The future of mathematics?

Kevin Buzzard

Imperial College London

MSR, 5th September 2019.
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• Clear: tools such as Lean will one day help us mathematicians search for theorems in the literature, and help us to prove theorems.
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Possible: tools such as Lean will begin to do research semi-autonomously, perhaps uncover problems in the literature.
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In April, Christian Szegedy from Google told me that he believes that computers will be beating humans at math within ten years.
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- The proof of Fermat's Last Theorem is long, and structurally extremely complex. The advent of the internet means that proofs are getting longer.
- Nervousness about the state of the mathematical literature was one reason I started to experiment with computer theorem provers.
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• So my personal main goal at this point is to bring other mathematicians into the area, so things begin to happen more quickly.
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- October 2019 – it’s going to be interesting.
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- Sylow’s theorems,
- the fundamental theorem of algebra,
- matrices and bilinear maps,
- the theory of localisation of rings,
- the sine, cosine and exponential functions,
- tensor products of modules,
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Sian Carey, Anca Ciobanu, Clara List and Ramon Fernandez Mir have all formalised mathematics in Lean as part of projects.

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In October 2019 all of the homework in my course will be in Lean format, and all of the course notes too. Like this.
\( (\exists \varphi : X/\sim \to Y, \forall x \in X, \varphi(\pi(x)) = f(x)) \Leftrightarrow (\forall x, x' \in X, x \sim x' \Rightarrow f(x) = f(x')). \)

**Theorem** passage au quotient \((X Y : Type) (s : \text{setoid } X) (f : X \to Y) : \)

\( (\exists \varphi : \text{quotient } s \to Y, \forall x : X, \varphi([x]) = f(x)) \Leftrightarrow (\forall x, x' : X, x \sim x' \Rightarrow f(x) = f(x')). \)

\[\text{Démonstration}\]

Montrons les deux implications.

On commence par supposer la condition de gauche.

Fixons un \( \varphi \) vérifiant cette propriété.

Soit \( x \) et \( x' \) des éléments équivalents de \( X \).

On veut montrer que \( f(x) = f(x') \). Vu la propriété supposée pour \( \varphi \), on peut réécrire le membre de gauche comme \( \varphi(\pi(x)) \) et celui de droite comme \( \varphi(\pi(x)) \).

Drw \( H \varphi \ x \), \( H \varphi \ x' \).

Le point clé est que, puisque \( x \sim x' \), le théorème fondamental de la théorie des quotients assure \( \pi(x) = \pi(x') \).

Drw \( clef : [x] = [x'] \), \{ exact \ quotient.sound \ Hxx' \},

On conclut en reportant cette égalité dans notre objectif, qui devient une tautologie.

Drw clef,

Réciproquement, supposons la condition de droite et construisons une fonction \( \varphi \) convenable.

Intro hyp,

Le théorème fondamental assure que \( \pi \) est surjectif.

Drw surj \( : \forall q, \exists x : X, [x] = q \), \{ apply quotient.exists_rep \},

L'axiome du choix donne alors une fonction \( \sigma : X/\sim \to X \) qui est un inverse à droite de \( \pi \).

Choose \( a H \sigma \) using surj,

Montrons que la fonction qui envoie \( q \) sur \( f(\sigma(q)) \) convient.

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Example of what I have learnt myself from using Lean:
First part of first question on first problem sheet of my course:
“True or false – if $x$ is a real number, and $x^2 - 3x + 2 = 0$, then $x = 1$.”
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$2^2 - 3 \times 2 + 2 = 0$ and that (b) $2 \neq 1$.
Me in 2017: “…”
A few weeks later, this was fixed by computer scientists, who
wrote a tactic which solved these goals.
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<table>
<thead>
<tr>
<th>Got author a Fields Medal?</th>
<th>Proof of odd order theorem</th>
<th>Perfectoid spaces</th>
</tr>
</thead>
<tbody>
<tr>
<td>High level mathematics?</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Lots of PhD students and post-docs working in the area?</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Talks happening about these things all over the world?</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Mathematicians interested in 2019?</td>
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A real manifold is a topological space which locally looks like a ball. For this to typecheck we need to know that a ball is a topological space. This is not difficult.

A perfectoid space is a locally ringed space which locally looks like an affinoid perfectoid space. For this to typecheck we need to show that affinoid perfectoid spaces are locally ringed spaces (or actually something slightly weaker).
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We mathematicians don’t see the modern complex mathematical objects which we use every day, in theorem provers. Yet. I just wrote some EU grant proposal to fund post-docs who will write a bunch of Lean code defining the objects which “make a mathematician tick”. And then (following Tom Hales) we can start to make a database, or a network, mapping out the state of the beliefs of the elders.
Conclusions:

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Thanks for coming!