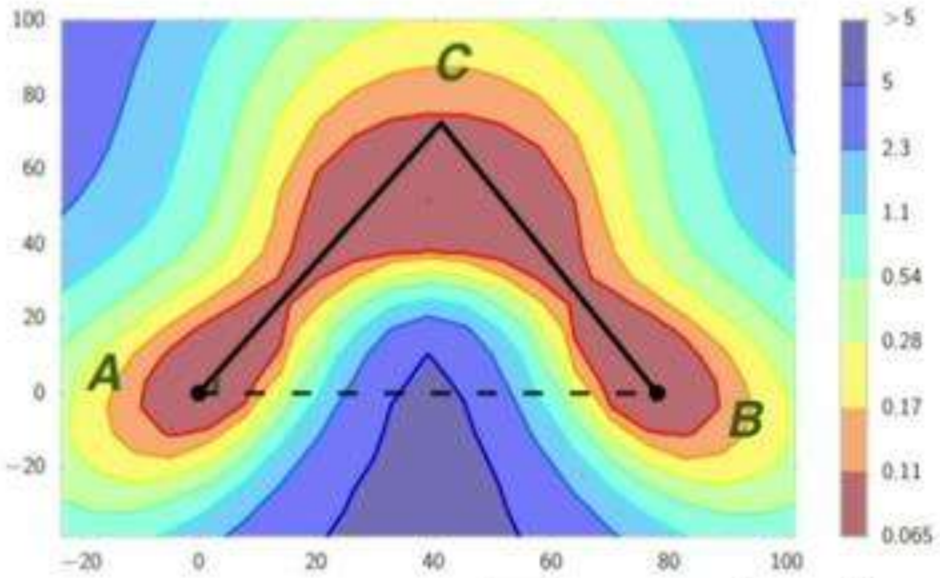
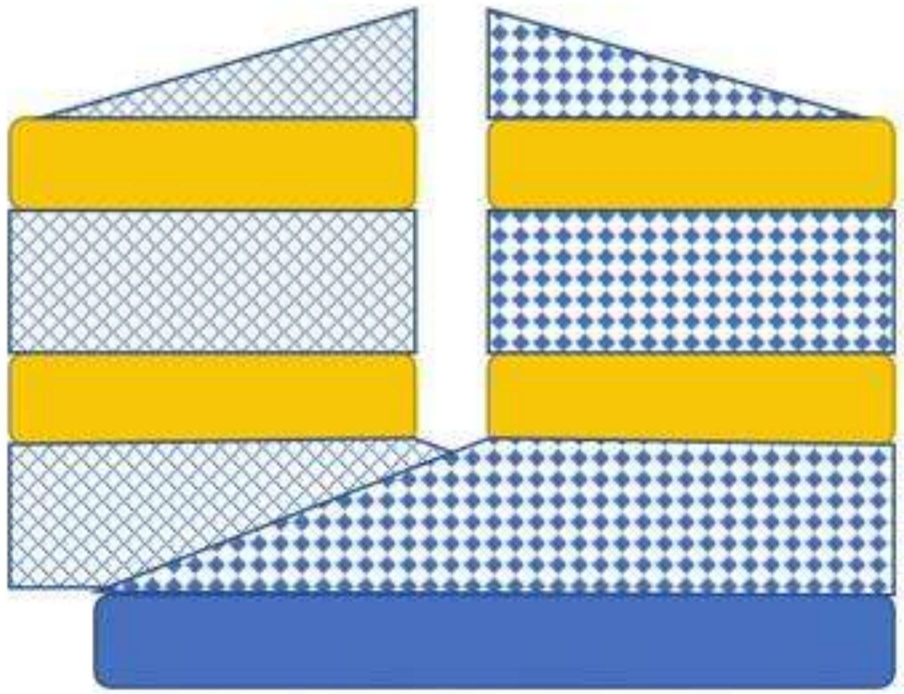


Explaining Landscape Connectivity of Low-cost Solutions for Multilayer Nets

Geometry of Deep Learning Workshop

Rong Ge, Duke University

Joint work with Rohith Kuditipudi, Xiang Wang (Duke)
Holden Lee, Yi Zhang, Zhiyuan Li, Wei Hu, Sanjeev Arora (Princeton)



Mode Connectivity [Freeman and Bruna 16, Garipov et al. 18, Draxler et al. 18]

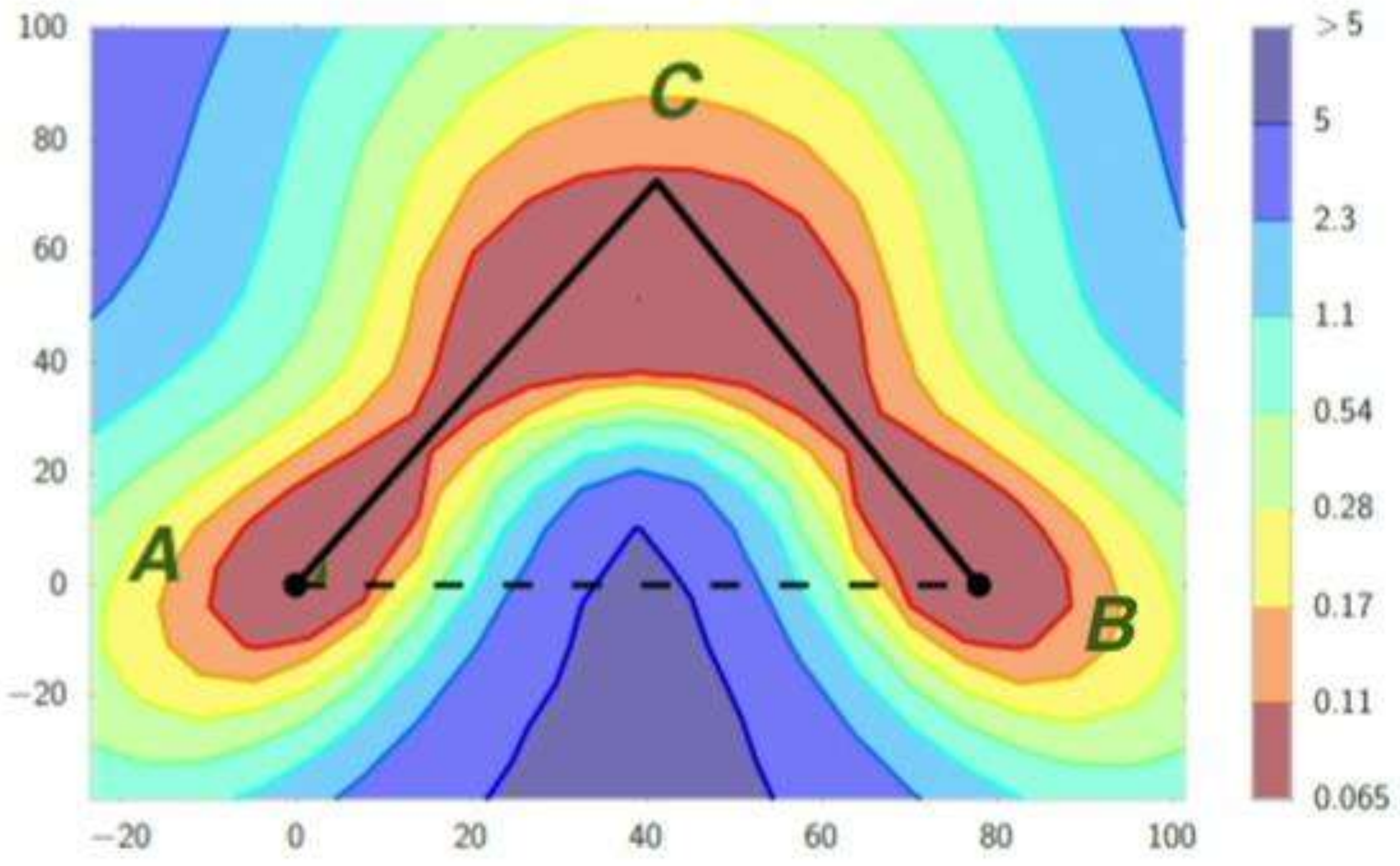


Image from [Garipov et al. 18]

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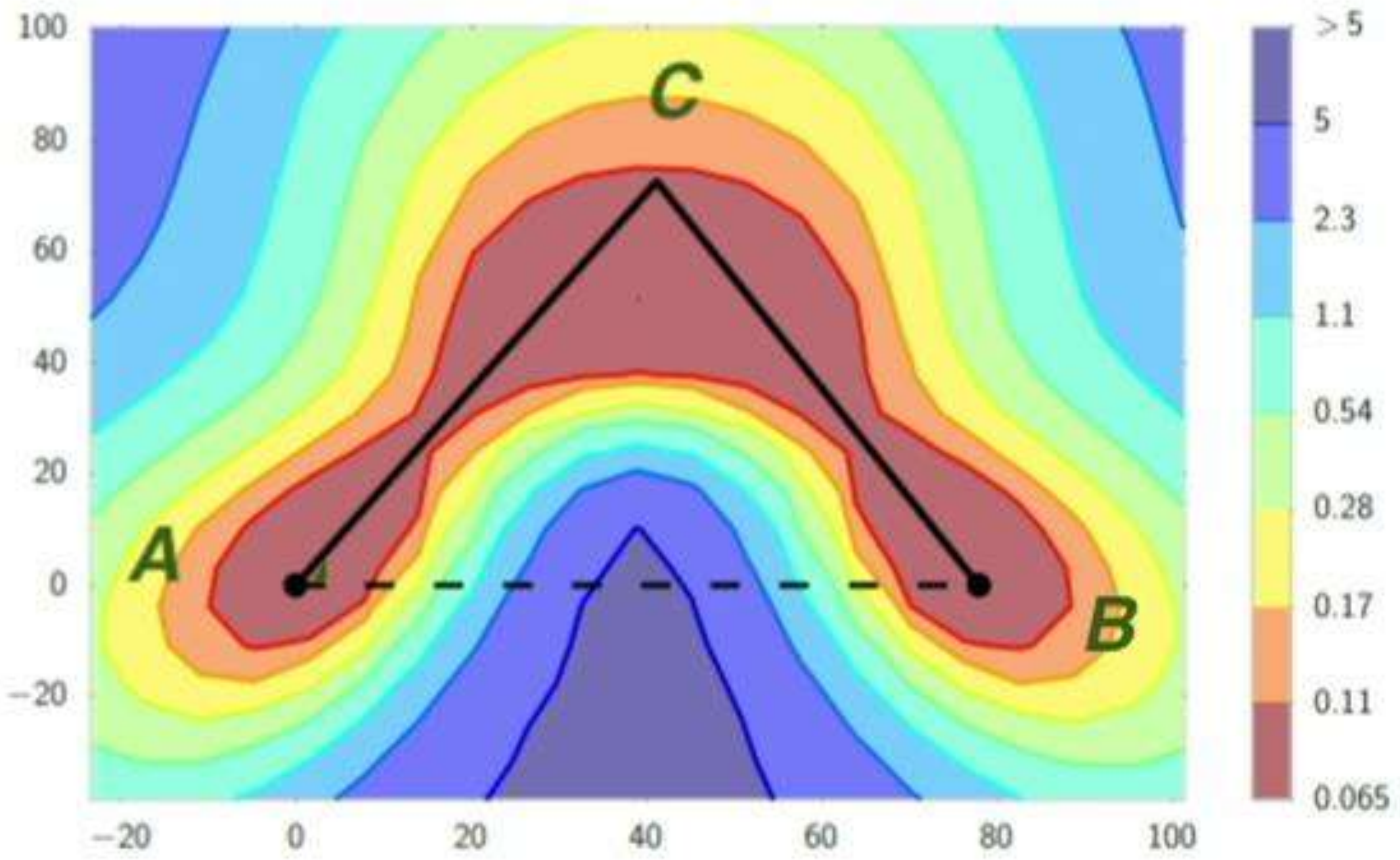

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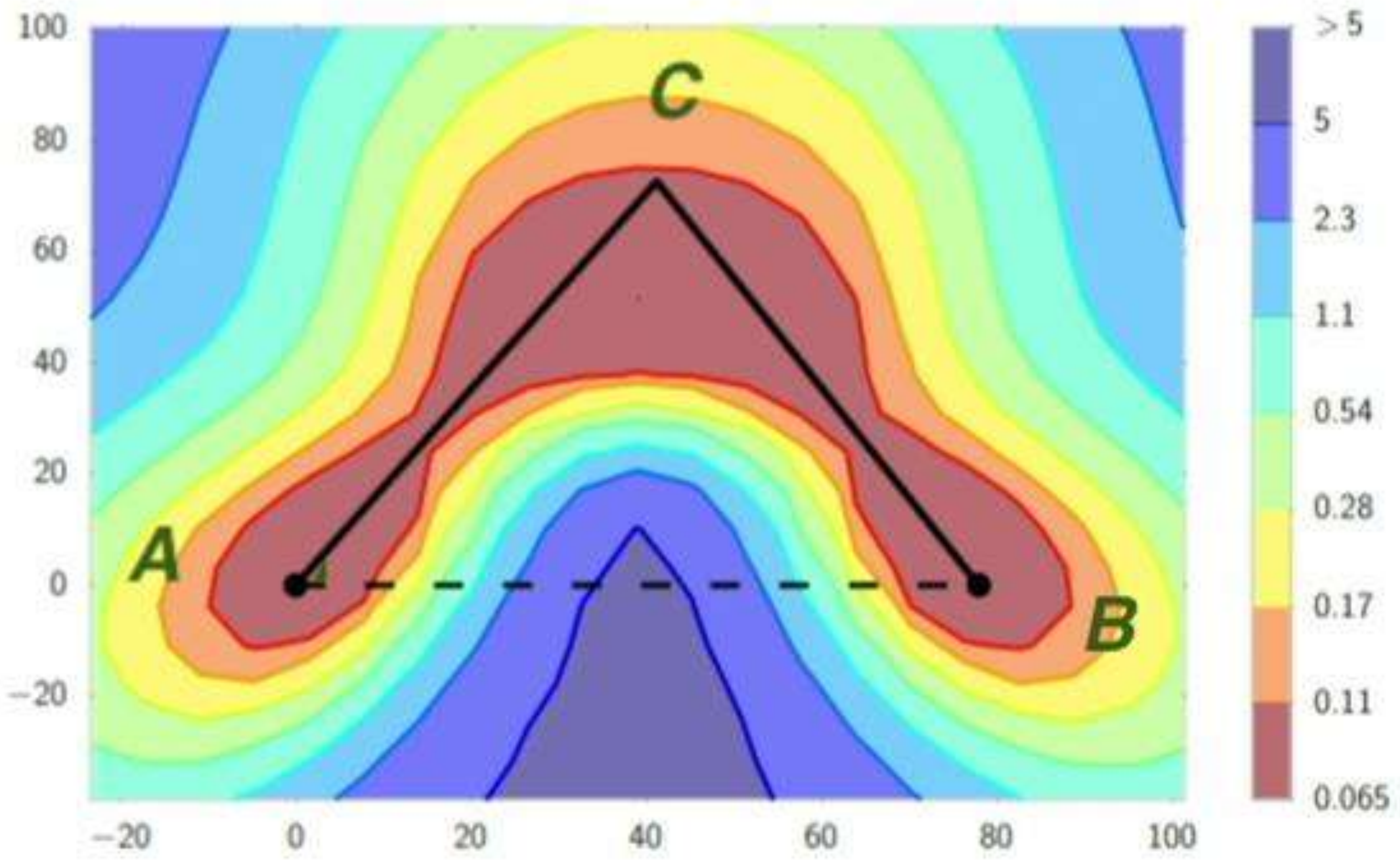
Mode Connectivity[Freeman and Bruna 16, Garipov et al. 18, Draxler et al. 18]

- For neural networks, local minima found via gradient descent are connected by simple paths in the parameter space
- Every point on the path is another solution of almost the same cost.



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Equivalent local minima and symmetry

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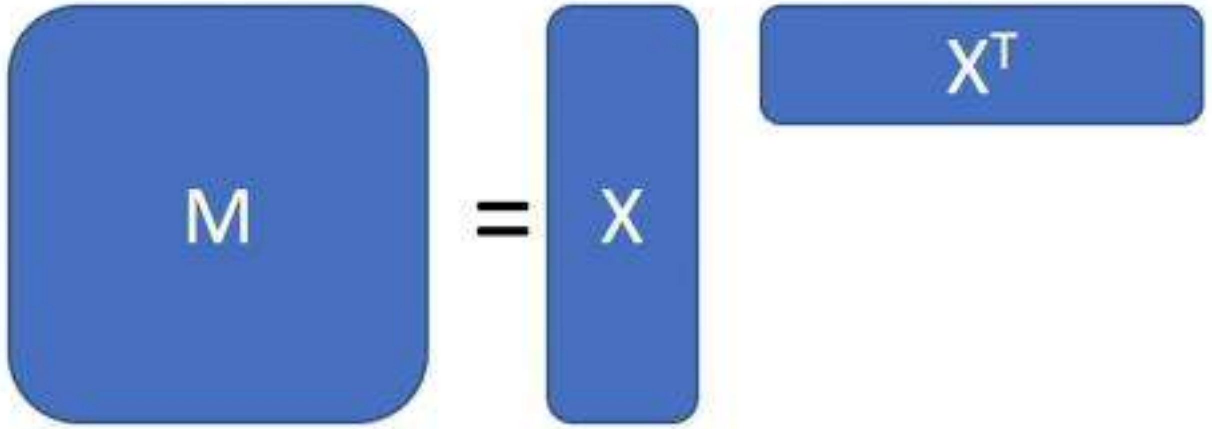
- Matrix Problems
- Goal: find low rank matrix M



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$$X = X^* R, RR^\top = I$$

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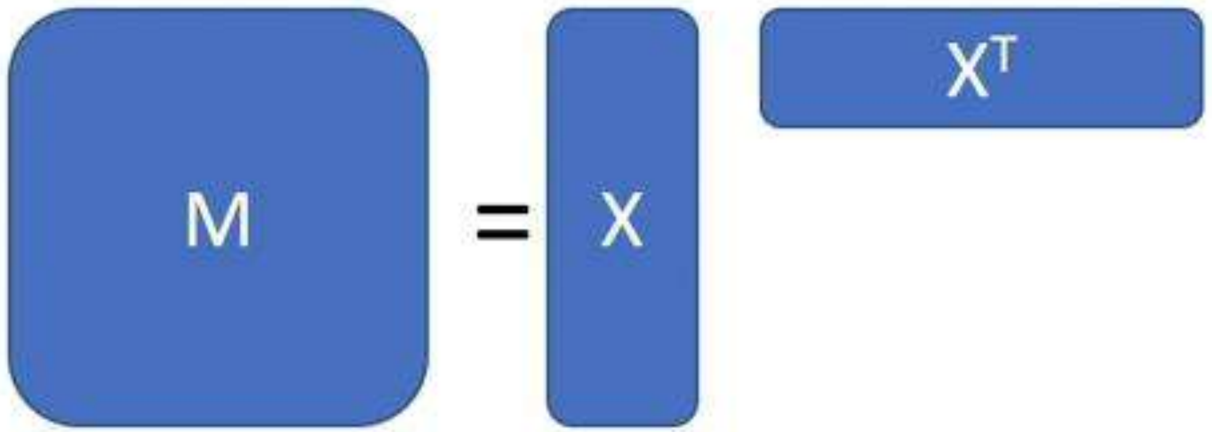

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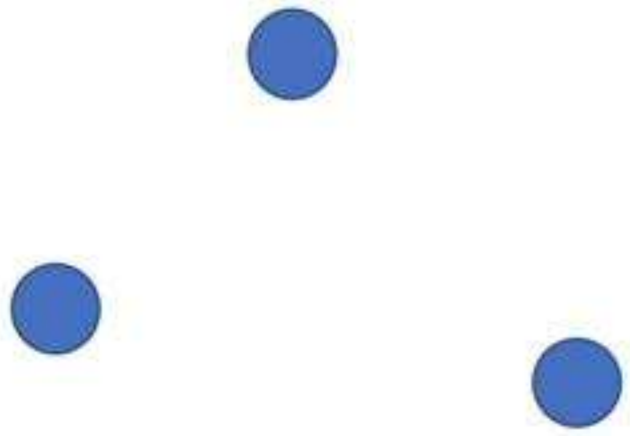

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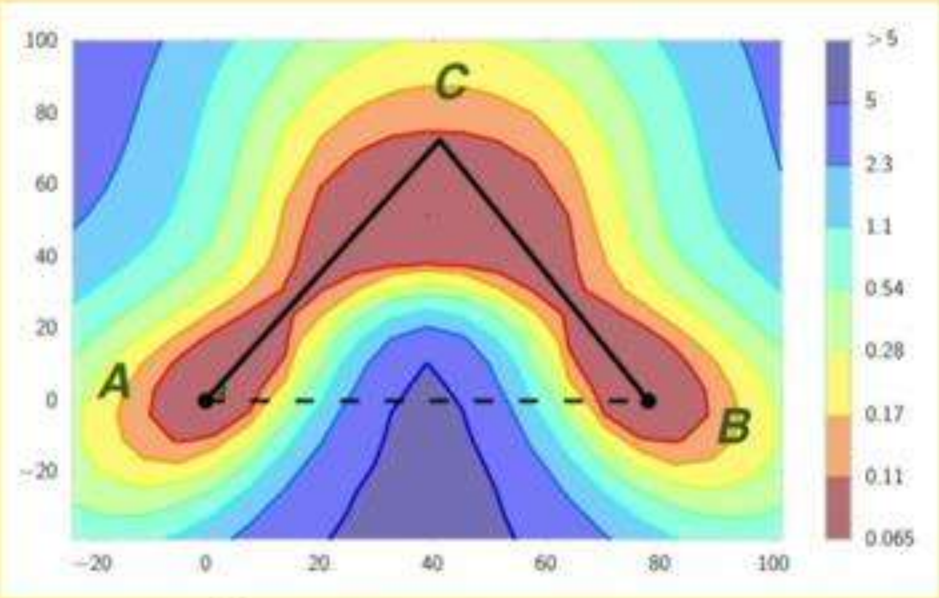

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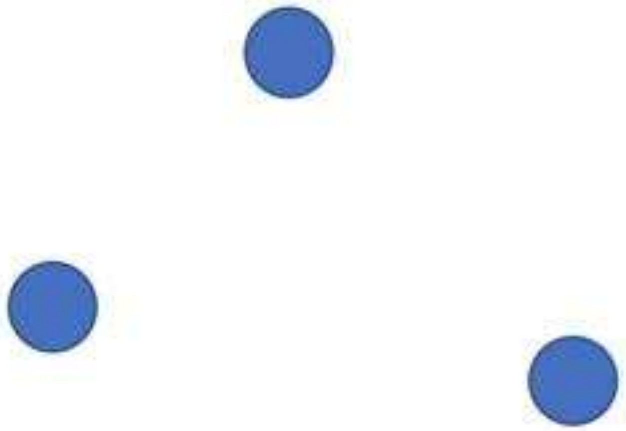

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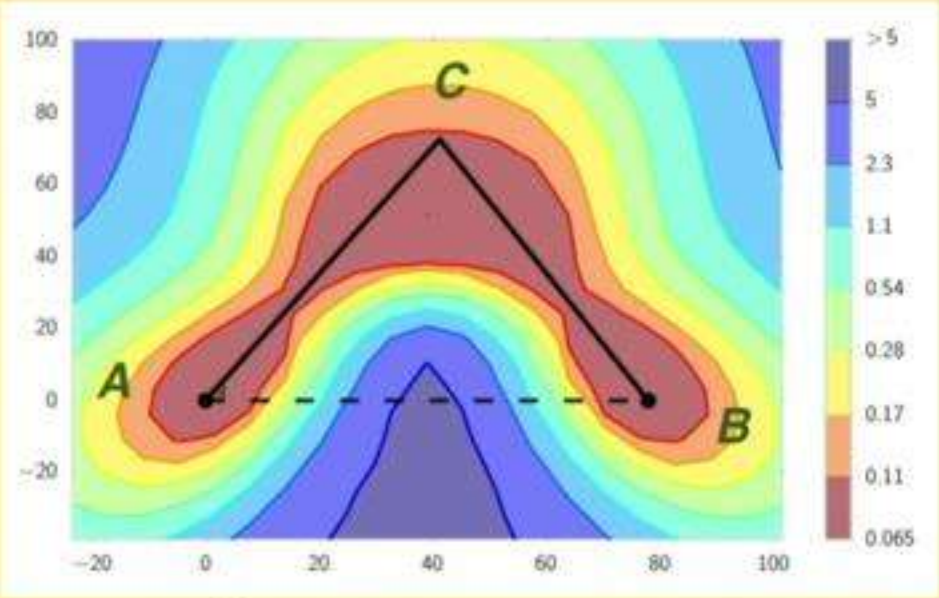

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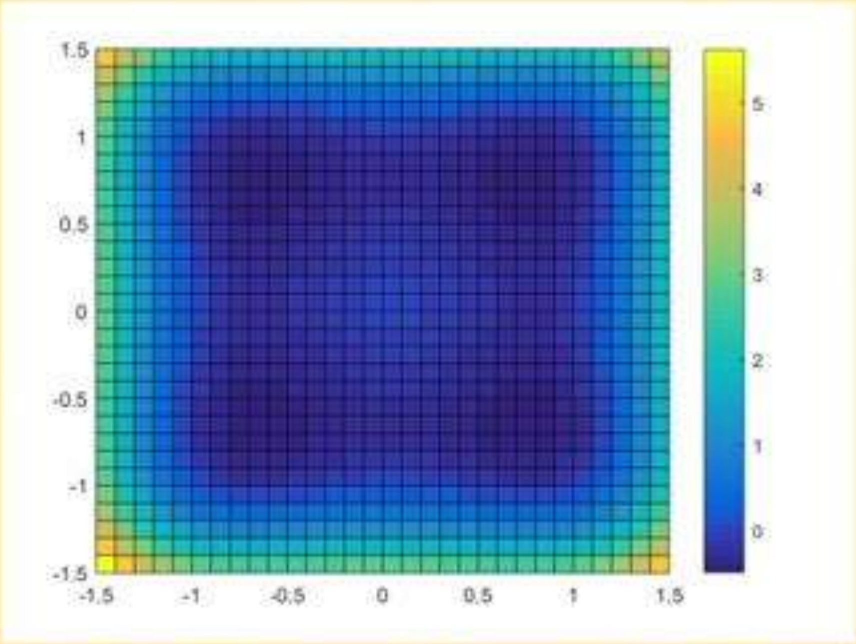

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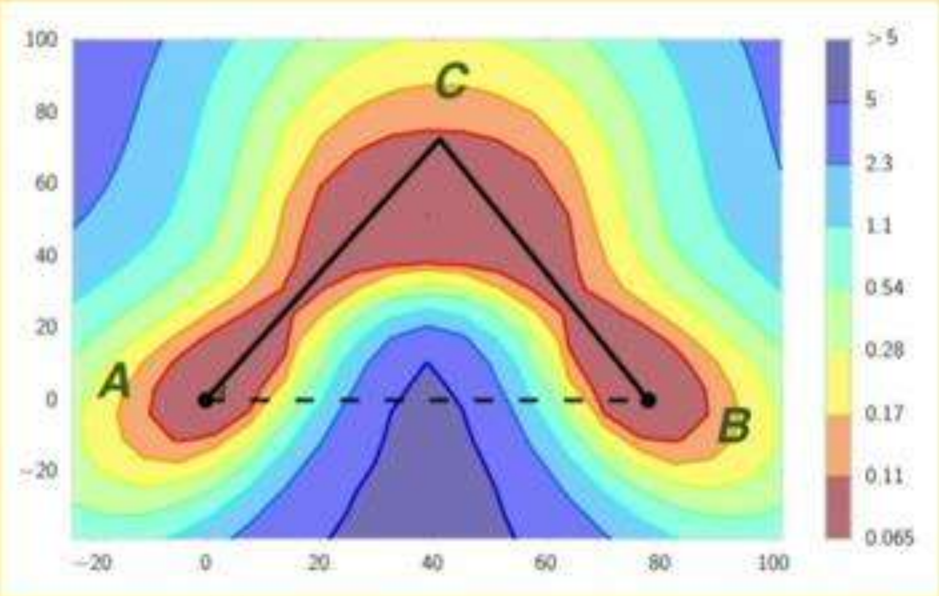

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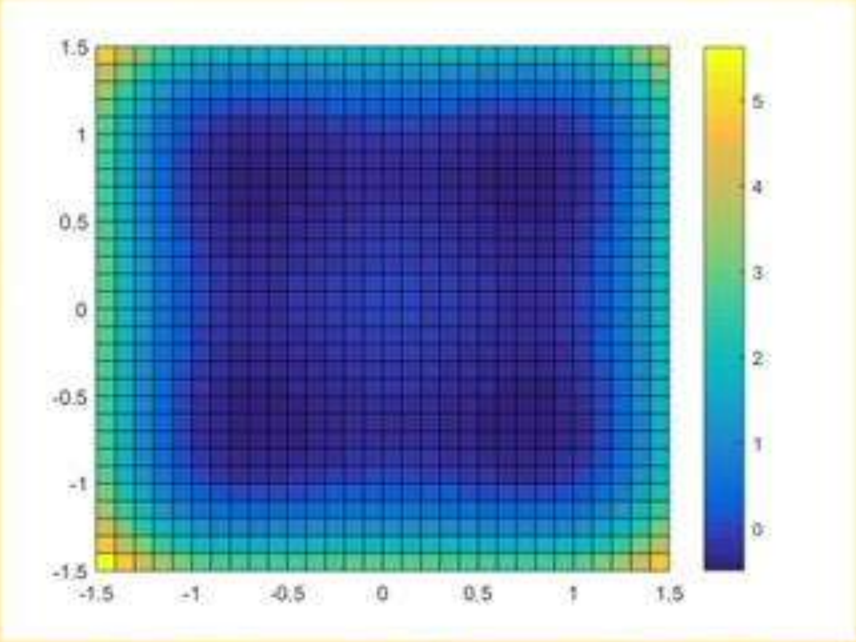

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Neural networks only have permutation symmetry,
why do they have connected local min?

(Partial) short answer: overparametrization

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- Can we prove similar results in mildly overparametrized regime?

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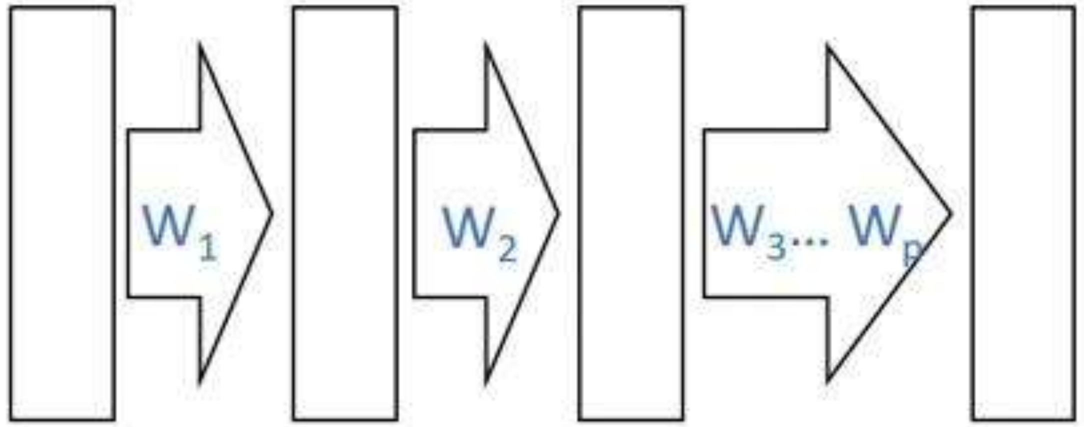
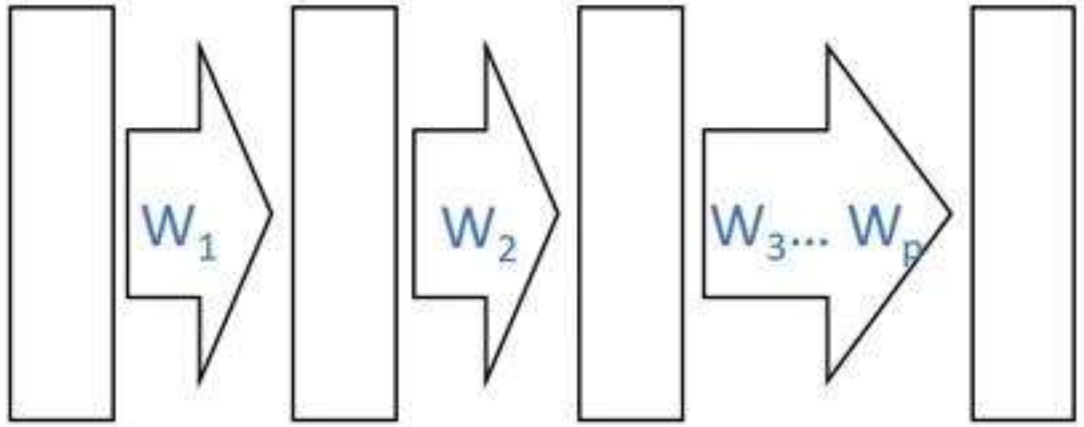
- Possible to switch dropout stability with noise stability (used for proving generalization bounds for neural nets)

Deep Neural Networks



Deep Neural Networks

- For simplicity: Fully Connected Networks



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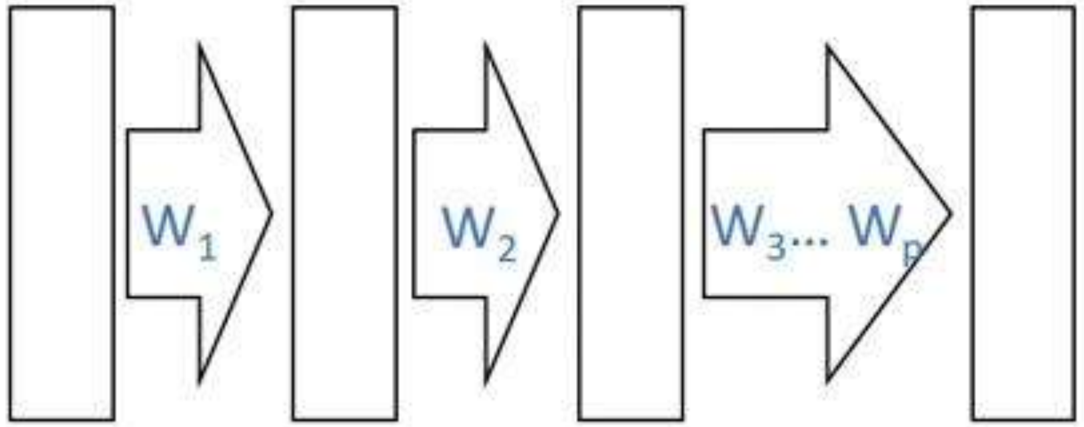

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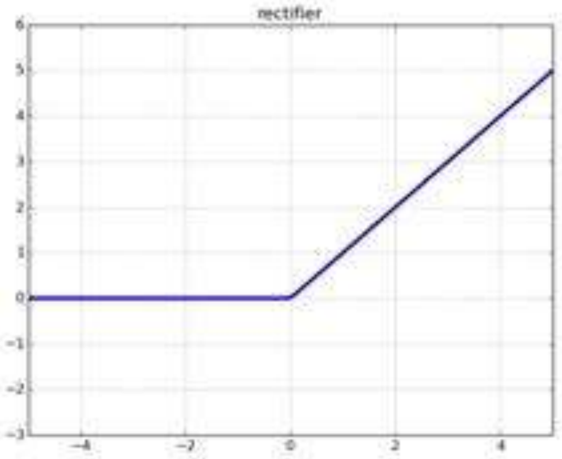

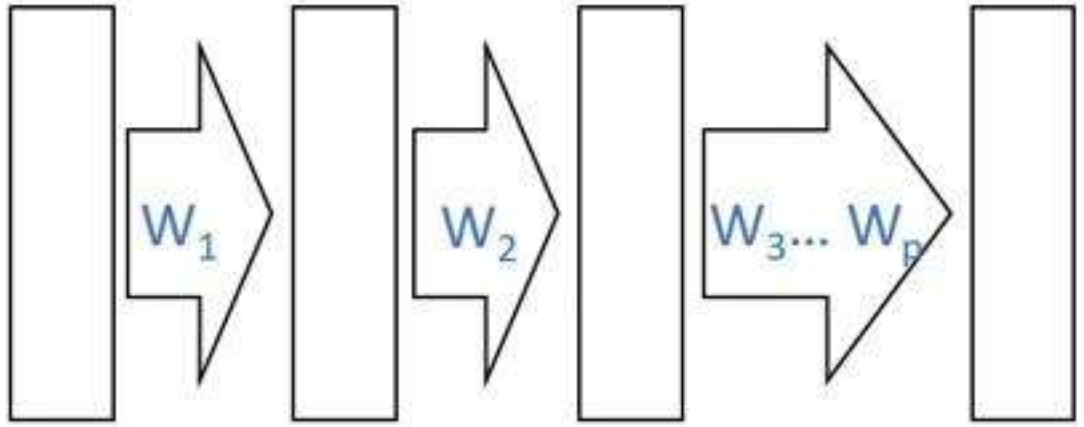

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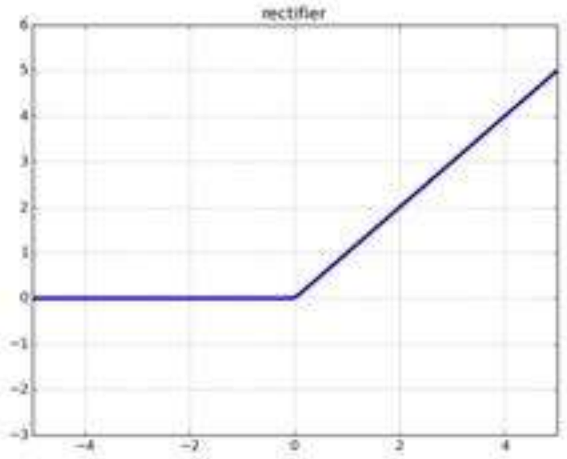

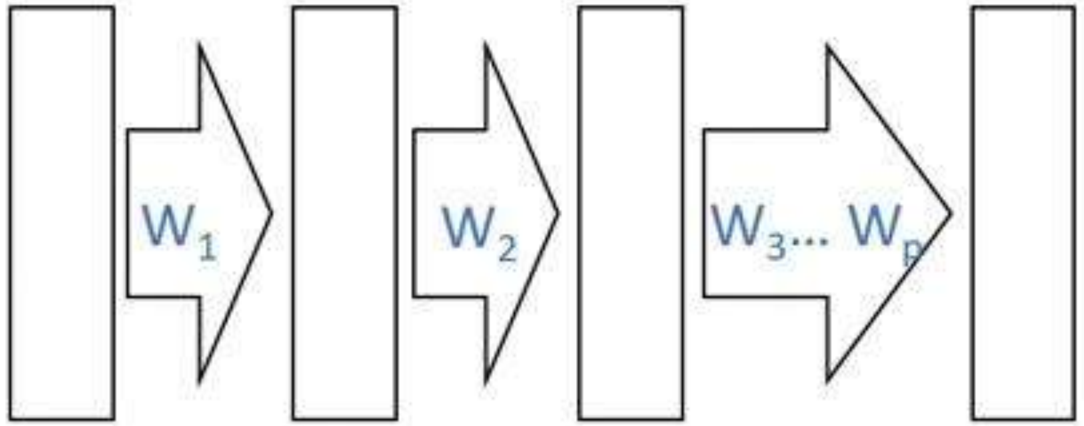

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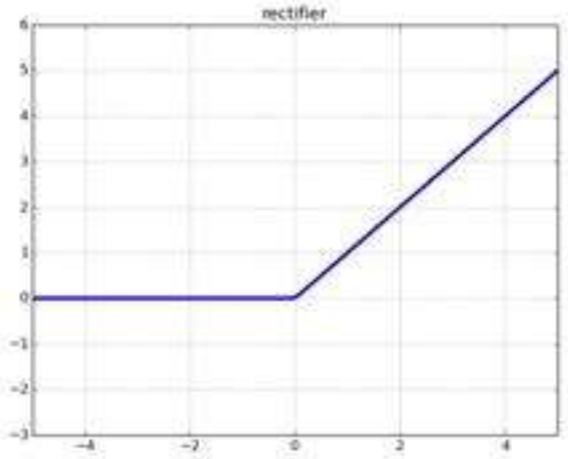

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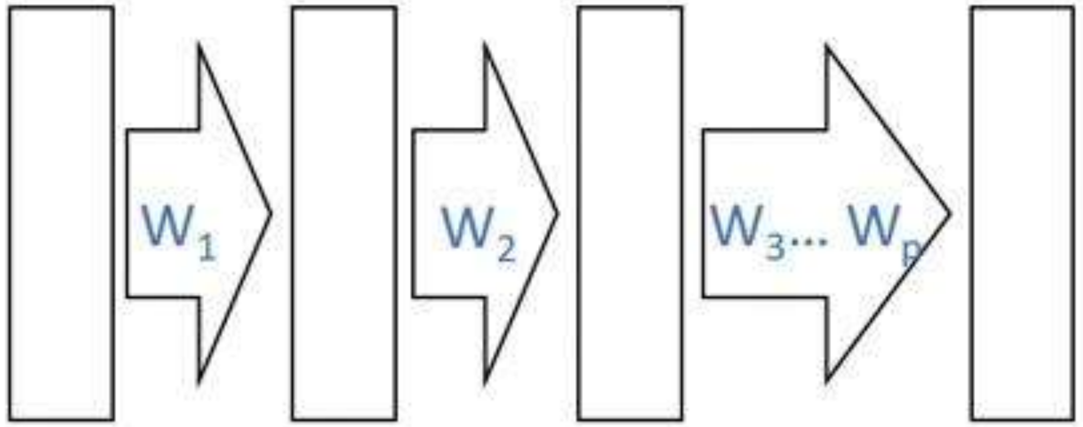

$x^{(0)} = x$ $\quad x^{(1)} = \sigma(W_1 x)$ $\qquad x^{(p)} = W_p x^{(p-1)}$

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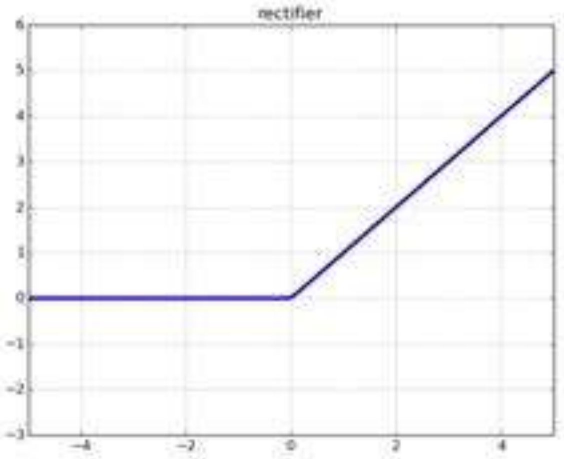

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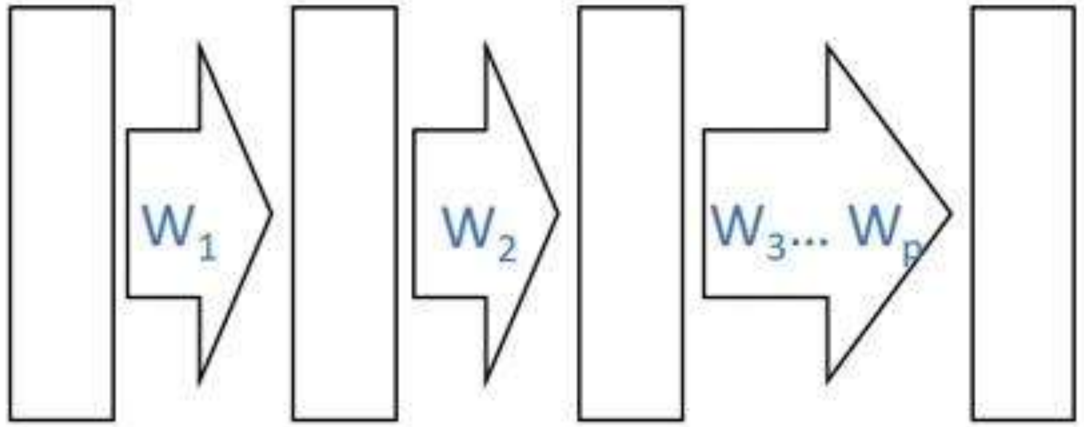

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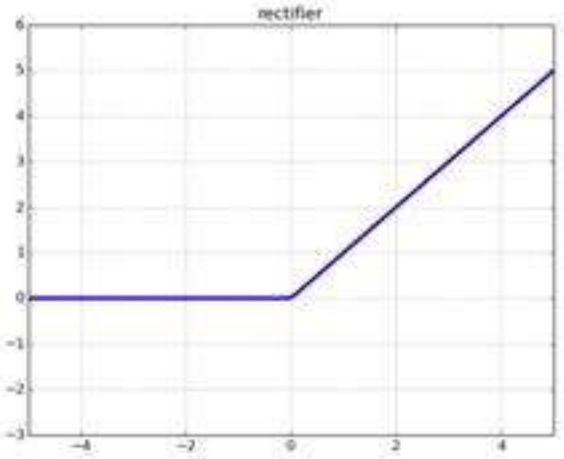

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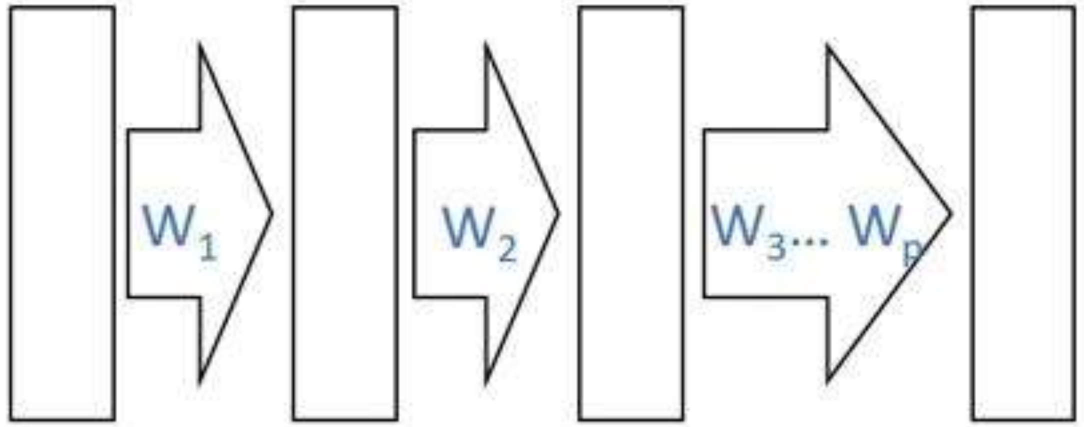

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Convex loss function

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- Overparametrization: consider optimization of a 2 layer neural network with h ($h >> 2$) hidden neurons.

- Theorem: For any $h > 2$, there exists a data-set with $h+2$ samples, such that the set of global minimizers are not connected.

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- This talk: Dropout stability

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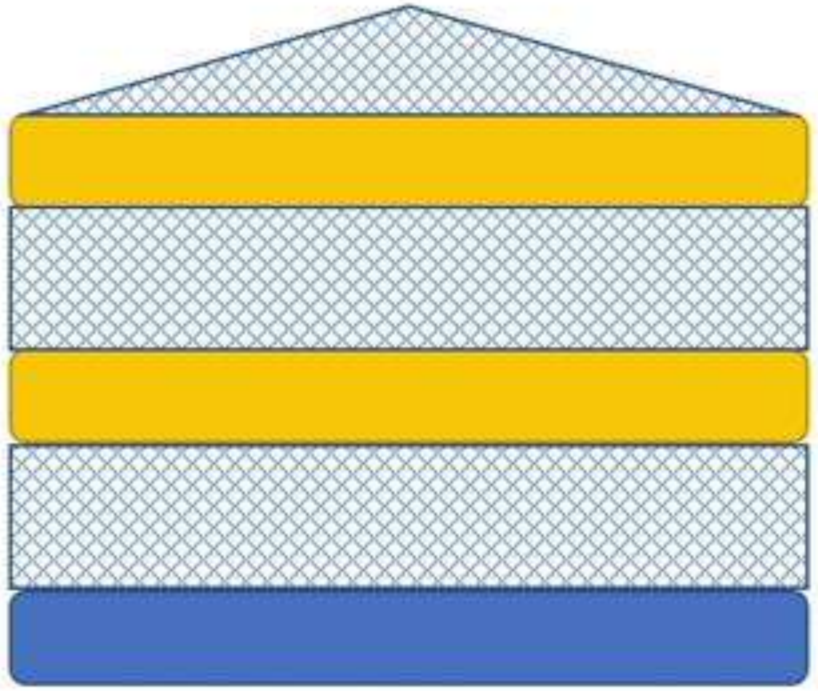

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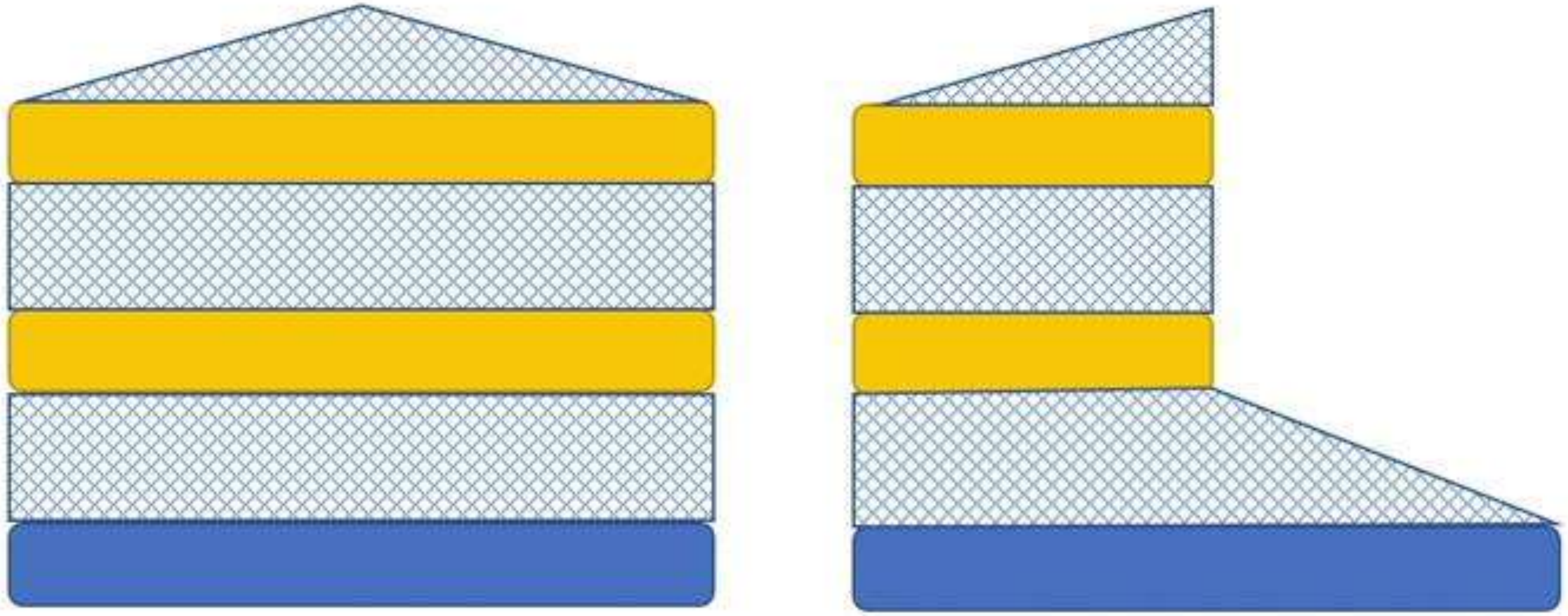

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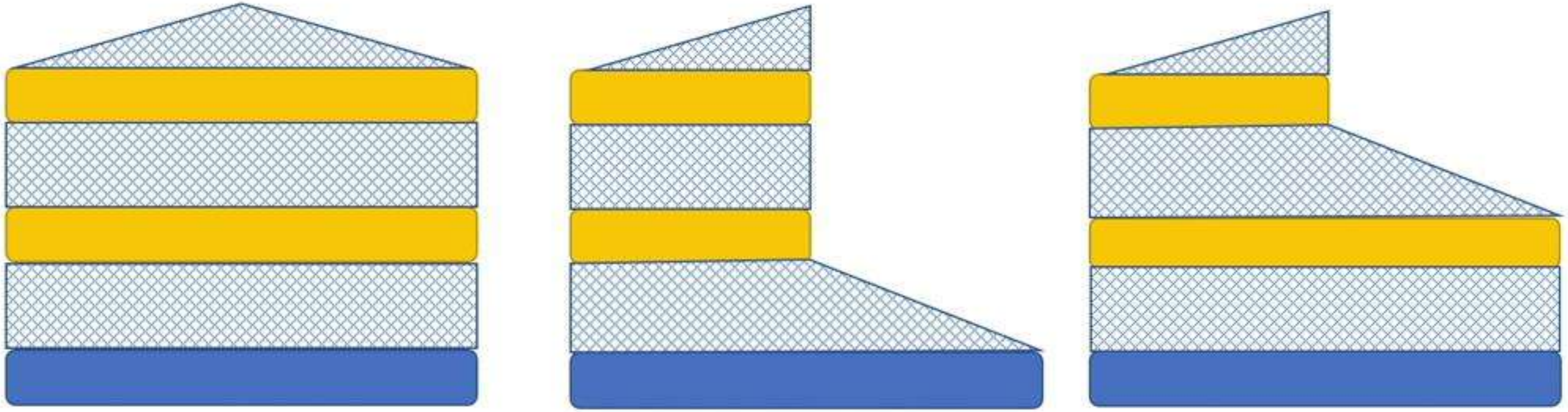

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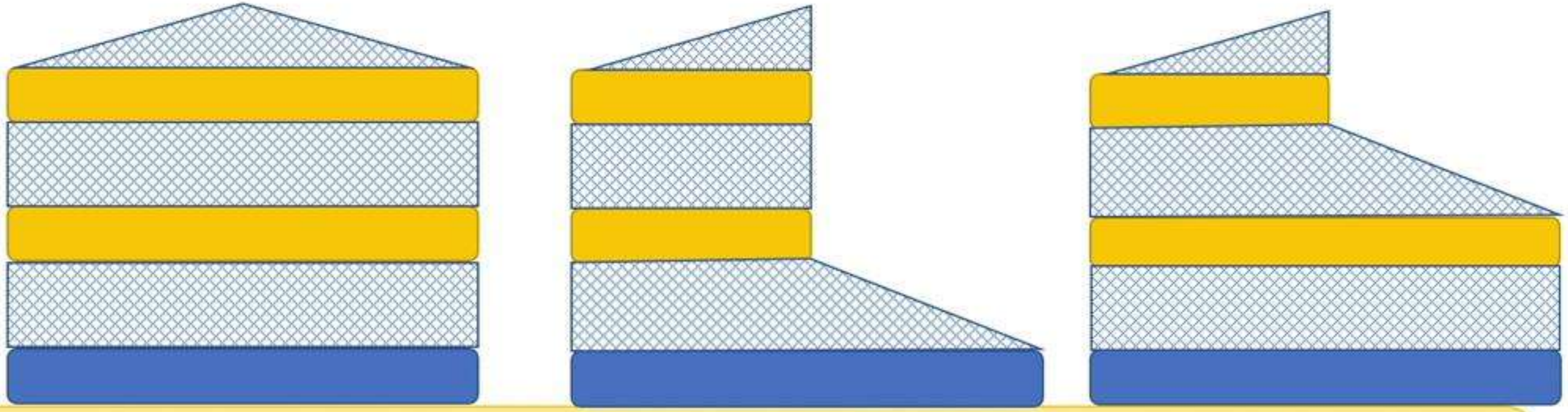
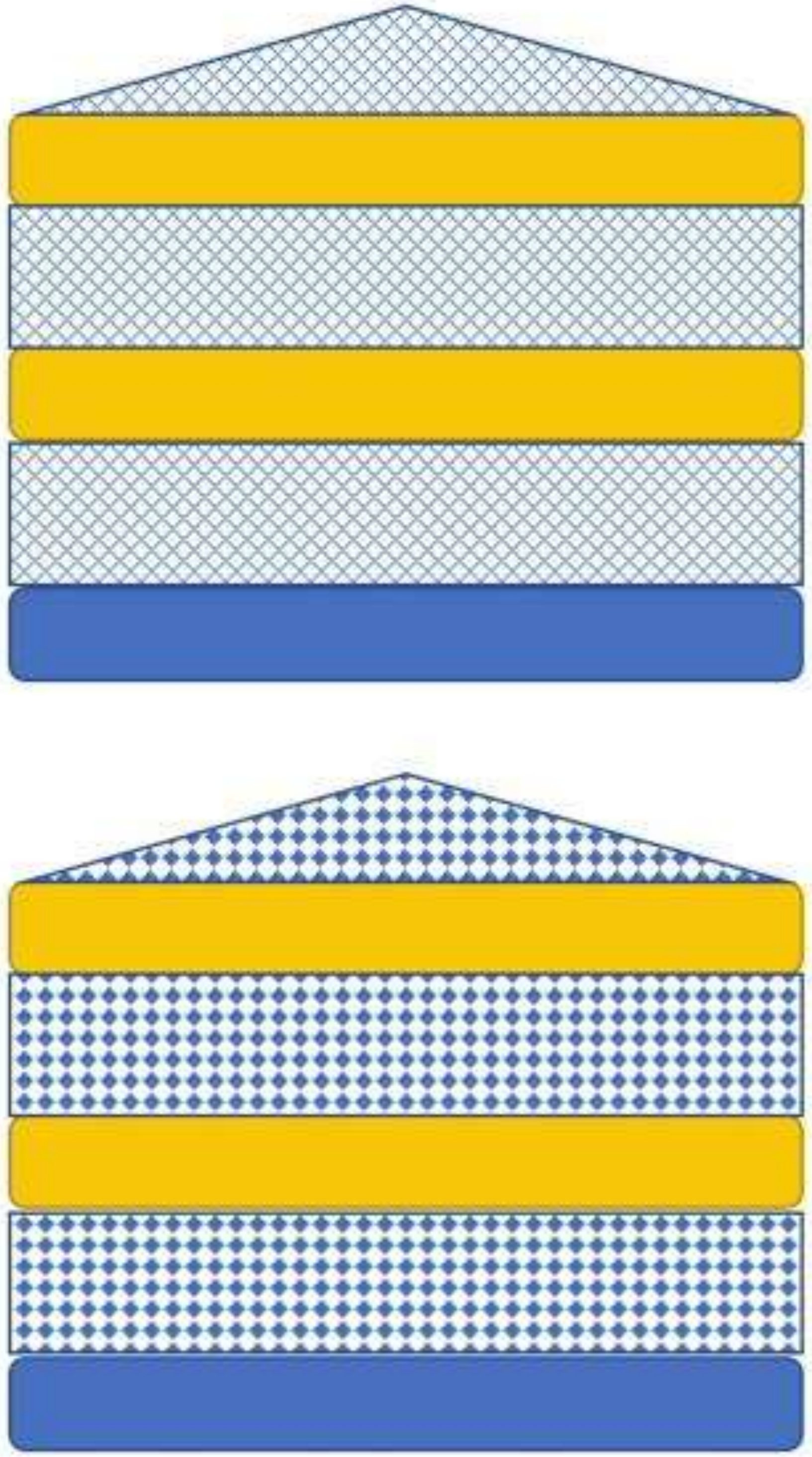


- Theorem: If both θ_A and θ_B are ε-dropout stable, then there exists a path between them with maximum loss $\leq \max\{L(\theta_A), L(\theta_B)\} + \varepsilon$

High level steps



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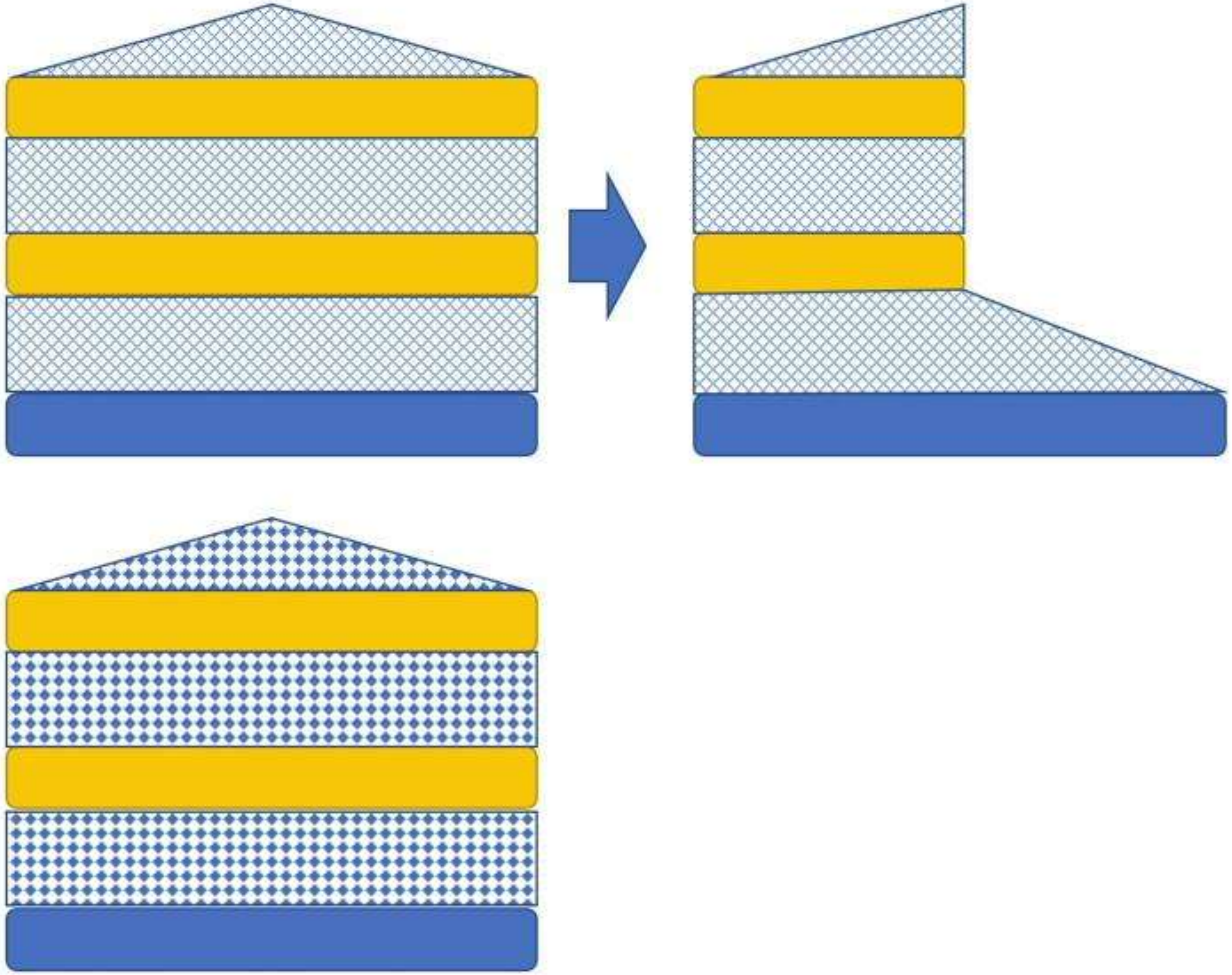

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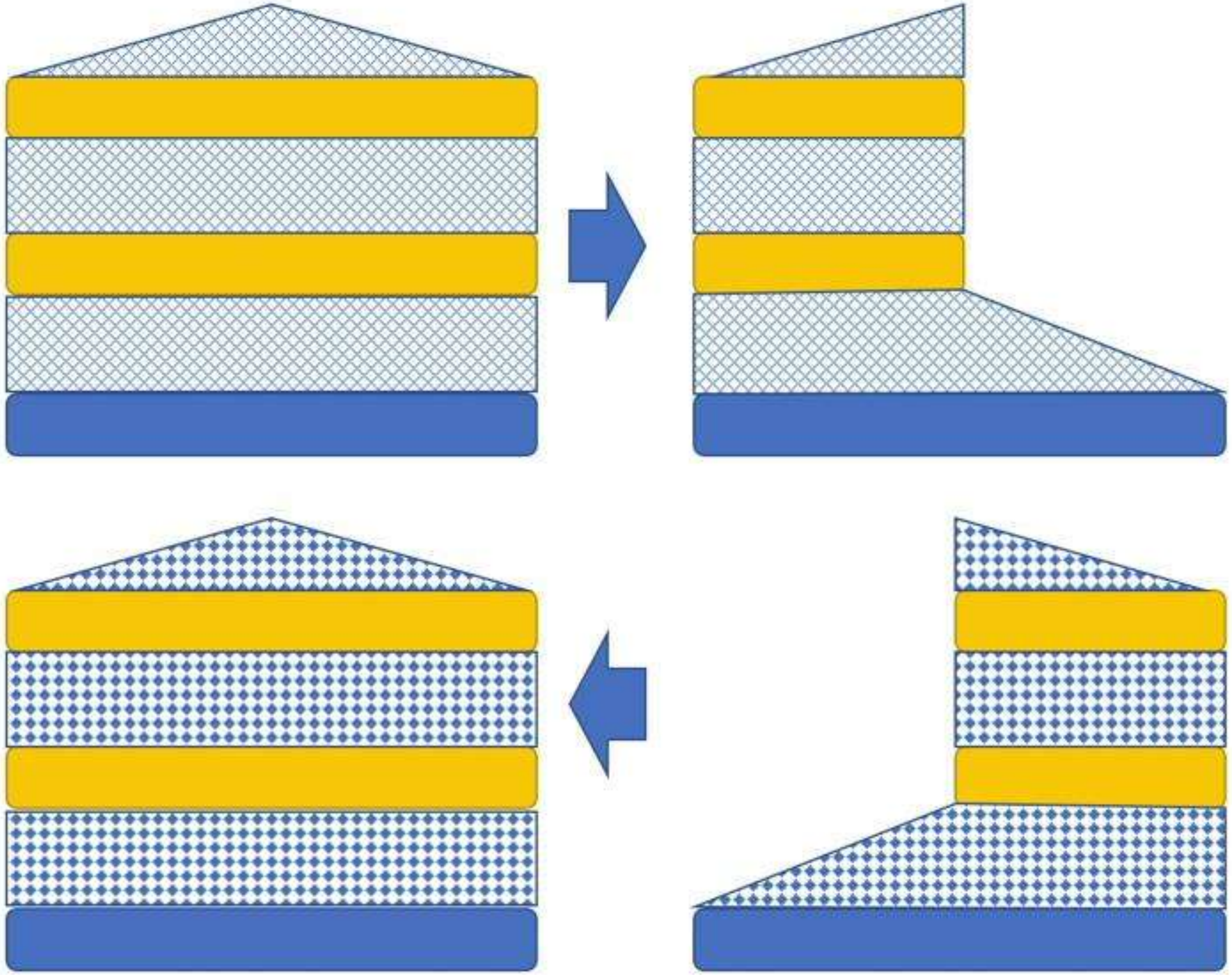

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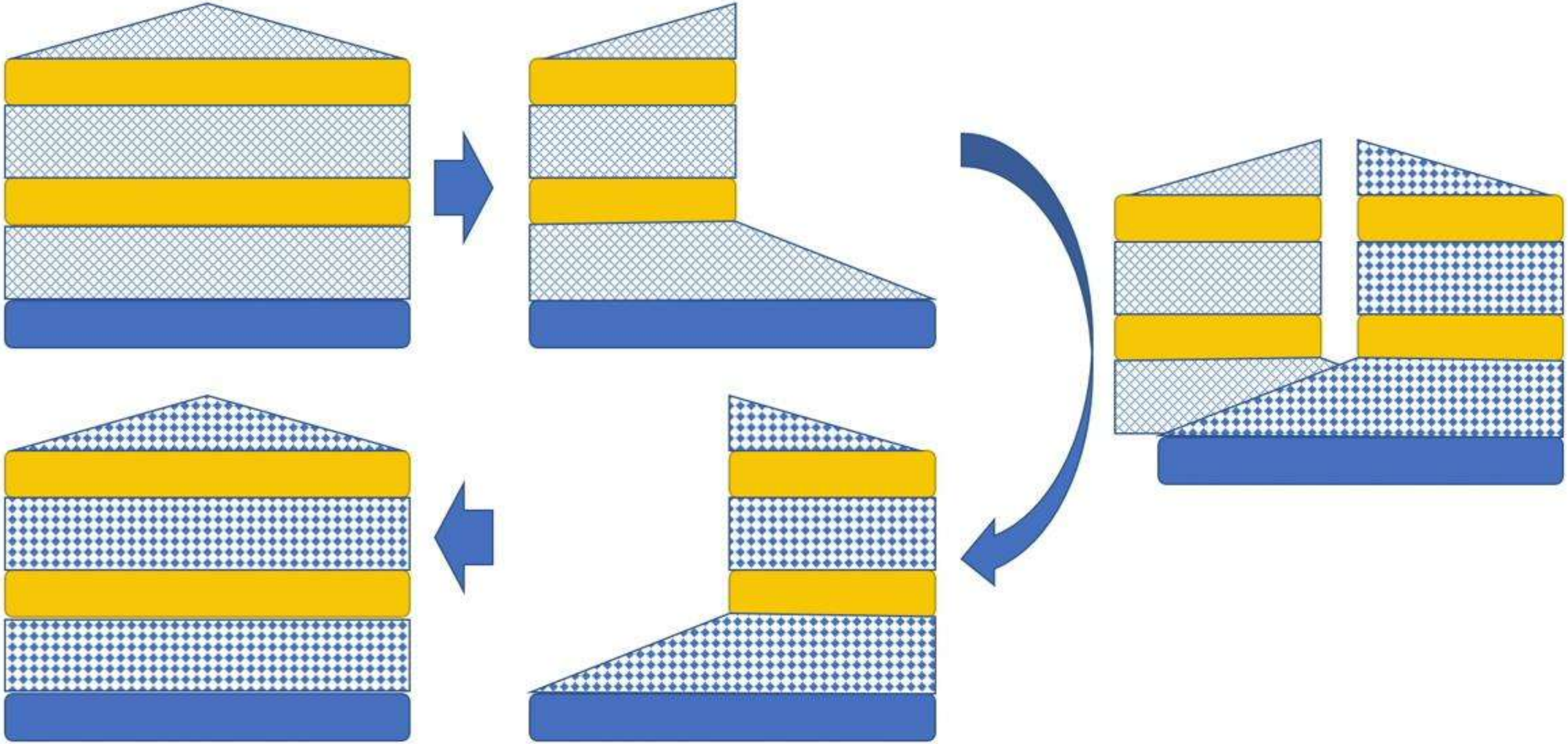
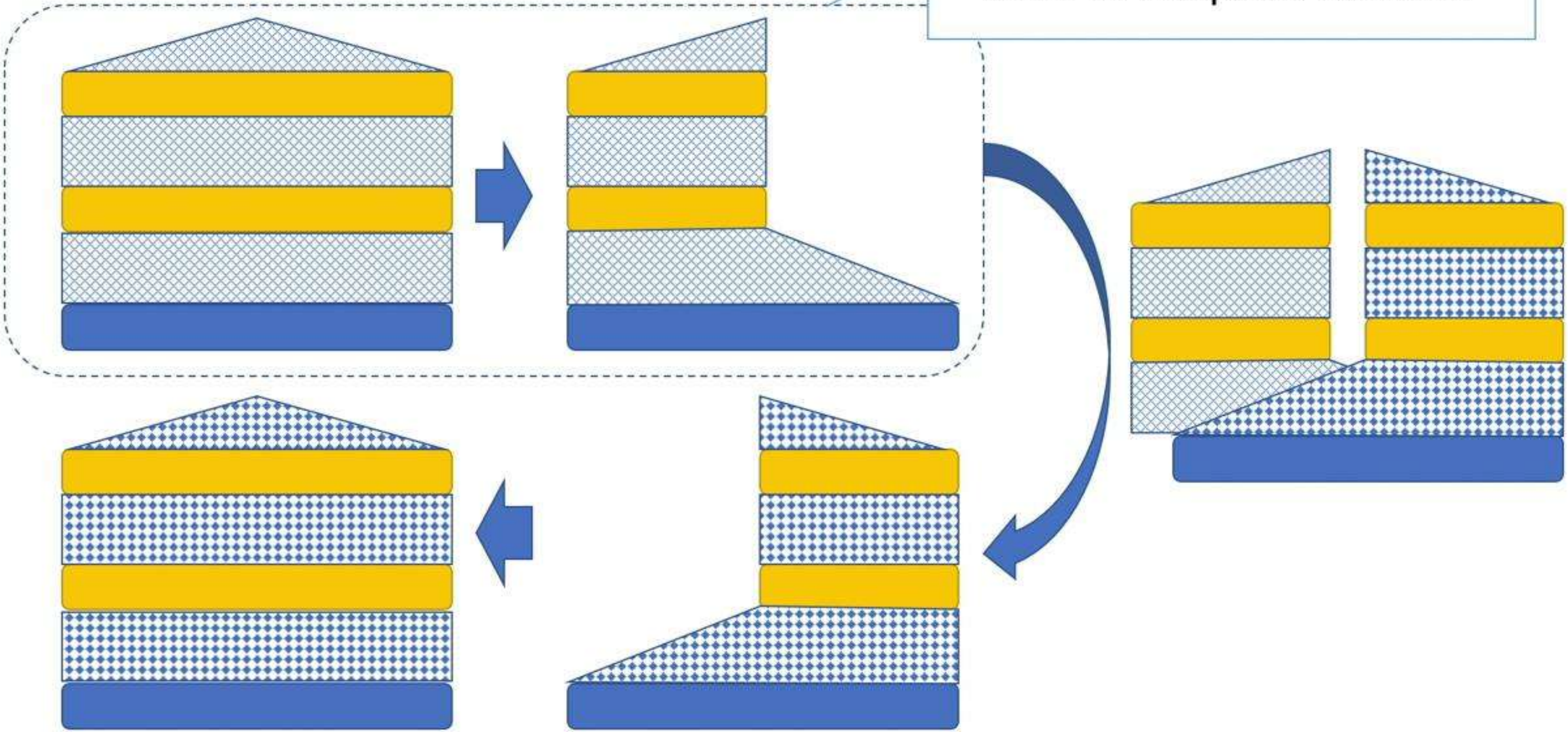


High level steps

How to connect a network with its dropout version?



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- Type (b): If a group of neurons do not have any outgoing edges, can change their incoming edges arbitrarily.
- Idea: Recurse from the top layer, use Type (b) moves to prepare for the next Type (a) move

An example path for 3 layer network

$$(1)\quad \left[\begin{array}{c|c} L_3 & R_3 \end{array}\right] \left(\begin{array}{c|c} L_2 & C_2 \\ \hline D_2 & R_2 \end{array}\right) \left(\begin{array}{c} L_1 \\ \hline B_1 \end{array}\right)$$

$$(2)\quad \left[\begin{array}{c|c} 2L_3 & 0 \end{array}\right] \left(\begin{array}{c|c} L_2 & C_2 \\ \hline D_2 & R_2 \end{array}\right) \left(\begin{array}{c} L_1 \\ \hline B_1 \end{array}\right) \quad (a)$$

$$(3)\quad \left[\begin{array}{c|c} 2L_3 & 0 \end{array}\right] \left(\begin{array}{c|c} L_2 & C_2 \\ \hline 2L_2 & 0 \end{array}\right) \left(\begin{array}{c} L_1 \\ \hline B_1 \end{array}\right) \quad (b)$$

$$(4)\quad \left[\begin{array}{c|c} 0 & 2L_3 \end{array}\right] \left(\begin{array}{c|c} L_2 & C_2 \\ \hline 2L_2 & 0 \end{array}\right) \left(\begin{array}{c} L_1 \\ \hline B_1 \end{array}\right) \quad (a)$$

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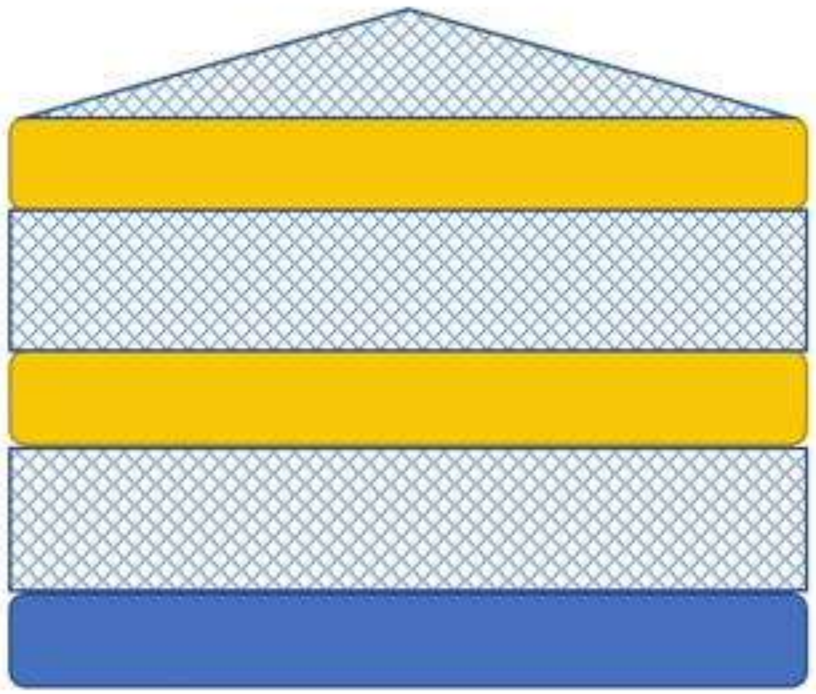
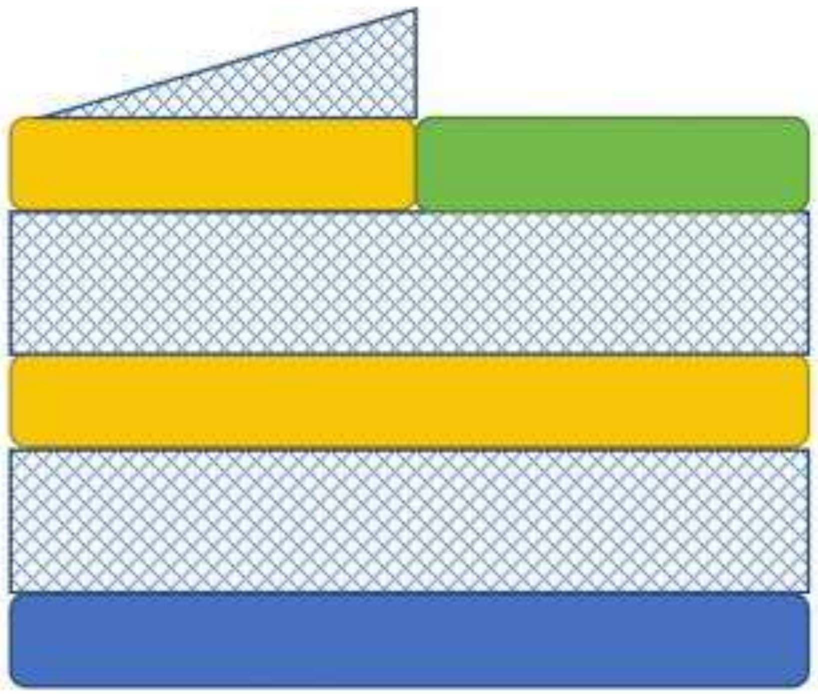
Example path explained (1) -> (2)





$$(1)\ \begin{pmatrix} L_3 & \big| & R_3 \end{pmatrix} \left(\begin{array}{c|c} L_2 & C_2 \\ \hline D_2 & R_2 \end{array}\right) \left(\begin{array}{c} L_1 \\ \hline B_1 \end{array}\right)$$

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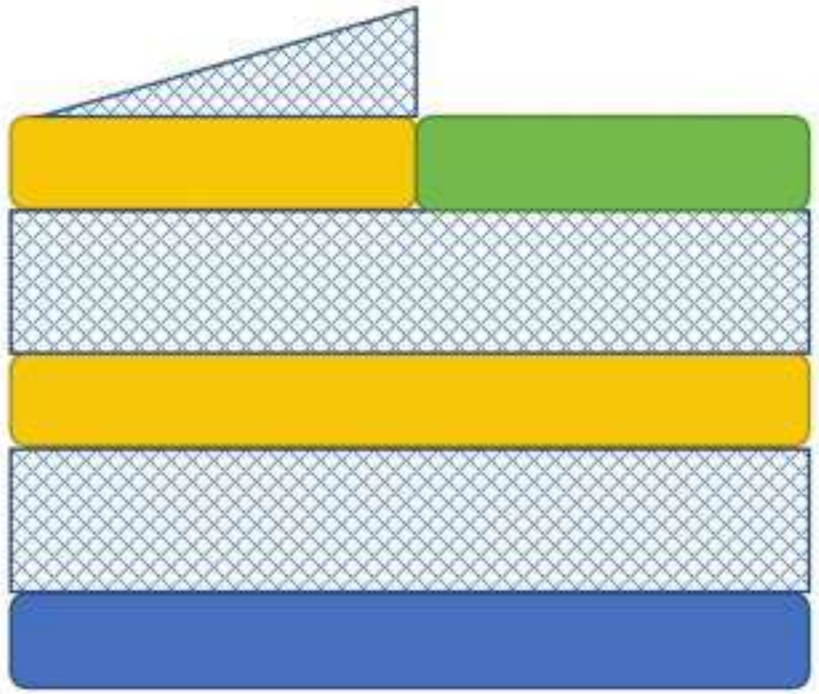

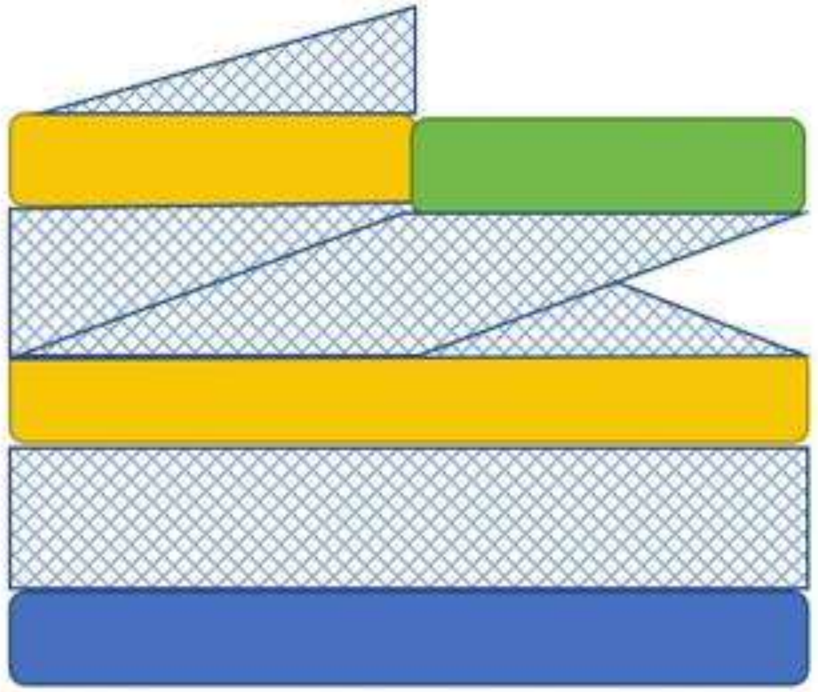

$$(2)\ \left(\begin{array}{c|c} 2L_3 & 0 \end{array}\right) \left(\begin{array}{c|c} L_2 & C_2 \\ \hline D_2 & R_2 \end{array}\right) \left(\begin{array}{c} L_1 \\ \hline B_1 \end{array}\right) \text{(a)} \quad (3)\ \left(\begin{array}{c|c} 2L_3 & 0 \end{array}\right) \left(\begin{array}{c|c} L_2 & C_2 \\ \hline 2L_2 & 0 \end{array}\right) \left(\begin{array}{c} L_1 \\ \hline B_1 \end{array}\right) \text{(b)}$$

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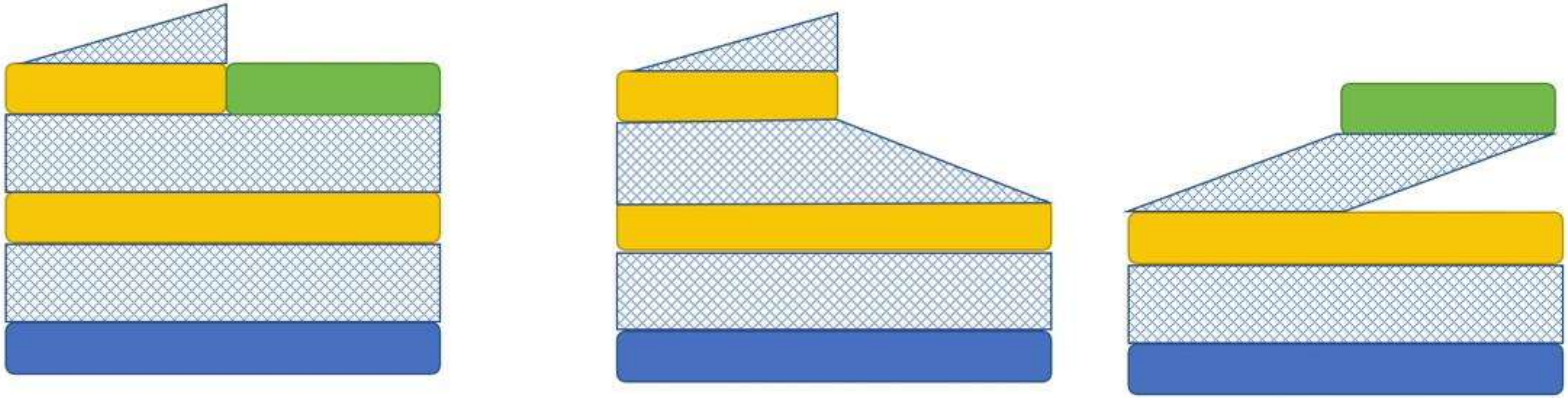
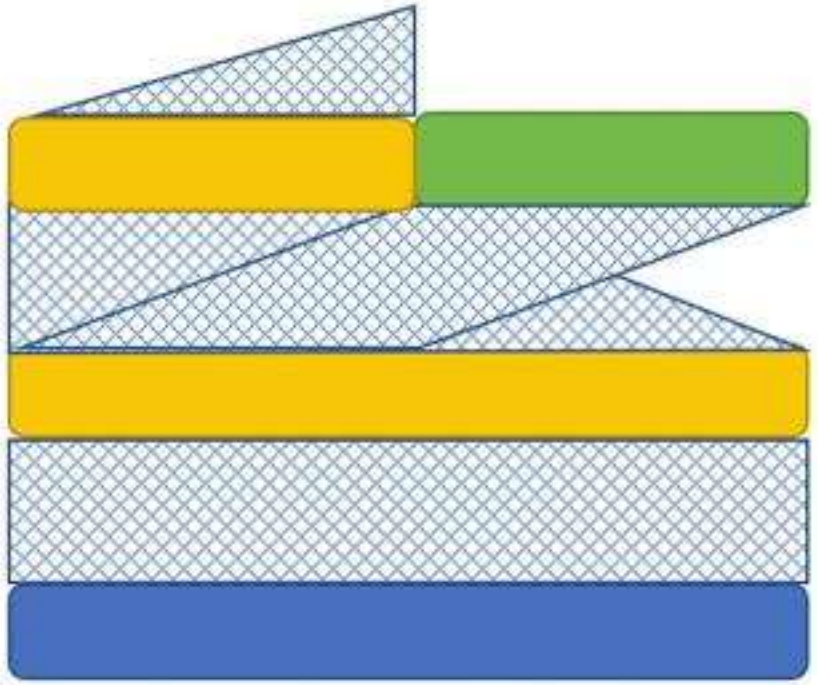
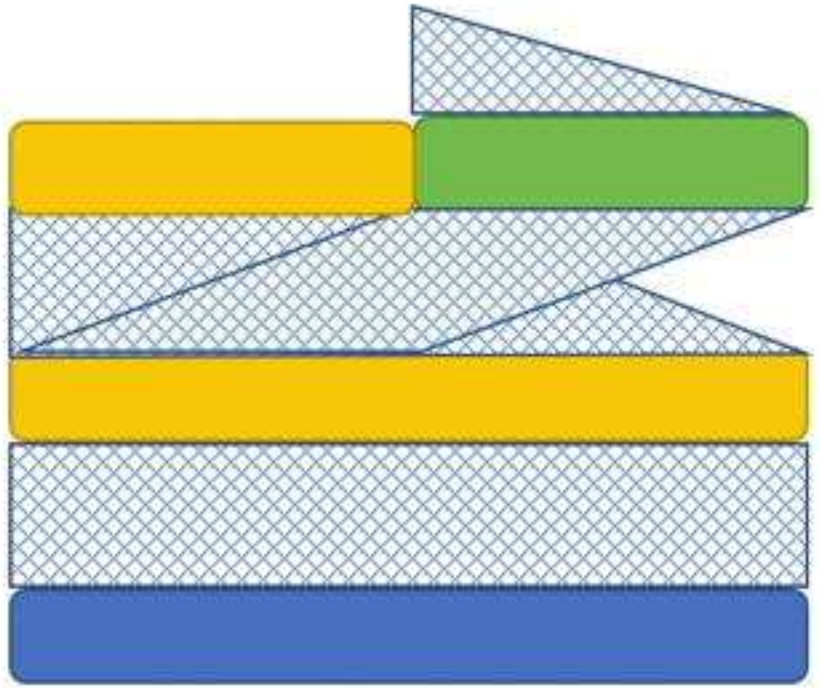


$$(2)\ \left(\begin{array}{c|c} 2L_3 & 0 \end{array}\right) \left(\begin{array}{c|c} L_2 & C_2 \\ \hline D_2 & R_2 \end{array}\right) \left(\begin{array}{c} L_1 \\ \hline B_1 \end{array}\right)\ \text{(a)} \quad (3)\ \left(\begin{array}{c|c} 2L_3 & 0 \end{array}\right) \left(\begin{array}{c|c} L_2 & C_2 \\ \hline 2L_2 & 0 \end{array}\right) \left(\begin{array}{c} L_1 \\ \hline B_1 \end{array}\right)\ \text{(b)}$$

Example path explained (3) -> (4)





(3) $\left(\begin{array}{c|c} 2L_3 & 0 \end{array}\right) \left(\begin{array}{c|c} L_2 & C_2 \\ \hline 2L_2 & 0 \end{array}\right) \left(\begin{array}{c} L_1 \\ \hline B_1 \end{array}\right)$ (b)

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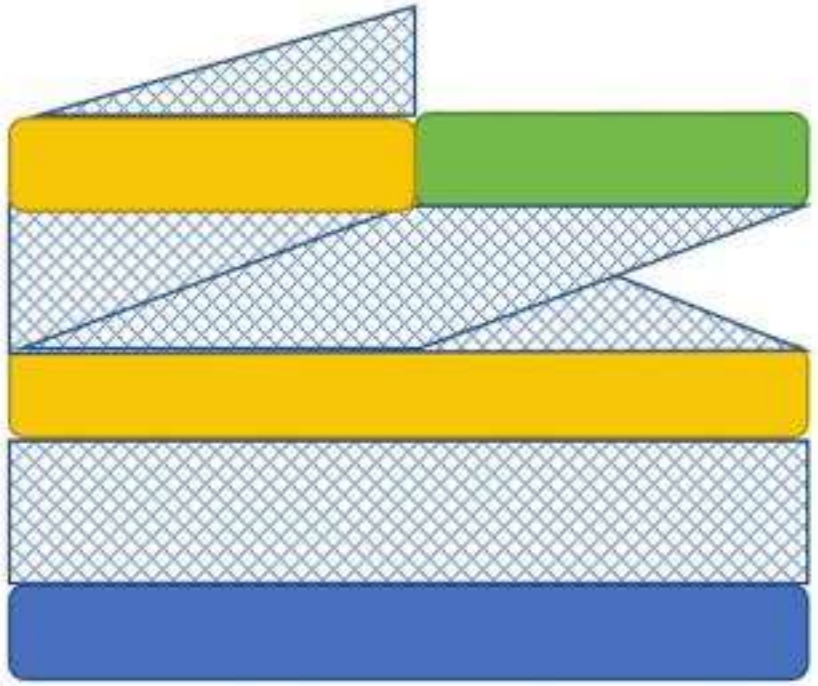

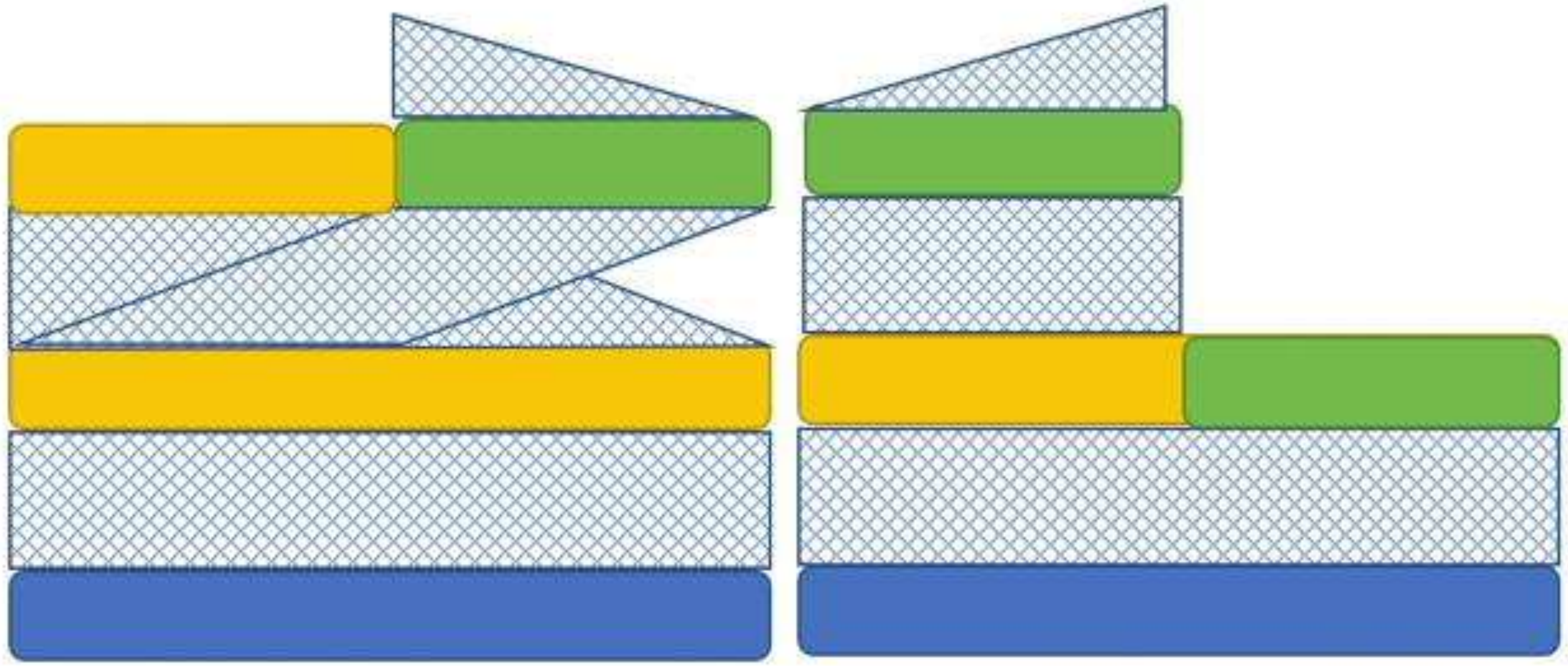

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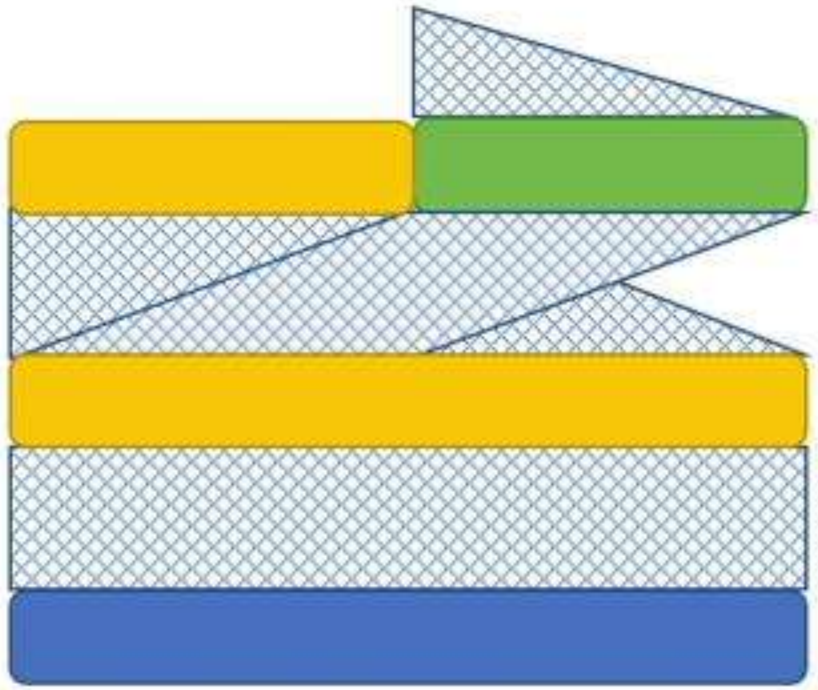

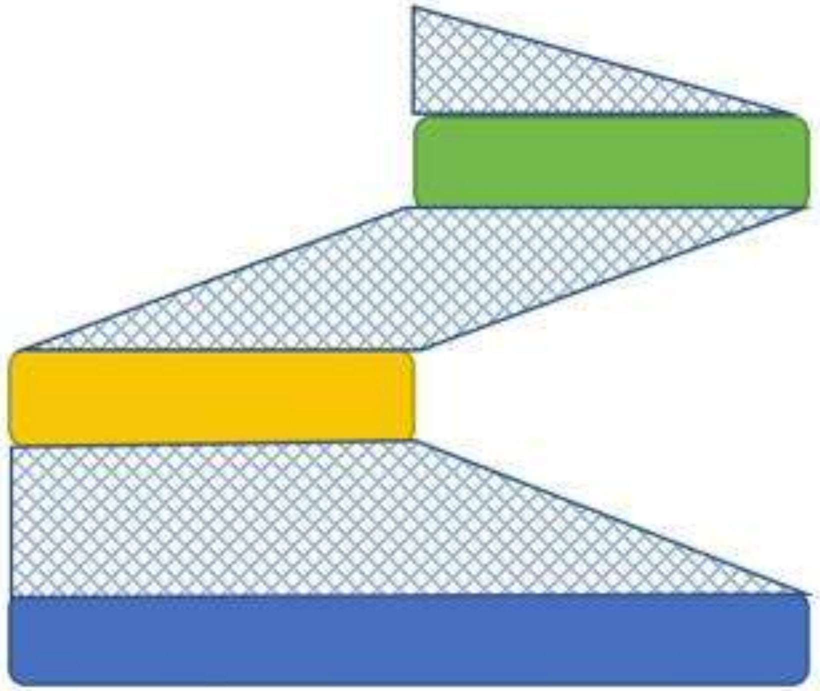

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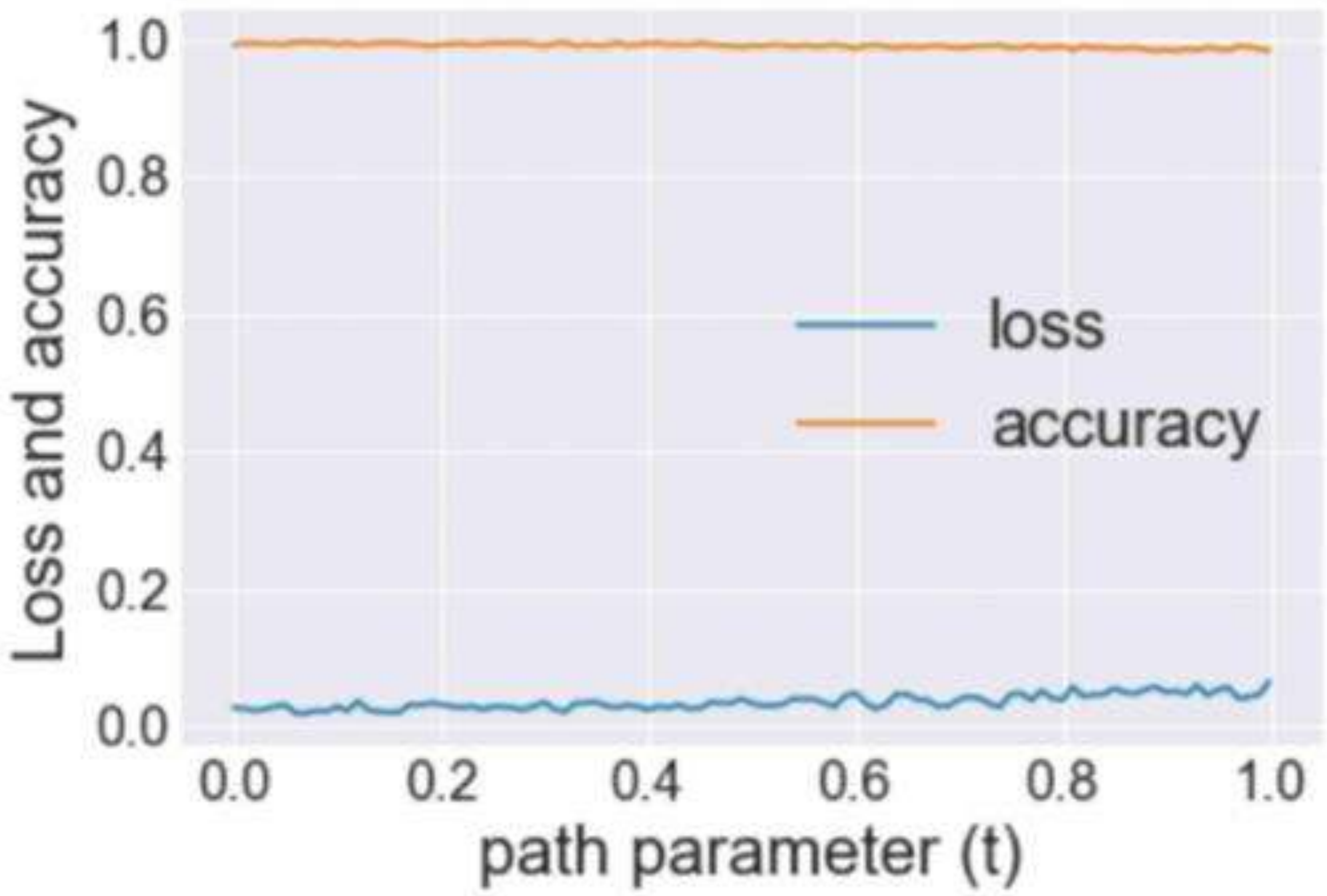
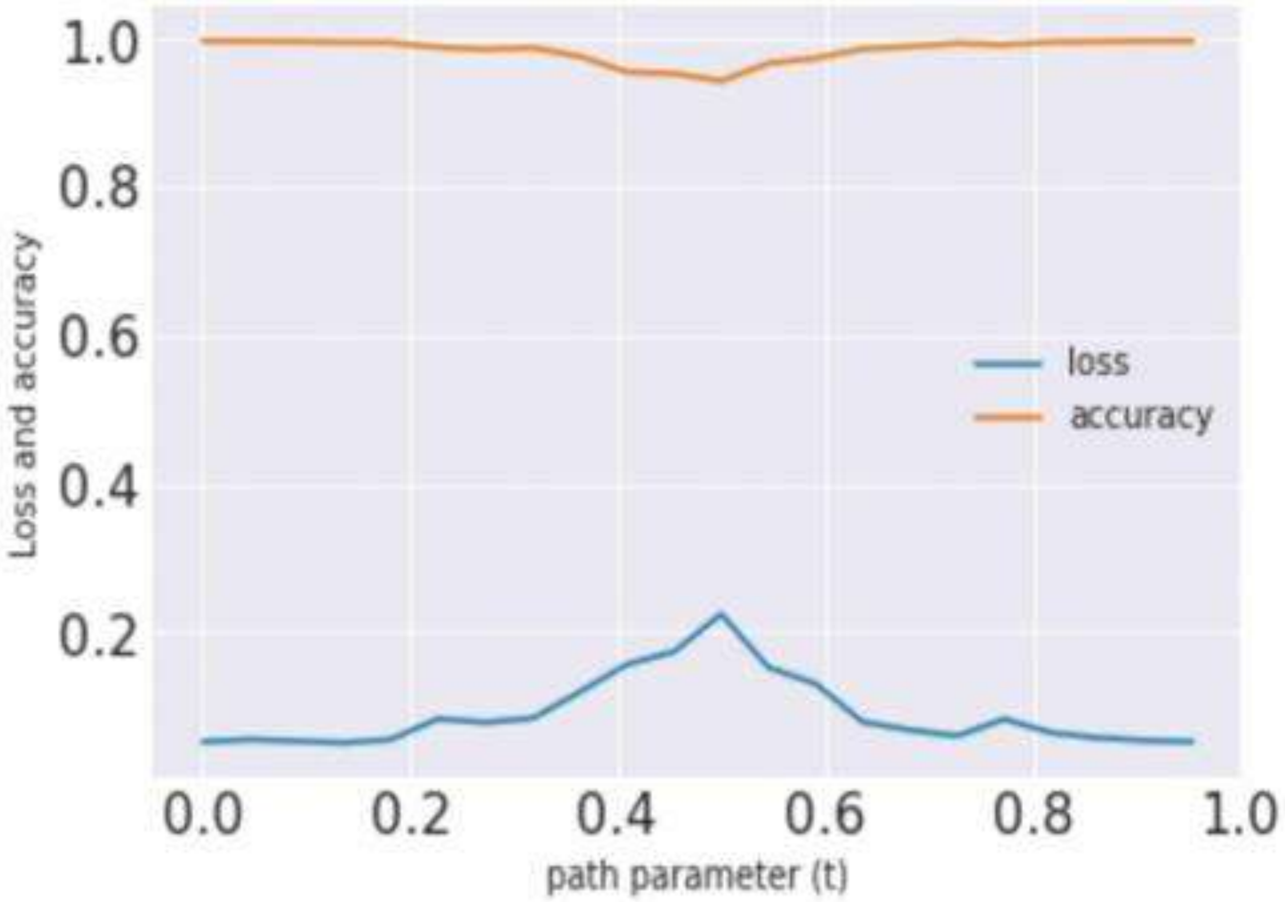
Experiments



MNIST, 3-layer CNN



CIFAR-10, VGG-11

Conclusions

For neural networks, not all local/global min are connected, even in the overparametrized setting.

Solutions that satisfy dropout/noise stability are connected.

Open Problems

- Path found by dropout/noise stability are still more complicated than the path found in practice.
- Path are known to exist in practice, even if the solutions are not as dropout stable as we hoped.

- Can we leverage mode connectivity to design better optimization algorithms? Maybe by proving the stronger convexity requirements as Leon talked about?

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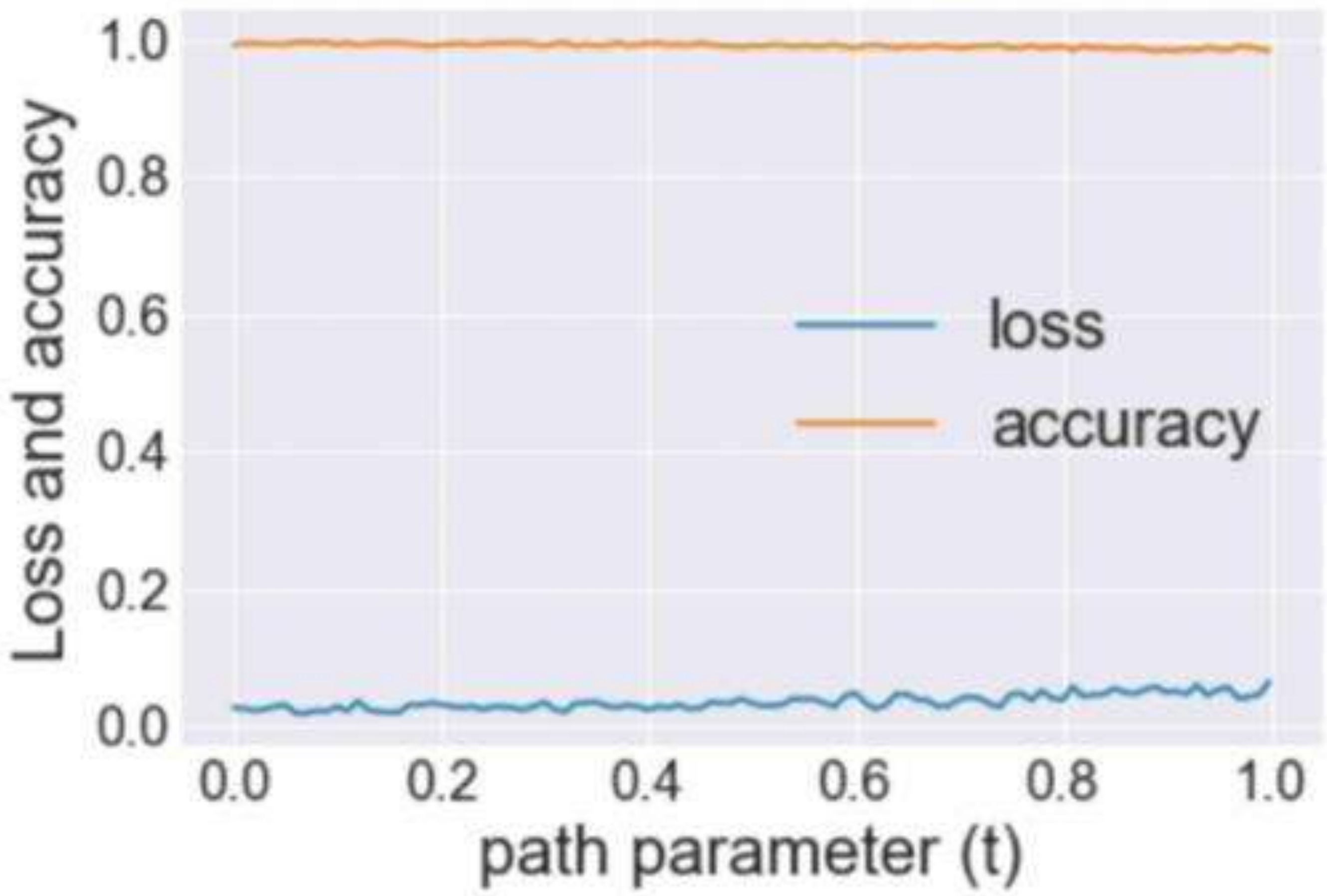

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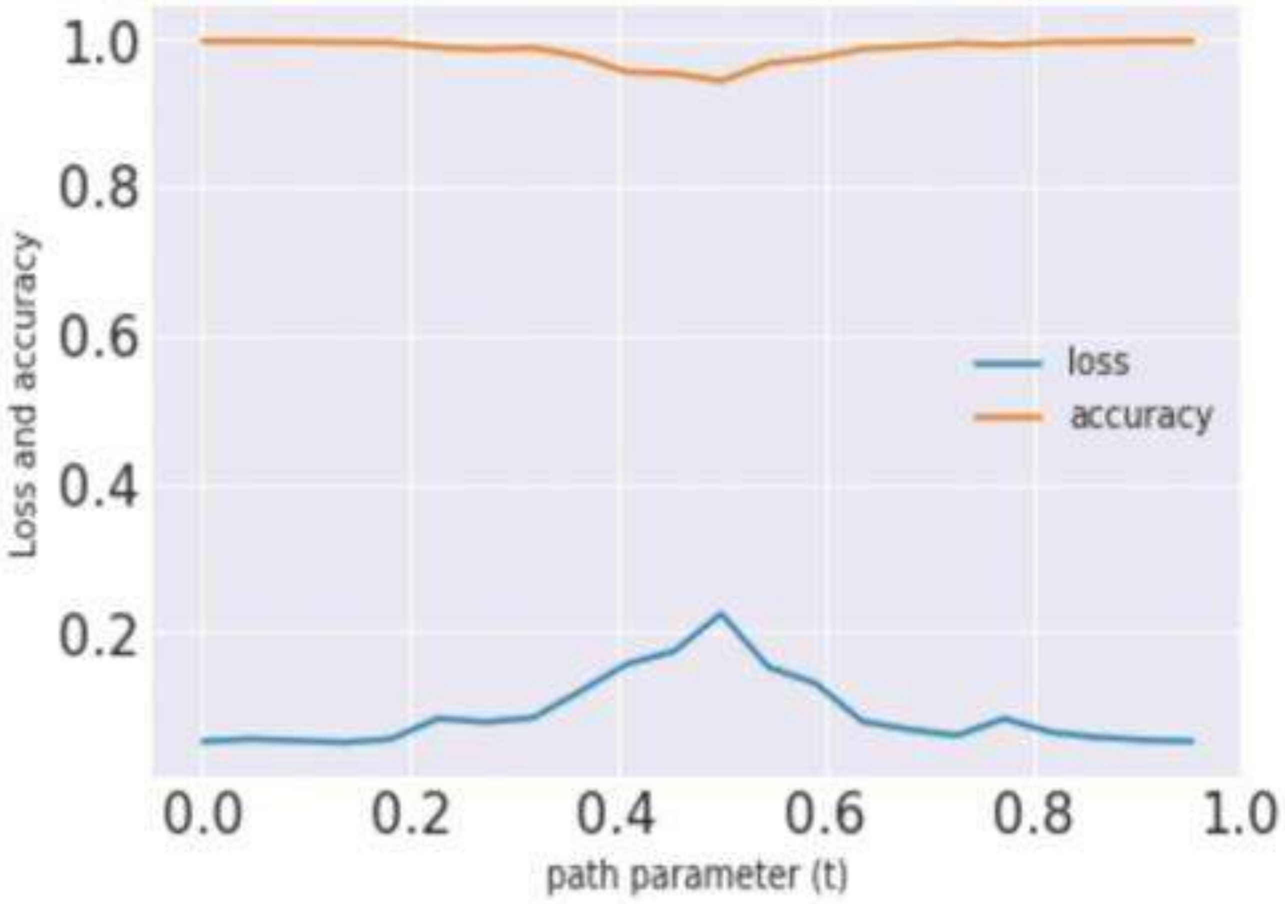

CIFAR-10, VGG-11

Deep learning theory: The first proof for the most simple example of over-parameterization

Yuanzhi Li

Stanford University

date: Today

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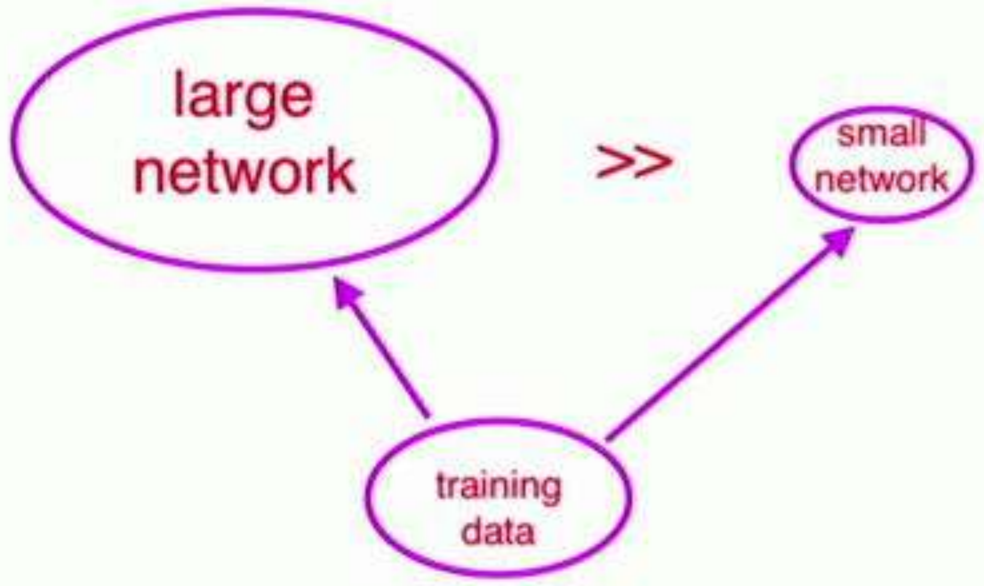

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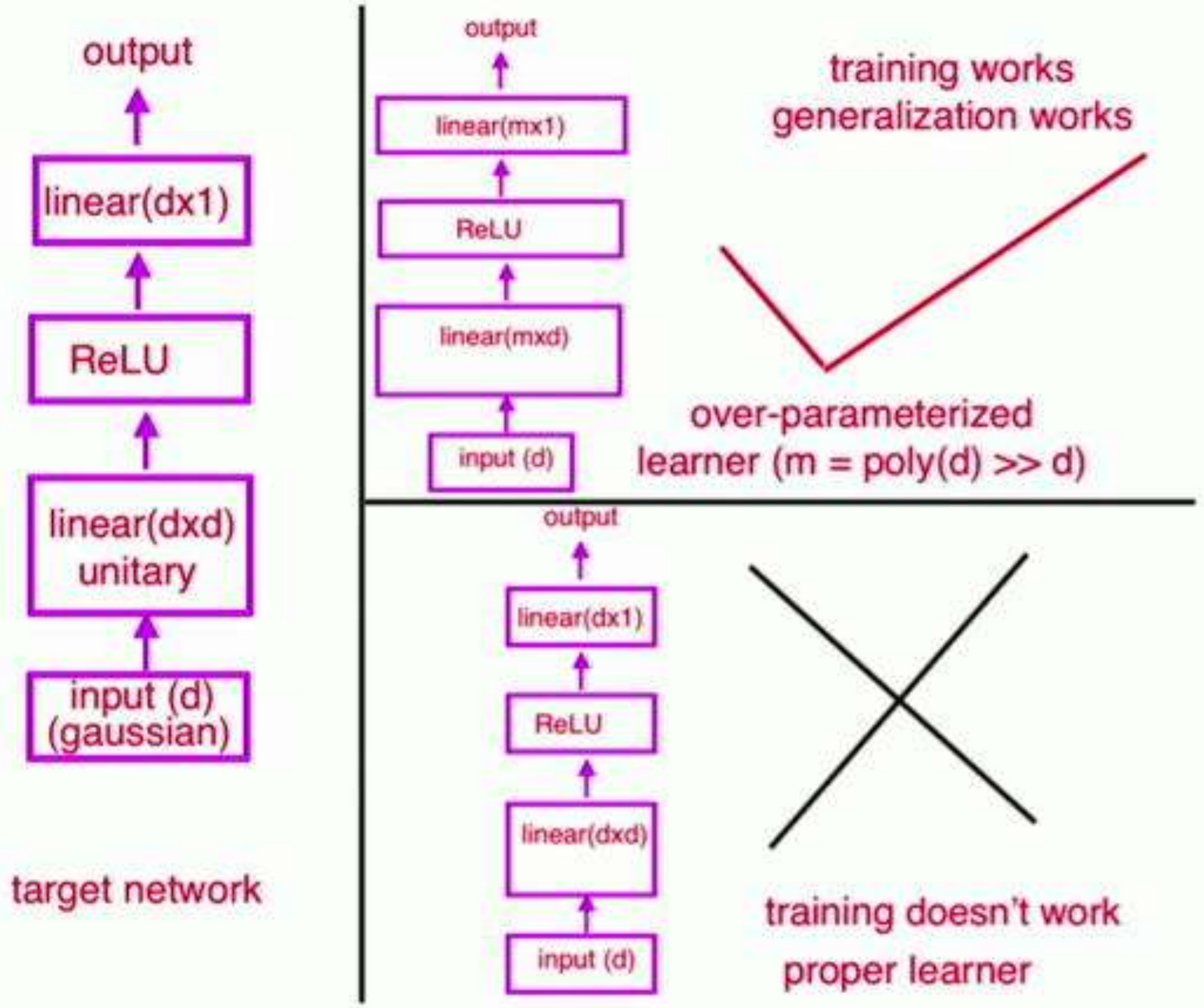

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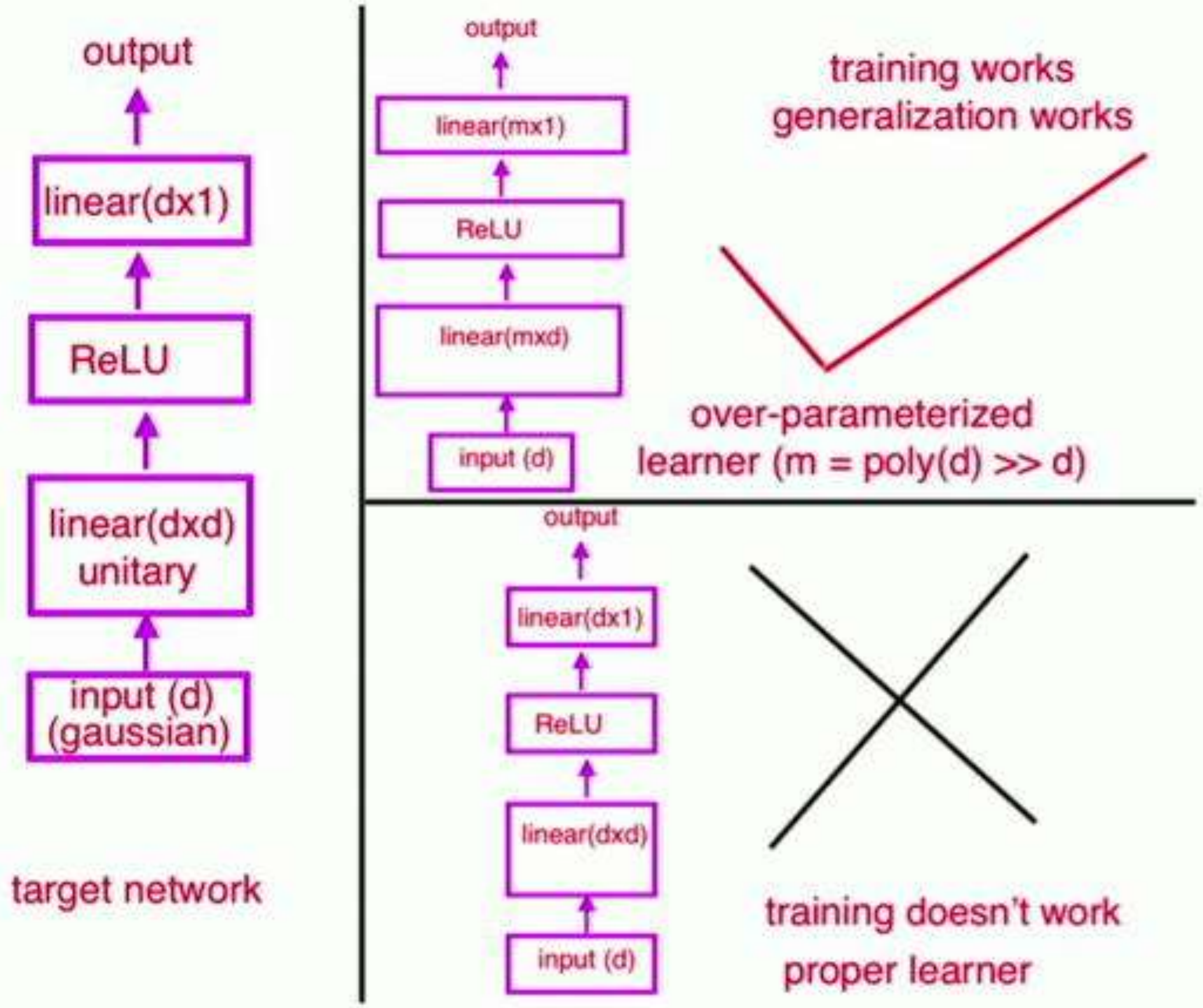

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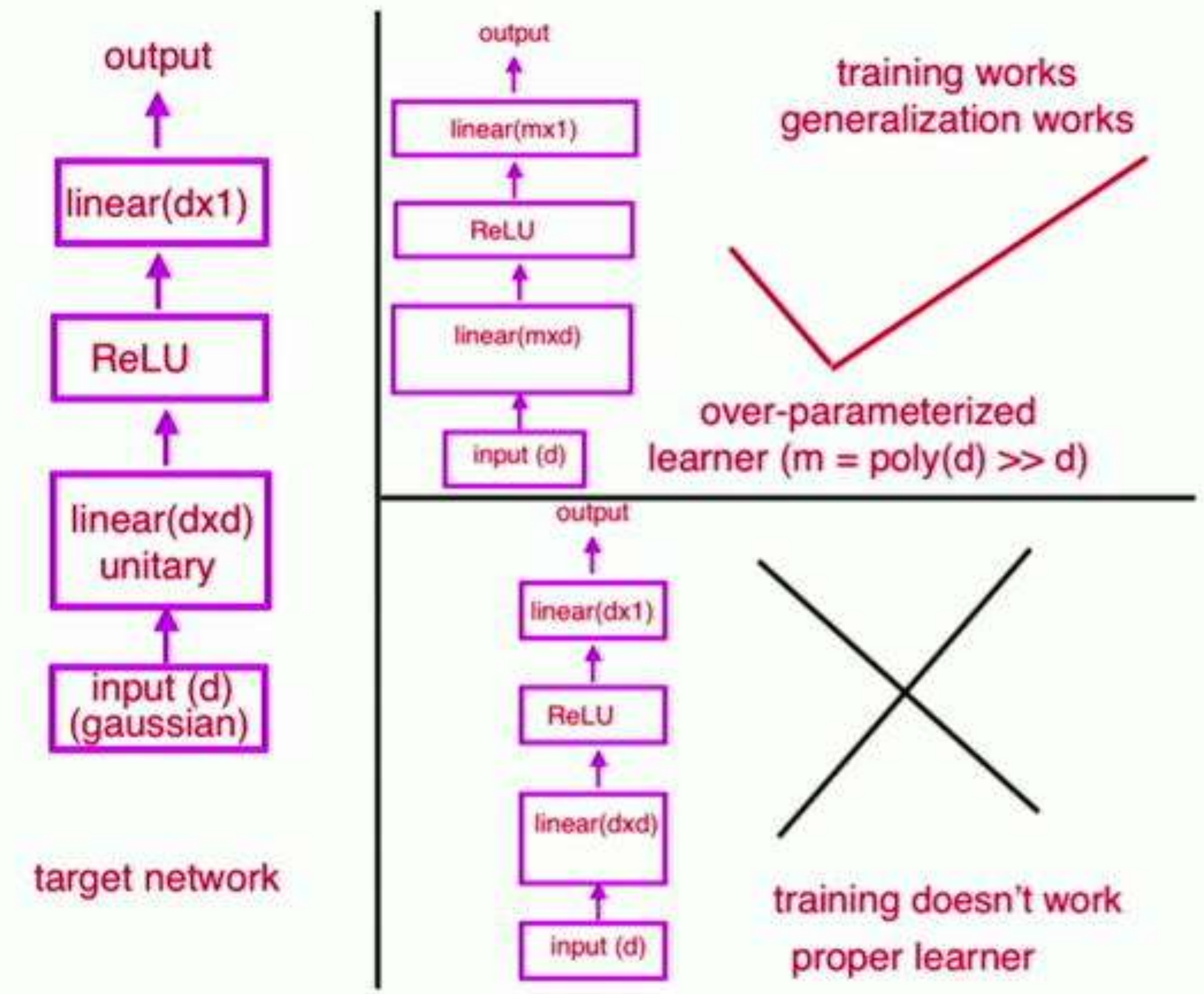

-
- Folklore example ([Lecun et al'14]), formally reported in for example [GLM'17], empirically.

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- What is the fundamental reason behind over-parameterization?
- Let us start with prior works on over-parameterization (which are fundamentally different from this work).

Prior works

- Over-parameterization and the neural tangent kernel (NTK). (More info: See my talk at the simons institute $https://simons.berkeley.edu/talks/tbd-70$ $https://arxiv.org/abs/1811.03962$).

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- So this simple example can not be explained by theorem of NTK.

Prior works differ from NTK

- $[LY'17]$ Given a properly parameterized network and a good initialization ($\|W - W^*\|_2 \leq 0.1$ and $a_i = a_i^*$), then running SGD over W works. (Note: W^* is unitary so $\|W^*\|_2 = 1$.)

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- $[LMW'19]$ Given an exponential amount of over-parameterization, noisy SGD works.
- However, (ℓ_1 constrained) linear regression over an n exponential size set of feature mappings also works – Moving from exponential to polynomial is the key advantage of using neural networks.

This work

Theorem (LMZ'19, sketched)

Given target network of form $f^*(x) = \langle v^*, x\rangle + \sum_{i=1}^{r} a_i^* ReLU(\langle w_i^*, x\rangle)$*, where* $w_i^* \in \mathbb{R}^d$ *are orthonormal (so* $r \le d$*), each* $a_i^* \in [1, \kappa] \cup [-\kappa, -1]$ *for constant* κ*.*
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- This is not the NTK regime due to the size of the random initialization (we use $w_i \sim \mathcal{N}(0, 1/m)$ so $|a_i| \approx \frac{1}{\sqrt{m}}$), NTK requires $|a_i| \approx 1$.

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- With sufficient over-parameterization, the optimization process of the network is actually simulating an infinite neuron optimization process, which is not convex (not NTK) but still have a benign landscape.
- Difficulty: why the infinite neuron process has a better training performance? How does the simulation work with only poly-size over-parameterization?

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- Without the linear term, the population loss $L = \mathbb{E}_{x \sim \mathcal{N}(0,I)}(f(x) - f^*(x))^2$ is given by

$$L = \sigma_0 \left\| \sum_{i=1}^{m} \|w_i\|_2^2 - \sum_{i=1}^{d} a_i^* \right\|_F^2 + \sum_{r=1}^{\infty} \sigma_{2r} \left\| \sum_{i=1}^{m} w_i^{\otimes 2} \bar{w}_i^{\otimes 2r-2} - \sum_{i=1}^{d} a_i^* (w_i^*)^{\otimes 2r} \right\|_F^2$$

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  - This observation allows us to simplify many cross terms when calculating the gradient update.

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  - So the entire infinite neuron process is a mixture of PCA and tensor decomposition, which is highly non-convex (not NTK).

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$$w \to T_1(w) \to T_2(w) \to \cdots \to T_t(w), \quad \text{via } w = w - \eta \nabla_w L_{\text{inf}}$$

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- Question: (ignoring the $\sqrt{m}$ multiplicative scaling between the norm of w_i and w) if $w = w_i$, then is $T_t(w)$ close to $\tilde{T}_t(w_i)$?

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$$|T_1(w_i) - \hat{T}_1(w_i)| \approx \frac{1}{\sqrt{m}}, \quad |\hat{T}_1(w_i) - \tilde{T}_1(w_i)| \approx \frac{1}{\sqrt{N}}$$

From infinite to finite, one step

- The error between $w \to T_1(w)$ and $w_i \to \tilde{T}_1(w_i)$:
- Three steps: $w_i \to T_1(w_i) \to \hat{T}_1(w_i) \to \tilde{T}_1(w_i)$.
- $T_1(w_i) \to \hat{T}_1(w_i)$: over-parameterization error: the error caused by moving from P to the "empirical version" $\{w_i\}_{i=1}^m$: e.g. the error between $\mathbb{E}_{w\sim P} w^{\otimes 2}$ and $\frac{1}{m}\sum_{i=1}^m w_i^{\otimes 2}$ where each w_i i.i.d. $w_i \sim P$.
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- $\frac{1}{\sqrt{m}}$: The role of (poly-size) over-parameterization.

From infinite to finite, the entire process

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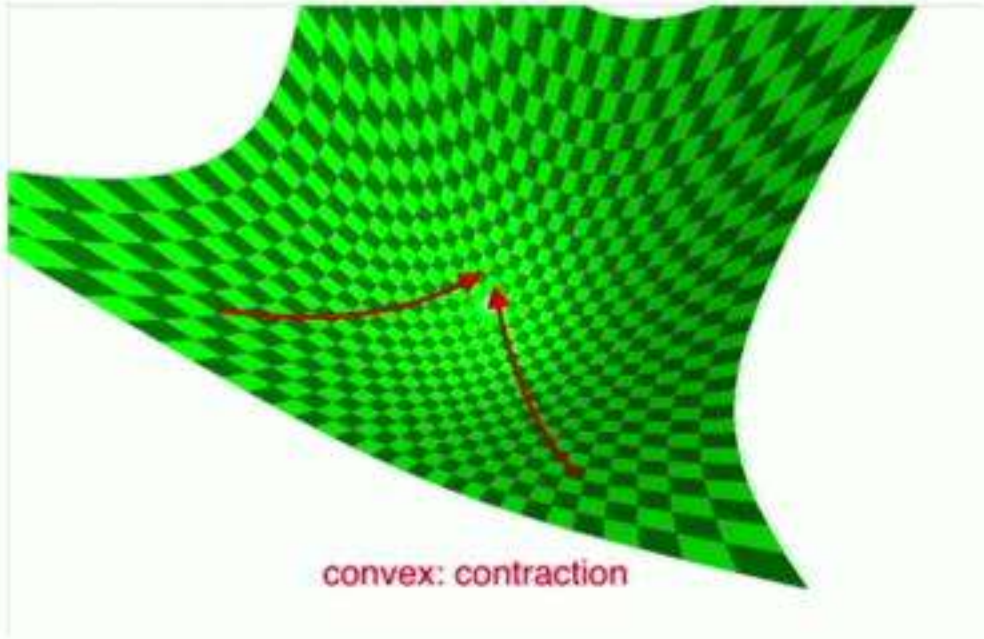

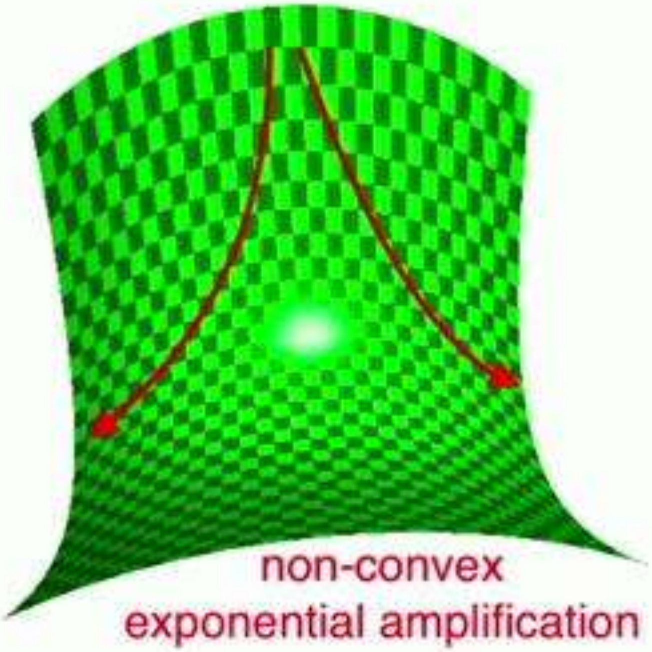

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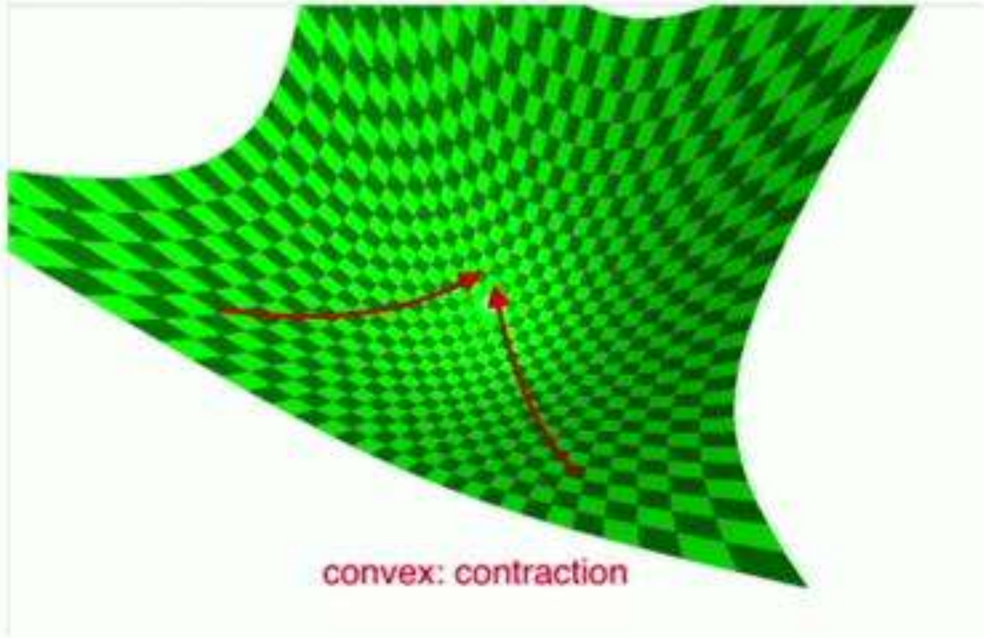

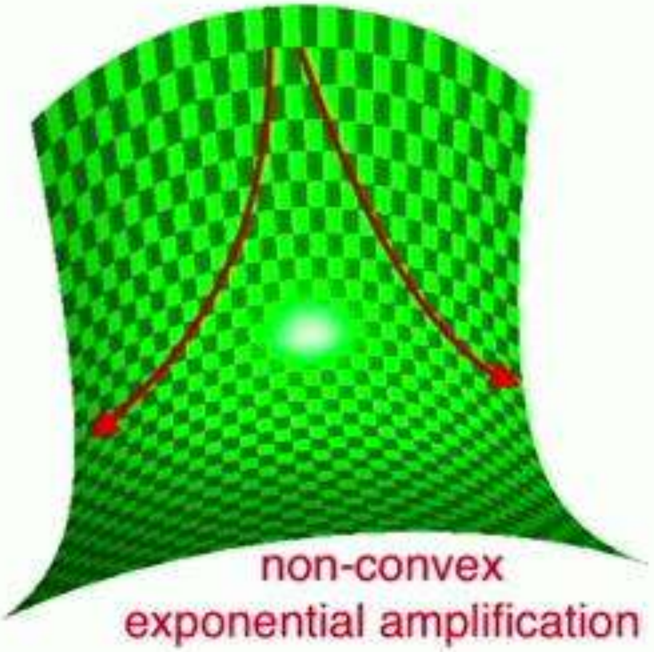

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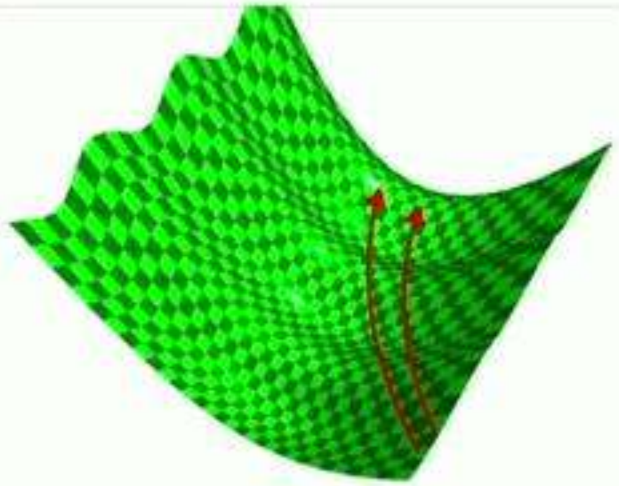
- 
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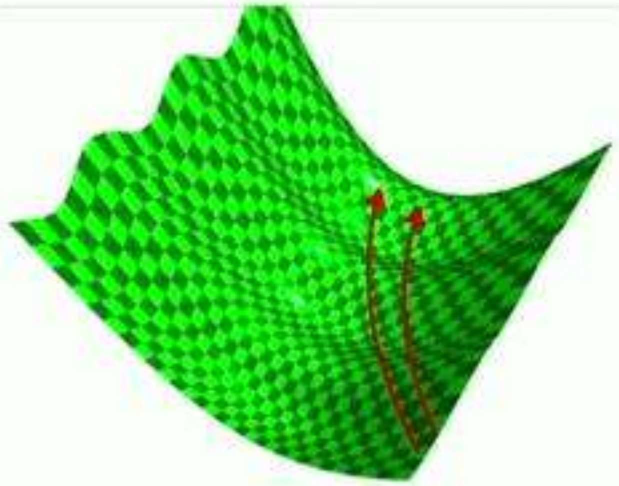

-
- Together with the infinite neuron process result, this gives the final theorem.

Summary: the most simple example of over-parameterization

- We show that running SGD on a over-parameterized two-layer neural network with ReLU activation can learn a smaller two-layer neural network with ReLU activation, where the hidden weights are orthonormal, and the data distribution is spherical Gaussian.

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  - Infinite neuron optimization process has a non-convex but benign landscape (a mixture of PCA and tensor decomposition, with symmetric property and mild-amplification property).
- When the hidden weights are not orthonormal, it can require $2^{\Omega(d)}$ queries in the worst case $[VW'19]$ for any SQ model.

The infinite neuron optimization process

- Infinite neuron optimization: Each step: update P via gradient descent (with learning rate η) over w on the infinite neuron loss:

$$w = w - \eta \nabla_w L_{\inf}$$

$$\nabla_w L_{\inf} = \sigma_0 \frac{\partial}{\partial v}\Big|_{v=w} \left\| \left(\|v\|_2^2 - \|w\|_2^2 \right) + \mathbb{E}_{w' \sim P} \|w'\|_2^2 - \textstyle\sum_{i=1}^d a_i^* \right\|_F^2$$
$$+ \textstyle\sum_{r=1}^{\infty} \sigma_{2r} \frac{\partial}{\partial v}\Big|_{v=w} \left\| \left(v^{\otimes 2} \bar{v}^{\otimes 2r-2} - w^{\otimes 2} \bar{w}^{\otimes 2r-2} \right) + \mathbb{E}_{w' \sim P} (w')^{\otimes 2} \bar{w}'^{\otimes 2r-2} - \textstyle\sum_{i=1}^d a_i^* (w_i^*)^{\otimes 2r} \right\|_F^2$$

- The loss contains: Zero and second order term:

$$G = \sigma_0 \left\| \mathbb{E}_{w \sim P} \|w\|_2^2 - \sum_{i=1}^d a_i^* \right\|_F^2 + \sigma_2 \left\| \mathbb{E}_{w \sim P} w^{\otimes 2} - \sum_{i=1}^d a_i^* (w_i^*)^{\otimes 2} \right\|_F^2$$

- These are PCA-type terms, can not identify w_i^* uniquely.
- Forth order term:

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Memorization Capacity of ReLU Nets

SUVRIT SRA
Laboratory for Information and Decision Systems
Massachusetts Institute of Technology

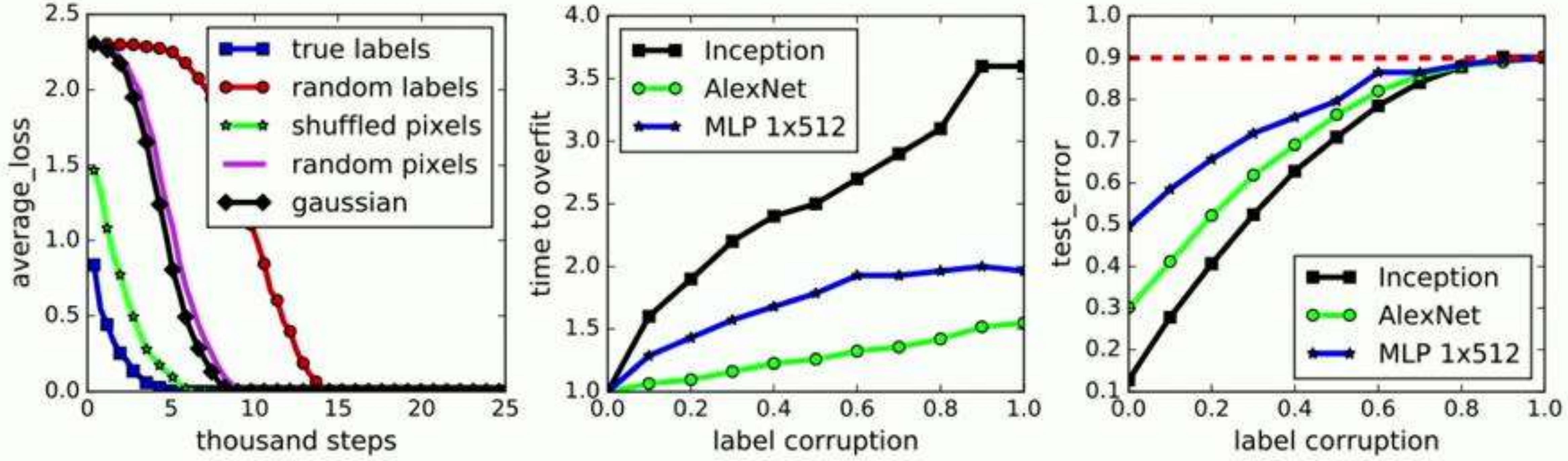
Chulhee Yun

Ali Jadbabaie

ml.mit.edu

The Memorization Phenomenon

- Overparametrized NNs trained with SGD can memorize even random noise *[Zhang et al., 2017]*

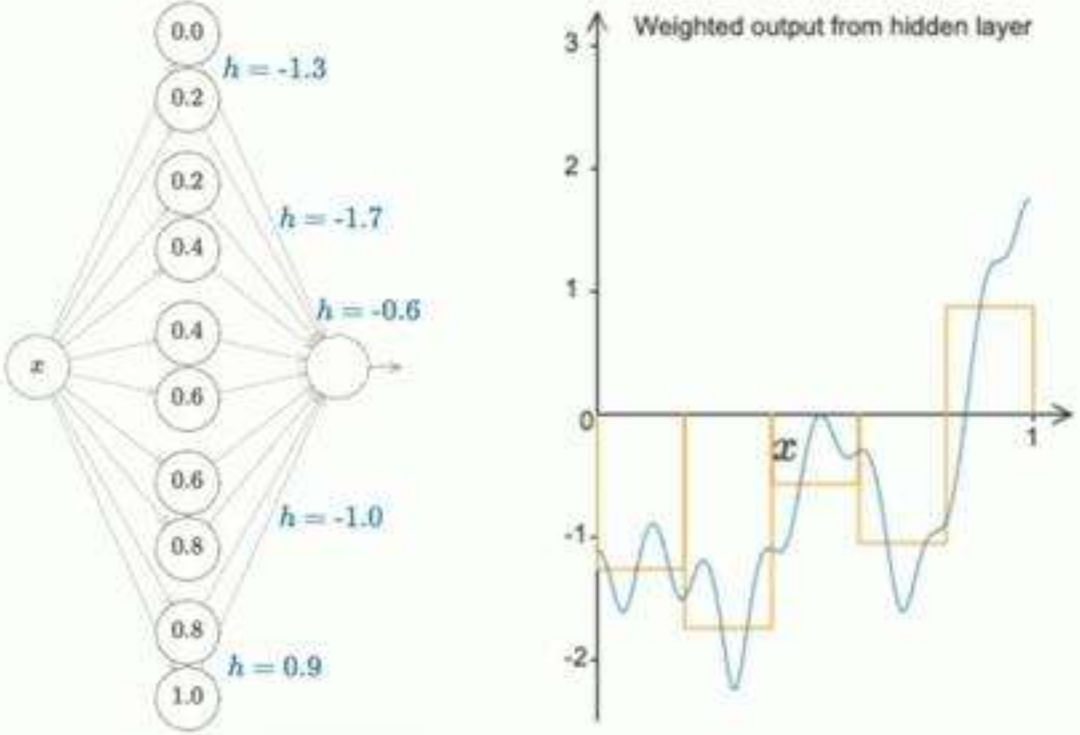


(a) learning curves (b) convergence slowdown (c) generalization error growth

Figure 1: Fitting random labels and random pixels on CIFAR10. (a) shows the training loss of various experiment settings decaying with the training steps. (b) shows the relative convergence time with different label corruption ratio. (c) shows the test error (also the generalization error since training error is 0) under different label corruptions.

Expressive power of NNs

- To understand memorization phenomenon, it is important to understand *expressive power*
- Expressive power a classic topic in NN theory; *universal approximation theory* *[Cybenko, '89, Hornik '91, Leshno '93, …]*



(http://neuralnetworksanddeeplearning.com/chap4.html)

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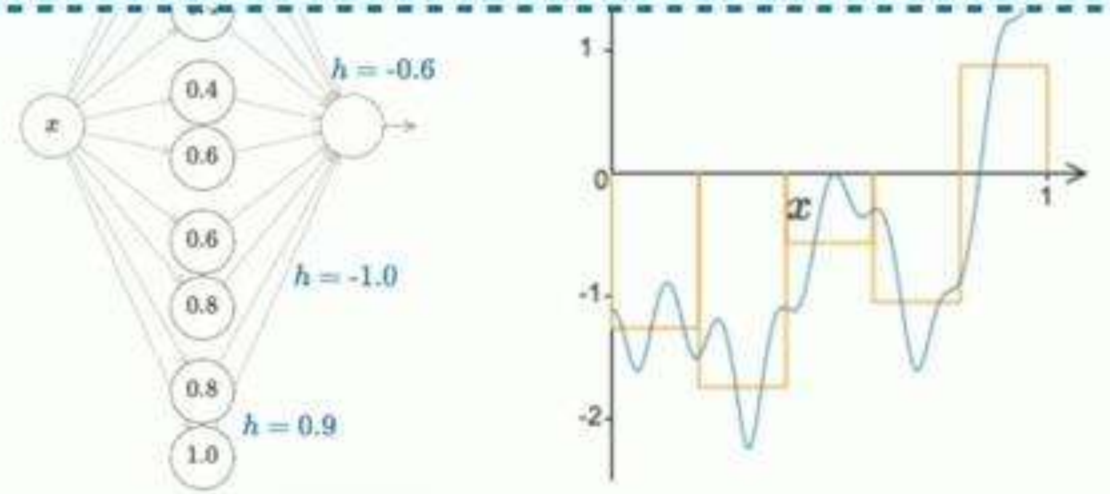
Majority of results consider function approximation (infinite points),
Not finite samples!



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Finite sample expressivity

Defn. We define (universal) **finite sample expressivity** of a neural network $f_{\boldsymbol{\theta}}(\cdot)$ as the network's ability to satisfy:

For arbitrary $\{(x_i, y_i) \in \mathbb{R}^d \times \mathbb{R}^p\}_{i=1}^N$ there exists a parameter $\boldsymbol{\theta}$ such that $f_{\boldsymbol{\theta}}(x_i) = y_i$ for $1 \leqslant i \leqslant N$.

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That is, the net can memorize an arbitrary dataset with *(input, output)* points.

Memorization capacity

Defn. For $p = 1$, we define **memorization capacity** to be the **maximum value** of N for which the network has finite sample expressivity.

That is, the maximum N, s.t. for any $\{(x_i, y_i) \in \mathbb{R}^d \times \mathbb{R}\}_{i=1}^N$, there exists a parameter $\boldsymbol{\theta}$ such that $f_{\boldsymbol{\theta}}(x_i) = y_i$.

Comparison to VC dimension

- **Memorization capacity**:

The maximum N such that **for all** $\{(x_i, y_i)\}_{i=1}^N$, there exists a parameter $\boldsymbol{\theta}$ such that $f_{\boldsymbol{\theta}}(x_i) = y_i$ *(recall 'p'=1 here).*

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⇒ **Memorization capacity ≤ VC dimension**

Related work on memorization

Classical works

- focus on memorization capacity of NNs with activations such as **linear threshold** or **sigmoid** *[Cover, 1965; Baum, 1988; Huang & Huang, 1991; Huang & Babri, 1998; Huang, 2003; etc…]*

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Recent Results

- ReLU fully-connected NNs (FNNs) *[Zhang et al., 2017]*
- Residual networks (ResNets) *[Hardt & Ma, 2017]*
- Convolutional neural networks (CNNs) *[Nguyen & Hein, 2017]*

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  [Hardt & Ma, 2017, Nguyen & Hein, 2017]

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Can we use depth to memorize with fewer hidden nodes?

Outline

Tight memorization capacity of FCNNs

number of hidden nodes necessary and sufficient for universal memorization

Memorization capacity of Resnets

*number of hidden nodes **sufficient** for universal memorization*

Behavior of SGD near memorizing global min

*Analysis of **without replacement** SGD near global min*

Finite sample expressivity of FCNNs

Setup

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FCNN architecture:

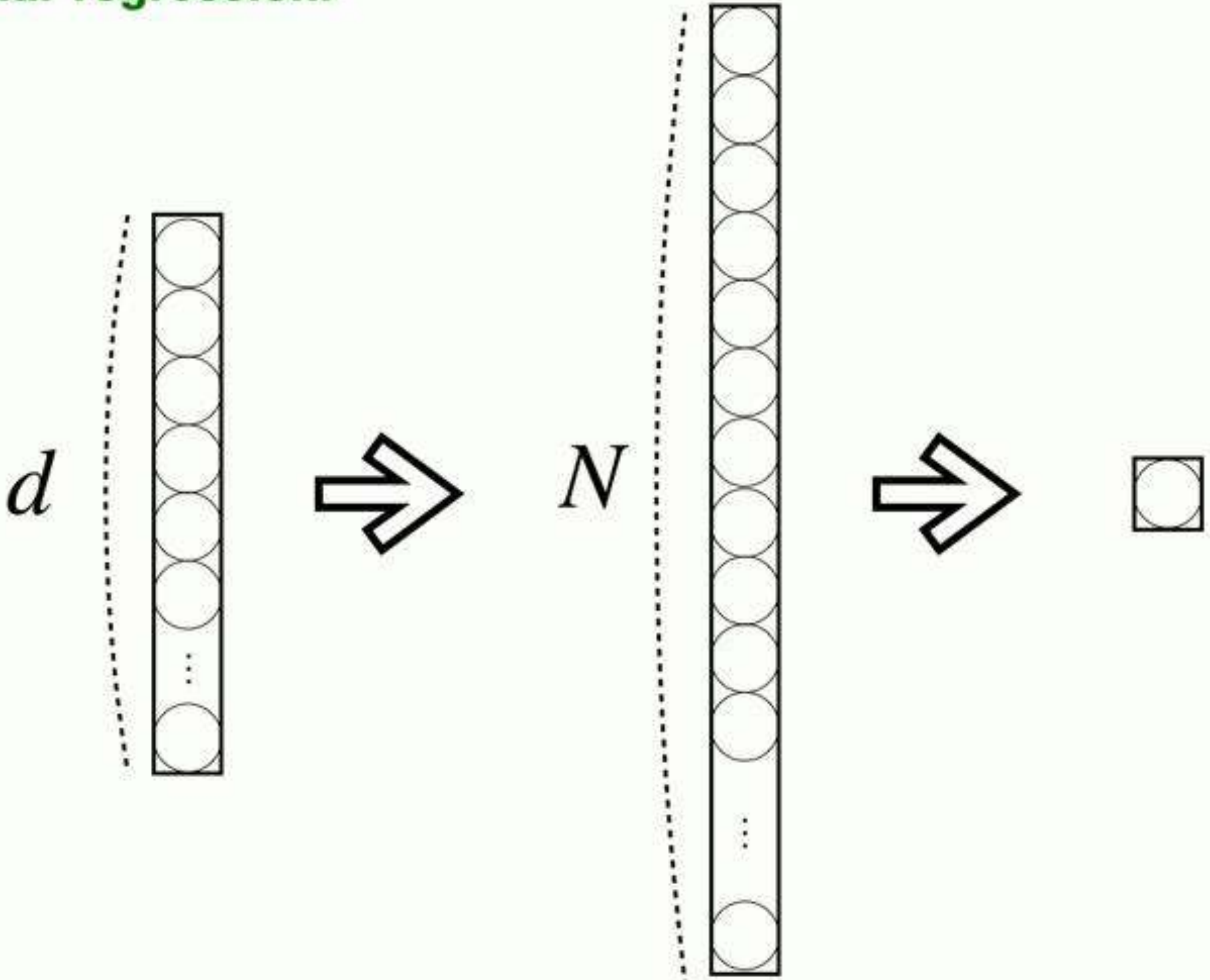
$$a^0(x) = x, \;\; a^l(x) = \sigma(W^l a^{l-1}(x) + b^l), \;\; l \in \{1,\ldots,L-1\}, \;\; f_{\theta}(x) = W^L a^{L-1}(x) + b^L$$

Activation function   Weight matrix   Bias vector

Activation $\sigma(t) = \max\{s_+ t, s_- t\}, \; s_+ > s_- \geq 0.$
(includes ReLU and Leaky ReLU)

Finite sample expressivity of FCNNs

For scalar regression:



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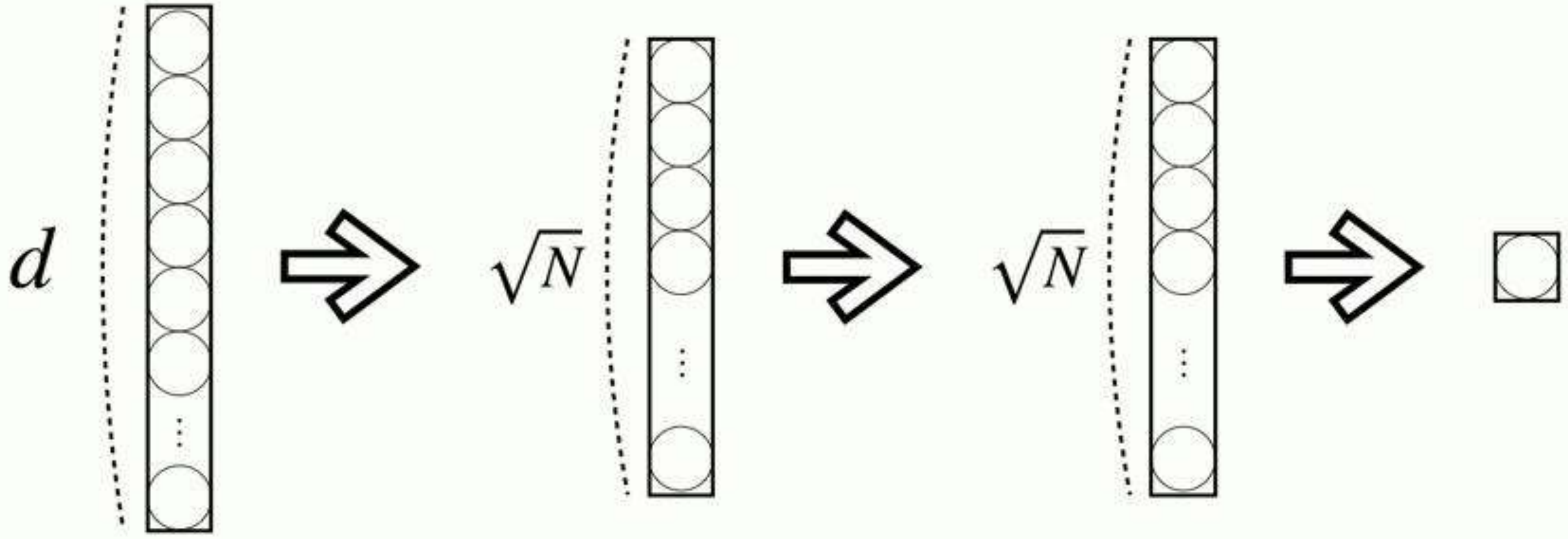

Memorization: sufficiency results

Theorem 1.

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Proposition 2 (classification).

A 3-hidden-layer ReLU network with hidden layer dimensions $d_1 d_2 \geq 4N$ and $d_3 \geq 4p$ can memorize any arbitrary classification dataset with N distinct points *(here 'p' is the number of classes)*

Memorization: necessity result

Theorem 3.

A 1-hidden-layer ReLU network with $d_1 + 2 < N$, or
a 2-hidden-layer ReLU network with $2d_1d_2 + d_2 + 2 < N$
cannot memorize arbitrary datasets ($p = 1$) with N points.
(i.e., there exist datasets that they fail to memorize)

Discussion

Depth-width tradeoff in finite sample expressivity

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Necessary and sufficient width for memorizing ($p = 1$): 1-hidden-layer $\Theta(N)$ vs 2-hidden-layers $\Theta(\sqrt{N})$

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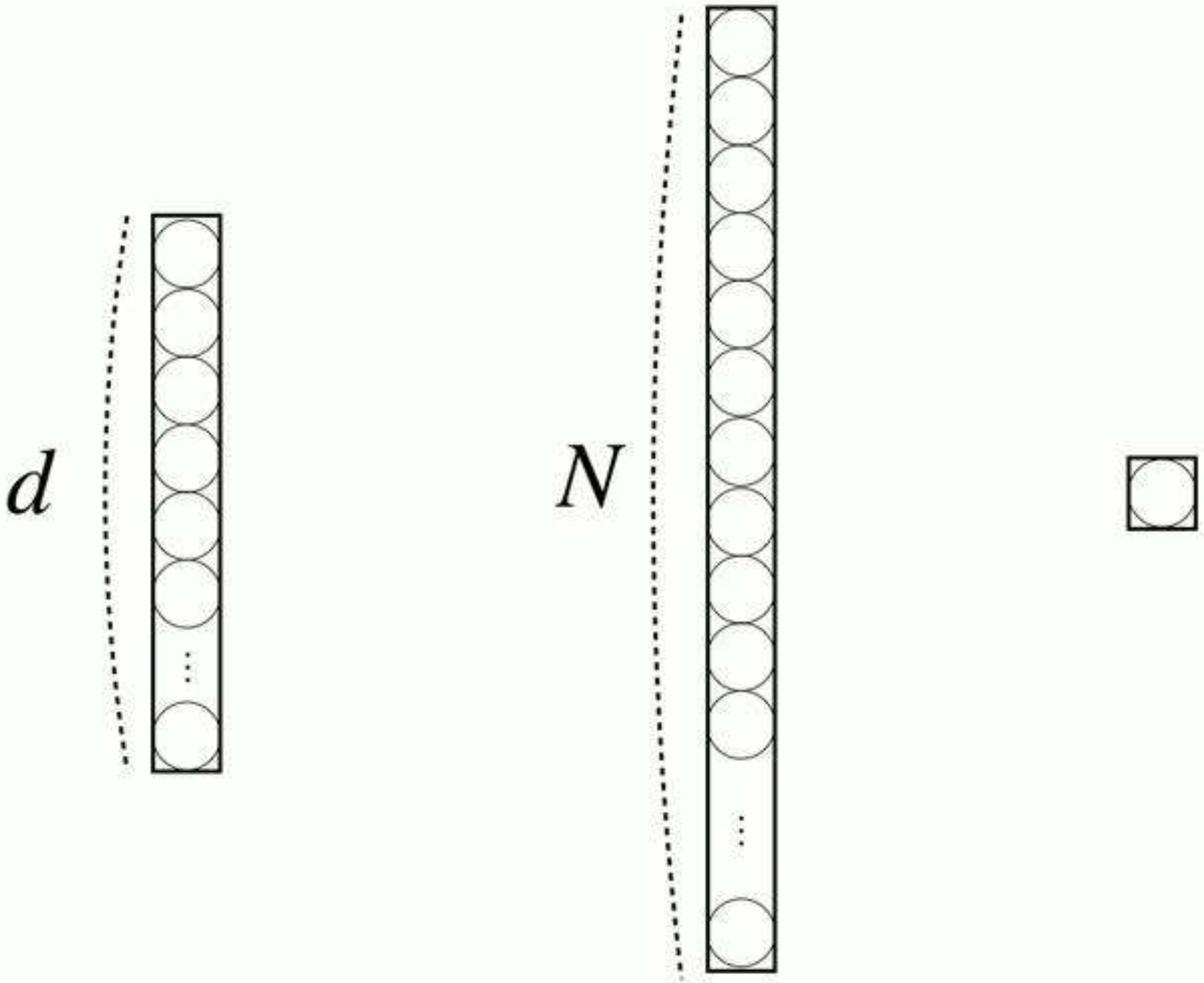
For p-class classification, $\Theta(\sqrt{Np})$ requirement of 2-hidden-layer improves to $\Theta(\sqrt{N} + p)$ by adding another layer

- ImageNet $N \approx 10^6, p = 10^3$ memorizable with 2k-2k-4k FCNN.
- Surprisingly small network size is required to memorize & <u>achieve zero training loss</u> at global minimum.

Proof ideas: setup

- **Training data**: $\{(x_i, y_i)\}_{i=1}^{N}, \;\; x_i \in \mathbb{R}^d, \;\; y_i \in \mathbb{R}$
- **Assumption**: all x_i's are distinct and all $y_i \in [-1,1]$.
- **Clipping region**: $\mathbb{R} \backslash [-1,1]$

Key ideas of proof



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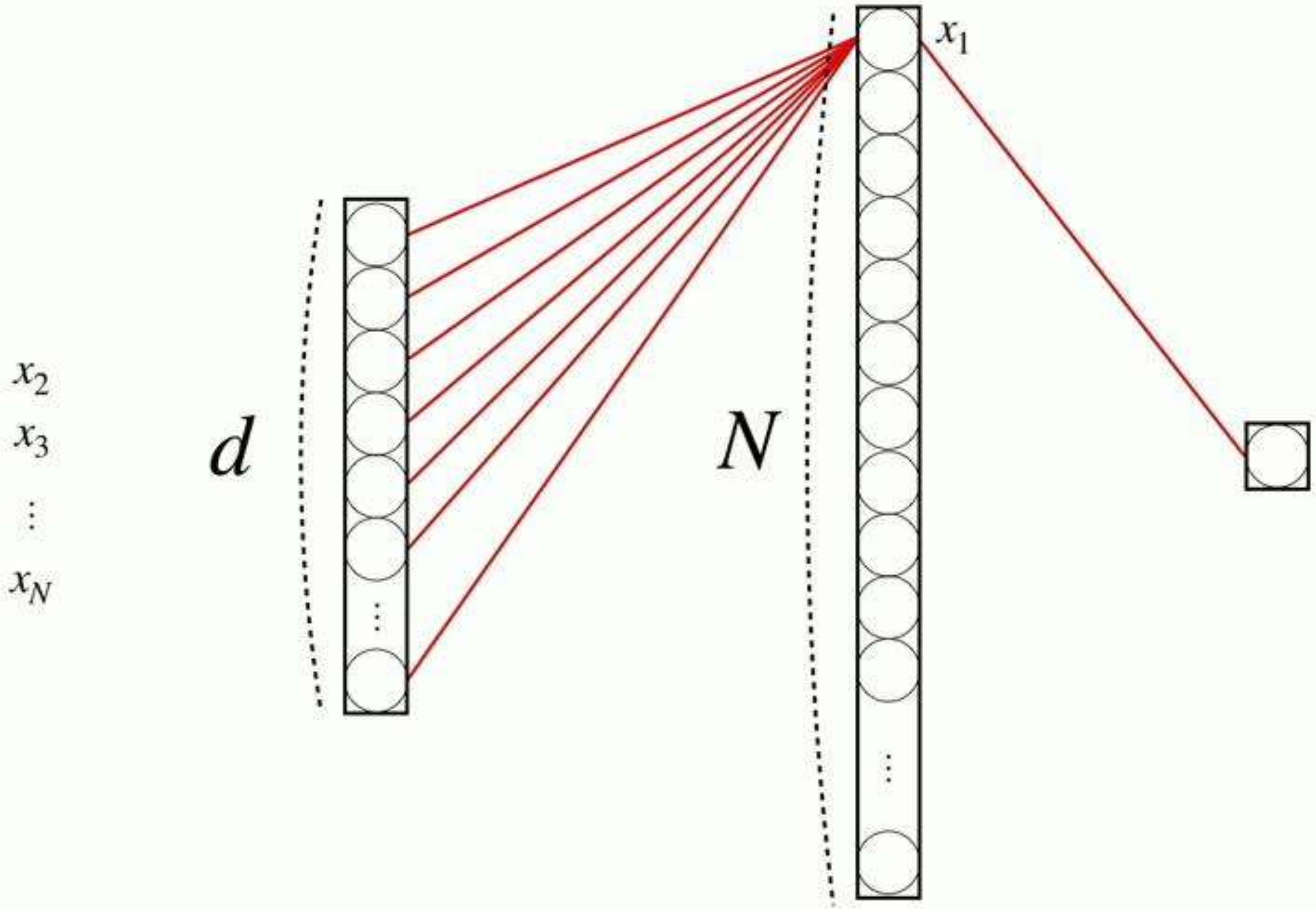

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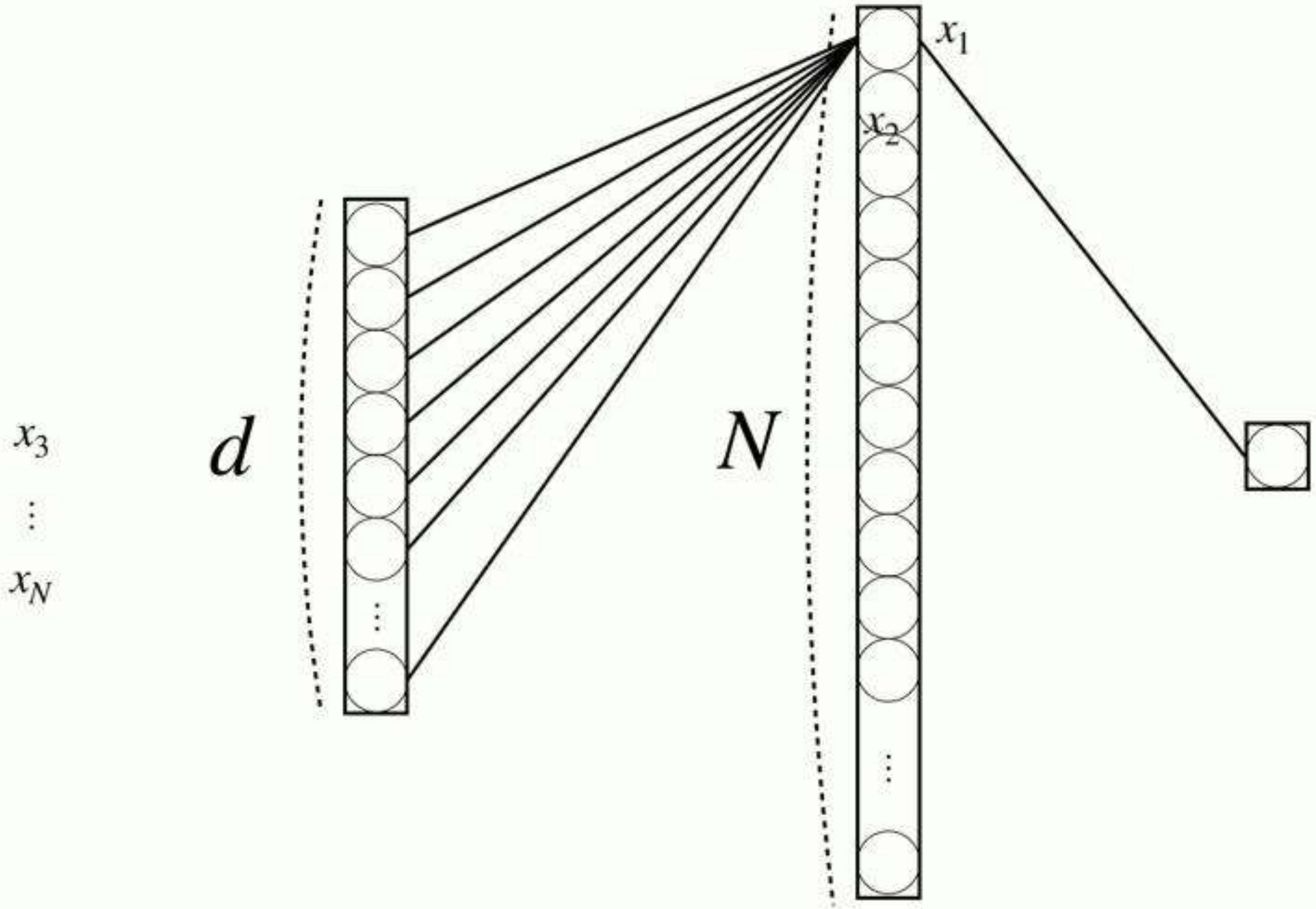

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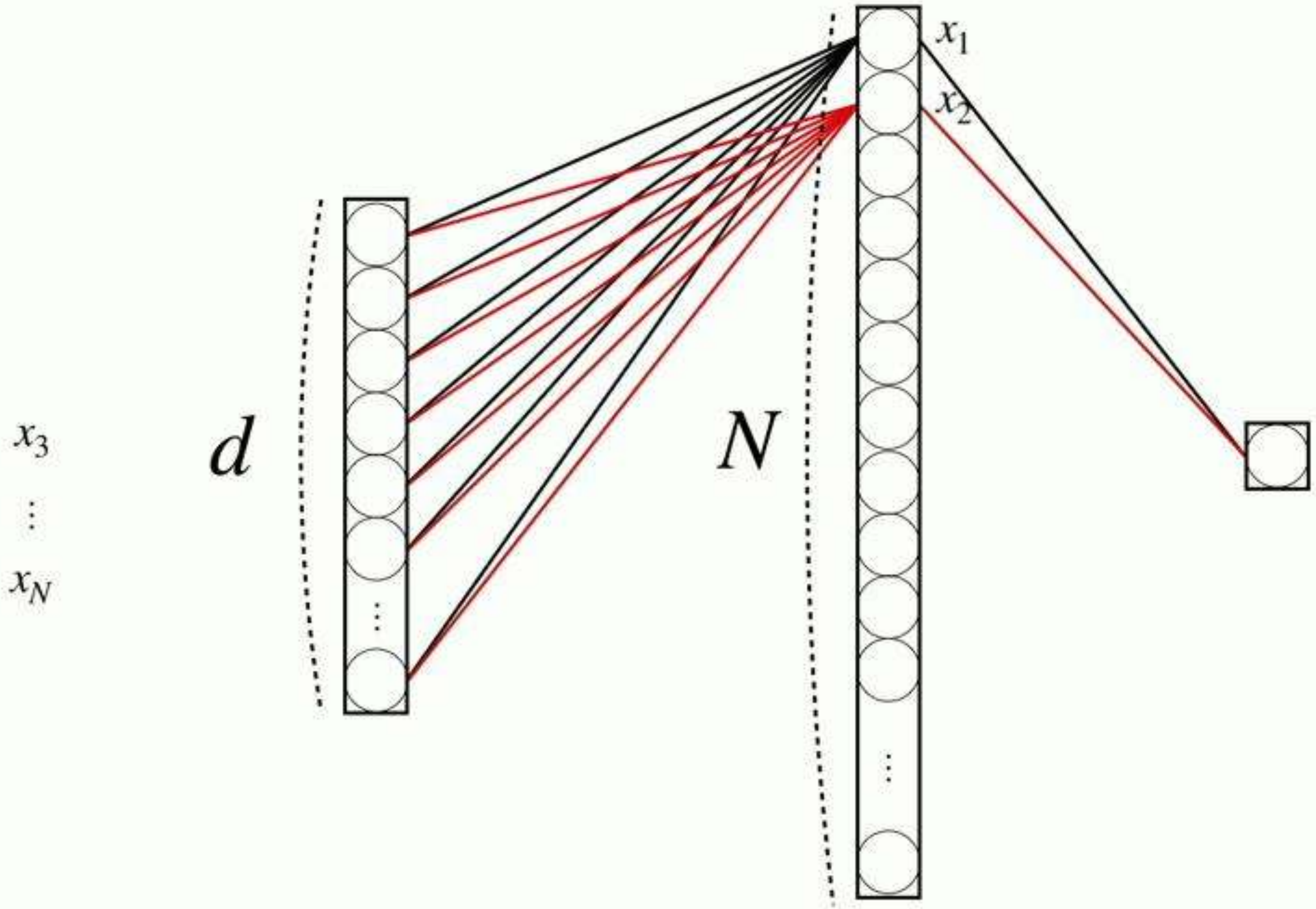

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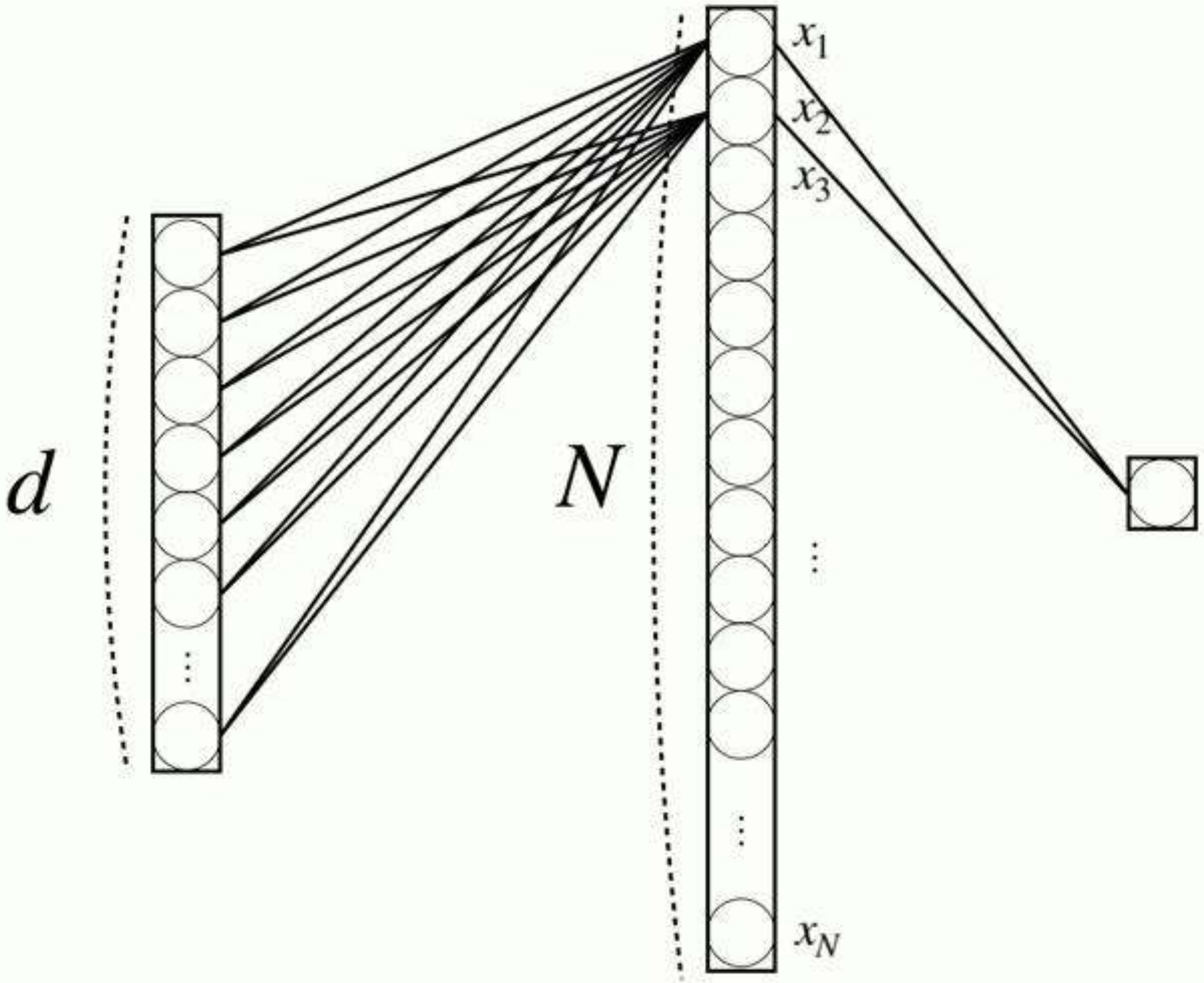

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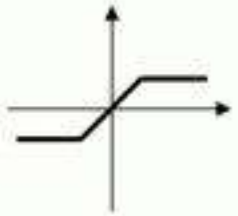

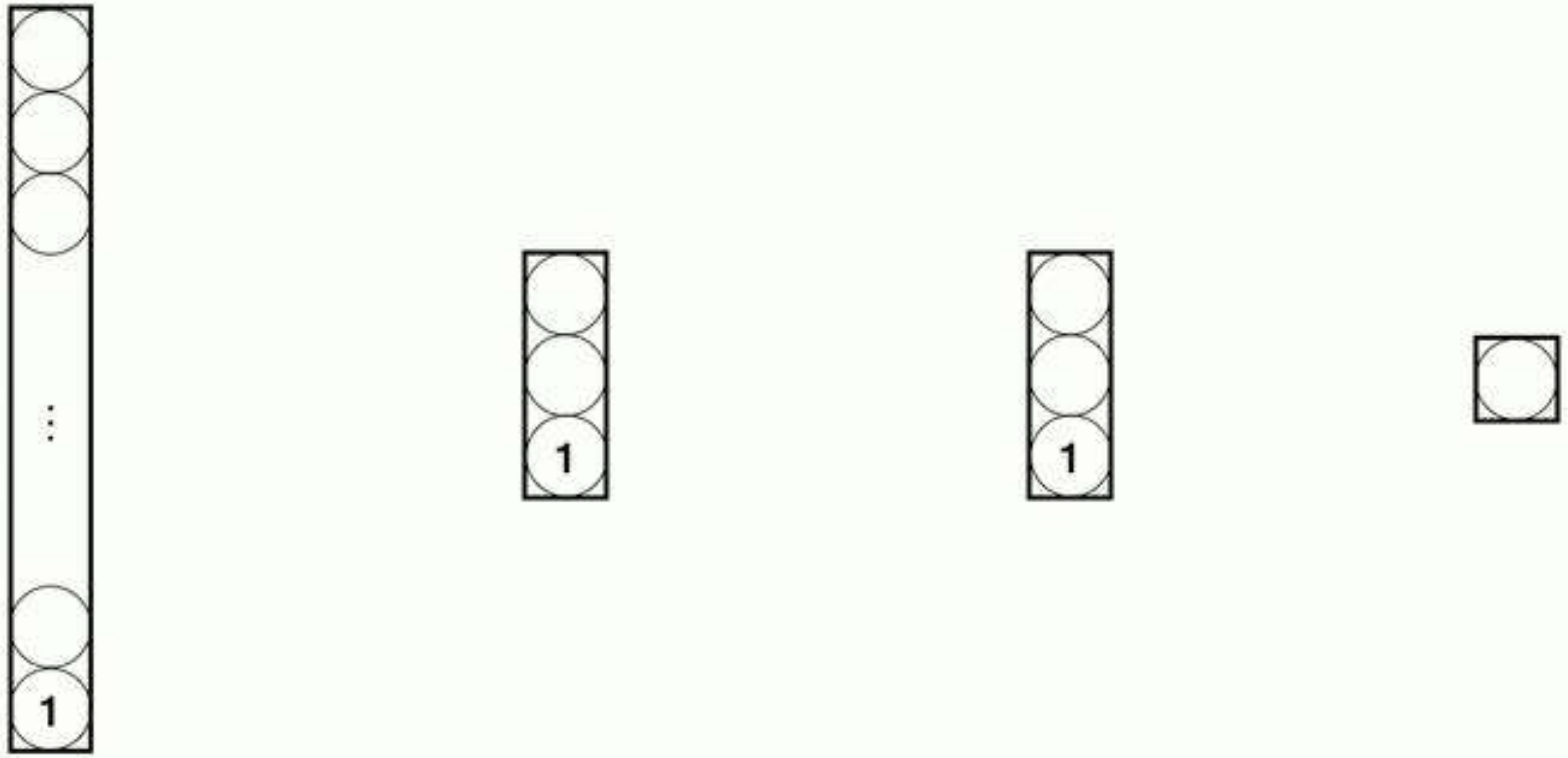

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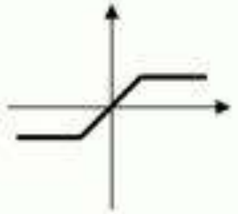

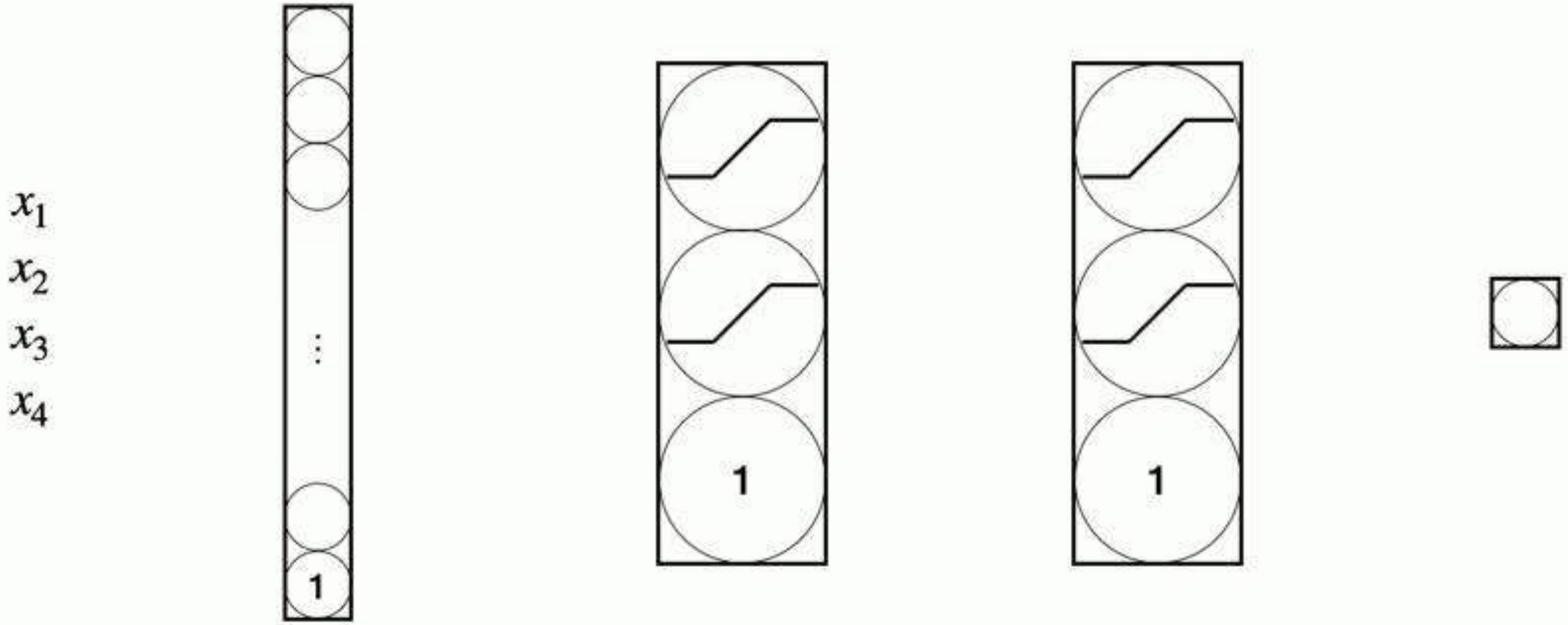

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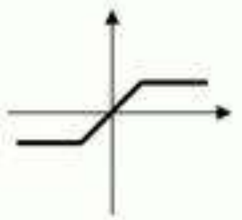

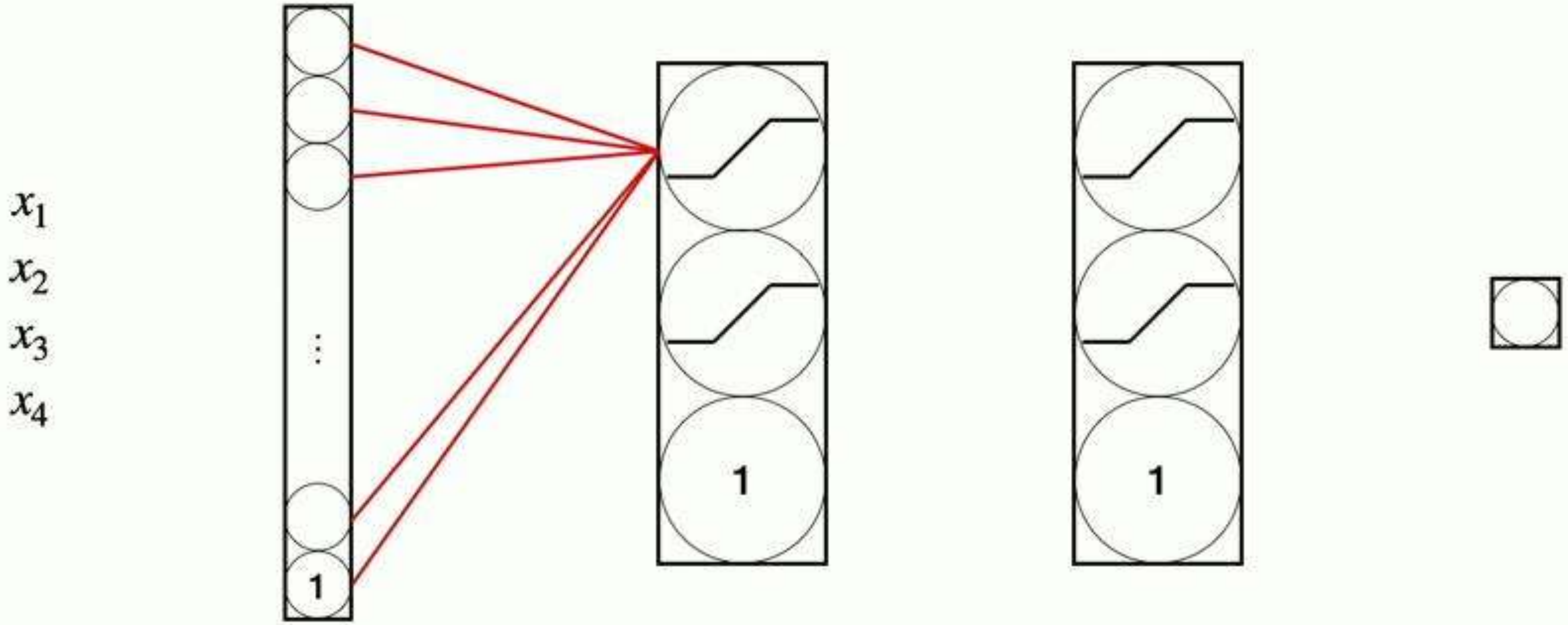

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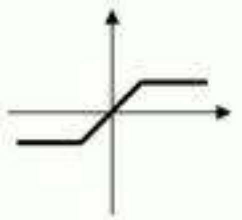

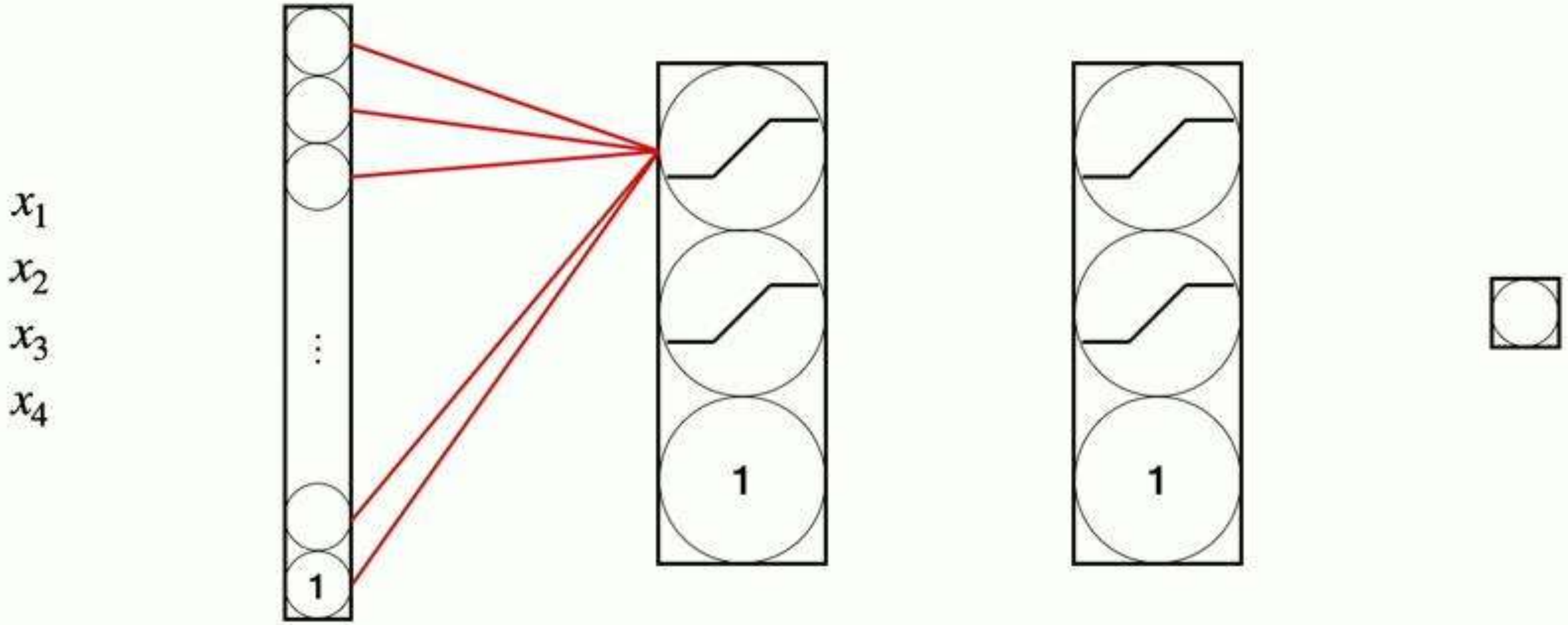

$u^T x_1 < u^T x_2 < u^T x_3 < u^T x_4$

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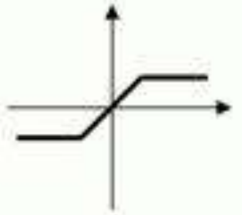

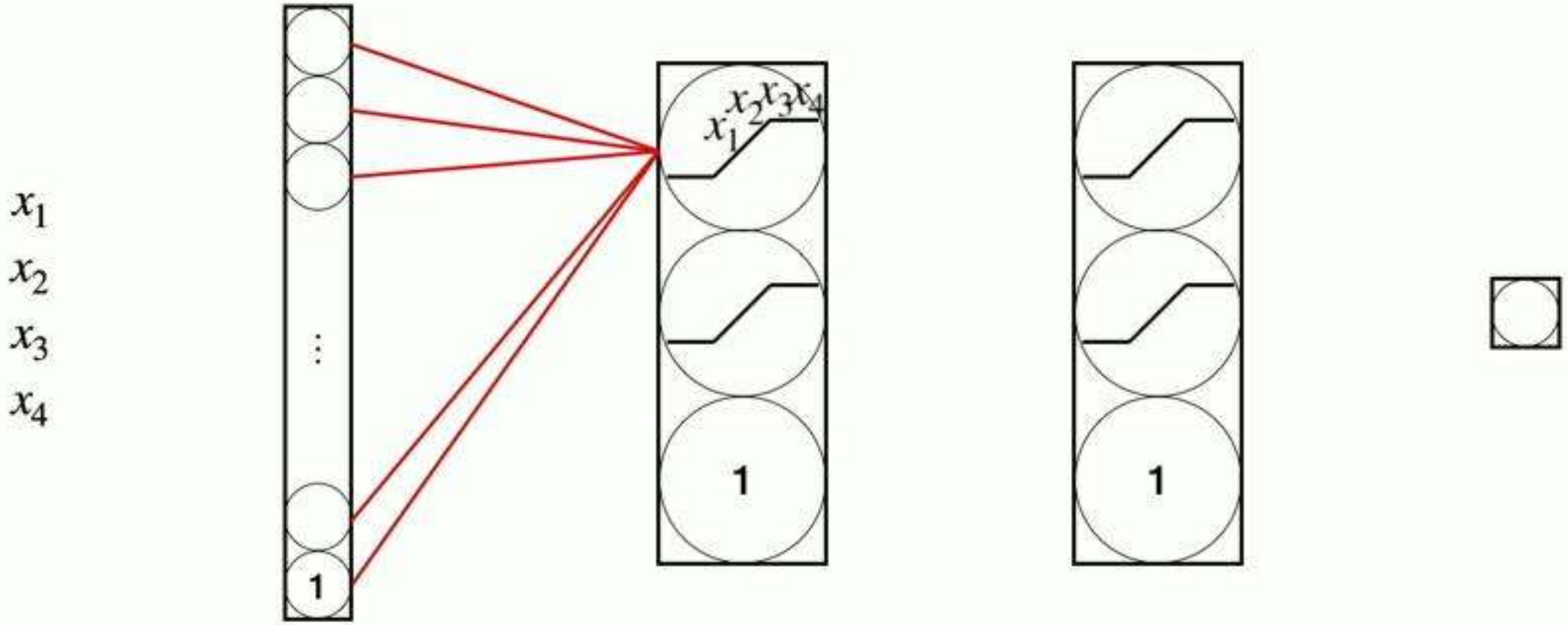

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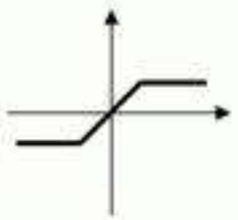
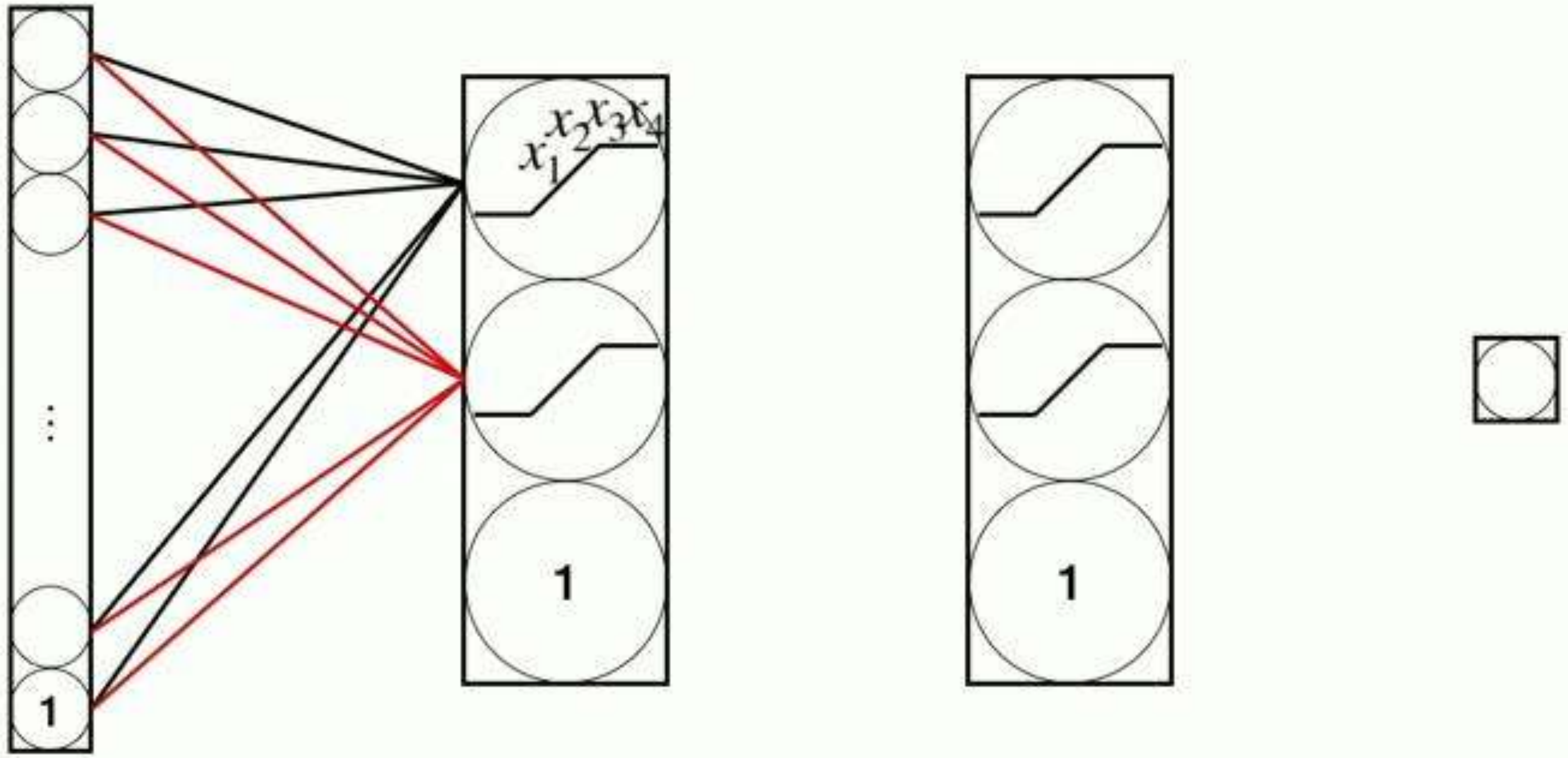
 = hard-tanh, $\sigma_{\mathrm{H}}(t) = \begin{cases} -1 & t < -1 \\ t & t \in [-1,1] \\ 1 & t > 1 \end{cases}$ 

x_1
x_2
x_3
x_4



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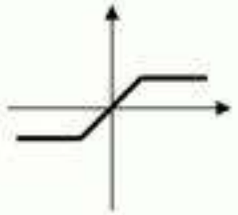

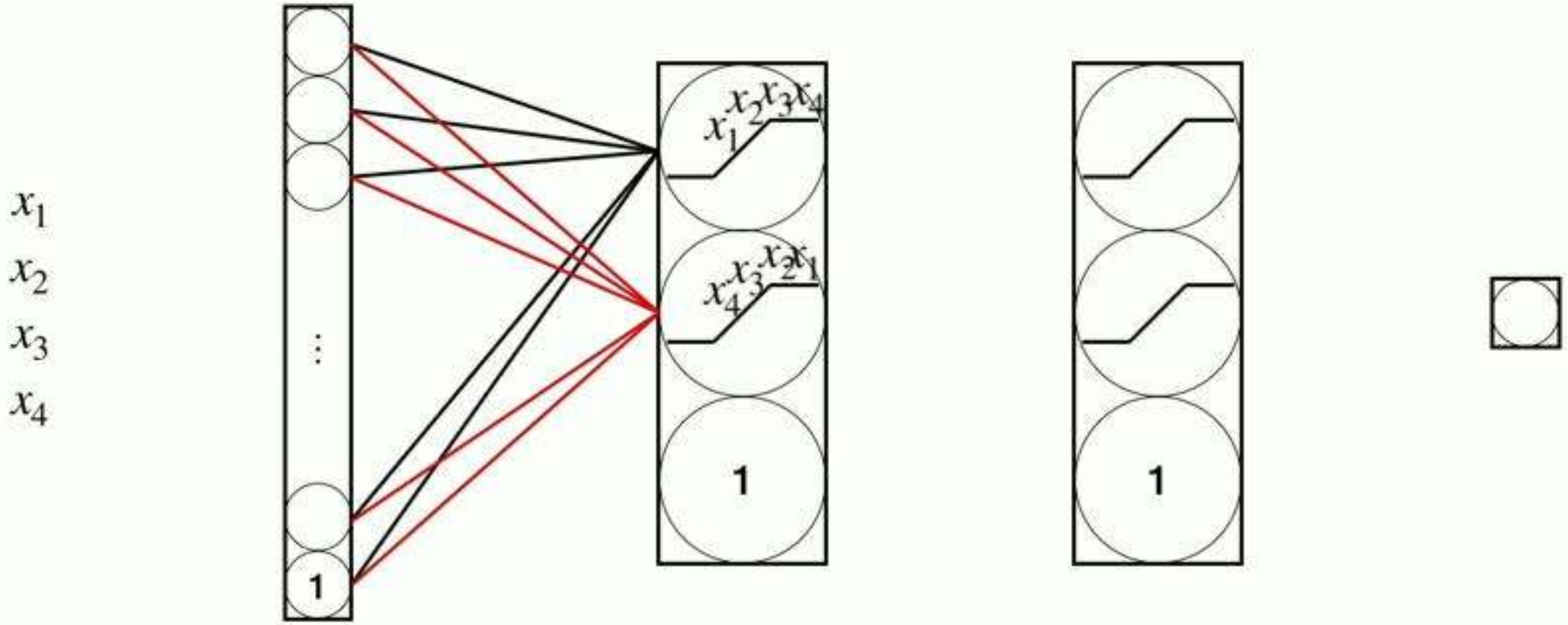

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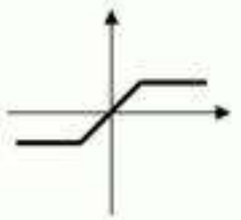

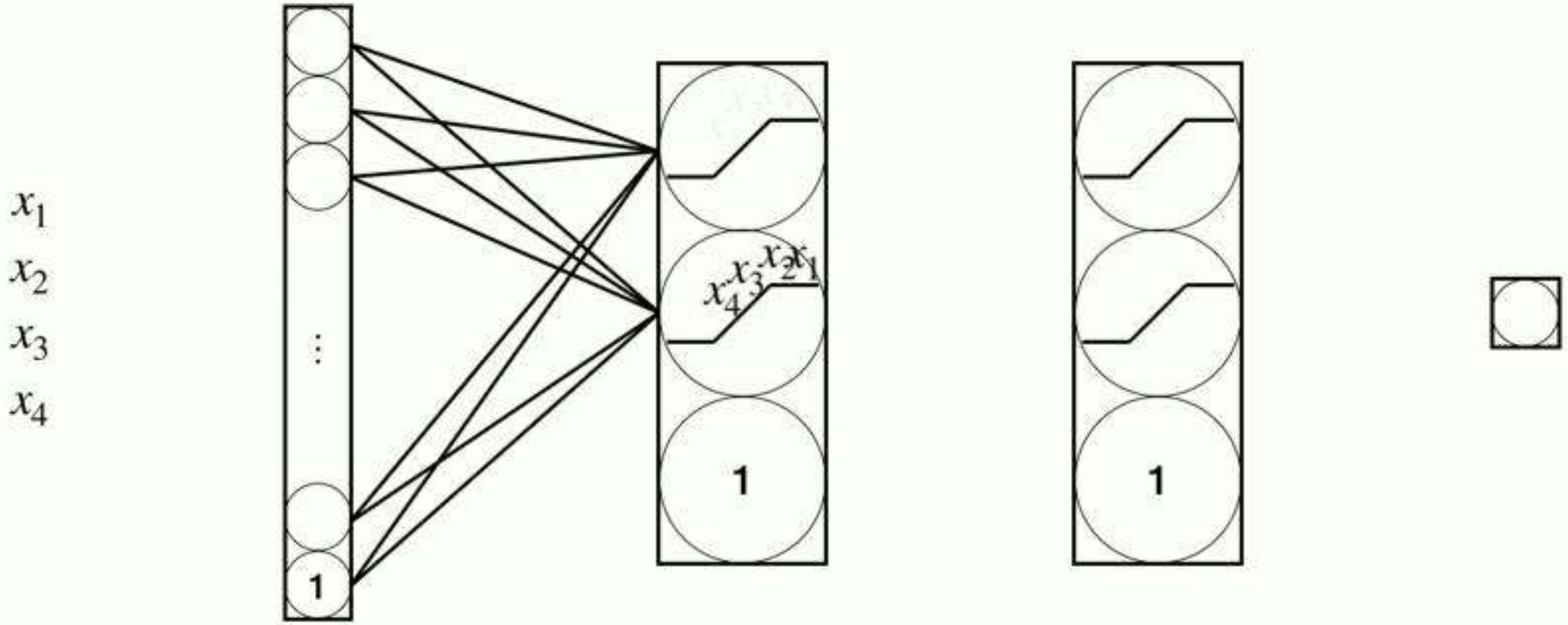

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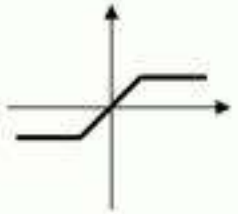

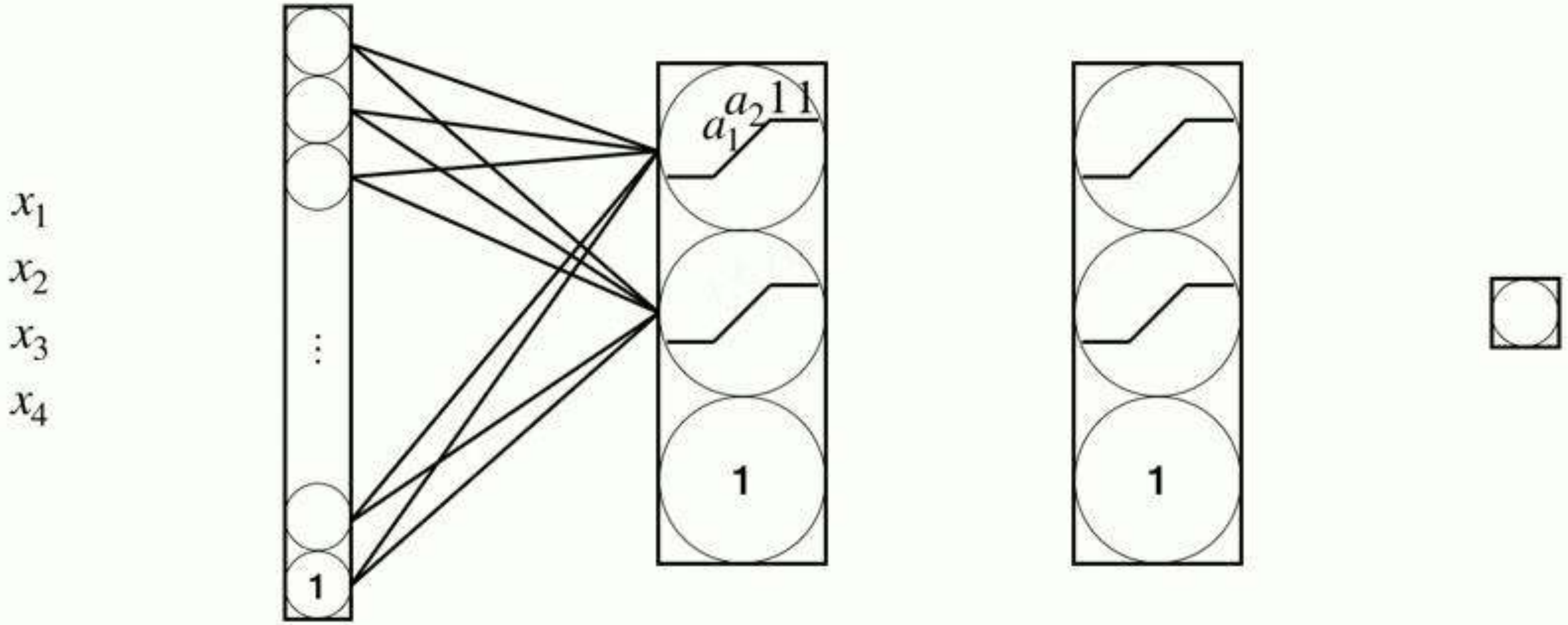

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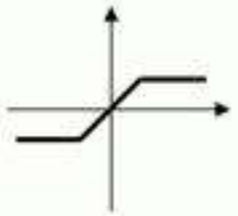

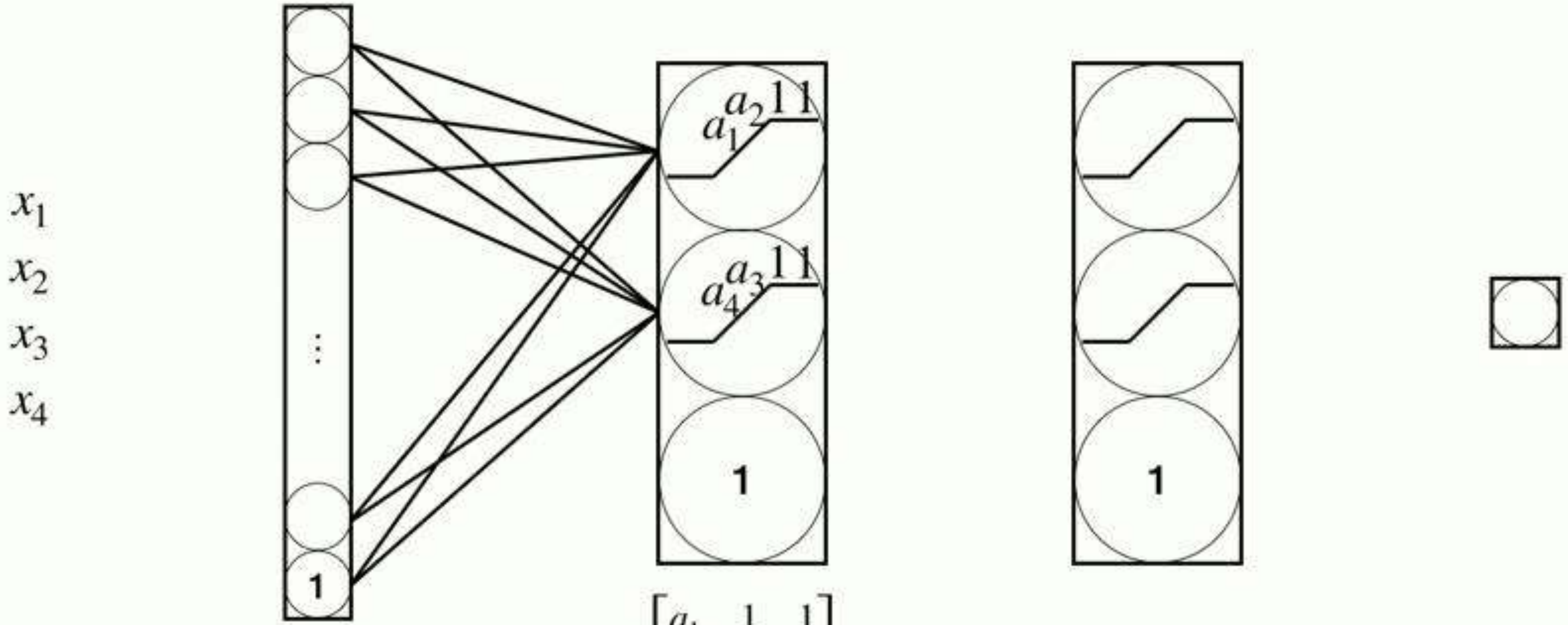

$$\begin{bmatrix} a_1 & 1 & 1 \\ a_2 & 1 & 1 \\ 1 & a_3 & 1 \\ 1 & a_4 & 1 \end{bmatrix}$$

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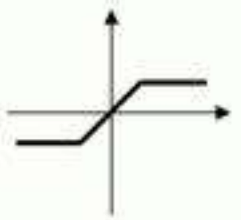

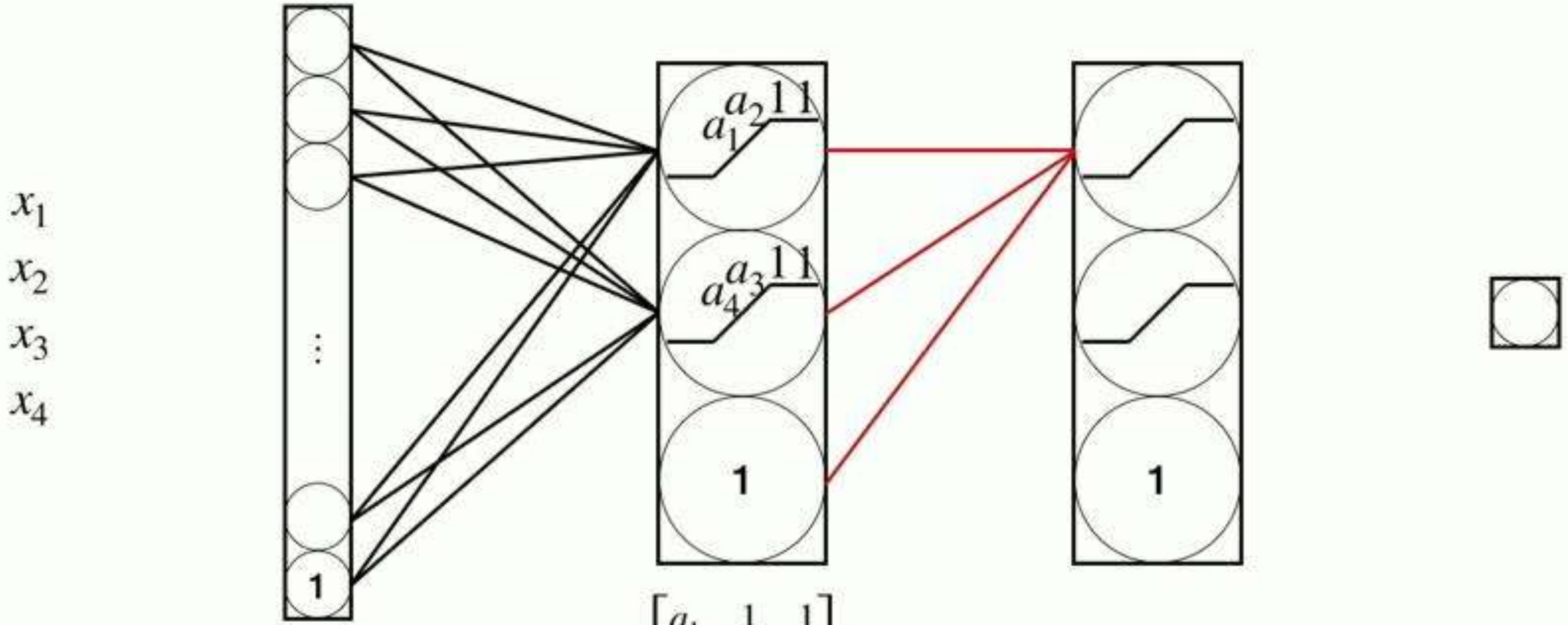

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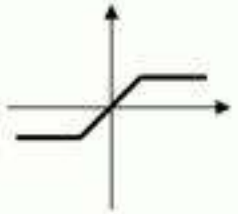
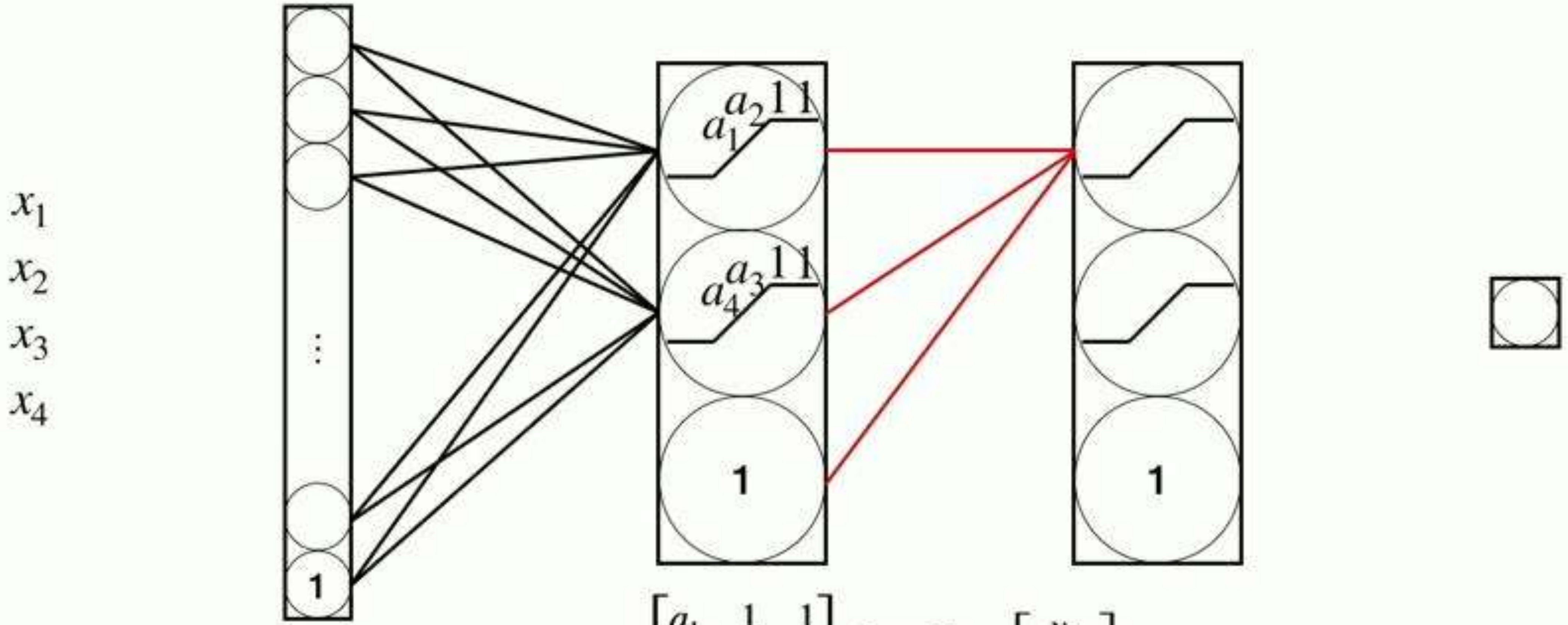
 = hard-tanh, $\sigma_H(t) = \begin{cases} -1 & t < -1 \\ t & t \in [-1,1] \\ 1 & t > 1 \end{cases}$ 



$$\begin{bmatrix} a_1 & 1 & 1 \\ a_2 & 1 & 1 \\ 1 & a_3 & 1 \\ 1 & a_4 & 1 \end{bmatrix} \begin{bmatrix} w_{11} \\ w_{12} \\ b_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ > 1 \\ > 1 \\ y_4 \end{bmatrix}$$

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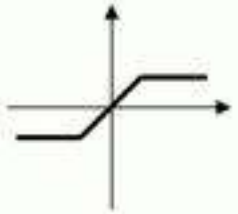

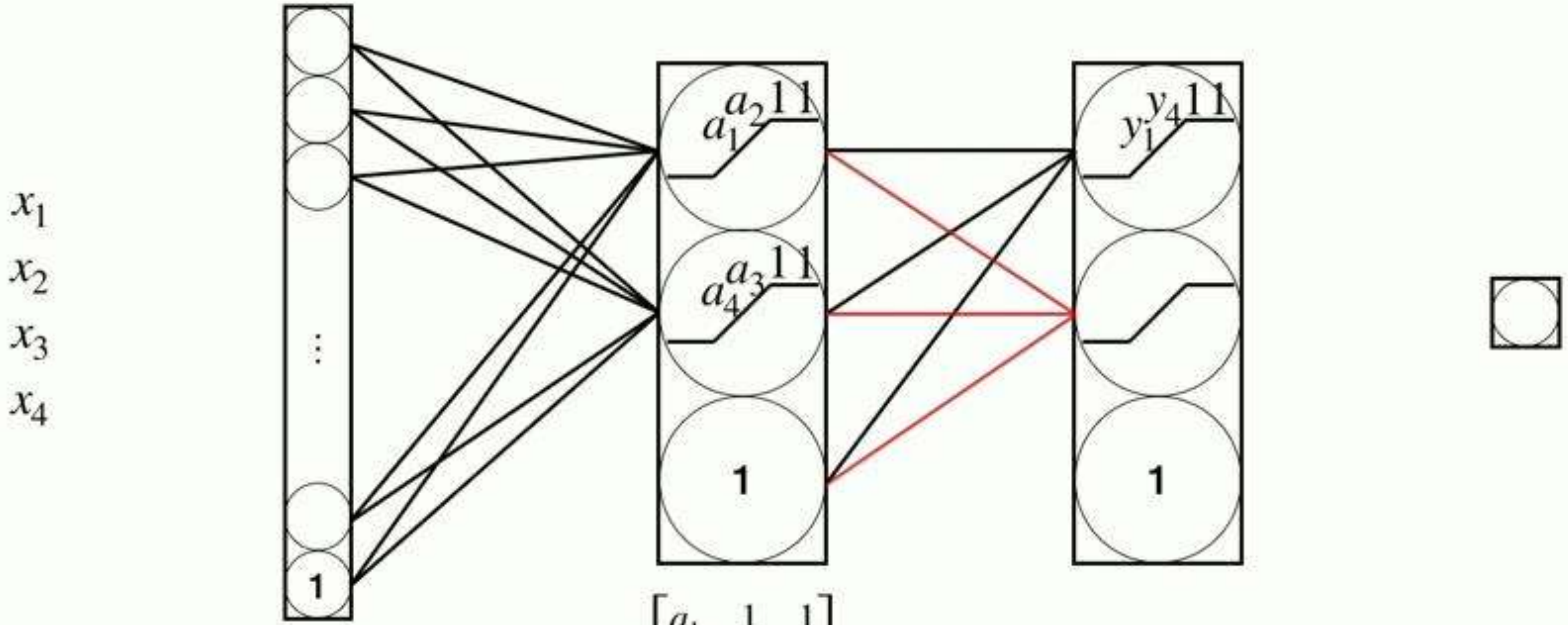

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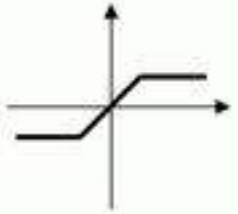

x_1
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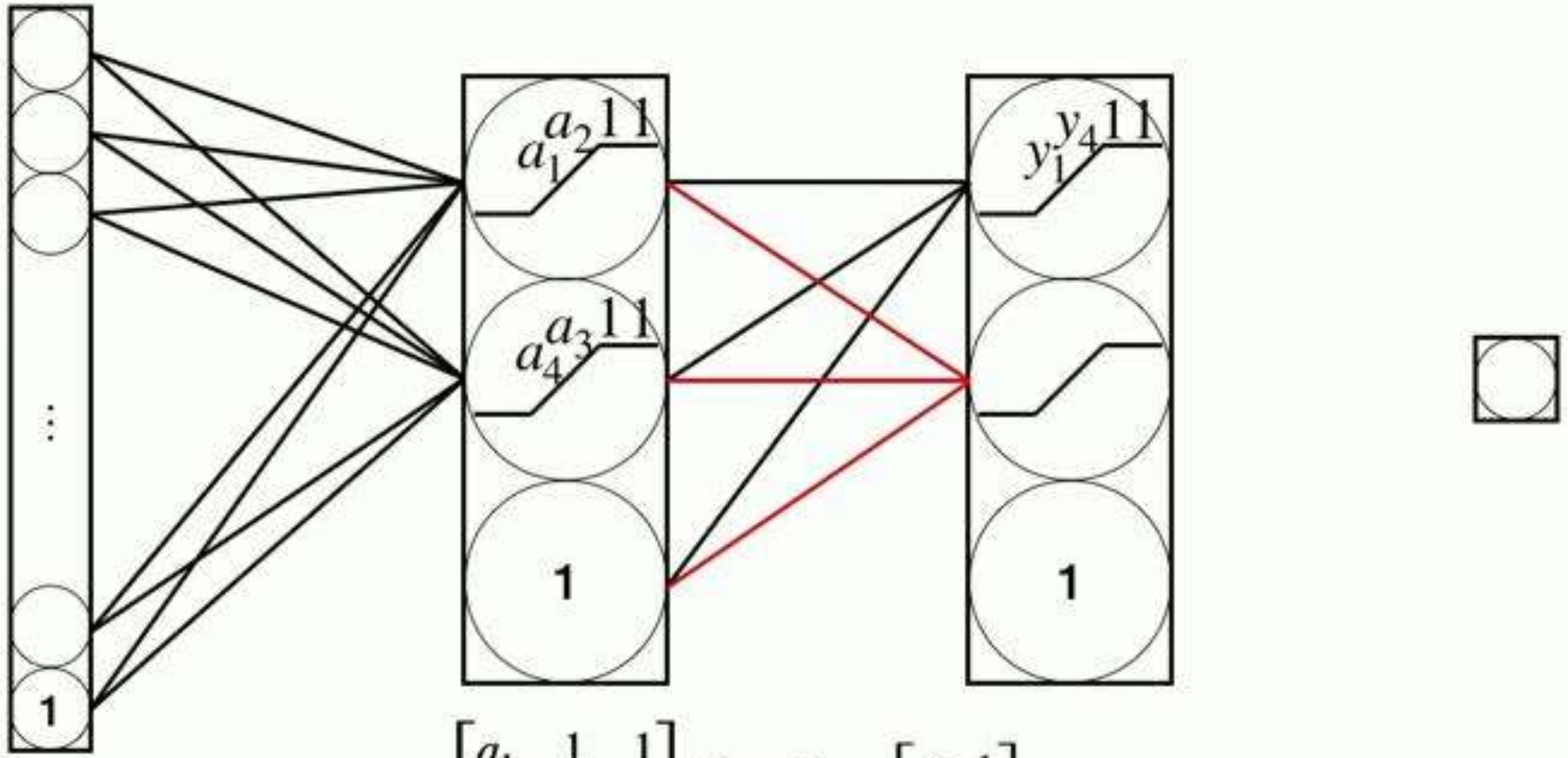

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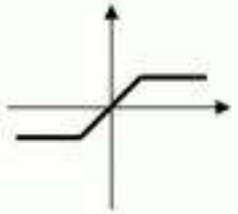

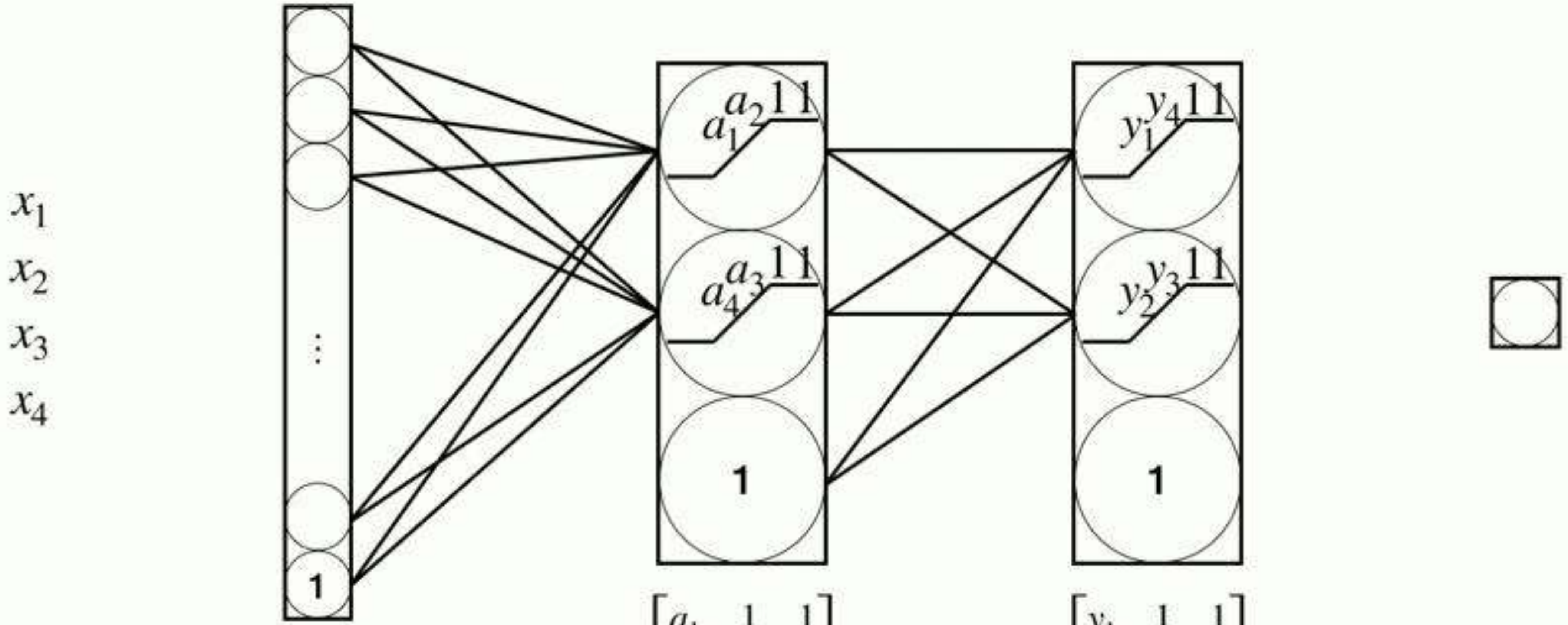

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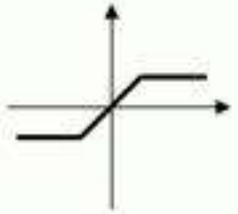

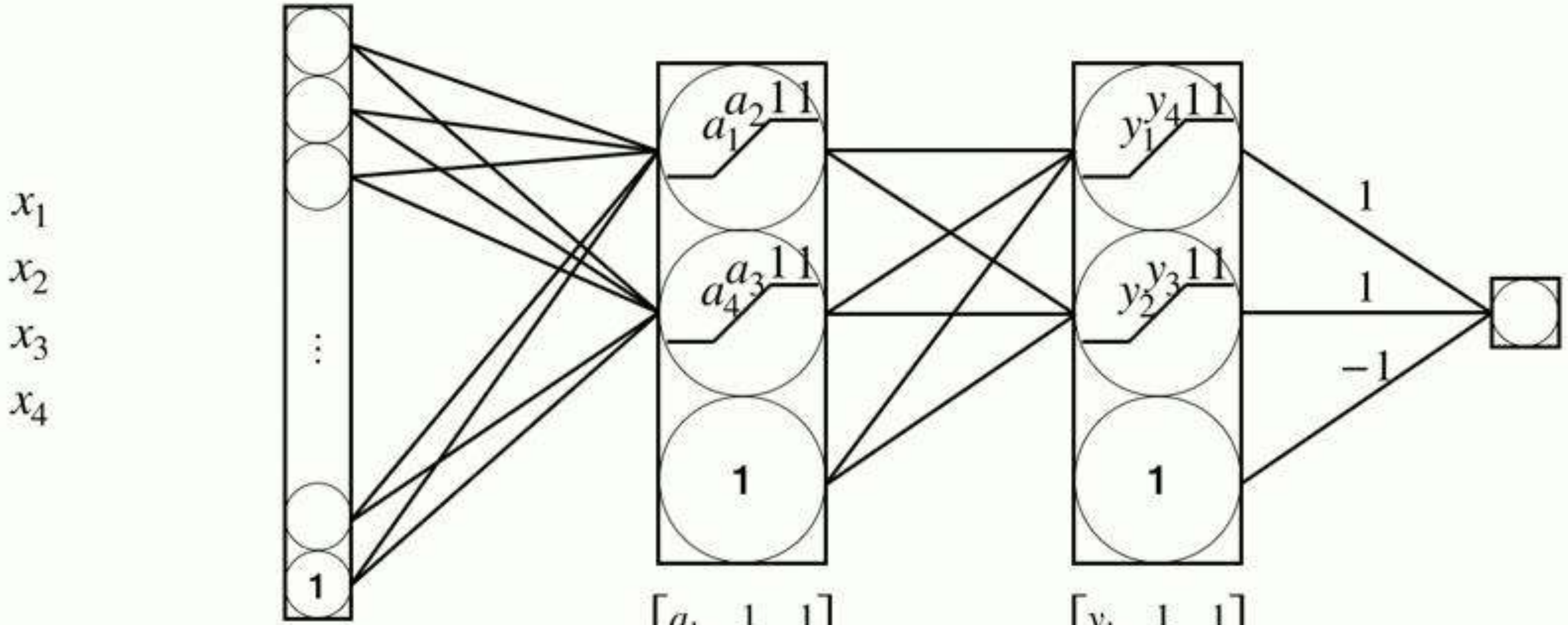

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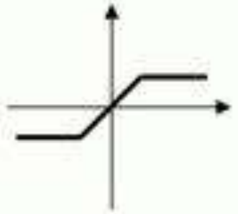

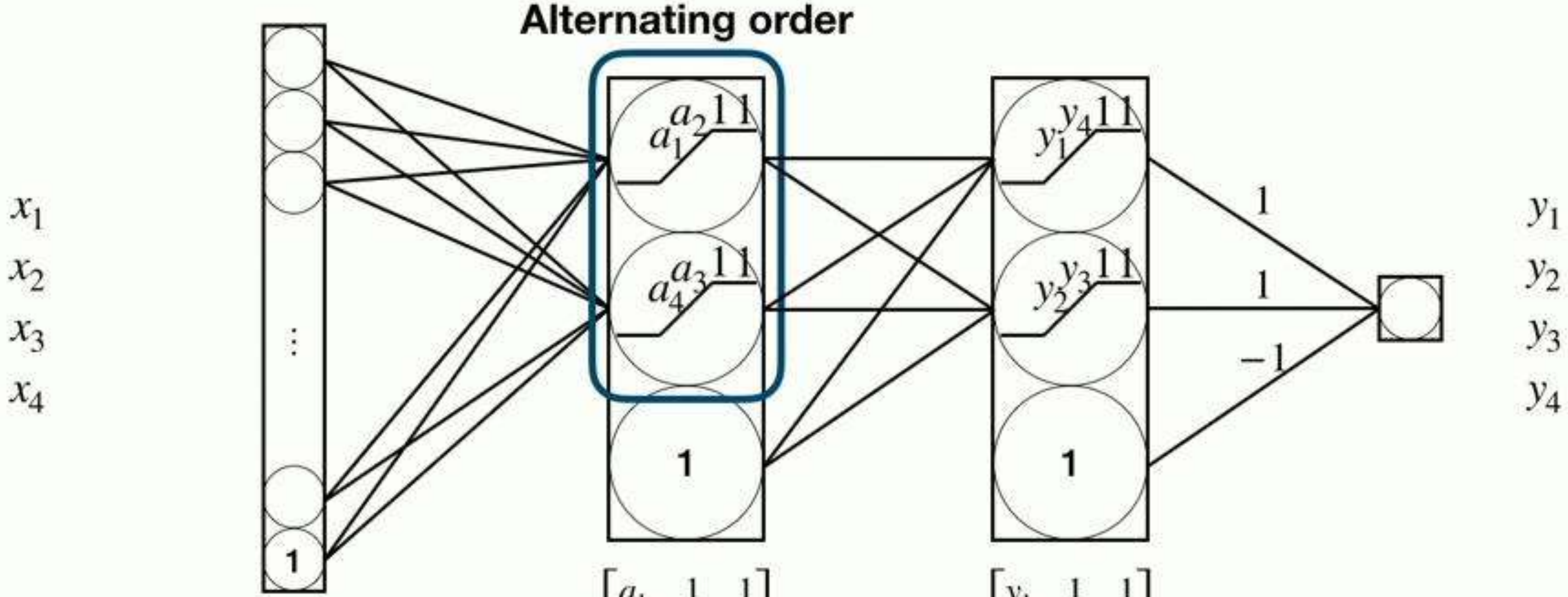

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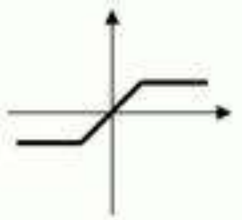

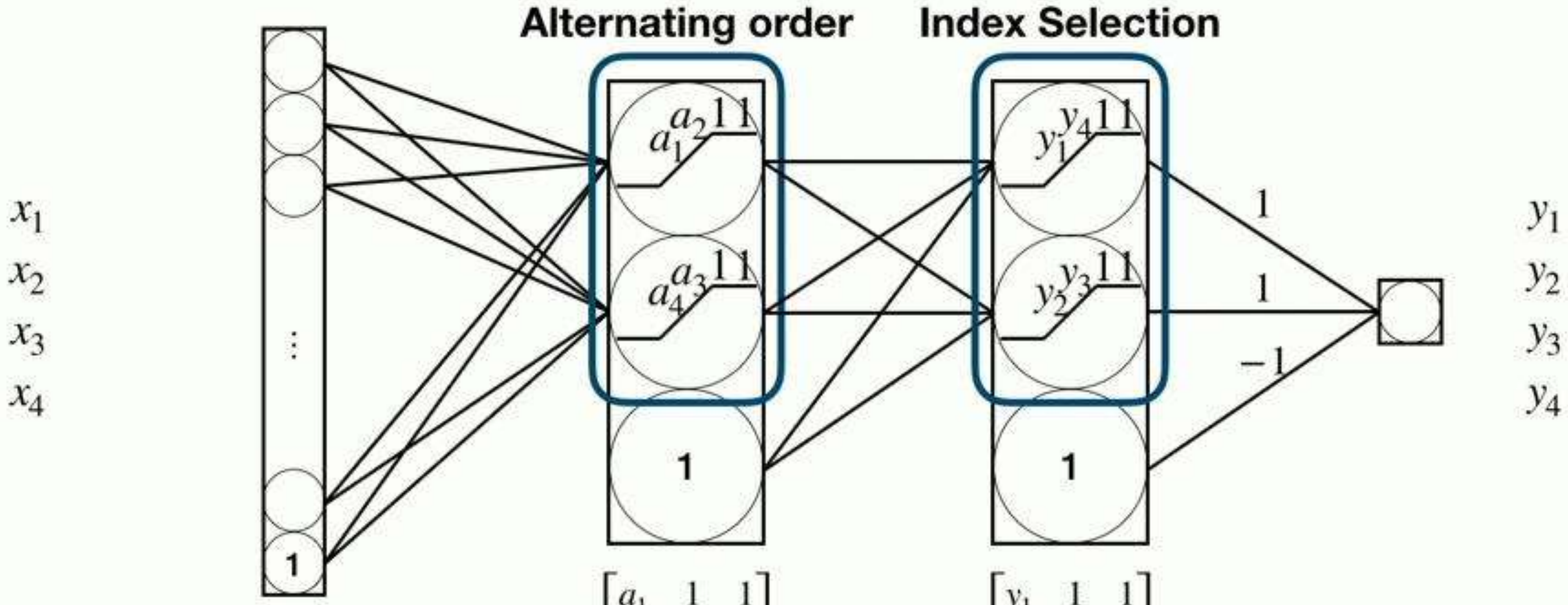

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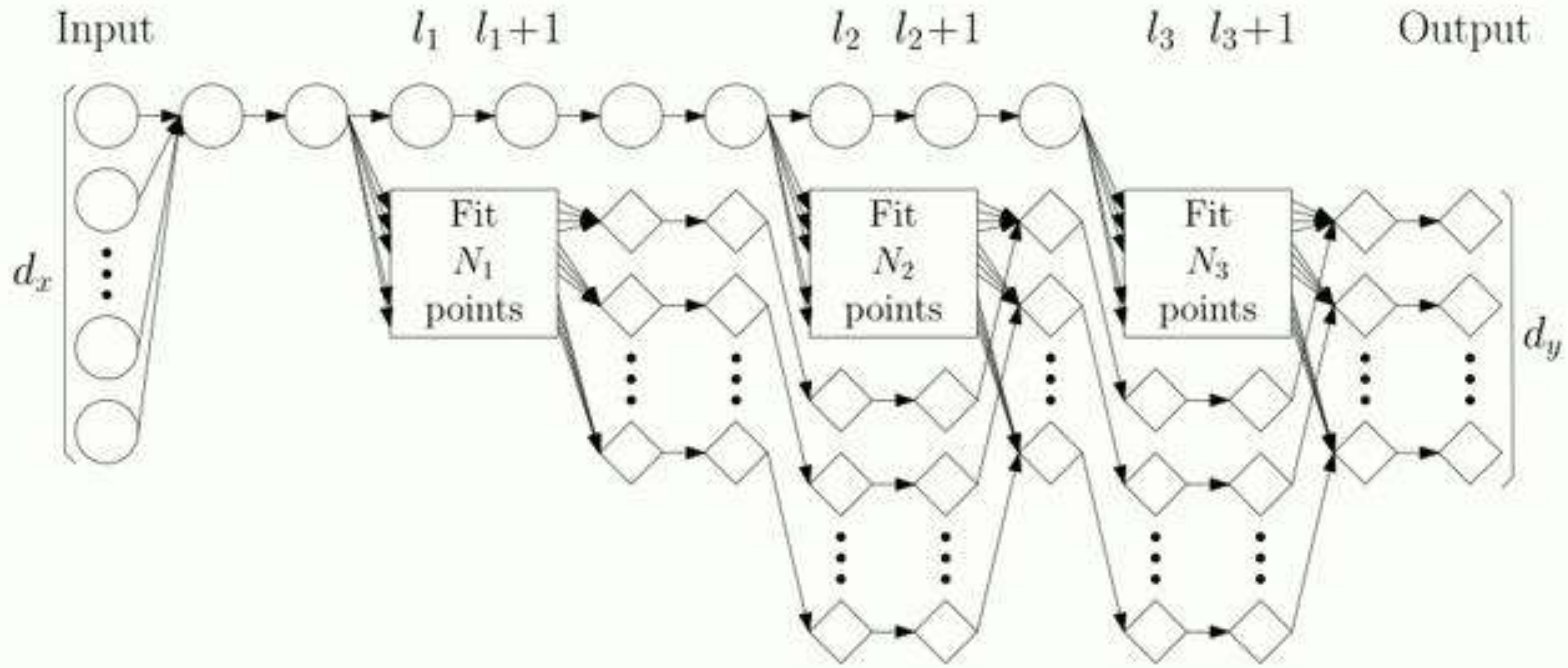
Extension to deeper networks

- Extension to deeper networks possible:
  if there are $\Omega(Np)$ parameters between hidden layers, the network can memorize N points.



(schematic above shows only nonzero weights; it's still an FCNN)

Lower and upper bounds

- If $p = 1$, $\Theta(N)$ parameters suffice to memorize N points $\Longrightarrow$ lower bound $\Omega(W)$ on memorization capacity

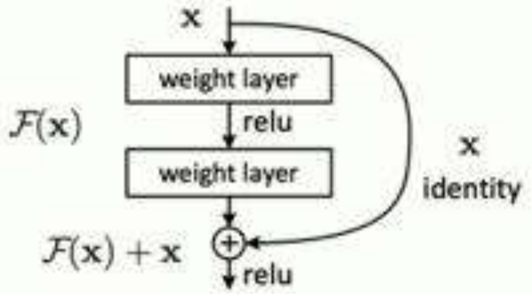
- Upper bound on VC-dim $O(WL \log W)$ *[Bartlett et al., 2019]* gives almost-tight upper bound on memorization capacity (assuming L is a constant).

Finite sample expressivity of ResNets

- **Assumption**: data points x_i's are in general position, i.e., no $d+1$ data points lie on the same affine hyperplane.

- **Assumption**: $y_i \in \{0,1\}^p$ is a one-hot encoding.

- Residual network (ResNet)



$$
\begin{aligned}
h^0(x) &= x, \\
h^l(x) &= h^{l-1}(x) + V^l \sigma(U^l h^{l-1}(x) + b^l) + c^l, \quad l \in \{1, \ldots, L-1\} \\
g_{\boldsymbol{\theta}}(x) &= V^L \sigma(U^L h^{L-1}(x) + b^L) + c^L
\end{aligned}
$$

- d_l is the number of hidden nodes in l-th residual layer

Sufficiency result for ResNets

Theorem 4.

ResNets with hidden layer dims $\sum_{l=1}^{L-1} d_l \geq \frac{4N}{d} + 4p$ and $d_L \geq 2p$ can memorize arbitrary N point classification datasets.

- Under a different assumption, we improve the requirement $N + p$ of *[Hardt & Ma, 2017]* to $\frac{4N}{d} + 6p$.
- For CIFAR-10 ($N = 50$k, $d = 3{,}072$, $p = 10$): 50,010 nodes vs 126 nodes

SGD near memorizers

- We want to solve the empirical risk minimization problem:

$$\text{minimize}_{\boldsymbol{\theta}} \quad \mathfrak{R}(\boldsymbol{\theta}) := \frac{1}{N}\sum_{i=1}^{N} \ell(f_{\boldsymbol{\theta}}(x_i); y_i)$$

- **Assumption.** The loss $\ell(z; y)$ is strictly convex and three times differentiable in z. For any y, there exists a global minimizer z of $\ell(z; y)$.

- **Defn.** A point $\boldsymbol{\theta}^*$ is a **memorizing global minimum** of $\mathfrak{R}(\boldsymbol{\theta})$ if $\nabla_z \ell(f_{\boldsymbol{\theta}^*}(x_i); y_i) = 0$ for all $1 \le i \le N$.

SGD near memorizers

- We analyze without-replacement SGD; mini-batch size B.

- At every $E = N/B$ steps, dataset reshuffled and partitioned into $B^{(kE)}, B^{(kE+1)}, \ldots, B^{(kE+E-1)}$

- SGD update

$$\boldsymbol{\theta}^{(t+1)} \leftarrow \boldsymbol{\theta}^{(t)} - \frac{\eta}{B} \sum_{i \in B^{(t)}} \nabla_{\boldsymbol{\theta}} \ell(f_{\boldsymbol{\theta}^{(t)}}(x_i); y_i)$$

SGD near memorizers

Theorem 5 (informal).

If the initialization $\boldsymbol{\theta}^{(0)}$ satisfies $\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^*\| \leq \rho$ for some memorizing global minimum $\boldsymbol{\theta}^*$ and small constant ρ, initialization satisfies $\mathfrak{R}(\boldsymbol{\theta}^{(0)}) - \mathfrak{R}(\boldsymbol{\theta}^*) \leq C\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^*\|^2$.
If we run SGD with small enough η, it finds a point $\boldsymbol{\theta}$ that satisfies

$$\mathfrak{R}(\boldsymbol{\theta}) - \mathfrak{R}(\boldsymbol{\theta}^*) \leq C'\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^*\|^4, \text{ and}$$
$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\| \leq 2\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^*\|.$$

SGD near memorizers

- Theorem restricted to initialization very close to memorizing global minima
- However, holds **without** any *width/depth requirement* on the network or *distributional assumption* on data— the only requirement: $\boldsymbol{\theta}^*$ **memorizes** the data.
- Completely **deterministic**, independent of the partition of dataset taken by SGD
- The behavior of SGD after finding $\boldsymbol{\theta}$ is not well understood

Theorem 5 (informal).

If the initialization $\boldsymbol{\theta}^{(0)}$ satisfies $\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^*\| \le \rho$ for some memorizing global minimum $\boldsymbol{\theta}^*$ and small constant ρ, initialization satisfies $\Re(\boldsymbol{\theta}^{(0)}) - \Re(\boldsymbol{\theta}^*) \le C\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^*\|^2$. If we run SGD with small enough η, it finds a point $\boldsymbol{\theta}$ that satisfies

$$\Re(\boldsymbol{\theta}) - \Re(\boldsymbol{\theta}^*) \le C'\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^*\|^4, \quad \text{and } \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\| \le 2\|\boldsymbol{\theta}^{(0)} - \boldsymbol{\theta}^*\|.$$

References and other works

Small ReLU networks are powerful memorizers: a tight analysis of memorization capacity *(Yun, S., Jadbabaie, https://arxiv.org/abs/1810.07770)*

Are Deep-ResNets provably better than linear predictors? *(Yun, S., Jadbabaie, https://arxiv.org/abs/1907.03922)*

Why is gradient-clipping faster: an analysis of adaptive-gradient methods under a new smoothness condition weaker than usual Lipschitz gradients *(Zhang, S., Jadbabaie, https://arxiv.org/abs/1905.11881)*

T H A N K S!