juxt}
\end{array}
$$

Types guarantee rectangularity.

Efficiency of composition

- Arrow composition is associative.
- Some associations are more efficient than others, so
  - Associate optimally.
  - Equivalent to *matrix chain multiplication* — $O(n \log n)$.
  - Choice determined by *types*, i.e., compile-time information.
- All-right: "forward mode AD" (FAD).
- All-left: "reverse mode AD" (RAD).
- RAD is much better for gradient-based optimization.

Left-associating composition (RAD)

- CPS-like category:
  - Represent $a \leadsto b$ by $(b \leadsto r) \to (a \leadsto r)$.
  - Meaning: $f \mapsto (\circ\, f)$.
  - Results in left-composition.
  - Initialize with $id :: r \leadsto r$.
  - Construct $h \circ \mathcal{D}\, f\ a$ directly, without $\mathcal{D}\, f\ a$.

Continuation category

$$\textbf{newtype}\ Cont^{r}_{(\leadsto)}\ a\ b = Cont\ ((b \leadsto r) \to (a \leadsto r))$$

$$\begin{aligned} &cont :: Category\ (\leadsto) \Rightarrow (a \leadsto b) \to Cont^{r}_{(\leadsto)}\ a\ b \\ &cont\ f = Cont\ ({\circ}\ f) \end{aligned}$$

Require *cont* to preserve structure. Solve for methods.

We'll use an isomorphism:

$$\begin{aligned} &join\ \ \ \, :: Cocartesian\ (\leadsto) \Rightarrow (c \leadsto a) \times (d \leadsto a) \to ((c \times d) \leadsto a) \\ &unjoin :: Cocartesian\ (\leadsto) \Rightarrow ((c \times d) \leadsto a) \to (c \leadsto a) \times (d \leadsto a) \end{aligned}$$

$$\begin{aligned} &join\ (f, g) = f \triangledown g \\ &unjoin\ h\ \ \ = (h \circ inl, h \circ inr) \end{aligned}$$

Continuation category (solution)

$$
\begin{aligned}
&\textbf{instance } \mathit{Category}\ (\leadsto) \Rightarrow \mathit{Category}\ \mathit{Cont}^{r}_{(\leadsto)} \textbf{ where}\\
&\quad \mathit{id} = \mathit{Cont}\ \mathit{id}\\
&\quad \mathit{Cont}\ g \circ \mathit{Cont}\ f = \mathit{Cont}\ (f \circ g)
\end{aligned}
$$

$$
\begin{aligned}
&\textbf{instance } \mathit{Cartesian}\ (\leadsto) \Rightarrow \mathit{Cartesian}\ \mathit{Cont}^{r}_{(\leadsto)} \textbf{ where}\\
&\quad \mathit{exl}\ = \mathit{Cont}\ (\mathit{join} \circ \mathit{inl})\\
&\quad \mathit{exr} = \mathit{Cont}\ (\mathit{join} \circ \mathit{inr})\\
&\quad (\vartriangle) = \mathit{inNew}_2\ (\lambda f\ g \rightarrow (f \triangledown g) \circ \mathit{unjoin})
\end{aligned}
$$

$$
\begin{aligned}
&\textbf{instance } \mathit{Cocartesian}\ (\leadsto) \Rightarrow \mathit{Cocartesian}\ \mathit{Cont}^{r}_{(\leadsto)} \textbf{ where}\\
&\quad \mathit{inl}\ = \mathit{Cont}\ (\mathit{exl} \circ \mathit{unjoin})\\
&\quad \mathit{inr} = \mathit{Cont}\ (\mathit{exr} \circ \mathit{unjoin})\\
&\quad (\triangledown) = \mathit{inNew}_2\ (\lambda f\ g \rightarrow \mathit{join} \circ (f \vartriangle g))
\end{aligned}
$$

$$
\begin{aligned}
&\textbf{instance } \mathit{ScalarCat}\ (\leadsto)\ a \Rightarrow \mathit{ScalarCat}\ \mathit{Cont}^{r}_{(\leadsto)}\ a \textbf{ where}\\
&\quad \mathit{scale}\ s = \mathit{Cont}\ (\mathit{scale}\ s)
\end{aligned}
$$

Reverse-mode AD without tears

$$D_{Cont^{r}_{M_s}}$$

Duality

- Vector space dual: $u \multimap s$, with u a vector space over s.
- If u has finite dimension, then $u \multimap s \cong u$.
- For $f :: u \multimap s$, $f = dot\ v$ for some $v :: u$.
- Gradients are derivatives of functions with scalar codomain.
- Represent $a \multimap b$ by $(b \multimap s) \to (a \multimap s)$ by $b \to a$.
- *Ideal* for extracting gradient vector. Just apply to 1 (id).

Duality

$$\textbf{newtype}\ Dual_{(\leadsto)}\ a\ b = Dual\ (b \leadsto a)$$

$$asDual :: Cont^{s}_{(\leadsto)}\ a\ b \rightarrow Dual_{(\leadsto)}\ a\ b$$
$$asDual\ (Cont\ f) = Dual\ (dot^{-1} \circ f \circ dot)$$

where

$$dot \quad :: u \rightarrow (u \multimap s)$$
$$dot^{-1} :: (u \multimap s) \rightarrow u$$

Require *asDual* to preserve structure. Solve for methods.

Duality (solution)

$$\textbf{newtype}\ \mathit{Dual}_{(\leadsto)}\ a\ b = \mathit{Dual}\ (b \leadsto a)$$

$$
\begin{array}{l}
\textbf{instance}\ \mathit{Category}\ (\leadsto) \Rightarrow \mathit{Category}\ \mathit{Dual}_{(\leadsto)}\ \textbf{where} \\
\quad \mathit{id}\ \ = \mathit{Dual}\ \mathit{id} \\
\quad (\circ) = \mathit{inNew}_2\ (\mathit{flip}\ (\circ))
\end{array}
$$

$$
\begin{array}{l}
\textbf{instance}\ \mathit{Cocartesian}\ (\leadsto) \Rightarrow \mathit{Cartesian}\ \mathit{Dual}_{(\leadsto)}\ \textbf{where} \\
\quad \mathit{exl}\ = \mathit{Dual}\ \mathit{inl} \\
\quad \mathit{exr} = \mathit{Dual}\ \mathit{inr} \\
\quad (\vartriangle)\ = \mathit{inNew}_2\ (\triangledown)
\end{array}
$$

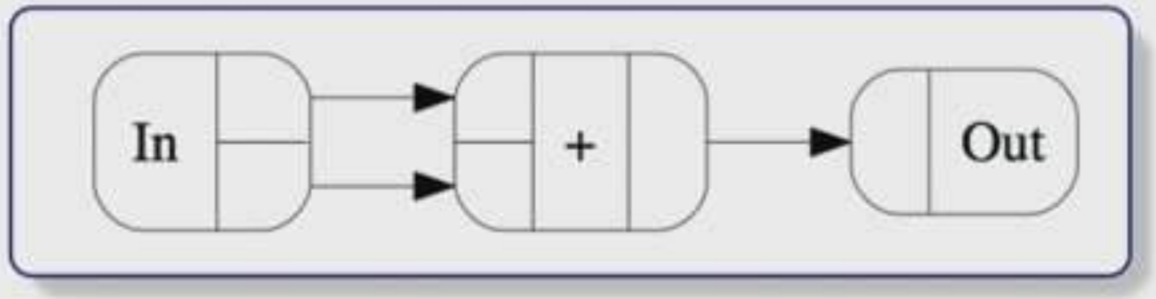
$$
\begin{array}{l}
\textbf{instance}\ \mathit{Cartesian}\ (\leadsto) \Rightarrow \mathit{Cocartesian}\ \mathit{Dual}_{(\leadsto)}\ \textbf{where} \\
\quad \mathit{inl}\ = \mathit{Dual}\ \mathit{exl} \\
\quad \mathit{inr} = \mathit{Dual}\ \mathit{exr} \\
\quad (\triangledown) = \mathit{inNew}_2\ (\vartriangle)
\end{array}
$$

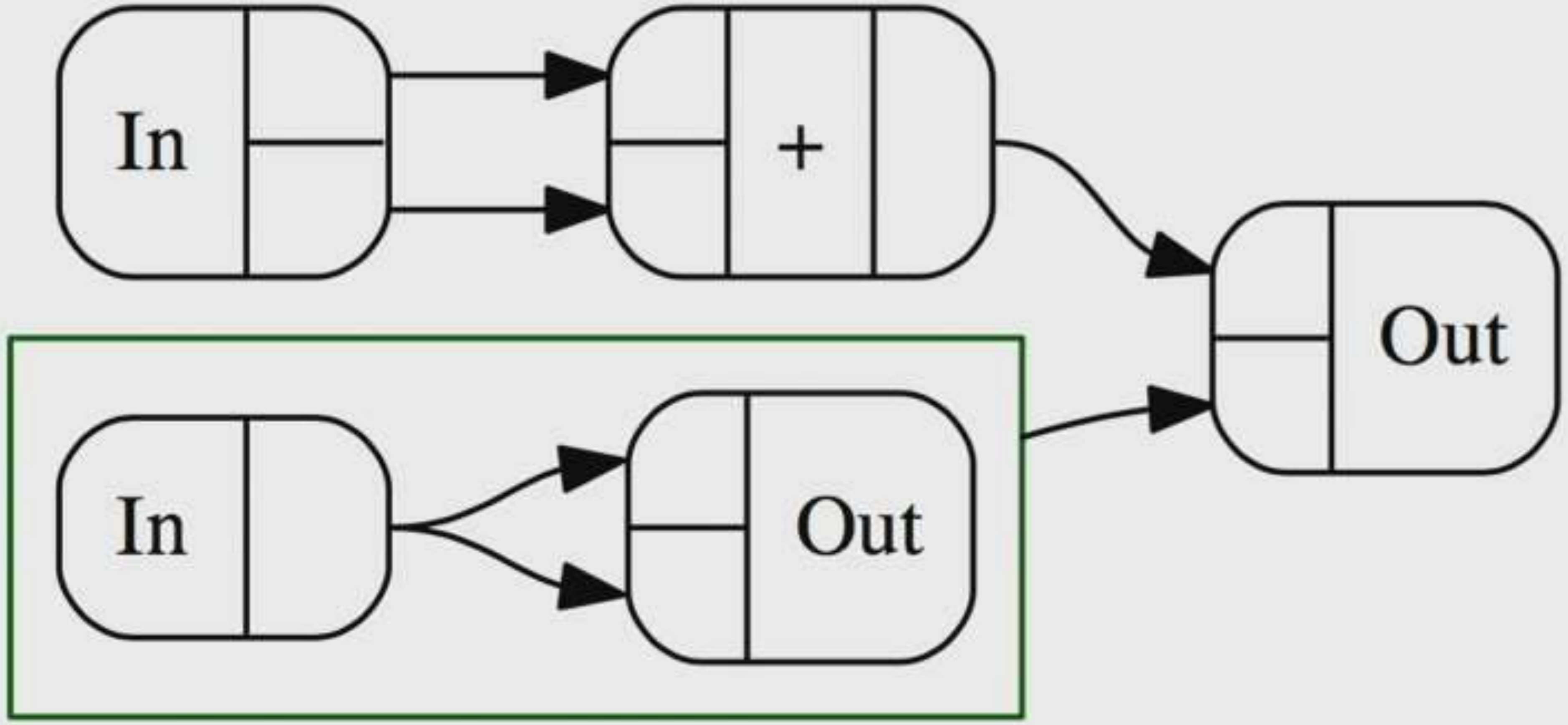
$$
\begin{array}{l}
\textbf{instance}\ \mathit{ScalarCat}\ (\leadsto)\ s \Rightarrow \mathit{ScalarCat}\ \mathit{Dual}_{(\leadsto)}\ s\ \textbf{where} \\
\quad \mathit{scale}\ s = \mathit{Dual}\ (\mathit{scale}\ s)
\end{array}
$$

Backpropagation

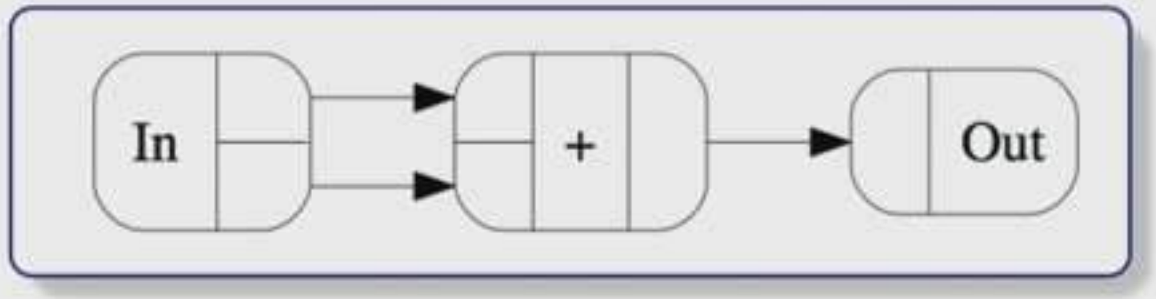
$$D_{Dual_{\multimap^+}}$$

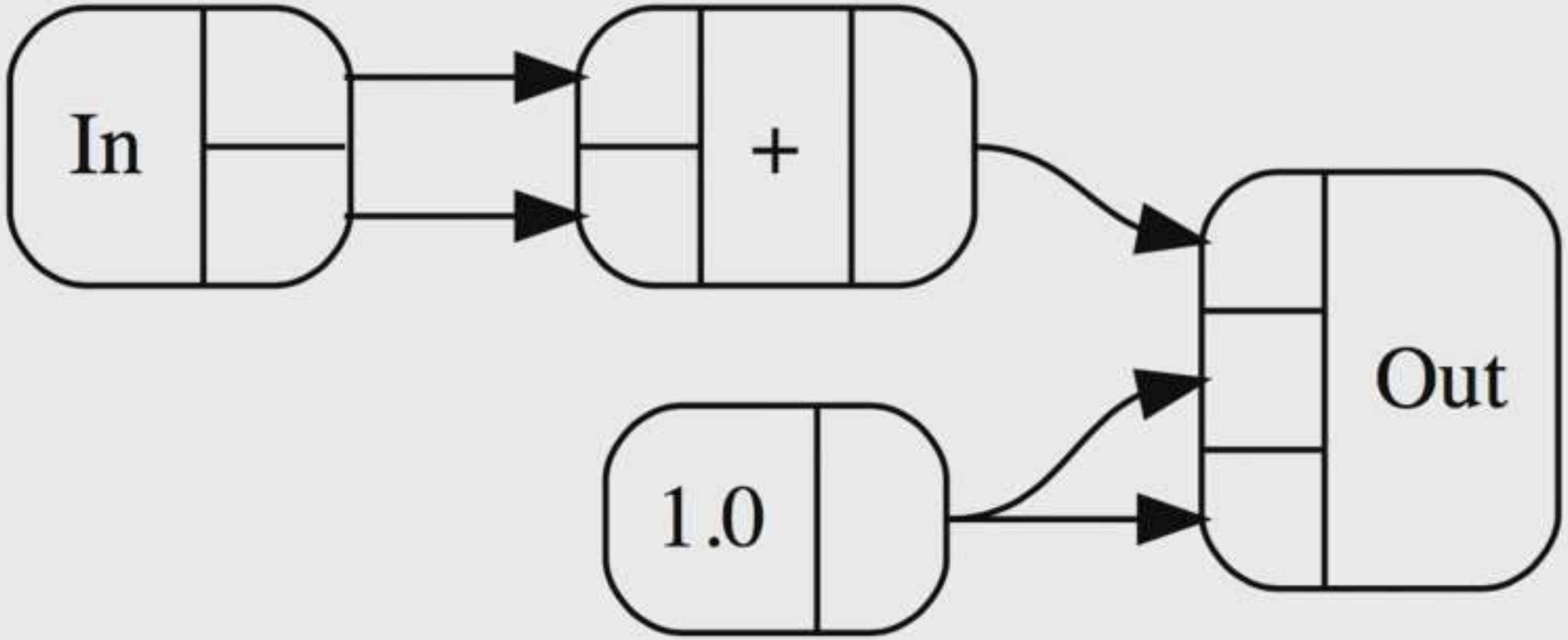
RAD example (dual function)





RAD example (dual vector)





Solution: simple automatic differentiation

$$\mathbf{newtype}\ D\ a\ b = D\ (a \rightarrow b \times (a \multimap b))$$

$$linearD\ f = D\ (\lambda a \rightarrow (f\ a, a))$$

$$\begin{aligned}
&\mathbf{instance}\ Category\ D\ \mathbf{where}\\
&\quad id = linearD\ id\\
&\quad D\ g \circ D\ f = D\ (\lambda a \rightarrow \mathbf{let}\ \{(b, f') = f\ a; (c, g') = g\ b\}\ \mathbf{in}\ (c, g' \circ f'))
\end{aligned}$$

$$\begin{aligned}
&\mathbf{instance}\ Cartesian\ D\ \mathbf{where}\\
&\quad exl = linearD\ exl\\
&\quad exr = linearD\ exr\\
&\quad D\ f \vartriangle D\ g = D\ (\lambda a \rightarrow \mathbf{let}\ \{(b, f') = f\ a; (c, g') = g\ a\}\ \mathbf{in}\ ((b, c), f' \vartriangle g'))
\end{aligned}$$

$$\begin{aligned}
&\mathbf{instance}\ NumCat\ D\ \mathbf{where}\\
&\quad negate = linearD\ negate\\
&\quad add = linearD\ add\\
&\quad mul = D\ (mul \vartriangle (\lambda(a, b) \rightarrow \lambda(da, db) \rightarrow b * da + a * db))
\end{aligned}$$

Generalizing AD

$\textbf{newtype}\ D\ a\ b = D\ (a \rightarrow b \times (a \multimap b))$

$linearD\ f = D\ (\lambda a \rightarrow (f\ a, f))$

$\textbf{instance}\ Category\ D\ \textbf{where}$
$\quad id = linearD\ id$
$\quad D\ g \circ D\ f = D\ (\lambda a \rightarrow \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ b\}\ \textbf{in}\ (c, g' \circ f'))$

$\textbf{instance}\ Cartesian\ D\ \textbf{where}$
$\quad exl = linearD\ exl$
$\quad exr = linearD\ exr$
$\quad D\ f \vartriangle D\ g = D\ (\lambda a \rightarrow \textbf{let}\ \{(b, f') = f\ a; (c, g') = g\ a\}\ \textbf{in}\ ((b, c), f' \vartriangle g'))$

Each D operation just uses corresponding $(\multimap)$ operation.

Generalize from $(\multimap)$ to other cartesian categories.