The KPZ fixed point

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joint work with
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Two-dimensional growth – Eden’s model

t(some shape) + t^x Fluctuations

1 + 1 d: Fluctuation exponent $\chi = 1/3$
Burning paper  Coffee Stains  Tumour growth
Kardar-Parisi-Zhang (KPZ) Equation 86

\[ \partial_t h = (\partial_x h)^2 + \partial_x^2 h + \xi \]

- lateral growth
- relaxation
- space–time white noise
- height at time \( t \), position \( x \)

\[ F(\partial_x h) \sim F(0) + F'(0)\partial_x h + \frac{1}{2} F''(0)(\partial_x h)^2 + \cdots \]

\[ \partial_x h = u \sim \text{stoch Burgers eqn} \]
\[ \partial_t u = 2u\partial_x u + \partial_x^2 u + \partial_x \xi \]

- Eden growth
- Ballistic aggregation
- ASEP

2.2. Snapshots illustrating surface and bulk properties of three distinct stochastic growth models: (a) Eden cluster, (b) ballistic deposit, and (c) RSOS solid. All belong to the KPZ universality class [HH93].
A special discretization of KPZ equation (TASEP)

\[ h(x + 1) = h(x) \pm 1, \ x \in \mathbb{Z} \]

local max \[\leftrightarrow\] local min at rate 1

\[-21 = \frac{1}{2} [(\nabla^- h)(\nabla^+ h) - 1 + \Delta h] \]

symmetric random walk invariant (except for height shift)

Lab mouse of non-equilibrium stat mech since the late 60's
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Ballistic aggregation

ASEP
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Asymptotic fluctuations depend on initial data

In TASEP special initial data could be computed by Johansson, Spohn, Borodin, Sasamoto,...

\[ h(t, x) \sim c_1 t - c_2 \frac{x^2}{t} + c_3 t^{1/3} F_{\text{GUE}} \]

Corner

\[ h(t, x) \sim c_3 t + c_4 t^{1/3} F_{\text{GOE}} \]

Flat

\[ F_{\text{GUE}}/F_{\text{GOE}} \] are the rescaled top eigenvalues of a matrix from the Gaussian Unitary/Orthogonal Ensembles
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Corner

\[ h(t, x) \sim c_1 t - c_2 \frac{x^2}{t} + c_3 t^{1/3} A_2(t^{-2/3} x) \]

Flat

\[ h(t, x) \sim c_3 t + c_4 t^{1/3} A_1(t^{-2/3} x) \]

Airy_2/Airy_1 are special stochastic processes
The Airy$_2$ process $A_2(x)$

Defined via its finite dimensional distributions, given by Fredholm determinants of “extended kernels”

\[ x_1 < \cdots < x_n \]

\[ \chi_g = 1_{z > g_j} \]

\[
\mathbb{P}(A_2(x_1) \leq g_1, \ldots, A_2(x_n) \leq g_n) = \det(I - \chi_g K_{Ai}^{ext}) L^2(\{x_1, \ldots, x_n\} \times \mathbb{R})
\]

\[
K_2^{ext}(x, z; x', z') = \begin{cases} 
\int_0^\infty d\lambda \ e^{-\lambda(x-x')} \ \text{Ai}(z + \lambda) \ \text{Ai}(z' + \lambda) & x \geq x' \\
-\int_{-\infty}^0 d\lambda \ e^{-\lambda(x-x')} \ \text{Ai}(z + \lambda) \ \text{Ai}(z' + \lambda) & x < x' 
\end{cases}
\]
Conjectural KPZ universality class

KPZ fixed point $h(t, x)$

- Determinantal:
  - Polynuclear growth
  - Last passage percolation
  - TASEP
- Non-determinantal:
  - ASEP
  - $q$-TASEP
  - $q$-Hahn TASEP
  - O'Connell-Yor semi-discrete polymer
  - Log-Gamma polymer
  - Stochastic higher spin vertex models

KPZ equation

$$\partial_t h = \partial_x^2 h + \lambda (\partial_x h)^2 + \xi$$

$$\varepsilon^{1/2} h(\varepsilon^{-3/2} t, \varepsilon^{-1} x)$$

- Bacterial colony boundaries
- Eden model
- Ballistic aggregation
- Stochastic reaction-diffusion fronts
- Stochastic Hamilton-Jacobi equations
- Exclusion processes
- First passage percolation
- Directed random polymers

Not exactly solvable

KPZ fixed point was a complete mystery, both in math and physics. All we had was a few self-similar solutions (Airy processes)
What is the fixed point?

KPZ 1:2:3 scaling \( h_\epsilon(t, x) = \epsilon^{1/2} h(\epsilon^{-3/2} t, \epsilon^{-1} x) \) takes KPZ eqn to

\[
\partial_t h_\epsilon = (\partial_x h_\epsilon)^2 + \epsilon^{1/2} \partial_x^2 h_\epsilon + \epsilon^{1/4} \xi
\]

Goal to understand the limit \( h \) of \( h_\epsilon \) as \( \epsilon \to 0 \)

Or \( u = \partial_x h \), \( u = \partial_x h \) stochastic Burgers fixed point

Weak solution of Burgers equation \( \partial_t u = \partial_x u^2 \), but not unique

\[
\text{Dissipationless limit} \quad \partial_t u = \partial_x u^2 + \epsilon \partial_x^2 u \quad \neq \quad \text{dispersionless limit} \quad \partial_t u = \partial_x u^2 + \epsilon \partial_x^3 u \quad (\text{Lax, Levermore, Venakidis, Deift, Zhou, ...})
\]

KPZ fixed point not dissipative limit because Hopf-Lax formula
\( h(t, x) = \sup_y \{ h_0(y) - \frac{1}{t} (x - y)^2 \} \) does not preserve Brownian motion

With Matetski and Remenik (2017) we solve TASEP and take the 1:2:3 scaling limit to find the KPZ fixed point
The KPZ fixed point

Markov process on state space $UC =$ upper semi-cont fns, with local Hausdorff topology

(eg. Narrow wedge $= \lim_{\varepsilon \to 0} -\varepsilon^{1/2}|\varepsilon^{-1}x|$)

Transition probabilities

$\mathbb{P}_{h_0} (h(t, x_1) \leq a_1, \ldots, h(t, x_M) \leq a_M) = \det(I - K_{h_0, t, x, a})$

Fredholm determinant of a compact (trace class) operator $K$ is

$\det(I + K)_{L^2(S, \mu)} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{S^n} \det[K(u_i, u_j)]_{i,j=1}^{n} d\mu(u_1) \cdots d\mu(u_n)$

Stochastic integrable system: dynamics linearized by a new type of transform
Brownian scattering transform

For $h_0 \in UC$ define for $B$ Brownian motion

$$P_{-L,L}^{h_0}(u_1, u_2) = \mathbb{P}_{B(-L)=u_1, B(L)=u_2}(B \text{ hits hypo}(h_0) \text{ on } [-L, L])$$

The Brownian scattering transform of $h_0$ is

$$K^{h_0} = \lim_{L \to \infty} e^{-L \partial^2} P_{-L,L}^{h_0} e^{-L \partial^2}$$

Looks terrible because of backwards heat equation, but we only ever use

$$U_t K^{h_0} U_t^{-1} \quad U_t = e^{t/3 \partial^3}$$

ok because $e^{x \partial^2 + t/3 \partial^3}$ is convolution with

$$t^{-1/3} e^{2x^3/3 - x} \text{Ai}(-t^{-1/3} z + t^{-4/3} x^2)$$
Properties

Brownian scattering transform \( K^{h_0} = \lim_{L \to \infty} e^{-L \partial^2} P_{-L,L}^{\text{hit \, } h_0} e^{-L \partial^2} \)

Extend \( K^{\text{ext}}_{h_0}(x_i, \cdots; x_j, \cdots) = -e^{(x_j-x_i)\partial^2} \mathbb{1}_{x_i < x_j} + e^{-x_i \partial^2} K^{h_0}_{e x_j \partial^2} \)

1. For \( t \neq 0 \) the map \( h_0 \mapsto \chi_a U_t K^{\text{ext}}_{h_0} U_t^{-1} \chi_a \) is continuous from UC into the trace class on \( L^2(\{x_1, \cdots, x_M\} \times \mathbb{R}) \). \( \chi_a(x_i, u) = \mathbb{1}_{u > a_i} \)

2. Inverted by the determinant

\[
\det \left( I - \chi_a U_t K^{\text{ext}}_{h_0} U_t^{-1} \chi_a \right)_{L^2(\{x_1, \cdots, x_M\} \times \mathbb{R})} \xrightarrow{t \to 0} \prod_{i=1}^{M} \mathbb{1}_{h_0(x_i) \leq a_i}
\]

3. The determinants define Markov transition probabilities on UC, i.e. the Chapman-Kolmogorov equations

\[
\int_{UC} P_{\text{hit}_0}(s, df) P_{f}(t, B) = P_{\text{hit}_0}(s + t, B).
\]

\[
P_{\text{hit}_0}(h(t, x_1) \leq a_1, \ldots, h(t, x_M) \leq a_M) = \det \left( I - \chi_a U_t K^{\text{ext}}_{h_0} U_t^{-1} \chi_a \right)
\]
KPZ fixed pt \( h(t, x; h_0) \) unique Markov process with these transition probs starting from \( h_0 \)

- Conjecturally unique local dynamics satisfying
  1. (1:2:3 scaling invariant) \( \alpha h(\alpha^{-3}t, \alpha^{-2}x; \alpha h_0(\alpha^{-2}x)) \overset{\text{dist}}{=} h(t, x; h_0) \)
  2. (Skew time reversible) \( \mathbb{P}(h(t, x; g) \leq -f(x)) = \mathbb{P}(h(t, x; f) \leq -g(x)) \)
  3. (Stationarity in space) \( h(t, x + u; h_0(x - u)) \overset{\text{dist}}{=} h(t, x; h_0) \)

- For TASEP (and a few related models) we know that if the rescaled height function converges to \( h_0 \) in UC. Then
  \[
  \varepsilon^{1/2}\left[h(2\varepsilon^{-3/2}t, 2\varepsilon^{-1}x) + \varepsilon^{-3/2}t\right] \xrightarrow{\varepsilon \rightarrow 0} h(t, x; h_0) \text{ in distr. in UC.}
  \]

- For \( t > 0 \), \( h(t, x) \) is locally Brownian and locally Hölder \( \frac{1}{2} \) in \( x \).
  For fixed \( x \), \( h(t, x) \) is locally Hölder \( \frac{1}{3} \) in \( t > 0 \).

- Although Brownian scattering transform looks daunting, in special cases like flat, narrow wedge, it is easy to compute and recovers the known formulas for the Airy processes. Also, new closed forms for cusps, parabolas.
A determinantal formula for TASEP

Solution to the master equation by using Bethe ansatz:

Schütz'97:

For $N \geq 1$ particles one has the determinantal formula

$$P(X_t = x | X_0 = y) = \det \left[ F_{i-j}(y_{N+1-i} - y_{N+1-j}, t) \right]_{1 \leq i,j \leq N}$$

with $x, y \in \{z_N < \cdots < z_1\} \subset \mathbb{Z}^N$ and

$$F_n(x, t) = \frac{(-1)^n}{2\pi i} \oint_{\Gamma_{0,1}} \frac{dw}{w^{x+1}} \left( \frac{w}{1-w} \right)^n e^{t(w-1)}$$

where $\Gamma_{0,1}$ is a simple loop around 0 and 1

This formula is not suitable for $N \to \infty$, since the matrix size goes to $\infty$. Sasamoto (2006) realized that because of special properties of $F$ it can be written as a (signed) determinantal point process on certain wired Gelfand-Tsetlin patterns.
Biorthogonalization

For one-sided i.d. $X(0) = X(-1) = X(-2) = \cdots = +\infty$:

**Borodin, Ferrari, Prähofer and Sasamoto’07:**

For $1 \leq n_1 < n_2 < \cdots < n_M$ one has

$$
\mathbb{P}_{X_0} \left( X_t(n_j) > a_j, \ j = 1, \ldots, M \right) = \det \left( I - 1_{x_i \leq a_i} K_t 1_{x_i \leq a_i} \right)_{\ell^2(\{ n_1, \ldots, n_M \} \times \mathbb{Z})}
$$

where the RHS is a Fredholm determinant,

$$
K_t(n_i, x_i; n_j, x_j) = -Q^{n_j-n_i}(x_i, x_j) 1_{n_i < n_j} + \sum_{k=1}^{n_j} \psi_{n_i-k}^{n_j}(x_i) \phi_{n_j-k}^{n_j}(x_j)
$$

$$
Q(x, y) = 1_{x > y} \quad \psi_k^n(x) = \frac{1}{2\pi i} \oint_{\Gamma_0} \frac{d\omega}{\omega^{x-x_0(n-k)+1}} \left( \frac{1-w}{w} \right)^k e^{\omega(w-1/2)}
$$

The functions $\phi_k^n$ are defined implicitly by

1. Biorthogonality: $\sum_{x \in \mathbb{Z}} \psi_k^n(x) \phi_k^n(x) = 1_{\ell=k}$
2. $\phi_k^n$ is a polynomial of degree $k$
Exact formulas for $\Phi_k^n$

The functions $\Phi_k^n$ has been computed only for special initial datum
[Borodin, Ferrari, Prähofer, Sasamoto]

- **Step initial data:** $X_0(i) = -i$, $i \geq 1$:

$$\Phi_k^n(x) = \frac{1}{2\pi i} \oint_{\Gamma_0} dv \frac{(1 - v)^{x+n}}{v^{k+1}} e^{t(v+1/2)}$$

- **Periodic initial data:** $X_0(i) = -di$, $i \geq 1$, $d \geq 2$:

$$\Phi_k^n(x) = \frac{1}{2\pi i} \oint_{\Gamma_0} dv \frac{(1 - dv)(2(1 - v))^x + d^{n-1}}{v(2^d(1 - v)^{d-1}v)^k} e^{t(v+1/2)}$$

In these cases $\Phi_k^n$ are essentially the same as the $\Psi_k^n$
Explicit biorthogonalization

We can write

\[ \Psi^n_k(x) = e^{\frac{\xi}{2} \nabla^-} Q^{-k}(X_0(n - k), x) \]

where \( Q^{-1} Q = I \) and \( Q(x, y) = I_{x > y} \)

Matetski, Q, Remenik'17:

The functions \( \Phi^n_k \) are given by

\[ \Phi^n_k(x) = e^{\frac{\xi}{2} \nabla^-} h^n_k(0, x) \]

where \( h^n_k(\ell, z) \) is the unique solution to the backwards heat equation

\[
\begin{cases}
(Q^*)^{-1} h^n_k(\ell, z) = h^n_k(\ell + 1, z) & \ell < k, \ z \in \mathbb{Z} \\
h^n_k(k, z) = 1 & z \in \mathbb{Z} \\
h^n_k(\ell, X_0(n - \ell)) = 0 & \ell < k
\end{cases}
\]

Note! \( h^n_k \) is well defined because \( \dim \ker(Q^*)^{-1} = 1 \)
A simple check

Recall:
\[
\begin{align*}
(Q^*)^{-1} h_k^n(\ell, z) &= h_k^n(\ell + 1, z) & \ell < k, z \in \mathbb{Z} \\
h_k^n(k, z) &= 1 & z \in \mathbb{Z} \\
h_k^n(\ell, X_0(n - \ell)) &= 0 & \ell < k
\end{align*}
\]

Then we have
\[
\sum_{z \in \mathbb{Z}} \psi^n_\ell(z) \Phi_k^n(z) = \sum_{z, z_1, z_2 \in \mathbb{Z}} e^{-\frac{1}{2} \nabla^-} (z, z_1) Q^{-\ell}(z_1, X_0(n - \ell)) h_k^n(0, z_2) e^{\frac{1}{2} \nabla^-} (z_2, z)
\]
\[
= (Q^*)^{-\ell} h_k^n(0, X_0(n - \ell))
\]
\[
= h_k^n(\ell, X_0(n - \ell))
\]
\[
= \mathbb{1}_{k=\ell}
\]

To show that $\Phi_k^n$ is a polynomial we use $Q^{-1} = I + 2 \nabla^+$ recursively