Stochastic explosions in branching processes and non-uniqueness for nonlinear PDE

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Incompressible Navier-Stokes Equations:
\[
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p - \Delta u &= 0 \\
\nabla \cdot u &= 0 \\
\end{align*}
\]
\[u(x, 0) = u_0(x) \text{ (& B.C.)}\]

Formally:
\[
\begin{align*}
u_t + Au + B(u, u) &= 0 \\
u(0) &= u_0
\end{align*}
\]

**Known:** via Approximations/Fixed Point arguments:

- local existence and uniqueness of *smooth* solutions;
- global existence and uniqueness for *small initial data*;
- global existence of *weak* (Leray-Hopf) solutions; uniqueness?

**Open Problems:**

- Global well-posedness: existence and uniqueness...
- Regularity problem: Can smooth solutions blow up?
Scaling and Regularity

\[ u(x, t) \text{ - soln. } \Rightarrow u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t) \text{ is also a soln.} \]

What is known globally:
"\[ \|u(t)\|^2 \leq \|u_0\|^2 < \infty \]" – not enough for regularity.

The most basic regularity criterion:
"\[ \|A^{\frac{1}{2}} u(t)\|^2 < \infty \Rightarrow \text{regularity.} \]

Basic small initial data result:
"\[ \|A^{\frac{1}{2}} u_0\| \|u_0\| < \epsilon \Rightarrow \text{regularity.} \]

Note:
\[ \|u_\lambda(t)\|^2 = \frac{1}{\lambda} \|u(\lambda^2 t)\|^2 - \text{"subcritical"} \]
\[ \|A^{\frac{1}{2}} u_\lambda(t)\|^2 = \int_{\mathbb{R}^3} \|(-\Delta)^{\frac{1}{2}} \lambda u(\lambda^2 t, \lambda x)\|^2 dx = \lambda \|A^{\frac{1}{2}} u(\lambda^2 t)\|^2 - \text{"supercritical"} \]
\[ \|A^{\frac{1}{2}} (u_0)_\lambda\| \|(u_0)_\lambda\| = \|A^{\frac{1}{2}} u_0\| \|u_0\| - \text{"critical"} \]
NSE - quick look

Incompressible Navier-Stokes Equations:

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\nabla \cdot u &= 0 \\
u(x, 0) &= u_0(x) \text{ (\& B.C.\ldots)}
\end{align*}
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\[ \|A^{\frac{1}{2}} (u_0)_\lambda\| \|(u_0)_\lambda\| = \|A^{\frac{1}{2}} u_0\| \|u_0\| \text{ - "critical"} \]
Self-similar solutions:

Definition

\( u \) is a self-similar solution if \( u(x, t) = u_\lambda(x, t) \forall \lambda > 0. \)

In Fourier space:

\[
    u(x, t) = \lambda u(\lambda x, \lambda^2 t) \Rightarrow \hat{u}(\xi, t) = \frac{1}{\lambda^2} \hat{u}\left(\frac{\xi}{\lambda}, \lambda^2 t\right).
\]

Can self-similar Soln's help understand the role of scaling in NSE problem?
Effect of self-similar scaling.

NSE in mild formulation in Fourier space:

$$
\hat{u}(\xi, t) = \hat{u}_0(\xi)e^{-|\xi|^2t} + \int_0^t e^{-|\xi|^2s} \frac{|\xi|}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{u}(\eta, t-s) \odot_{\xi} \hat{u}(\xi-\eta, t-s) \, d\eta \, ds,
$$

with $\xi \cdot \hat{u}(\xi) = 0$, $\hat{u}(-\xi) = \overline{\hat{u}(\xi)}$, $e_\xi = \xi/|\xi|$,

$$
v \odot_{e_\xi} w = -i(e_\xi \cdot w) \pi_{e_\xi \perp} v.
$$

In self-similar case, re-scale using $\hat{u}(\xi, t) = \frac{1}{\lambda^2} \hat{u}\left(\frac{\xi}{\lambda}, \lambda^2 t\right)$ with $\lambda = |\xi|:

$$
\hat{u}(e_\xi, |\xi|^2 t) = \hat{u}_0(e_\xi) e^{-|\xi|^2t}
+ \frac{1}{(2\pi)^{3/2}} \int_0^\tau \int_{\mathbb{R}^3} e^{-|\xi|^2s} \hat{u}(e_\eta, |\eta|^2(t-s)) \odot_{\xi} \hat{u}(e_{\xi-\eta}, |\xi-\eta|^2(t-s)) \frac{|\xi|^3 d\eta ds}{|\eta|^2|\xi-\eta|^2}.
$$

After a change of vars. (with $\tau = |\xi|^2 t - "similarity \ horizon")...
Self-Similar rescaling of NSE

For $\tau = |\xi|^2 t$:

$$\hat{u}(e_\xi, \tau) = \hat{u}_0(e_\xi) e^{-\tau}$$

$$+ \frac{1}{(2\pi)^{\frac{3}{2}}} \int_0^\tau e^{-\sigma} \int_{\text{Exp}(1)_{\mathbb{R}^3}} \hat{u}(e_\eta, |\eta|^2(\tau - \sigma)) \otimes_{\xi} \hat{u}(e_{\xi - \eta}, |e_\xi - \eta|^2(\tau - \sigma)) \frac{d\eta d\sigma}{|\eta|^2|e_\xi - \eta|^2 \pi^3 H(\eta|e_\xi) d\eta d\sigma}$$

In non-ss case the same re-scaling $v(\xi, t) = c \frac{\hat{u}(\xi, t)}{|\xi|^2}$ leads to:

$$v(\xi, t) = v_0(\xi) e^{-|\xi|^2 t} + \int_0^t |\xi|^2 e^{-|\xi|^2 s} \int_{\mathbb{R}^3} v(\eta, t-s) \otimes_{e_\xi} v(\xi - \eta, t-s) H(\eta|\xi) d\eta ds,$$

$$H(\eta|\xi) = \frac{|\xi|}{\pi^3 |\eta|^2 |\xi - \eta|^2}.$$
Probabilistic interpretation.

Abuse notation $\hat{u} \sim v...$

$$v(\mathbf{e}_\xi, \tau) = v_0(\mathbf{e}_\xi) e^{-\tau} +$$

$$\int_0^\tau e^{-\sigma} \int_{\mathbb{R}^3} v(\mathbf{e}_\eta, |\eta|^2(\tau - \sigma)) \otimes_{\mathbf{e}_\xi} v(\mathbf{e}_{\mathbf{e}_\xi - \eta}, |\mathbf{e}_\xi - \eta|^2(\tau - \sigma)) \frac{d\eta d\sigma}{\pi^3 |\eta|^2 |\mathbf{e}_\xi - \eta|^2}$$

“Solution” process:

$$X(\mathbf{e}, \tau) = v_0(\mathbf{e}) 1_{T_0 \geq \tau} + X^{(1)}(\mathbf{e}_{w_1}, \tau_1) \otimes_{\mathbf{e}} X^{(2)}(\mathbf{e}_{w_2}, \tau_2) 1_{T_0 < \tau},$$

where $T_0 \sim \text{Exp}(1)$, $W_1 \sim H(\cdot | \mathbf{e})$, $W_2 = \mathbf{e} - W_1$, $\tau_j = |W_j|^2(\tau - T_0)$.

If $\mathbb{E}(|X(\mathbf{e}_\xi, \tau)|) < \infty$, then:

$$v(\mathbf{e}_\xi, \tau) = \mathbb{E}(X(\mathbf{e}_\xi, \tau))$$

is a solution to ssmNSE.
Self-Similar Cascade.

- $T_b$ - i.i.d. $\text{Exp}(1)$, $e_b = e_{w_b}$
- $W_{b1} \sim H(\cdot | e_b)$, $W_{b2} = e_b - W_{b1}$
- Similarity horizon changes:
  $\tau_{bj} = |W_{bj}|^2 (\tau_b - T_b)$
- $X(t, e_{\xi})$ is the $\odot$-product of $v_0(e_b)$ s.t. $T_b \geq \tau_b$.
- $v(t, e_{\xi}) = \mathbb{E}X(t, e_{\xi})$ solves ssmNSE if $\mathbb{E}|X(t, \xi)| < \infty$
- Non-explosion $\Rightarrow$ uniqueness.

Note: if $\tau_{121} \geq T_{121}$ (the dotted branch in the picture ends):

$$X(e_0, \tau_0) = [v_0(e_{11}) \odot e_1 (v_0(e_{121}) \odot e_{12} v_0(e_{122}))] \odot e_0 (v_0(e_{21}) \odot e_2 v_0(e_{22})).$$
Probabilistic interpretation.

Abuse notation $\hat{u} \sim v$...

$$v(e_\xi, \tau) = v_0(e_\xi) e^{-\tau} +$$

$$+ \int_0^\tau e^{-\sigma} \int_{\mathbb{R}^3} v(e_\eta, |\eta|^2(\tau - \sigma)) \otimes_{e_\xi} v(e_{e_\xi - \eta}, |e_\xi -\eta|^2(\tau - \sigma)) \frac{d\eta d\sigma}{\pi^3 |\eta|^2 |e_\xi - \eta|^2}$$

“Solution” process:

$$X(e, \tau) = v_0(e) \mathbb{1}_{T_\theta \geq \tau} + X^{(1)}(e_{w_1}, \tau_1) \otimes_{e} X^{(2)}(e_{w_2}, \tau_2) \mathbb{1}_{T_\theta < \tau},$$

where $T_\theta \sim \text{Exp}(1)$, $W_1 \sim H(\cdot | e)$, $W_2 = e - W_1$, $\tau_j = |W_j|^2(\tau - T_\theta)$.

If $\mathbb{E}(|X(e_\xi, \tau)|) < \infty$, then:

$$v(e_\xi, \tau) = \mathbb{E}(X(e_\xi, \tau))$$ is a solution to ssmNSE.
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- $T_b$ - i.i.d. $\text{Exp}(1)$, $e_b = e_{wb}$
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\[
v(e_\xi, \tau) = v_0(e_\xi) e^{-\tau} + \\
+ \int_0^\tau e^{-\sigma} \int_{\mathbb{R}^3} v(e_\eta, |\eta|^2(\tau - \sigma)) \odot_{e_\xi} v(e_{e_\xi - \eta}, |e_\xi - \eta|^2(\tau - \sigma)) \frac{d\eta d\sigma}{\pi^3 |e_\xi - \eta|^2}
\]

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Non-explosion \( \Rightarrow \) uniqueness.

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Self-similar explosion

Recall: cascade continues at $b \in \{1, 2\}^n$ if $\tau_b - T_b > 0$. But:

$$\tau_b - T_b = |W_b|^2(\tau_{b|n-1} - T_{b|n-1}) - T_b = \ldots$$

$$= \prod_{k=1}^{n} |W_{b|k}|^2 \left( \tau_0 - \sum_{j=0}^{n} \frac{T_{b|j}}{\prod_{k=0}^{j} |W_{b|k}|^2} \right)$$

$\Rightarrow$ "explosion horizon / shortest path" R.V:

$$S(e_\xi) = \lim_{n \to \infty} \inf_{|b|=n} \sum_{j=0}^{n} \frac{T_{b|j}}{\prod_{k=0}^{j} |W_{b|k}|^2_{b}}$$

Explosion event: $E = \{S < \infty\}$. 
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⇒ "explosion horizon / shortest path" R.V:

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Explosion event: \( E = \{ S < \infty \} \).
A simplification: the $\alpha$-Riccati equation

Recall ssmNSE:

$$\nu(e, \tau) = \nu_0(e) e^{-\tau} + \int_0^\tau e^{-\sigma} \int_{\mathbb{R}^3} \nu(e_{\eta_1}, |\eta_1|^2(\tau - \sigma)) \otimes e \nu(e_{\eta_2}, |\eta_2|^2(\tau - \sigma)) \ H(\eta_1 | e) \ d\eta_1 \ d\sigma.$$ 

Simplifications:

- $H(\eta_1 | e) \sim \delta_{\sqrt{\alpha}}$
- Ignore geometry: $\otimes \sim$ product of magnitudes.
- Ignore the difference between $|\eta_1|$ and $|\eta_2|$.
- Ignore rotation $|\nu(e, \tau)| \sim \nu(\tau)$.

$\Rightarrow \alpha$-Riccati equation:

$$\nu(\tau) = \nu_0 e^{-\tau} + \int_0^\tau e^{-\sigma} \nu^2(\alpha(\tau - \sigma)) \ d\sigma.$$ 

Differential form: $\nu'(\tau) = -\nu(\tau) + \nu^2(\alpha \tau)\ldots$
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A simplification: the $\alpha$-Riccati equation

Recall ssmNSE:

$$\nu(e, \tau) = \nu_0(e) e^{-\tau} + \int_0^\tau e^{-\sigma} \int_0^{\mathbb{R}^3} \nu(e_{\eta_1}, |\eta_1|^2(\tau - \sigma)) \otimes e \nu(e_{\eta_2}, |\eta_2|^2(\tau - \sigma)) H(\eta_1 | e) d\eta_1 d\sigma.$$

Simplifications:

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$$\nu(\tau) = \nu_0 e^{-\tau} + \int_0^\tau e^{-\sigma} \nu^2(\alpha(\tau - \sigma)) d\sigma.$$

Differential form: $\nu'(\tau) = -\nu(\tau) + \nu^2(\alpha \tau)\ldots$
Cascade set-up for $\alpha$-Riccati:

$$v(\tau) = v_0 \ e^{-\tau} + \int_0^{\tau} e^{-\sigma} v^2(\alpha(\tau - \sigma)) \ d\sigma.$$ 

Solution Process = Process with inherited property:

$$X(\tau) = X(0) \mathbb{1}_{T_\theta \geq \tau} + X^{(1)}(\alpha \tau_1) X^{(2)}(\alpha \tau_2) \mathbb{1}_{T_\theta < \tau},$$

- $\mathbb{E}(|X(\tau)|) < \infty \Rightarrow v(\tau) = \mathbb{E}(X(\tau))$ a solution, $v_0 = \mathbb{E}(X(0))$.
- $T_b$ – i.i.d. Exp(1).
- $\tau_{bj} = \alpha(\tau_b - T_b), \ \tau_\theta = \tau$.
- $T_b \geq \tau_b$ – process at branch $b$ stops.
- Non-explosion $\Rightarrow$ uniqueness.

Note: if $\tau_{121} \geq T_{121}$: $X(\tau_\theta) = v_0^5$. 
Some inherited events

Define:
Shortest path: \( S = \lim_{n \to \infty} \min_{|b|=n} \sum_{k=0}^{n} \frac{T_{b|k}}{\alpha^k} \),

Longest path: \( L = \lim_{n \to \infty} \max_{|b|=n} \sum_{k=0}^{n} \frac{T_{b|k}}{\alpha^k} \),

explosion events: \( \{S < \infty\} – \text{explosion}; \{L < \infty\} – \text{hyper-explosion} \).

Note:
- no-explosion and hyper-explosion are inherited events.
- \( X(t) = \mathbb{1}_{S \geq \tau}, X(t) = \mathbb{1}_{L < \tau} \) – processes with inherited property.

Conclusion:
\( \nu = P(S = \infty), \nu = P(L < \infty), \nu(\tau) = P(S \geq \tau), \nu(\tau) = P(L < \tau) \)
solve \( \alpha \)-Riccati.
The case $\alpha \leq 1$.

**Theorem**

*When $\alpha \in [0, 1]$ the cascade is non-exploding:* $\mathbb{P}(S = \infty) = 1$.

**Proof.**

Compare to Yule process. \hfill \Box

Consequence: $X(t) = v_0^{N(\tau)}$, $N(\tau)$—number of branches;

**Theorem**

*Let $\alpha < 1$. $\forall v_0 > 0 \exists$! solution $v(t) = \mathbb{E}(v_0^{N(\tau)}) < \infty$:

$$
\lim_{n \to \infty} \mathbb{E}(v_0^{N(\tau)}) = \begin{cases} 
0, & 0 \leq v_0 < 1, \\
1, & v_0 = 1, \\
\infty, & v_0 > 1
\end{cases}
$$

**Proof.**

Estimate $\mathbb{P}(N(\tau) = n)$. \hfill \Box
Special Cases.

- $\alpha = 0$: equation is $v'(\tau) = -v(\tau) + v_0^2$.
  
  $P(N(\tau) = 1) = e^{-\tau}$, \quad $P(N(\tau) = 2) = 1 - e^{-\tau}$,

  \[
  v(\tau) = \mathbb{E}(v_0^N(\tau)) = v_0 e^{-\tau} + v_0^2 (1 - e^{-\tau}).
  \]

- $\alpha = 1$: equation is $v'(\tau) = -v(\tau) + v^2(\tau)$.
  
  $P(N(\tau) = n) = e^{-\tau} (1 - e^{-\tau})^{n-1}$, \quad $n \in \mathbb{N}$.

  \[
  v(\tau) = \mathbb{E}(v_0^N(\tau)) = \sum_{n=1}^{\infty} v_0^n e^{-\tau} (1 - e^{-\tau})^{n-1} = \frac{v_0 e^{-\tau}}{1 - v_0 (1 - e^{-\tau})}.
  \]

- $\alpha = 1/2$: equation is $v'(\tau) = -v(\tau) + v^2(\tau/2)$.
  
  $P(N(\tau) = n) = e^{-\tau} \tau^{n-1} / (n - 1)!$, \quad $n \in \mathbb{N}$.

  \[
  v(\tau) = \mathbb{E}(v_0^N(\tau)) = \sum_{n=1}^{\infty} v_0^n \frac{\tau^{n-1}}{(n - 1)!} e^{-\tau} = v_0 e^{v_0 \tau - 1}.
  \]
The case $\alpha > 1$: explosion.

**Theorem (Maximal-path Explosion Theorem)**

Suppose $\exists C < 1$ s.t. in the self-similar cascade for any $b$,

$$Z_b = \max\{|W_{b1}|, |W_{b2}|\}$$ satisfies $\mathbb{E}(Z_b^{-2}) \leq C$. Then the cascade is exploding: $\mathbb{P}(S < \infty) = 1$.

...In the $\alpha$-Riccati case $|W_b| = \alpha \forall b$, so the cascade is exploding when $\alpha > 1$.

In fact:

**Theorem**

*When $\alpha > 1$ the cascade for $\alpha$-Riccati is hyper-explosive: $\mathbb{P}(L < \infty) = 1$.*

**Question**: How to build process $X(t)$ in the presence exploding branches?
\( \alpha > 1 \). Iterative Process.

Start with an \( X_0(\tau) \geq 0 \), and set

\[
X_n(\tau) = \begin{cases} 
V_0, & T_0 \geq \tau, \\
X_n^{(1)}(\alpha(\tau - T_0)) X_n^{(2)}(\alpha(\tau - T_0)), & T_0 < \tau, 
\end{cases}, \quad n \in \mathbb{N},
\]

If \( y_n(\tau) = \mathbb{E}(X_n(\tau)) < \infty \):

\[
y_n(\tau) = u_0 e^{-\tau} + \int_0^{\tau} e^{-\sigma} y_{n-1}^{2}(\alpha(\tau - \sigma)) \, d\sigma,
\]

**Question:** For which \( X_0 \) we can pass to the limit?
\[ \alpha > 1. \text{ Minimal Solution: Existence.} \]

Choose \( X_0 = 0 \). Then \( X_n(\tau) \to X(\tau) = v_0^{N(t)} \mathbb{1}_{S > t} \).

Note: \( v_0 = 1, X(\tau) = \mathbb{1}_{S > t} \to 0, \tau \to \infty; \quad v_0 = 0, X = 0 \) a.s.

**Theorem**

- If \( v(\tau) = \mathbb{E}(X(\tau)) < \infty \), then \( v(\tau) \) is a solution;
- If \( v(\tau) \) is another solution for the same initial data, then \( v(\tau) < v(\tau) \).

**Theorem**

Let \( \alpha > 1 \).

- If \( v_0 \leq \max\{1, (2\alpha - 1)/4\} \), then \( v(\tau) = \mathbb{E}(X(\tau)) \) is a (global) solution.
- If \( v_0 > 2\alpha - 1 \) then local solutions blow up in finite time.
$\alpha > 1$: Uniqueness Problem for $0 \leq \nu_0 \leq 1$.

**Theorem**

Let $0 \leq \nu_0 < 1$. Then $\nu(\tau)$ is the unique solution in the class $\|\nu(\tau)\|_\infty < 1$.

**Proof.**

Let $\nu(\tau)$ be a solution. Initialize iterations with $X_0 = \nu(\tau)$ and compare to $X$...

“Upper Solution”: use $X_0 = 1$: $X_n(\tau) \to \overline{X}(\tau) = \nu_0^{N(\tau)}$. Set $\nu(\tau) = E(\overline{X}(\tau))$.

Note: $\nu_0 = 0$, $\overline{X}(\tau) = \mathbb{1}_{L \leq t} \to 1$, $\tau \to \infty$; $\nu_0 = 1$, $\overline{X} = 1$ a.s.

**Theorem**

Let $\alpha > 1$ and $\nu_0 \in [0, 1]$. There are at least two solutions:

- $\nu(\tau) \to 0$ as $\tau \to \infty$,
- $\overline{\nu}(\tau) \to 1$ as $\tau \to \infty$.

Moreover, if $\nu(\tau) \in [0, 1]$, then $\nu(\tau) \leq \underline{\nu}(\tau) \leq \overline{\nu}(\tau)$. 
$\alpha > 1$. Iterative Process.

Start with an $X_0(\tau) \geq 0$, and set

$$X_n(\tau) = \begin{cases} v_0, & T_0 \geq \tau, \\ X_{n-1}^{(1)}(\alpha(\tau - T_0)) X_{n-1}^{(2)}(\alpha(\tau - T_0)), & T_0 < \tau \end{cases}, \quad n \in \mathbb{N},$$

If $y_n(\tau) = \mathbb{E}(X_n(\tau)) < \infty$:

$$y_n(\tau) = u_0 e^{-\tau} + \int_0^{\tau} e^{-\sigma} y_{n-1}^2(\alpha(\tau - \sigma)) d\sigma,$$

**Question:** For which $X_0$ we can pass to the limit?
\( \alpha > 1 \): Uniqueness Problem for \( 0 \leq \nu_0 \leq 1 \).

Theorem

Let \( 0 \leq \nu_0 < 1 \). Then \( \nu(\tau) \) is the unique solution in the class \( \| \nu(\tau) \|_\infty < 1 \).

Proof.

Let \( \nu(\tau) \) be a solution. Initialize iterations with \( X_0 = \nu(\tau) \) and compare to \( X \ldots \)

“Upper Solution”: use \( X_0 = 1 \): \( X_n(\tau) \rightarrow \overline{X}(\tau) = \nu_0^{N(\tau)} \). Set \( \overline{\nu}(\tau) = \mathbb{E}(\overline{X}(\tau)). \)

Note: \( \nu_0 = 0, \overline{X}(\tau) = \mathbb{1}_{L \leq \tau} \rightarrow 1, \tau \rightarrow \infty; \quad \nu_0 = 1, \overline{X} = 1 \) a.s.

Theorem

Let \( \alpha > 1 \) and \( \nu_0 \in [0, 1] \). Three are at least two solutions:

- \( \nu(\tau) \rightarrow 0 \) as \( \tau \rightarrow \infty \),
- \( \overline{\nu}(\tau) \rightarrow 1 \) as \( \tau \rightarrow \infty \).

Moreover, if \( \nu(\tau) \in [0, 1] \), then \( \nu(\tau) \leq \nu(\tau) \leq \overline{\nu}(\tau) \).
$\alpha > 1$. Non-uniqueness for big initial data.

**Theorem**

Let $\alpha > 5/2$ and $1 < u_0 \leq \frac{2\alpha-1}{4} - \phi_\infty$, $(\phi_\infty(\alpha) < 3/4)$. Then

$$v(\tau) < \overline{V}(\tau) \leq w + (1 - e^{-\tau}),$$

where $w$ is the minimal solution corresponding to initial data $w_0 \in [v_0 + \phi_\infty, (2\alpha - 1)/4]$.

Also can be shown: local in time solutions are not unique.
A Random Initialization.

\[ X_0(t) = \begin{cases} 
  0, & T_0 \geq \tau, \\
  G(\tau - T_0), & T_0 < \tau,
\end{cases} \]

Denote

\[ F(t) := \mathbb{E}(X_0(\tau)) = \int_0^\tau e^{-\sigma} G(\tau - \sigma) \, d\sigma \quad \text{or,} \quad F'(\tau) + F(\tau) = G(\tau) \quad F(0) = 0. \]

If \( L_n := \max_{|b| = n} \sum_{k=0}^n \frac{T_b |k\rangle}{\alpha^k} < \tau \) then \( X_n(\tau) = \prod_{|\tau| = n} G(\alpha^n(\tau - \theta_b)). \)

and so

\[ \mathbb{E} \left( X_{n+1}(\tau) \mathbb{1}_{L_{n+1} \leq \tau} \mid T_b, |b| \leq n \right) = \prod_{|b| = n} F^2(\alpha^{n+1}(\tau - \theta_b)). \]

Note: if \( F^2(\alpha \tau) \leq G(\tau) \), then \( X_n(\tau) \mathbb{1}_{L_n \leq \tau} \) is a super-martingale.
Athreya’s Solution.

For $\alpha > 1$ and $\nu_0 = 0$, Choose

$$F(\tau) = e^{-\tau - b}, \quad b = \frac{\ln 2}{\ln \alpha}.$$  

Then $X_n(\tau) = X_n(\tau)1_{L_n < \tau}$ is an uniformly integrable super-martingale.

Conclusion: $\exists X(t) = \lim_n X_n(t)$ and $\nu_A(t) = E(X(t))$ is yet another solution to $\alpha$-Riccati.
Existence & Uniqueness regions for $\alpha$-Riccati.

- **Non-existence**
- **Existence unknown**
- **Existence, uniqueness**
- **Existence, non-uniqueness**

- $u_0 = 2\alpha - 1$
- $u_0 = \frac{2\alpha - 1}{4} - \phi_\infty$

- **Existence, non-uniqueness** (uniqueness in the class $|u|_\infty < 1$)

- Axes: $\alpha$ and $u_0$
Back to NSE: explosion?

Some properties of $H(\eta|e_\xi) = \frac{1}{\pi^3} \frac{1}{|\eta|^2 |e_\xi - \eta|}$:

- Rotational invariance;
- $\forall b, \{|W_{b|j}|\} \sim \text{iid } \frac{2}{\pi^2} \frac{1}{r} \ln \left| \frac{1+r}{1-r} \right|$.
- In polar coordinates at each branching $b$:
  - azimuth angle for $W_{b1}$ is $\text{Unif}(0, 2\pi)$;
  - longitudinal angles for $W_{b1}$ and $W_{b2}$ are jointly $\text{Unif}\{0 \leq \phi_1, \phi_2 \leq \pi, \phi_1 + \phi_2 \leq \pi\}$.

Consequence:

$$\max\{|W_{b1}|, |W_{b2}|\} \text{ is distributed } z(r) = \frac{4}{\pi} \frac{1}{r} \ln \left| \frac{r}{r-1} \right|.$$  

Note:

if $Z \sim z(r)$, then $\mathbb{E}(Z^{-2}) = \frac{2}{\pi} < 1$.

Thus, by the Maximal-Path Explosion Theorem ssmNSE cascade is exploding!
Maximal-Path Explosion

Recall the Maximal-Path Explosion Theorem:

Theorem
Suppose \( \exists C < 1 \) s.t. in the self-similar cascade for any \( b \),
\( Z_b = \max\{|W_{b1}|, |W_{b2}|\} \) satisfies \( \mathbb{E}(Z_b^{-2}) \leq C \). Then the cascade is exploding \( \mathbb{P}(S < \infty) = 1 \).

Idea of Proof:
Recall: Explosion event \( \{S = \infty\} \), where \( S = \lim_{n \to \infty} \min_{|b| = n} \frac{\sum_{j=0}^{n} T_{b_{j+1}}}{\prod_{k=0}^{j} |W_{b_{k+1}}|^2} \)
Consider “maximal-descendant” path \( b_m \in \{1, 2\}^\mathbb{N} \) that at each junction \( b = b_m \) follows the path of maximal \( |W_{bi}|, i = 1, 2 \). Note:

\[
S \leq \sum_{j=0}^{\infty} \frac{T_{b_m_{j+1}}}{\prod_{k=0}^{j} |W_{b_m_{k+1}}|^2} \quad \text{a.s.}
\]
What could be proven for NSE?

Existence and uniqueness for $v_0$ via minimal solution:

**Theorem**
Suppose $\|v_0(e_\xi)\|_\infty < 1$. Then there exists a unique ssmNSE solution in the class $\|v(e_\xi, \tau)\|_\infty < 1$.

(In fact existence can be proven for $\|v_0(e_\xi)\|_\infty \leq C$ for a $C > 1$.)
Conclusions/Challenges

- Can iterative procedure be used to prove non-uniqueness for NSE? Lack of hyper-explosion is a challenge.

- Can we prove existence for NSE for arbitrary large initial data?
  
  **Good**: NSE nonlinearity – $\otimes$-product along a tree – contains repeated projection of the same vector on random planes, which should deplete non-linearity.
  
  **Bad**: A large tree generates long $\otimes$-products: enough $\nu_0$ may survive the depletion and make $X$ large.

- Can this explosion approach be connected to Kolmogorov backwards equation... Develop a general theory of explosion-nonuniqueness for nonlinear PDE?

- Can cascade structure be employed to describe energy transfer between scales with consequences in turbulence/regularity?