Benign Overfitting

Peter Bartlett
CS and Statistics
UC Berkeley

August 26, 2019

Phil Long

Gábor Lugosi

Alexander Tsigler

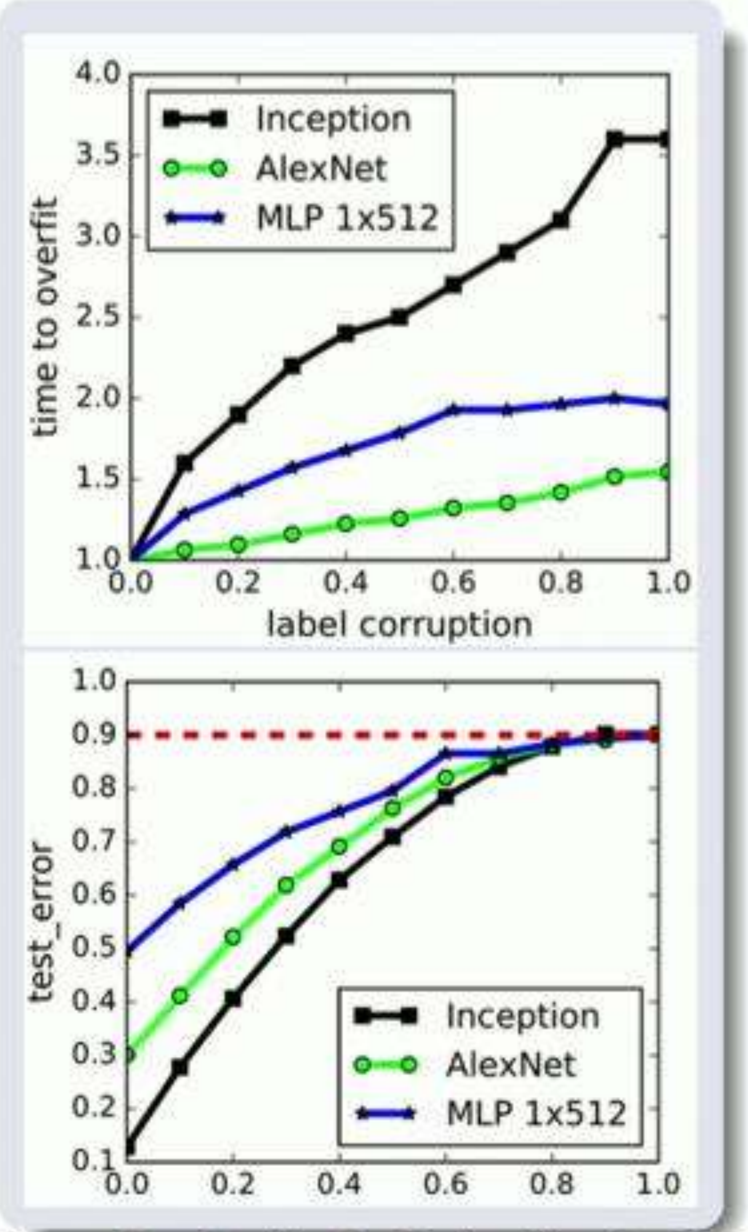
Overfitting in Deep Networks

- Deep networks can be trained to zero training error (for *regression* loss)

Overfitting in Deep Networks

- Deep networks can be trained to zero training error (for *regression* loss)
- ... with near state-of-the-art performance

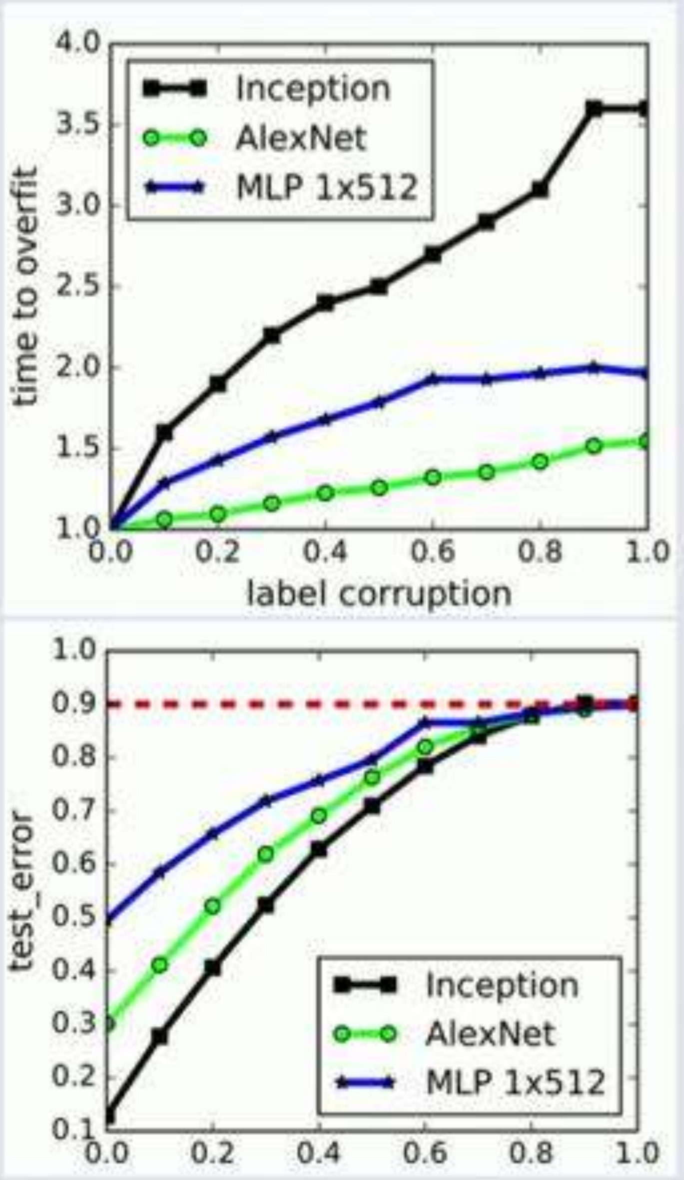
Overfitting in Deep Networks



(Zhang, Bengio, Hardt, Recht, Vinyals, 2017)

- Deep networks can be trained to zero training error (for *regression* loss)
- ... with near state-of-the-art performance
- ... even for noisy problems.

Overfitting in Deep Networks

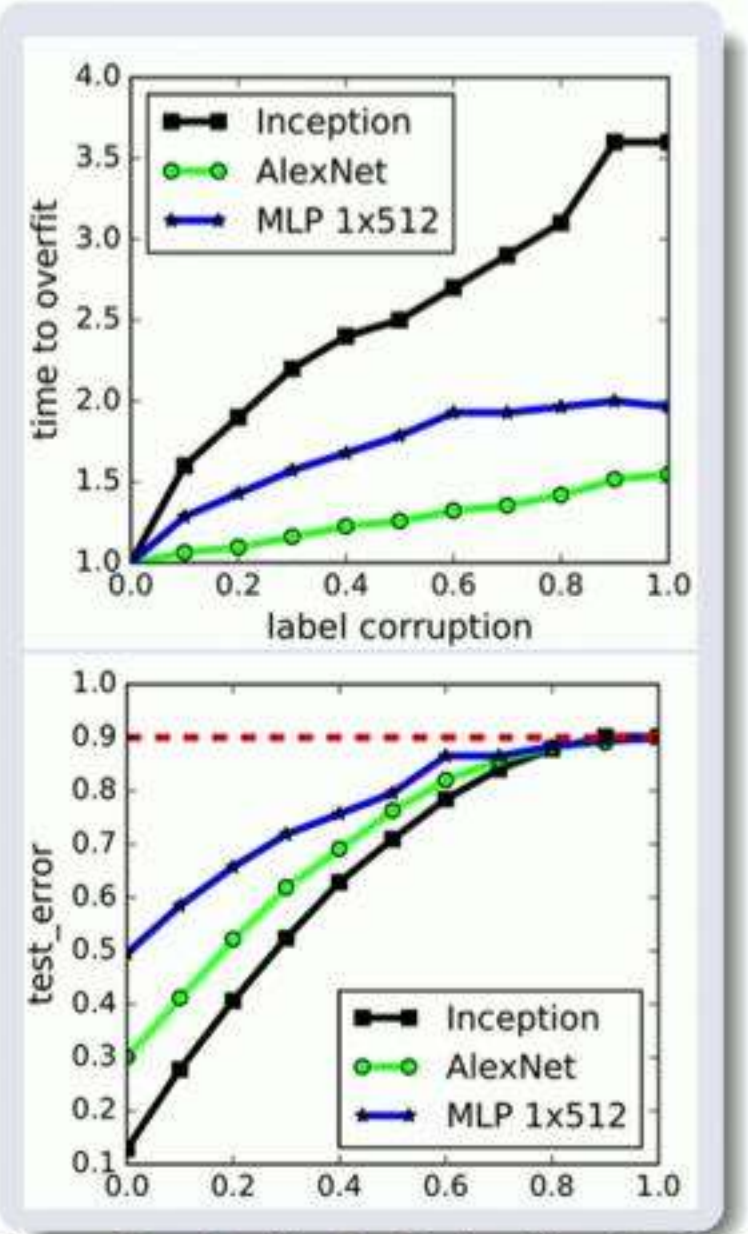


(Zhang, Bengio, Hardt, Recht, Vinyals, 2017)

- Deep networks can be trained to zero training error (for *regression* loss)
- ... with near state-of-the-art performance
- ... even for noisy problems.

also (Belkin, Hsu, Ma, Mandal, 2018)

Overfitting in Deep Networks

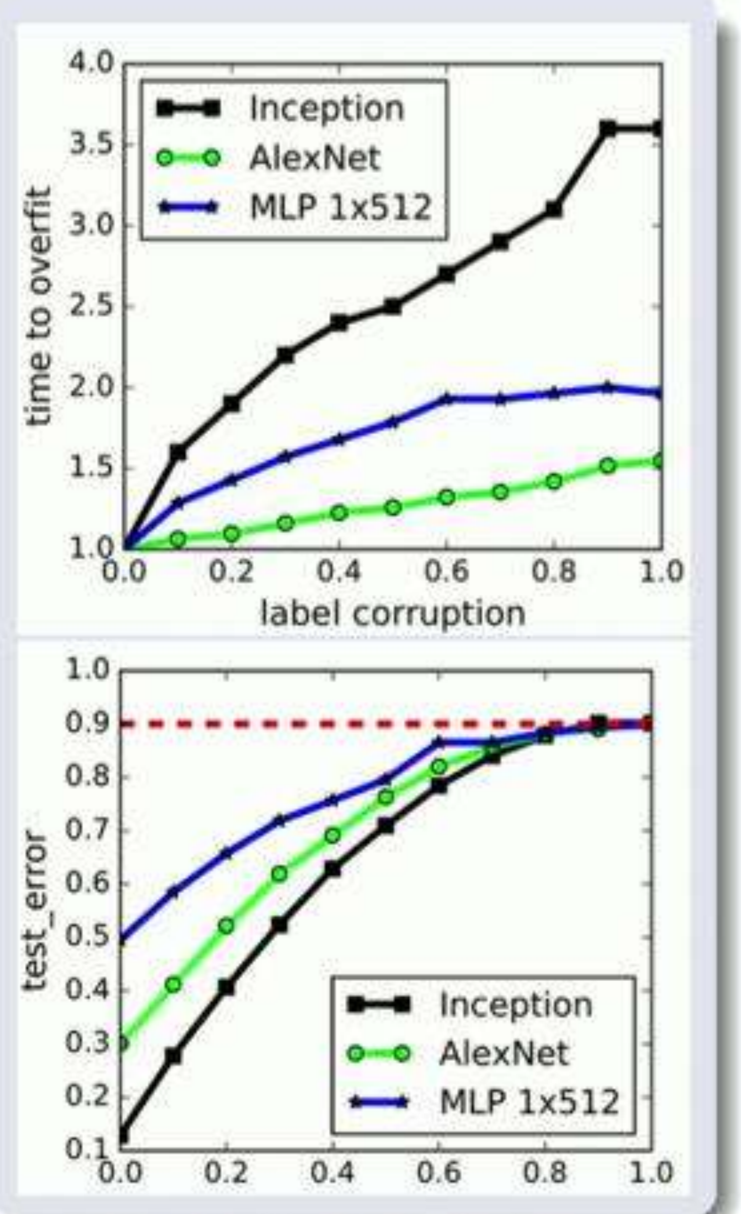


(Zhang, Bengio, Hardt, Recht, Vinyals, 2017)

- Deep networks can be trained to zero training error (for *regression* loss)
- ... with near state-of-the-art performance
- ... even for noisy problems.
- No tradeoff between fit to training data and complexity!

also (Belkin, Hsu, Ma, Mandal, 2018)

Overfitting in Deep Networks



(Zhang, Bengio, Hardt, Recht, Vinyals, 2017)

- Deep networks can be trained to zero training error (for *regression* loss)
- ... with near state-of-the-art performance
- ... even for noisy problems.
- No tradeoff between fit to training data and complexity!
- *Benign overfitting.*

also (Belkin, Hsu, Ma, Mandal, 2018)

Statistical Wisdom and Overfitting

Classical approaches to prediction

Typically, we aim for a trade-off between

- Fit to the training data,

- Complexity of a prediction rule

Statistical Wisdom and Overfitting

Classical approaches to prediction

Typically, we aim for a trade-off between

- Fit to the training data, e.g.,

$$\frac{1}{n}\sum_{i=1}^{n}\left(\hat{f}(x_i)-y_i\right)^2,$$

- Complexity of a prediction rule

Statistical Wisdom and Overfitting

Classical approaches to prediction

Typically, we aim for a trade-off between

- Fit to the training data, e.g.,

$$\frac{1}{n}\sum_{i=1}^{n}\left(\hat{f}(x_i) - y_i\right)^2,$$

- Complexity of a prediction rule, e.g.,
  - Number of parameters
  - Norm of parameter vector
  - Norm of function in a reproducing kernel Hilbert space,
  - Bandwidth of smoothing kernel,
  - ...

Statistical Wisdom and Overfitting

Classical approaches to prediction

Typically, we aim for a trade-off between

- Fit to the training data, e.g.,

$$\frac{1}{n}\sum_{i=1}^{n}\left(\hat{f}(x_i) - y_i\right)^2,$$

- Complexity of a prediction rule, e.g.,
  - Number of parameters
  - Norm of parameter vector
  - Norm of function in a reproducing kernel Hilbert space,
  - Bandwidth of smoothing kernel,
  - ...

This is especially important for nonparametric methods, that is, those for which the number of parameters grows with the sample size.

Statistical Wisdom and Overfitting

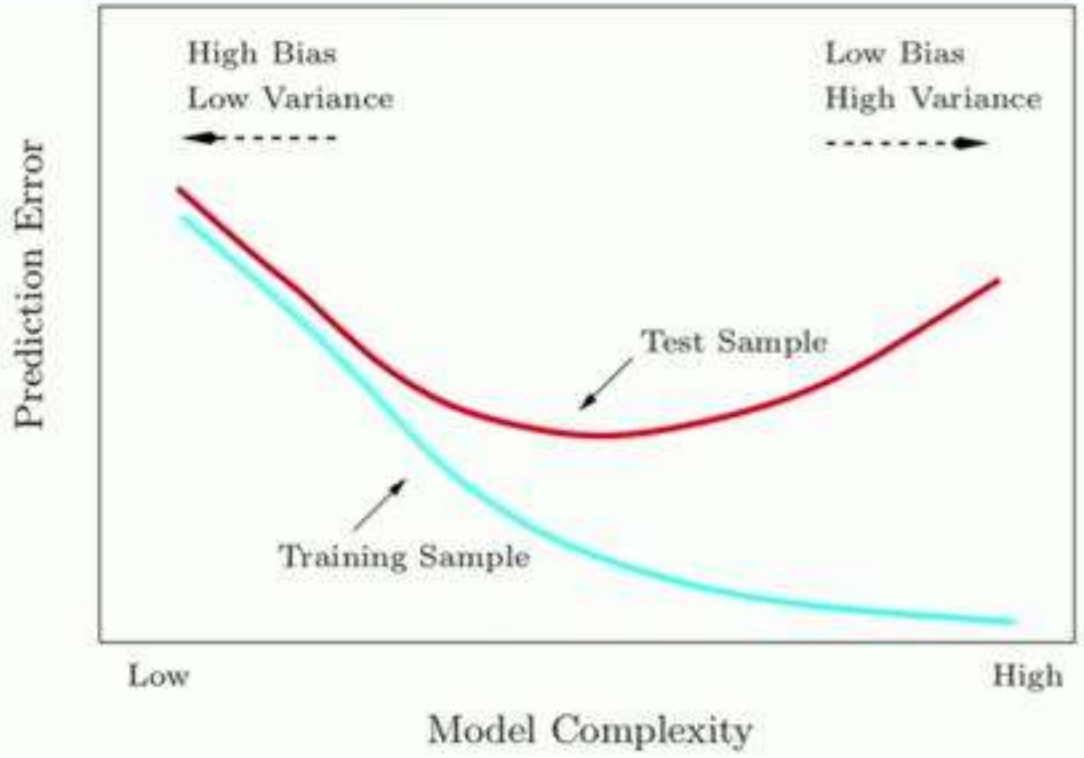
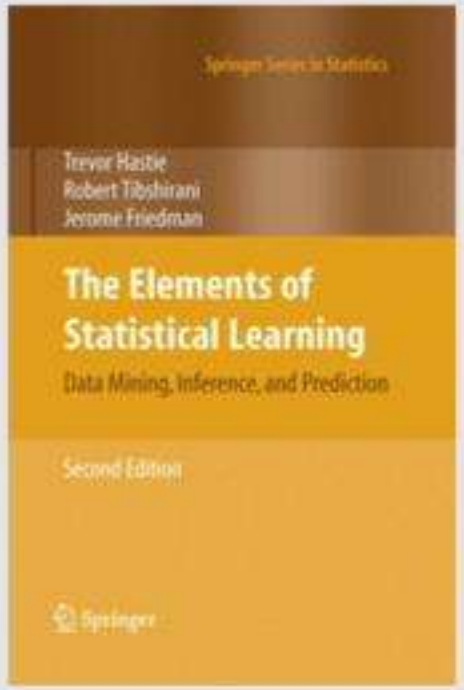


FIGURE 2.11. *Test and training error as a function of model complexity.*

Figure 2.11 shows the typical behavior of the test and training error, as model complexity is varied. The training error tends to decrease whenever we increase the model complexity, that is, whenever we fit the data harder. However with too much fitting, the model adapts itself too closely to the training data, and will not generalize well (i.e., have large test error). In

"... interpolating fits... [are] unlikely to predict future data well at all."



see also (B. and Rakhlin, Simons Institute, May 2019)

Statistical Wisdom and Overfitting

22 *2. How to Construct Nonparametric Regression Estimates?*

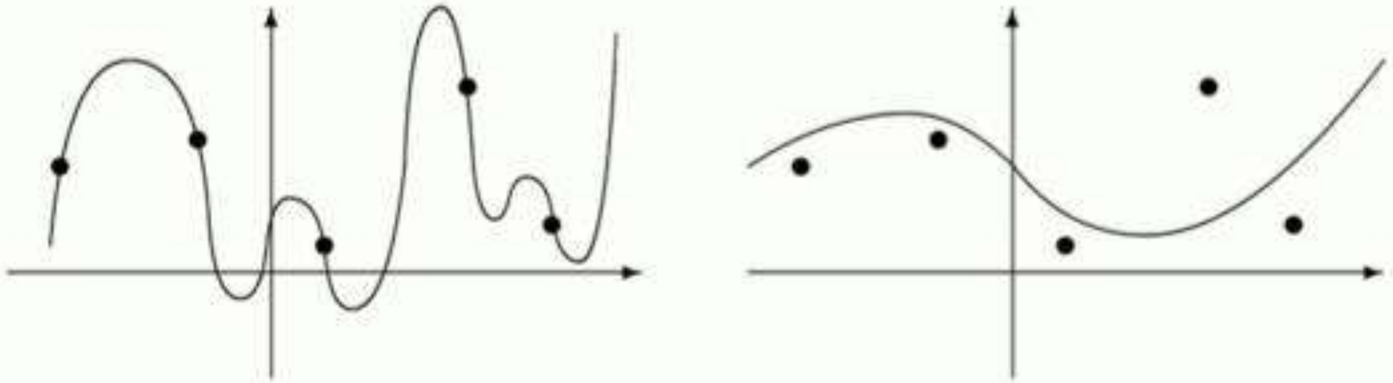
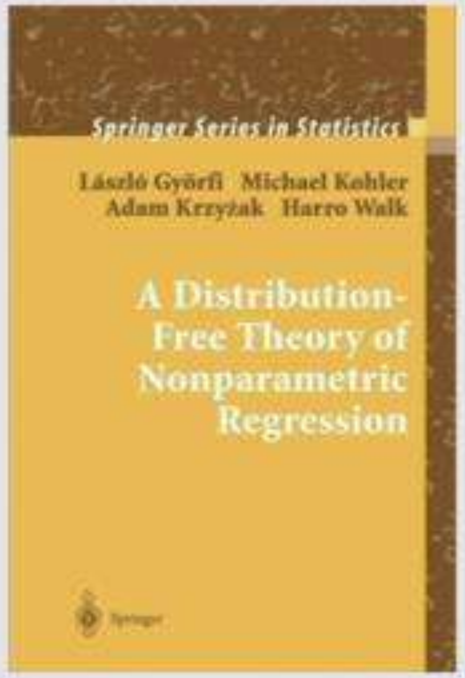


Figure 2.3. The estimate on the right seems to be more reasonable than the estimate on the left, which interpolates the data.

over $\mathcal{F}_n$. Least squares estimates are defined by minimizing the empirical L_2 risk over a general set of functions $\mathcal{F}_n$ (instead of (2.7)). Observe that it doesn't make sense to minimize (2.9) over all (measurable) functions f, because this may lead to a function which interpolates the data and hence is not a reasonable estimate. Thus one has to restrict the set of functions over

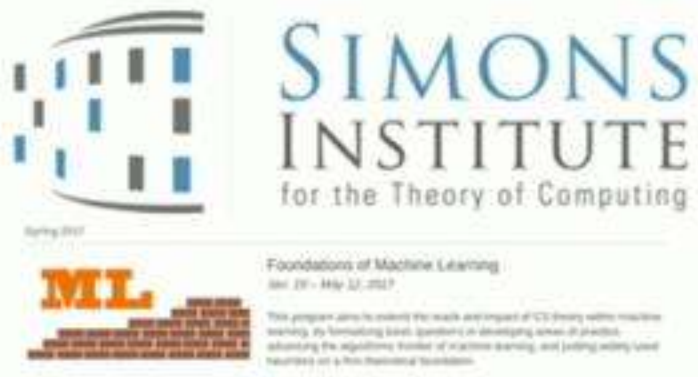


Benign Overfitting

**A new statistical phenomenon:
good prediction with zero training error for regression loss**

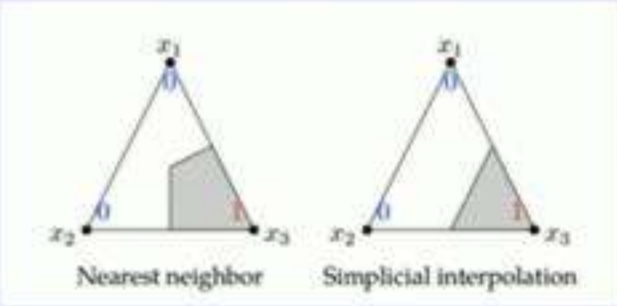
- Statistical wisdom says a prediction rule should not fit too well.
- But deep networks are trained to fit noisy data perfectly, and they predict well.

Progress on Interpolating Prediction Rules



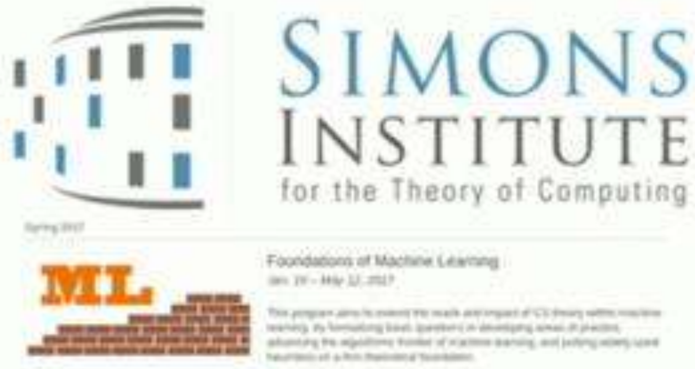
Progress on Interpolating Prediction Rules

Simplicial interpolation

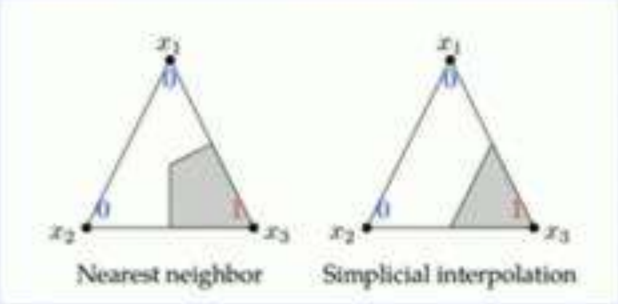


(Belkin, Hsu, Mitra, 2018)

Progress on Interpolating Prediction Rules



Simplicial interpolation



(Belkin, Hsu, Mitra, 2018)

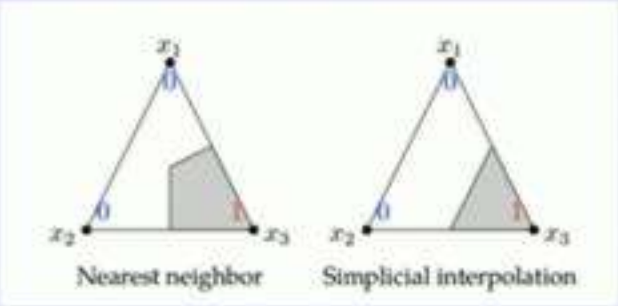
Kernel smoothing with singular kernels

$$\hat{f}(x) = \sum_{i=1}^{n} \frac{y_i K_h(x - x_i)}{\sum_{i=1}^{n} K_h(x - x_i)} \qquad \text{with } K_h(x) = \frac{1}{h\|x\|^{\alpha}}.$$

Minimax rates possible (with suitable h). (Belkin, Rakhlin, Tsybakov, 2018)

Progress on Interpolating Prediction Rules

Simplicial interpolation



(Belkin, Hsu, Mitra, 2018)

Kernel smoothing with singular kernels

$$\hat{f}(x) = \sum_{i=1}^{n} \frac{y_i K_h(x - x_i)}{\sum_{i=1}^{n} K_h(x - x_i)} \qquad \text{with } K_h(x) = \frac{1}{h\|x\|^\alpha}.$$

Minimax rates possible (with suitable h). (Belkin, Rakhlin, Tsybakov, 2018)

Linear regression with $d \asymp n$

- Kernels defined in terms of the Euclidean inner product

  (Liang and Rakhlin, 2018)

- Linear regression with $d, n \to \infty$, $d/n \to \gamma$ (Hastie, Montanari, Rosset, Tibshirani, 2019)

Outline

- *Linear regression*
- Characterizing benign overfitting
- Deep learning
- Adversarial examples

Definitions

Simple Prediction Setting: Linear Regression

Definitions

Simple Prediction Setting: Linear Regression

- Covariate $x \in \mathbb{H}$ (Hilbert space); response $y \in \mathbb{R}$.

Definitions

Simple Prediction Setting: Linear Regression

- Covariate $x \in \mathbb{H}$ (Hilbert space); response $y \in \mathbb{R}$.
- (x, y) Gaussian, mean zero. (or subgaussian, well-specified)

Definitions

Simple Prediction Setting: Linear Regression

- Covariate $x \in \mathbb{H}$ (Hilbert space); response $y \in \mathbb{R}$.
- (x, y) Gaussian, mean zero. (or subgaussian, well-specified)
- Define:

$$\Sigma := \mathbb{E}xx^\top = \sum_i \lambda_i v_i v_i^\top, \qquad (\text{assume } \lambda_1 \geq \lambda_2 \geq \cdots)$$

Definitions

Simple Prediction Setting: Linear Regression

- Covariate $x \in \mathbb{H}$ (Hilbert space); response $y \in \mathbb{R}$.
- (x, y) Gaussian, mean zero. (or subgaussian, well-specified)
- Define:

$$\Sigma := \mathbb{E}xx^\top = \sum_i \lambda_i v_i v_i^\top, \qquad \text{(assume } \lambda_1 \geq \lambda_2 \geq \cdots\text{)}$$

$$\theta^* := \arg\min_\theta \mathbb{E}\left(y - x^\top \theta\right)^2,$$

Definitions

Simple Prediction Setting: Linear Regression

- Covariate $x \in \mathbb{H}$ (Hilbert space); response $y \in \mathbb{R}$.
- (x, y) Gaussian, mean zero. (or subgaussian, well-specified)
- Define:

$$\Sigma := \mathbb{E}xx^\top = \sum_i \lambda_i v_i v_i^\top, \qquad \text{(assume } \lambda_1 \geq \lambda_2 \geq \cdots\text{)}$$

$$\theta^* := \arg\min_\theta \mathbb{E}\left(y - x^\top\theta\right)^2,$$

$$\sigma^2 := \mathbb{E}(y - x^\top\theta^*)^2.$$

Definitions

Minimum norm estimator

Definitions

Minimum norm estimator

- Data: $X \in \mathbb{H}^n$, $y \in \mathbb{R}^n$.

Definitions

Minimum norm estimator

- Data: $X \in \mathbb{H}^n$, $y \in \mathbb{R}^n$.
- Estimator $\hat{\theta} = \left(X^\top X\right)^\dagger X^\top y$, which solves

$$\begin{aligned} \min_{\theta \in \mathbb{H}} \quad & \|\theta\|^2 \\ \text{s.t.} \quad & \|X\theta - y\|^2 = \min_{\beta} \|X\beta - y\|^2 . \end{aligned}$$

Definitions

Excess prediction error

$$R(\hat{\theta}) := \mathbb{E}_{(x,y)}\left(y - x^\top \hat{\theta}\right)^2 - \underbrace{\min_{\theta} \mathbb{E}\left(y - x^\top \theta\right)^2}_{\text{optimal prediction error}}$$

Definitions

Excess prediction error

$$R(\hat{\theta}) := \mathbb{E}_{(x,y)}\left(y - x^\top\hat{\theta}\right)^2 - \underbrace{\min_{\theta}\mathbb{E}\left(y - x^\top\theta\right)^2}_{\text{optimal prediction error}}$$

$$= \mathbb{E}_{(x,y)}\left[\left(y - x^\top\hat{\theta}\right)^2 - \left(y - x^\top\theta^*\right)^2\right]$$

Definitions

Excess prediction error

$$
\begin{aligned}
R(\hat{\theta}) &:= \mathbb{E}_{(x,y)}\left(y - x^\top \hat{\theta}\right)^2 - \underbrace{\min_{\theta} \mathbb{E}\left(y - x^\top \theta\right)^2}_{\text{optimal prediction error}} \\
&= \mathbb{E}_{(x,y)}\left[\left(y - x^\top \hat{\theta}\right)^2 - \left(y - x^\top \theta^*\right)^2\right] \\
&= \left(\hat{\theta} - \theta^*\right)^\top \Sigma \left(\hat{\theta} - \theta^*\right).
\end{aligned}
$$

Definitions

Excess prediction error

$$R(\hat{\theta}) := \mathbb{E}_{(x,y)}\left(y - x^\top\hat{\theta}\right)^2 - \underbrace{\min_\theta \mathbb{E}\left(y - x^\top\theta\right)^2}_{\text{optimal prediction error}}$$

$$= \mathbb{E}_{(x,y)}\left[\left(y - x^\top\hat{\theta}\right)^2 - \left(y - x^\top\theta^*\right)^2\right]$$

$$= \left(\hat{\theta} - \theta^*\right)^\top \Sigma \left(\hat{\theta} - \theta^*\right).$$

So Σ determines the importance of parameter directions.

(Recall that $\Sigma = \sum_i \lambda_i v_i v_i^\top$ for orthonormal v_i, $\lambda_1 \geq \lambda_2 \geq \cdots$.)

Outline

- Linear regression
- *Characterizing benign overfitting*
- Deep learning
- Adversarial examples

Interpolating Linear Regression

Overfitting regime

- We consider situations where $\min_\beta \|X\beta - y\|^2 = 0$.

Interpolating Linear Regression

Overfitting regime

- We consider situations where $\min_\beta \|X\beta - y\|^2 = 0$.
- Hence, $y_1 = x_1^\top \hat{\theta}, \ldots, y_n = x_n^\top \hat{\theta}$.

Interpolating Linear Regression

Overfitting regime

- We consider situations where $\min_\beta \|X\beta - y\|^2 = 0$.
- Hence, $y_1 = x_1^\top \hat{\theta}, \ldots, y_n = x_n^\top \hat{\theta}$.
- When can the label noise be hidden in $\hat{\theta}$ without hurting predictive accuracy?

Benign Overfitting: A Characterization

Theorem

For universal constants b, c, and any linear regression problem $(\theta^*, \sigma^2, \Sigma)$ with $\lambda_n > 0$,

Benign Overfitting: A Characterization

Theorem

For universal constants b, c, and any linear regression problem $(\theta^*, \sigma^2, \Sigma)$ with $\lambda_n > 0$, if $k^* = \min\{k \geq 0 : r_k(\Sigma) \geq bn\}$,

Benign Overfitting: A Characterization

Theorem

For universal constants b, c, and any linear regression problem $(\theta^*, \sigma^2, \Sigma)$ with $\lambda_n > 0$, if $k^* = \min\{k \geq 0 : r_k(\Sigma) \geq bn\}$,

1. With high probability,

$$R(\hat{\theta}) \leq c \left(\|\theta^*\|^2 \|\Sigma\| \sqrt{\frac{r_0(\Sigma)}{n}} + \sigma^2 \left(\frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)} \right) \right),$$

Benign Overfitting: A Characterization

Theorem

For universal constants b, c, and any linear regression problem $(\theta^*, \sigma^2, \Sigma)$ with $\lambda_n > 0$, if $k^* = \min\{k \geq 0 : r_k(\Sigma) \geq bn\}$,

1. With high probability,

$$R(\hat{\theta}) \leq c \left(\|\theta^*\|^2 \|\Sigma\| \sqrt{\frac{r_0(\Sigma)}{n}} + \sigma^2 \left(\frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)} \right) \right),$$

2. $\mathbb{E} R(\hat{\theta}) \geq \frac{\sigma^2}{c} \min\left\{ \frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)}, 1 \right\}.$

   Also, $\frac{r_0(\Sigma)}{\ln(1 + r_0(\Sigma))} \geq \kappa n$ implies for some θ^*, $\Pr(R(\hat{\theta}) \geq 1/c) \geq 1/4$.

Notions of Effective Rank

Definition (Effective Ranks)

Recall that $\lambda_1 \geq \lambda_2 \geq \cdots$ are the eigenvalues of Σ.
For $k \geq 0$, if $\lambda_{k+1} > 0$, define the effective ranks

$$r_k(\Sigma) = \frac{\sum_{i>k} \lambda_i}{\lambda_{k+1}}, \qquad R_k(\Sigma) = \frac{\left(\sum_{i>k} \lambda_i\right)^2}{\sum_{i>k} \lambda_i^2}.$$

Notions of Effective Rank

Definition (Effective Ranks)

Recall that $\lambda_1 \geq \lambda_2 \geq \cdots$ are the eigenvalues of Σ.
For $k \geq 0$, if $\lambda_{k+1} > 0$, define the effective ranks

$$r_k(\Sigma) = \frac{\sum_{i>k} \lambda_i}{\lambda_{k+1}}, \qquad R_k(\Sigma) = \frac{\left(\sum_{i>k} \lambda_i\right)^2}{\sum_{i>k} \lambda_i^2}.$$

Lemma

$$1 \leq r_k(\Sigma) \leq R_k(\Sigma) \leq r_k^2(\Sigma).$$

Notions of Effective Rank

$$r_k(\Sigma) = \frac{\sum_{i>k} \lambda_i}{\lambda_{k+1}}, \qquad R_k(\Sigma) = \frac{\left(\sum_{i>k} \lambda_i\right)^2}{\sum_{i>k} \lambda_i^2}.$$

Examples

1. $r_0(I_p) = R_0(I_p) = p$.

Notions of Effective Rank

$$r_k(\Sigma) = \frac{\sum_{i>k} \lambda_i}{\lambda_{k+1}}, \qquad R_k(\Sigma) = \frac{\left(\sum_{i>k} \lambda_i\right)^2}{\sum_{i>k} \lambda_i^2}.$$

Examples

1. $r_0(I_p) = R_0(I_p) = p$.
2. If $\mathrm{rank}(\Sigma) = p$, we can write

$$r_0(\Sigma) = \mathrm{rank}(\Sigma) s(\Sigma), \qquad R_0(\Sigma) = \mathrm{rank}(\Sigma) S(\Sigma),$$

$$\text{with} \quad s(\Sigma) = \frac{1/p \sum_{i=1}^{p} \lambda_i}{\lambda_1}, \qquad S(\Sigma) = \frac{\left(1/p \sum_{i=1}^{p} \lambda_i\right)^2}{1/p \sum_{i=1}^{p} \lambda_i^2}.$$

Both s and S lie between $1/p$ ($\lambda_2 \approx 0$) and 1 (λ_i all equal).

Benign Overfitting: A Characterization

Theorem

For universal constants b, c, and any linear regression problem $(\theta^*, \sigma^2, \Sigma)$ with $\lambda_n > 0$, if $k^* = \min\{k \geq 0 : r_k(\Sigma) \geq bn\}$,

1. With high probability,

$$R(\hat{\theta}) \leq c\left(\|\theta^*\|^2\|\Sigma\|\sqrt{\frac{r_0(\Sigma)}{n}} + \sigma^2\left(\frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)}\right)\right),$$

2. $\mathbb{E}R(\hat{\theta}) \geq \frac{\sigma^2}{c}\min\left\{\frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)}, 1\right\}.$

   Also, $\frac{r_0(\Sigma)}{\ln(1 + r_0(\Sigma))} \geq \kappa n$ implies for some θ^*, $\Pr(R(\hat{\theta}) \geq 1/c) \geq 1/4$.

Notions of Effective Rank

$$r_k(\Sigma) = \frac{\sum_{i>k} \lambda_i}{\lambda_{k+1}}, \qquad R_k(\Sigma) = \frac{\left(\sum_{i>k} \lambda_i\right)^2}{\sum_{i>k} \lambda_i^2}.$$

Examples

Benign Overfitting: A Characterization

Theorem

For universal constants b, c, and any linear regression problem $(\theta^*, \sigma^2, \Sigma)$ with $\lambda_n > 0$, if $k^* = \min\{k \geq 0 : r_k(\Sigma) \geq bn\}$,

1. With high probability,

$$R(\hat{\theta}) \leq c \left(\|\theta^*\|^2 \|\Sigma\| \sqrt{\frac{r_0(\Sigma)}{n}} + \sigma^2 \left(\frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)} \right) \right),$$

2. $\mathbb{E}R(\hat{\theta}) \geq \dfrac{\sigma^2}{c} \min\left\{ \dfrac{k^*}{n} + \dfrac{n}{R_{k^*}(\Sigma)}, 1 \right\}.$

   Also, $\dfrac{r_0(\Sigma)}{\ln(1 + r_0(\Sigma))} \geq \kappa n$ implies for some θ^*, $\Pr(R(\hat{\theta}) \geq 1/c) \geq 1/4$.

Notions of Effective Rank

$$r_k(\Sigma) = \frac{\sum_{i>k} \lambda_i}{\lambda_{k+1}}, \qquad R_k(\Sigma) = \frac{\left(\sum_{i>k} \lambda_i\right)^2}{\sum_{i>k} \lambda_i^2}.$$

Examples

1. $r_0(I_p) = R_0(I_p) = p$.

Benign Overfitting: A Characterization

Theorem

For universal constants b, c, and any linear regression problem $(\theta^*, \sigma^2, \Sigma)$ with $\lambda_n > 0$, if $k^* = \min\{k \geq 0 : r_k(\Sigma) \geq bn\}$,

1. With high probability,

$$R(\hat{\theta}) \leq c\left(\|\theta^*\|^2\|\Sigma\|\sqrt{\frac{r_0(\Sigma)}{n}} + \sigma^2\left(\frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)}\right)\right),$$

2. $\mathbb{E}R(\hat{\theta}) \geq \frac{\sigma^2}{c}\min\left\{\frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)}, 1\right\}.$

   Also, $\frac{r_0(\Sigma)}{\ln(1 + r_0(\Sigma))} \geq \kappa n$ implies for some θ^*, $\Pr(R(\hat{\theta}) \geq 1/c) \geq 1/4$.

Benign Overfitting: A Characterization

Intuition

- The mix of eigenvalues of Σ determines:
  1. how the label noise is distributed in $\hat{\theta}$, and

Benign Overfitting: A Characterization

Intuition

- The mix of eigenvalues of Σ determines:
  1. how the label noise is distributed in $\hat{\theta}$, and
  2. how errors in $\hat{\theta}$ affect prediction accuracy.

Benign Overfitting: A Characterization

Intuition

- The mix of eigenvalues of Σ determines:
  1. how the label noise is distributed in $\hat{\theta}$, and
  2. how errors in $\hat{\theta}$ affect prediction accuracy.
- To avoid harming prediction accuracy, the noise energy must be distributed across many unimportant directions.

Benign Overfitting: A Characterization

Intuition

- The mix of eigenvalues of Σ determines:
  1. how the label noise is distributed in $\hat{\theta}$, and
  2. how errors in $\hat{\theta}$ affect prediction accuracy.
- To avoid harming prediction accuracy, the noise energy must be distributed across many unimportant directions.
- Overparameterization is essential for benign overfitting

Benign Overfitting: A Characterization

Intuition

- The mix of eigenvalues of Σ determines:
  1. how the label noise is distributed in $\hat{\theta}$, and
  2. how errors in $\hat{\theta}$ affect prediction accuracy.
- To avoid harming prediction accuracy, the noise energy must be distributed across many unimportant directions.
- Overparameterization is essential for benign overfitting
  - Number of non-zero eigenvalues: large compared to n,

Benign Overfitting: A Characterization

Intuition

- The mix of eigenvalues of Σ determines:
  1. how the label noise is distributed in $\hat{\theta}$, and
  2. how errors in $\hat{\theta}$ affect prediction accuracy.
- To avoid harming prediction accuracy, the noise energy must be distributed across many unimportant directions.
- Overparameterization is essential for benign overfitting
  - Number of non-zero eigenvalues: large compared to n,
  - Their sum: small compared to n,

Benign Overfitting: A Characterization

Intuition

- The mix of eigenvalues of Σ determines:
  1. how the label noise is distributed in $\hat{\theta}$, and
  2. how errors in $\hat{\theta}$ affect prediction accuracy.
- To avoid harming prediction accuracy, the noise energy must be distributed across many unimportant directions.
- Overparameterization is essential for benign overfitting
  - Number of non-zero eigenvalues: large compared to n,
  - Their sum: small compared to n,
  - Number of 'small' eigenvalues: large compared to n,

Benign Overfitting: A Characterization

Intuition

- The mix of eigenvalues of Σ determines:
  1. how the label noise is distributed in $\hat{\theta}$, and
  2. how errors in $\hat{\theta}$ affect prediction accuracy.
- To avoid harming prediction accuracy, the noise energy must be distributed across many unimportant directions.
- Overparameterization is essential for benign overfitting
  - Number of non-zero eigenvalues: large compared to n,
  - Their sum: small compared to n,
  - Number of 'small' eigenvalues: large compared to n,
  - Small eigenvalues: roughly equal (but they can be more assymmetric if there are many more than n of them).

Benign Overfitting: Proof Ideas

Interpolation for linear prediction

- Excess expected loss, has two components: (corresponding to $x^\top \theta^*$ and $y - x^\top \theta^*$)

Benign Overfitting: Proof Ideas

Interpolation for linear prediction

- Excess expected loss, has two components: (corresponding to $x^\top \theta^*$ and $y - x^\top \theta^*$)
  1. $\hat{\theta}$ is a distorted version of θ^*, because the sample $x_1, \ldots, x_n$ distorts our view of the covariance of x.

Benign Overfitting: Proof Ideas

Interpolation for linear prediction

- Excess expected loss, has two components: (corresponding to $x^\top \theta^*$ and $y - x^\top \theta^*$)
  1. $\hat{\theta}$ is a distorted version of θ^*, because the sample $x_1, \ldots, x_n$ distorts our view of the covariance of x.
  2. $\hat{\theta}$ is corrupted by the noise in $y_1, \ldots, y_n$.

Benign Overfitting: Proof Ideas

Interpolation for linear prediction

- Excess expected loss, has two components: (corresponding to $x^\top \theta^*$ and $y - x^\top \theta^*$)
  1. $\hat{\theta}$ is a distorted version of θ^*, because the sample $x_1, \ldots, x_n$ distorts our view of the covariance of x.

     Not a problem, even in high dimensions ($p > n$).
  2. $\hat{\theta}$ is corrupted by the noise in $y_1, \ldots, y_n$.

Benign Overfitting: Proof Ideas

Interpolation for linear prediction

- Excess expected loss, has two components: (corresponding to $x^\top \theta^*$ and $y - x^\top \theta^*$)
  1. $\hat{\theta}$ is a distorted version of θ^*, because the sample $x_1, \ldots, x_n$ distorts our view of the covariance of x.

     Not a problem, even in high dimensions ($p > n$).
  2. $\hat{\theta}$ is corrupted by the noise in $y_1, \ldots, y_n$.

     Problematic.
- When can the label noise be hidden in $\hat{\theta}$ without hurting predictive accuracy?

Benign Overfitting: Proof Ideas

Bias-variance decomposition

Define the noise vector ϵ by $y = X\theta^* + \epsilon$.

Benign Overfitting: Proof Ideas

Bias-variance decomposition

Define the noise vector ϵ by $y = X\theta^* + \epsilon$.

Estimator: $$\hat{\theta} = (X^\top X)^\dagger X^\top y$$

Benign Overfitting: Proof Ideas

Bias-variance decomposition

Define the noise vector ϵ by $y = X\theta^* + \epsilon$.

$$\text{Estimator:} \qquad \hat{\theta} = (X^\top X)^\dagger X^\top y = (X^\top X)^\dagger X^\top (X\theta^* + \epsilon),$$

Benign Overfitting: Proof Ideas

Bias-variance decomposition

Define the noise vector ϵ by $y = X\theta^* + \epsilon$.

$$\text{Estimator:} \qquad \hat{\theta} = (X^\top X)^\dagger X^\top y = (X^\top X)^\dagger X^\top (X\theta^* + \epsilon),$$

$$\text{Excess risk:} \qquad R(\hat{\theta}) = \left(\hat{\theta} - \theta^*\right)^\top \Sigma \left(\hat{\theta} - \theta^*\right)$$

Benign Overfitting: Proof Ideas

Bias-variance decomposition

Define the noise vector ϵ by $y = X\theta^* + \epsilon$.

$$\text{Estimator:} \qquad \hat{\theta} = (X^\top X)^\dagger X^\top y = (X^\top X)^\dagger X^\top (X\theta^* + \epsilon),$$

$$\begin{aligned}
\text{Excess risk:} \qquad R(\hat{\theta}) &= \left(\hat{\theta} - \theta^*\right)^\top \Sigma \left(\hat{\theta} - \theta^*\right) \\
&\approx \theta^{*\top} \left(I - \hat{\Sigma}\hat{\Sigma}^\dagger\right) \left(\Sigma - \hat{\Sigma}\right) \left(I - \hat{\Sigma}^\dagger\hat{\Sigma}\right) \theta^* \\
&\quad + \sigma^2 \mathrm{tr}\left(\left(X^\top X\right)^\dagger \Sigma\right).
\end{aligned}$$

Benign Overfitting: Proof Ideas

The excess risk

$$R(\hat{\theta}) = \left(\hat{\theta} - \theta^*\right)^\top \Sigma \left(\hat{\theta} - \theta^*\right).$$

Benign Overfitting: Proof Ideas

The excess risk

$$R(\hat{\theta}) = \left(\hat{\theta} - \theta^*\right)^\top \Sigma \left(\hat{\theta} - \theta^*\right).$$

- Write $\Sigma = \sum_i \lambda_i v_i v_i^\top$.

Benign Overfitting: Proof Ideas

The excess risk

$$R(\hat{\theta}) = \left(\hat{\theta} - \theta^*\right)^\top \Sigma \left(\hat{\theta} - \theta^*\right).$$

- Write $\Sigma = \sum_i \lambda_i v_i v_i^\top$.
- Split the v_i into "heavy" directions (corresponding to $\lambda_1 \geq \cdots \geq \lambda_k$) and "light" ones (corresponding to $\lambda_{k+1}, \ldots$).

Benign Overfitting: Proof Ideas

The excess risk

$$R(\hat{\theta}) = \left(\hat{\theta} - \theta^*\right)^\top \Sigma \left(\hat{\theta} - \theta^*\right).$$

- Write $\Sigma = \sum_i \lambda_i v_i v_i^\top$.
- Split the v_i into "heavy" directions (corresponding to $\lambda_1 \geq \cdots \geq \lambda_k$) and "light" ones (corresponding to $\lambda_{k+1}, \ldots$).
- If $r_k(\Sigma) \geq n$, the smallest positive ($(k+1)$-th to n-th) eigenvalues of $X^\top X$ are all concentrated (around $\rho := \sum_{i>k} \lambda_i$).

Benign Overfitting: Proof Ideas

The excess risk

$$R(\hat{\theta}) = \left(\hat{\theta} - \theta^*\right)^\top \Sigma \left(\hat{\theta} - \theta^*\right).$$

- Write $\Sigma = \sum_i \lambda_i v_i v_i^\top$.
- Split the v_i into "heavy" directions (corresponding to $\lambda_1 \geq \cdots \geq \lambda_k$) and "light" ones (corresponding to $\lambda_{k+1}, \ldots$).
- If $r_k(\Sigma) \geq n$, the smallest positive (($k+1$)-th to n-th) eigenvalues of $X^\top X$ are all concentrated (around $\rho := \sum_{i>k} \lambda_i$).
- So $XX^\top \succeq \rho I$.

$$\hat{\theta} = (X^\top X)^\dagger X^\top y$$

c.f. ridge regression: $\hat{\theta} = \left(X^\top X + \rho I\right)^{-1} X^\top y$.

Benign Overfitting: Proof Ideas

The minimum norm estimator

$\hat{\theta} = (X^\top X)^\dagger X^\top y = (X^\top X)^\dagger X^\top \epsilon + \cdots.$

Benign Overfitting: Proof Ideas

The minimum norm estimator

$\hat{\theta} = (X^\top X)^\dagger X^\top y = (X^\top X)^\dagger X^\top \epsilon + \cdots.$

Where does the energy from the noise go?

- A direction v_i sees noise energy (from $X^\top \epsilon$) proportional to $n\lambda_i$.

Benign Overfitting: Proof Ideas

The minimum norm estimator

$$\hat{\theta} = (X^\top X)^\dagger X^\top y = (X^\top X)^\dagger X^\top \epsilon + \cdots.$$

Where does the energy from the noise go?

- A direction v_i sees noise energy (from $X^\top \epsilon$) proportional to $n\lambda_i$.
- This is scaled by no more than ρ^{-2}.

Benign Overfitting: Proof Ideas

The minimum norm estimator

$$\hat{\theta} = (X^\top X)^\dagger X^\top y = (X^\top X)^\dagger X^\top \epsilon + \cdots . \qquad R(\hat{\theta}) = \left(\hat{\theta} - \theta^*\right)^\top \Sigma \left(\hat{\theta} - \theta^*\right).$$

Where does the energy from the noise go?

- A direction v_i sees noise energy (from $X^\top \epsilon$) proportional to $n\lambda_i$.
- This is scaled by no more than ρ^{-2}.
- Its impact on the prediction error is scaled by another factor of λ_i.

Benign Overfitting: Proof Ideas

The minimum norm estimator

$$\hat{\theta} = (X^\top X)^\dagger X^\top y = (X^\top X)^\dagger X^\top \epsilon + \cdots. \qquad R(\hat{\theta}) = \left(\hat{\theta} - \theta^*\right)^\top \Sigma \left(\hat{\theta} - \theta^*\right).$$

Where does the energy from the noise go?

- A direction v_i sees noise energy (from $X^\top \epsilon$) proportional to $n\lambda_i$.
- This is scaled by no more than ρ^{-2}.
- Its impact on the prediction error is scaled by another factor of λ_i.
- Bound on prediction error: $n\lambda_i^2 \rho^{-2}$.

Benign Overfitting: Proof Ideas

The minimum norm estimator

$$\hat{\theta} = (X^\top X)^\dagger X^\top y = (X^\top X)^\dagger X^\top \epsilon + \cdots . \quad R(\hat{\theta}) = \left(\hat{\theta} - \theta^*\right)^\top \Sigma \left(\hat{\theta} - \theta^*\right).$$

Where does the energy from the noise go?

- A direction v_i sees noise energy (from $X^\top \epsilon$) proportional to $n\lambda_i$.
- This is scaled by no more than ρ^{-2}.
- Its impact on the prediction error is scaled by another factor of λ_i.
- Bound on prediction error: $n\lambda_i^2 \rho^{-2}$.
- (We can do better in the "heavy" directions: $\leq 1/n$.)

Benign Overfitting: Proof Ideas

The minimum norm estimator

$$\hat{\theta} = (X^\top X)^\dagger X^\top y = (X^\top X)^\dagger X^\top \epsilon + \cdots. \qquad R(\hat{\theta}) = \left(\hat{\theta} - \theta^*\right)^\top \Sigma \left(\hat{\theta} - \theta^*\right).$$

Where does the energy from the noise go?

- A direction v_i sees noise energy (from $X^\top \epsilon$) proportional to $n\lambda_i$.
- This is scaled by no more than ρ^{-2}.
- Its impact on the prediction error is scaled by another factor of λ_i.
- Bound on prediction error: $n\lambda_i^2 \rho^{-2}$.
- (We can do better in the "heavy" directions: $\leq 1/n$.)

Total prediction error bound: $\dfrac{k}{n} + n \displaystyle\sum_{i>k} \lambda_i^2 \rho^{-2} = \dfrac{k}{n} + \dfrac{n}{R_k(\Sigma)}$.

Benign Overfitting: Proof Ideas

Lower bound

- The excess expected loss is at least as big as the same trace term, $\mathrm{tr}\left((X^\top X)^\dagger \Sigma\right)$.

Benign Overfitting: Proof Ideas

Lower bound

- The excess expected loss is at least as big as the same trace term, $\operatorname{tr}\left((X^\top X)^\dagger \Sigma\right)$.
- When the eigenvalues of $XX^\top$ are concentrated, the same split gives a lower bound within a constant factor of the upper bound.

Benign Overfitting: Proof Ideas

Lower bound

- The excess expected loss is at least as big as the same trace term, $\mathrm{tr}\left((X^\top X)^\dagger \Sigma\right)$.
- When the eigenvalues of $XX^\top$ are concentrated, the same split gives a lower bound within a constant factor of the upper bound.
- And otherwise, the excess expected loss is at least a constant.

Benign Overfitting: A Characterization

Theorem

For universal constants b, c, and any linear regression problem $(\theta^*, \sigma^2, \Sigma)$ with $\lambda_n > 0$, if $k^* = \min\{k \geq 0 : r_k(\Sigma) \geq bn\}$,

1. With high probability,
$$R(\hat{\theta}) \leq c\left(\|\theta^*\|^2\|\Sigma\|\sqrt{\frac{r_0(\Sigma)}{n}} + \sigma^2\left(\frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)}\right)\right),$$
2. $\mathbb{E}R(\hat{\theta}) \geq \frac{\sigma^2}{c}\min\left\{\frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)}, 1\right\}$.

   Also, $\frac{r_0(\Sigma)}{\ln(1 + r_0(\Sigma))} \geq \kappa n$ implies for some θ^*, $\Pr(R(\hat{\theta}) \geq 1/c) \geq 1/4$.

What kinds of eigenvalues?

We say Σ is *asymptotically benign* if

$$\lim_{n\to\infty}\left(\|\Sigma\|\sqrt{\frac{r_0(\Sigma)}{n}}+\frac{k_n^*}{n}+\frac{n}{R_{k_n^*}(\Sigma)}\right)=0,$$

where $k_n^* = \min\{k \geq 0 : r_k(\Sigma) \geq bn\}$.

Benign Overfitting: A Characterization

Theorem

For universal constants b, c, and any linear regression problem $(\theta^*, \sigma^2, \Sigma)$ with $\lambda_n > 0$, if $k^* = \min\{k \geq 0 : r_k(\Sigma) \geq bn\}$,

1. With high probability,

$$R(\hat{\theta}) \leq c \left(\|\theta^*\|^2 \|\Sigma\| \sqrt{\frac{r_0(\Sigma)}{n}} + \sigma^2 \left(\frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)} \right) \right),$$

2. $\mathbb{E}R(\hat{\theta}) \geq \dfrac{\sigma^2}{c} \min\left\{ \dfrac{k^*}{n} + \dfrac{n}{R_{k^*}(\Sigma)}, 1 \right\}.$

   Also, $\dfrac{r_0(\Sigma)}{\ln(1 + r_0(\Sigma))} \geq \kappa n$ implies for some θ^*, $\Pr(R(\hat{\theta}) \geq 1/c) \geq 1/4$.

What kinds of eigenvalues?

We say Σ is *asymptotically benign* if

$$\lim_{n\to\infty}\left(\|\Sigma\|\sqrt{\frac{r_0(\Sigma)}{n}}+\frac{k_n^*}{n}+\frac{n}{R_{k_n^*}(\Sigma)}\right)=0,$$

where $k_n^* = \min\{k \geq 0 : r_k(\Sigma) \geq bn\}$.

What kinds of eigenvalues?

We say Σ is *asymptotically benign* if

$$\lim_{n\to\infty}\left(\|\Sigma\|\sqrt{\frac{r_0(\Sigma)}{n}}+\frac{k_n^*}{n}+\frac{n}{R_{k_n^*}(\Sigma)}\right)=0,$$

where $k_n^* = \min\{k \geq 0 : r_k(\Sigma) \geq bn\}$.

Example

If $\lambda_i = i^{-\alpha}\ln^{-\beta}(i+1)$, then Σ is benign iff $\alpha = 1$ and $\beta > 1$.

Benign Overfitting: A Characterization

Theorem

For universal constants b, c, and any linear regression problem $(\theta^*, \sigma^2, \Sigma)$ with $\lambda_n > 0$, if $k^* = \min\{k \geq 0 : r_k(\Sigma) \geq bn\}$,

1. With high probability,

$$R(\hat{\theta}) \leq c \left(\|\theta^*\|^2 \|\Sigma\| \sqrt{\frac{r_0(\Sigma)}{n}} + \sigma^2 \left(\frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)} \right) \right),$$

2. $\mathbb{E}R(\hat{\theta}) \geq \frac{\sigma^2}{c} \min\left\{ \frac{k^*}{n} + \frac{n}{R_{k^*}(\Sigma)}, 1 \right\}.$

   Also, $\frac{r_0(\Sigma)}{\ln(1 + r_0(\Sigma))} \geq \kappa n$ implies for some θ^*, $\Pr(R(\hat{\theta}) \geq 1/c) \geq 1/4$.

What kinds of eigenvalues?

We say Σ is *asymptotically benign* if

$$\lim_{n\to\infty}\left(\|\Sigma\|\sqrt{\frac{r_0(\Sigma)}{n}}+\frac{k_n^*}{n}+\frac{n}{R_{k_n^*}(\Sigma)}\right)=0,$$

where $k_n^* = \min\{k \geq 0 : r_k(\Sigma) \geq bn\}$.

What kinds of eigenvalues?

We say Σ is *asymptotically benign* if

$$\lim_{n\to\infty}\left(\|\Sigma\|\sqrt{\frac{r_0(\Sigma)}{n}}+\frac{k_n^*}{n}+\frac{n}{R_{k_n^*}(\Sigma)}\right)=0,$$

where $k_n^* = \min\{k \geq 0 : r_k(\Sigma) \geq bn\}$.

Example

If $\lambda_i = i^{-\alpha}\ln^{-\beta}(i+1)$, then Σ is benign iff $\alpha = 1$ and $\beta > 1$.

What kinds of eigenvalues?

We say Σ is *asymptotically benign* if

$$\lim_{n\to\infty}\left(\|\Sigma\|\sqrt{\frac{r_0(\Sigma)}{n}}+\frac{k_n^*}{n}+\frac{n}{R_{k_n^*}(\Sigma)}\right)=0,$$

where $k_n^* = \min\{k \geq 0 : r_k(\Sigma) \geq bn\}$.

Example

If $\lambda_i = i^{-\alpha}\ln^{-\beta}(i+1)$, then Σ is benign iff $\alpha = 1$ and $\beta > 1$.

The λ_i must be almost diverging!!?!?

What kinds of eigenvalues?

Example: Finite dimension, plus isotropic noise

If

$$\lambda_{k,n} = \begin{cases} e^{-k} + \epsilon_n & \text{if } k \le p_n, \\ 0 & \text{otherwise,} \end{cases}$$

then Σ_n is benign iff

- $p_n = \omega(n)$,
- $\epsilon_n p_n = o(n)$ and $\epsilon_n p_n = \omega(ne^{-n})$.

What kinds of eigenvalues?

Example: Finite dimension, plus isotropic noise

If

$$\lambda_{k,n} = \begin{cases} e^{-k} + \epsilon_n & \text{if } k \le p_n, \\ 0 & \text{otherwise,} \end{cases}$$

then Σ_n is benign iff

- $p_n = \omega(n)$,
- $\epsilon_n p_n = o(n)$ and $\epsilon_n p_n = \omega(ne^{-n})$.

Furthermore, for $p_n = \Omega(n)$ and $\epsilon_n p_n = \omega(ne^{-n})$,

$$R(\hat{\theta}) = O\left(\frac{\epsilon_n p_n}{n} + \max\left\{\frac{1}{n}, \frac{n}{p_n}\right\}\right).$$

What kinds of eigenvalues?

Example: Finite dimension, plus isotropic noise

If

$$\lambda_{k,n} = \begin{cases} e^{-k} + \epsilon_n & \text{if } k \le p_n, \\ 0 & \text{otherwise,} \end{cases}$$

then Σ_n is benign iff

- $p_n = \omega(n)$,
- $\epsilon_n p_n = o(n)$ and $\epsilon_n p_n = \omega(ne^{-n})$. $(n \ge 40 \implies ne^{-n} < 2^{-52})$

Furthermore, for $p_n = \Omega(n)$ and $\epsilon_n p_n = \omega(ne^{-n})$,

$$R(\hat{\theta}) = O\left(\frac{\epsilon_n p_n}{n} + \max\left\{\frac{1}{n}, \frac{n}{p_n}\right\}\right).$$

Universal phenomenon: fast converging λ_i, $p_n \gg n$, noise in all directions.

Extensions

Beyond Gaussian

Extensions

Beyond Gaussian

1. Linear model: $\mathbb{E}[y|x] = x^\top \theta^*.$

Extensions

Beyond Gaussian

1. Linear model: $$\mathbb{E}[y|x] = x^\top \theta^*.$$
2. Noise is subgaussian: $$\mathbb{E}\left[\exp\left(\lambda\left(y - x^\top \theta^*\right)\right)|x\right] \leq \exp\left(\sigma_y^2 \lambda^2/2\right).$$
3. Components of $\Sigma^{-1/2}x$ are *independent* subgaussian: $$\mathbb{E}\left[\exp(\lambda^\top \Sigma^{-1/2} x)\right] \leq \exp\left(\sigma_x^2 \|\lambda\|^2/2\right).$$

Extensions

Beyond Gaussian

1. Linear model: $\mathbb{E}[y|x] = x^\top \theta^*.$
2. Noise is subgaussian:
$$\mathbb{E}\left[\exp\left(\lambda\left(y - x^\top\theta^*\right)\right)|x\right] \le \exp\left(\sigma_y^2\lambda^2/2\right).$$
3. Components of $\Sigma^{-1/2}x$ are *independent* subgaussian:
$$\mathbb{E}\left[\exp(\lambda^\top\Sigma^{-1/2}x)\right] \le \exp\left(\sigma_x^2\|\lambda\|^2/2\right).$$

Open questions

- Misspecified?

Extensions

Beyond Gaussian

1. Linear model: $\mathbb{E}[y|x] = x^\top \theta^*$.
2. Noise is subgaussian:
$$\mathbb{E}\left[\exp\left(\lambda\left(y - x^\top \theta^*\right)\right)|x\right] \le \exp\left(\sigma_y^2 \lambda^2/2\right).$$
3. Components of $\Sigma^{-1/2}x$ are *independent* subgaussian:
$$\mathbb{E}\left[\exp(\lambda^\top \Sigma^{-1/2} x)\right] \le \exp\left(\sigma_x^2 \|\lambda\|^2/2\right).$$

Open questions

- Misspecified?
- Less independence? e.g., $k(x, \cdot) \in \mathbb{H}$?

Extensions

Beyond Gaussian

1. Linear model: $\mathbb{E}[y|x] = x^\top \theta^*$.
2. Noise is subgaussian:
$$\mathbb{E}\left[\exp\left(\lambda\left(y - x^\top\theta^*\right)\right)|x\right] \leq \exp\left(\sigma_y^2\lambda^2/2\right).$$
3. Components of $\Sigma^{-1/2}x$ are *independent* subgaussian:
$$\mathbb{E}\left[\exp(\lambda^\top\Sigma^{-1/2}x)\right] \leq \exp\left(\sigma_x^2\|\lambda\|^2/2\right).$$

Open questions

- Misspecified?
- Less independence? e.g., $k(x,\cdot) \in \mathbb{H}$?

e.g., see (Rakhlin and Zhai, 2018)

Outline

- Linear regression
- Characterizing benign overfitting
- *Deep learning*
- Adversarial examples

Implications for deep learning

Neural networks versus linear prediction

For wide enough randomly initialized neural networks, gradient descent dynamics quickly converge to (approximately) a *min-norm interpolating solution* with respect to a certain kernel.

Implications for deep learning

Neural networks versus linear prediction

For wide enough randomly initialized neural networks, gradient descent dynamics quickly converge to (approximately) a *min-norm interpolating solution* with respect to a certain kernel.

For example, for

$$f(x) = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} a_i \sigma\left(\langle w_i, x\rangle\right),$$

the corresponding (random) kernel is

$$K^m(x, x_j) := \frac{1}{m} \sum_{i=1}^{m} a_i^2 \sigma'\left(\langle w_i, x\rangle\right) \sigma'\left(\langle w_i, x_j\rangle\right) \langle x, x_j\rangle.$$

(Xie, Liang, Song, '16), (Jacot, Gabriel, Hongler '18), (Li and Liang, 2018), (Du, Poczós, Zhai, Singh, 2018), (Du, Lee, Li, Wang, Zhai, 2018), (Arora, Du, Hu, Li, Wang, 2019).

(No generalization results for prediction rules that interpolate noisy data.)

Implications for deep learning

Neural networks versus linear prediction

- What can we say about realistic deep networks?
- The characterization of benign overfitting in linear regression requires $x = \Sigma^{1/2} z$ for a vector z with *independent* components.

Outline

- Linear regression
- Characterizing benign overfitting
- Deep learning
- *Adversarial examples*

Implications for adversarial examples

Label noise appears in $\hat{\theta}$

Implications for adversarial examples

Label noise appears in $\hat{\theta}$

We can find a unit norm Δ $\Delta \propto X^\top (XX^\top)^{-1}\epsilon$
such that perturbing an input x by Δ changes the output enormously: even if $\Delta^\top \theta^* = 0$,

$$\left\| (x+\Delta)^\top \hat{\theta} - x^\top \hat{\theta} \right\|^2 \geq \frac{\sigma}{\sqrt{\lambda_{k^*+1}}} \geq \sqrt{\frac{n}{\mathrm{tr}(\Sigma)}}\sigma.$$

Implications for adversarial examples

Label noise appears in $\hat{\theta}$

We can find a unit norm Δ $\Delta \propto X^\top (XX^\top)^{-1}\epsilon$
such that perturbing an input x by Δ changes the output enormously:
even if $\Delta^\top \theta^* = 0$,

$$\left\| (x+\Delta)^\top \hat{\theta} - x^\top \hat{\theta} \right\|^2 \geq \frac{\sigma}{\sqrt{\lambda_{k^*+1}}} \geq \sqrt{\frac{n}{\mathrm{tr}(\Sigma)}}\sigma.$$

Benign overfitting leads to huge sensitivity.

Interpolating prediction: Future directions

- Between interpolation and regularization?

Implications for adversarial examples

Label noise appears in $\hat{\theta}$

We can find a unit norm Δ $\Delta \propto X^\top (XX^\top)^{-1}\epsilon$
such that perturbing an input x by Δ changes the output enormously:
even if $\Delta^\top \theta^* = 0$,

$$\left\|(x+\Delta)^\top \hat{\theta} - x^\top \hat{\theta}\right\|^2 \geq \frac{\sigma}{\sqrt{\lambda_{k^*+1}}} \geq \sqrt{\frac{n}{\operatorname{tr}(\Sigma)}}\sigma.$$

Benign overfitting leads to huge sensitivity.

Interpolating prediction: Future directions

- Between interpolation and regularization?

Interpolating prediction: Future directions

- Between interpolation and regularization?
- Can we extend these results to interpolating deep networks?

Interpolating prediction: Future directions

- Between interpolation and regularization?
- Can we extend these results to interpolating deep networks?
  - What is the analog of the minimum norm linear prediction rule?

Interpolating prediction: Future directions

- Between interpolation and regularization?
- Can we extend these results to interpolating deep networks?
  - What is the analog of the minimum norm linear prediction rule?
  - What role does the optimization method play?

Interpolating prediction: Future directions

- Between interpolation and regularization?
- Can we extend these results to interpolating deep networks?
  - What is the analog of the minimum norm linear prediction rule?
  - What role does the optimization method play?
  - Implications for regularization methods?

Interpolating prediction: Future directions

- Between interpolation and regularization?
- Can we extend these results to interpolating deep networks?
  - What is the analog of the minimum norm linear prediction rule?
  - What role does the optimization method play?
  - Implications for regularization methods?
  - Implications for robustness?

Interpolating prediction: Future directions

- Between interpolation and regularization?
- Can we extend these results to interpolating deep networks?
  - What is the analog of the minimum norm linear prediction rule?
  - What role does the optimization method play?
  - Implications for regularization methods?
  - Implications for robustness?

Benign Overfitting in Linear Regression

- Interpolation: far from the regime of a tradeoff between fit to training data and complexity.

Benign Overfitting in Linear Regression

- Interpolation: far from the regime of a tradeoff between fit to training data and complexity.
- In linear regression, a long, flat tail of the covariance eigenvalues is necessary and sufficient for the minimum norm interpolant to predict well: The noise is hidden in many unimportant directions.

Benign Overfitting in Linear Regression

- Interpolation: far from the regime of a tradeoff between fit to training data and complexity.
- In linear regression, a long, flat tail of the covariance eigenvalues is necessary and sufficient for the minimum norm interpolant to predict well: The noise is hidden in many unimportant directions.
  - Relies on many (roughly equally) unimportant parameters

Benign Overfitting in Linear Regression

- Interpolation: far from the regime of a tradeoff between fit to training data and complexity.
- In linear regression, a long, flat tail of the covariance eigenvalues is necessary and sufficient for the minimum norm interpolant to predict well: The noise is hidden in many unimportant directions.
  - Relies on many (roughly equally) unimportant parameters
- But it leads to huge sensitivity to (adversarial) perturbations.