A Graph-Theoretical Basis of Stochastic-Cascading Network Influence: Characterizations of Influence-Based Centrality

Wei Chen
Microsoft Research
weic@microsoft.com

Shang-Hua Teng*
University of Southern California
shanghua@usc.edu

Hanrui Zhang†
Duke University
hrzhang@cs.duke.edu

Abstract

Ker-I Ko (葛可一, 1950-2018) made multiple pioneering and milestone contributions to structural complexity theory and real-number computation. His sharp perception of both discrete and continuous structures had lead him to study fundamental concepts from one-way functions [29], to polynomial-time hierarchy [21], from integral equations [24], to real functions [28], from fractals [27] to fixed-points [25], from optimization [20] to relativization [22], from instance complexity [36] to real complexity [28], from NP-completeness [31] to exponential-time completeness [23]. In his final two years, he turned his focus to game-theoretical designs for dynamic processes over social networks [12, 33].

For this *honorary special issue in memory of Professor Ker-I Ko*, we are pleased to be able to contribute a paper with scope intersecting the areas that drew his last attention, and with results connecting discrete and continuous formulations of network dynamics. We prove that the complex stochastic network-influence processes have a simple graph-theoretical basis: *Every stochastic-cascading influence profile can be written as a linear combination of breadth-first-search-based broadcast-propagations over layered-graphs.* This graph-theoretical basis of stochastic network influence provides us with a systematic framework for studying the following fundamental question in network analysis:

> *How should one assess the centralities of nodes in an information/influence propagation process over a social network?*

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Our framework systematically extends a family of classical graph-theoretical centrality formulations — including degree centrality, harmonic centrality, and various notions of "sphere-of-influence" — to influence-based network centralities. Given that group cooperation is essential in social influences, we further extend natural group centralities from graph models to influence models, enabling us to assess individuals' centralities in group influence settings by applying the fundamental concept of Shapley value from cooperative game theory. Mathematically, using the property that these centrality formulations are Bayesian,[1] we prove an *axiomatic characterization* theorem: Every influence-based centrality formulation in this family is the *unique Bayesian centrality* that conforms with its corresponding graph-theoretical centrality. Moreover, the uniqueness is fully determined by the centrality formulation on the class of layered graphs, which is derived from a beautiful algebraic structure of influence instances modeled by cascading sequences. We further provide an algorithmic framework for efficient approximation of these influence-based centrality measures.

Our study provides us with a systematic road map for comparative analyses of different influence-based centrality formulations. The fact that layered graphs form a basis for the space of influence-cascading-sequence profiles could also be useful in other studies of network influences.

Keywords: Network centrality; influence-based centrality; axiomatic characterization; graph-theoretical basis.

1 In Remembrance of Ker-I Ko (葛可一)

A Gentle and Sharp Mind: Ker-I Ko was a quiet scholar with deep insight, who contributed profoundly to the theory of computing. Through the complexity-theoretical lens, he discovered many elegant mathematical structures:

- Some are *grand*, conceptually connecting computation of real functions with optimization of discrete objectives [28, 20] , while others are *intricate*, meticulously separating levels of polynomial-time hierarchy [21].
- Some are *abstract*, differentiating Kolmogorov complexity from instance complexity [36] and connecting one-way functions with the Isomorphism Conjecture [29], while others are *concrete*, exploring paths in two-dimensional domains [14] and learning string/tree patterns from examples [30].
- Some are *theoretical*, highlighting the "paradox" of polynomial-time computable curves with

[1] That is, they are linear to the convex combination of influence instances.

nonrecursive interior measure, [26] while other are *practical*, addressing Pooling design (in DNA testing) [13] and fault-tolerant computing (in wireless networks) [45].

Ker-I Ko's journey — at times solo — covered a vast areas of TCS, and left multiple landmarks — often at complex corners and crossroads – that helped to transform the landscape.

One Author's First and Last Encounters with Ker-I Ko: In the early summmer of 1986, Professor Gary Miller — my advisor at USC — invited me to join him to drive to Berkeley for *the 18th Annual ACM Symposium on Theory of Computing* (STOC). At the time — in the first decade after the joint communiqué on the establishment of diplomatic relations between the United States and China — there were a very small number of mainland Chinese students/scholars studying (theoretical) computer science in the United States. Gary quickly introduced me to Yanjun Zhang, a UC Berkeley Ph.D. candidate studying parallel combinatorial search under the guidance of Professor Dick Karp, and mentioned to him that I was a first-year student attending my first conference. Before leaving me with Yanjun, Gary whispered to me, "To start a conversation when meeting new people, just ask 'what are you working on?'. Don't be afraid if you don't understand anything they say." Within an hour, Yanjun introduced me to ALL other "mainland Chinese" theoretical computer scientists at the conference: Jin-Yi Cai (Cornell), Ding-Zhu Du (UC. Berkeley), and Ming Li (Ohio State).

Ding-Zhu Du took me to his office at Berkeley's Mathematical Sciences Research Institute and introduced me to another Chinese-speaking theoretician Professor Ker-I Ko (葛可一). He was a coauthor of Ding-Zhu's STOC paper, "A note on one-way functions and polynomial-time isomorphisms." Ker-I was extraordinarily generous when I asked him, "what are you working on?" Over the course of the conference, he explained to me not just about his result on the Berman-Hartmanis Isomorphism Conjecture but also his work on computational complexity of real functions, on circuit complexity and polynomial-time hierarchy, and on query complexity for identifying permutations. These subjects were way beyond my scope of comprehension, and it took years for me to acquire enough knowledge to understand the meaning of his theorems and appreciate the significance of his results. But Ker-I's patience with a beginner, ability to paint big pictures, voice of encouragement, and sincerity towards a newcomer of the field had left a long-lasting mark in my mind. After STOC'86, whenever I ran into his papers while reading journals and conference proceedings in libraries[2], I would give a genuine read without concerning whether they were directly connected with my immediate research projects; whenever I saw him in the conferences, I would ask him , "what are you working on?" not to start a conversation, but to enjoy informative scientific

[2]During the pre-Web time, I spent most of my days in libraries reading TCS papers from journals, conference proceedings, technical reports, and books.

exchanges. He always answered that magic question in the same manner as he did the first time when we met: generous, patient, and elaborative.

My last extentive interaction with Ker-I was in 2008,[3] when I arrived at Andy Yao's institute at Tsinghua University (for a sabbatical). Ker-I, who was visiting Tsinghua that day, invited me to a dinner, "Let's go to the Taiwanese restaurant just outside the campus", knowing about my past family tie with Taiwan. After we sat down and ordered traditional Taiwanese food, Ker-I pointed to the restaurant name, "8°1", and said with a jovial tone, "If you can figure out what this name stands for, the dinner will be on me." Seeing that I was still focusing on deciphering "8°1", he assured me with a smile, "The dinner is on me anyway." He was thrilled when I finally decoded "8°1" as Taiwanese (Hokkien) "belly hungry (八豆吆/巴豆吆)." Geekily, he exclaimed in delight, "8°1 is a pretty-good pseudo-random sequence that Andy and I didn't figure out its meaning." I was glad that the result of my familiarity with the Beijing dialect (hence knowing that the number "1" in Chinese could be pronounced as both as "Yī" and "Yāo") and my sparse knowledge of daily-life Taiwanese dialect drew such an affirmative response from the author of, "On the notion of infinite pseudorandom sequences." In fact, moment before learning (on December 15, 2018) from the ISAAC 2019 PC chair, Pinyan Lu, about the news of Ker-I's passing, I shared this very story with the ISAAC 2018 PC Chair, D.T. Lee and his wife, at Yilan Taiwan. I cherish the memory of that evening for learning not just about Ker-I's research, but also about his multifaceted life as a computer scientist, mathematician, artist, and story writer; I cherish the opportunity of knowing him and being impacted by his deep insight into mathematics and into life.

Dear 可一, Rest in Peace! by Shang-Hua Teng

Ker-I Ko's Last Projects: Games and Influence Dynamics on Social Networks: After his retirement from Stony Brook University in 2012, Ker-I became a global scholar, with an active research/educational profile at King Abdulaziz University (Saudi Arabia), National Chiao Tung University (his native Taiwan), Tsinghua University (China), and Ocean University of China. In the process, Ker-I shifted his focus from complexity theory to approxmiation algorithms for graph-theoretical problems, which then led him to his last project, studying game-theoretical and dynamics processes on social networks. Working with Professor Qizhi Fang's research team at the Ocean University of China, he published two papers:

a *Competitive Profit Maximization in Social Networks*

b *Centralized and Decentralized Rumor Blocking Problems*

[3] Afterwards, I only briefly saw him once when I was visiting Taiwan to attend the NCTU Spectral Graph Theory Workshop organized by Salil Vadhan in 2015.

The former [33] analyzed competitive multi-player network games with individual influence maximization objectives. The later [12] studied a two-player decentralized rumor blocking problem, under Kempe-Kleinberg-Tardos' network influence models, in which each player (protector) aims to "restrain the rumor by his own ability and maximize his own personal utility." In both papers, Ker-I and coauthors proved that Vetta's *valid utility system* can be used to characterize these competitive influence games, providing efficiency guarantees on all Nash equilibria of the underlying network games.

The Focus of This Paper: We prove that the complex stochastic network-influence processes have a simple graph-theoretical basis: *Every stochastic-cascading influence profile can be written as a linear combination of breadth-first-search-based broadcast-propagations over layered-graphs.* This graph-theoretical basis of stochastic network influence provides us with a systematic framework for studying the following fundamental question in network analysis:

> *How should one assess the centralities of nodes in an information/influence propagation process over a social network?*

2 Introduction: Network Influence and Axiomatic Characterizations of Influence-based Centrality

Network influence is a fundamental subject in network sciences [38, 15, 18, 4]. It arises from vast real-world backgrounds, ranging from epidemic spreading/control, to viral marketing, to innovation, and to political campaign. It also provides a family of concrete and illustrative examples for studying network phenomena — particularly regarding the interplay between network dynamics and graph structures — which require solution concepts beyond traditional graph theory [7]. As a result, network influence is a captivating subject for theoretical modeling, mathematical characterization, and algorithmic analysis [38, 15, 18, 6].

2.1 Network Influence: Models and Influence Maximization

In contrast to some graph-theoretical processes such as random walks, network influence is defined not solely by the static graph structures. It is fundamentally defined by the interaction between the *dynamic* influence models and the *static* network structures. Stochastic-diffusion influence models describe how information or influence are propagated through a network. A number of models have been well-studied (cf. [18, 6]), and among them independent cascade (IC) and linear threshold (LT) models are most popular ones. Here, we illustrate with the *triggering model*[4] of

[4] Triggering model will also be the subject of our algorithmic study.

Kempe-Kleinberg-Tardos [18], which includes IC and LT models as special cases. In a triggering model, the static network structure is modeled as a directed graph $G = (V, E)$ with $n = |V|$. Each node $v \in V$ has a random *triggering set* $T(v)$ drawn from distribution $D(v)$ over subsets of v's in-neighbors $N^-(v)$. At time $t = 0$, triggering sets of all nodes are sampled from their distributions, and nodes in a given seed set $S \subseteq V$ are activated. At any time $t \geq 1$, a node v is activated if some nodes in its triggering set $T(v)$ was activated at time $t - 1$. The propagation continues until no new nodes are activated in a step. Influence propagation is an important topic in network science and network mining. One well-studied problem on influence propagation is the influence maximization problem [16, 39, 18], which is to find k seed nodes that generate the largest influence in the network. Influence maximization has been extensively studied for improving its efficiency or extending it to various other settings (e.g. [19, 32, 3, 9, 10]).

Even on the same static network, different influence propagation models — such as the popular *independent cascade* and *linear-threshold* models — induce different underlying relationships among nodes in the network. The characterization of this interplay thus requires us to reformulate various fundamental graph-theoretical concepts such as centrality, closeness, distance, neighborhood (sphere-of-influence), and clusterability, as well as to identify new concepts fundamental to emerging network phenomena.

In this paper, we will study the following basic question in network science with focusing on *influence-based network centrality.*

> *Is there a systematic framework to expand graph-theoretical concepts in network sciences?*

2.2 Network Centrality

Network centrality — a basic concept in network analysis — measures the importance and the criticality of nodes or edges within a given network. Naturally, as network applications vary — being Web search, internet routing, or social interactions — centrality formulations should adapt as well. Thus, numerous centrality measures have been proposed, based on degree, closeness, betweenness, and random-walks (e.g., PageRank) (cf. [35]) to capture the significance of nodes on the Web, in the Internet, and within social networks. Most of these centrality measures depend only on the static graph structures of the networks. Thus, these traditional centrality formulations could be inadequate for many real-world applications — including social influence, viral marketing, and epidemics control — in which static structures are only part of the network data that define the dynamic processes. Our research will focus on the following basic questions:

How should we summarize influence data to capture the significance of nodes in the dynamic propagation process defined by an influence model? How should we extend graph-theoretic centralities to the influence-based centralities? What does each centrality formulation capture? How should we comparatively evaluate different centrality formulations?

2.3 An Earlier Axiomatic Characterization of Influence-Based Centrality

At WWW'17, Chen and Teng [7] presented an axiomatic framework for characterizing influence-based network centralities. Their work is motivated by studies in multiple disciplines, including social-choice theory [2], cooperative game theory [41], data mining [17], and particularly by [37] on measures of intellectual influence and [1] on PageRank. They present axiomatic characterizations for two basic centrality measures: (a) *Single Node Influence* (SNI) *centrality*, which measures each node's significance by its influence spread;[5] (b) *Shapley Centrality*, which uses the Shapley value of the influence spread function — formulated based on a fundamental cooperative-game-theoretical concept. Mathematically, the axioms are structured into two categories.

- **Principle Axioms**: The set of axioms that all desirable influenced-based centrality formulations should satisfy. In [7], two principle axioms, Anonymity and Bayesian, are identified. Anonymity is an ubiquitous and exemplary principle axiom, which states that centrality measures should be preserved under isomorphisms among influence instances. Bayesian states that influence-based centrality is a linear measure for mixtures of influence instances.
- **Choice axioms:** A (minimal) set of axioms that together with the principle axioms uniquely determine a given centrality formulation.

Such characterizations and the taxonomy of axioms precisely capture the essence of centrality formulations as well as their fundamental differences. In particular, the choice axioms succinctly distill the comparative differences between different centrality formulations.

To comparatively study vast different influence models under a unified axiomatic framework, Chen and Teng consider the following basic *stochastic profiles* — which we refer to as the *seed-target (probabilistic) profile* — resulting from the interplay between dynamic processes and static graph structures in network influence: Each influence model induces a probability distribution of the final influenced nodes given any initial seed set. Note that over an n node groundset V, the *seed-target (probabilistic) profile* can be specified by a family of $2^{|V|} - 1$ distributions, one for each seed set

[5] The influence spread — as defined in [18] — of a group is the expected number of nodes this group can activate as the initial active set, called *seed set*.

in V. For each $S \in 2^V - \emptyset$, the support of the distribution with seed set S consists of all sets $\{T \subseteq V | S \subseteq T\}$. The *seed-target profiles* also introduce a basic *equivalent relation* among influence models, and the axiomatic characterization of [7] respects this equivalence.

2.4 Motivations and Highlights of Our Contributions

The axiomatic characterization in [7] has two major limitations preventing it to be generalized to study more influence-based centralities. First, the *seed-target profiles* upon which the axiomatic characterization is based significantly compressed out all intermediate steps in a cascading process and only takes the initial seed nodes and the final target nodes into account. This simplification is enough to study centrality measures concerning the final influence spread of the diffusion model, but is inadequate for characterizing influenced-based centrality measures that can capture the propagation details of network influences, such as neighborhood, closeness, sphere-of-influence centralities. Second, its choice axioms are based on a family of *critical set instances*, which do not have a graph-theoretical interpretation, making it less powerful in explaining the connection between graph-theoretical centralities and influence-based centralities.

In this paper, we address both of the above issues in [7] and significantly expand the characterization of influence-based network centrality.

- Capturing Propagation Details: Mathematically, we consider more complex *stochastic cascading profiles* — instead of seed-target profiles as in [7] — in order to capture propagation details. The *stochastic cascading profiles* — See Section 3.1 for formal definition — are distributions of influence activation sequences (i.e. the time-series of stochastic activations of the influence process) generated by an influence model from all seeding sets.

- Algebraic Characterization of Network Influences: As the key technical contribution of the paper, we prove that the *vector space* of stochastic cascading profiles has a graph-theoretical basis. Specifically, for every influence model, its stochastic cascading profile can be expressed as a linear combination of the (determinstic) cascading profiles of Breadth-First-Search (BFS) cascading sequences in a simply family of directed graphs.

- Systematic Extention of Graph-Theoretical Centrality to Influence-based Centrality: This *algebraic characterization of network influences* then serves as the theoretical foundation for systematic extention of several important measures of graph-theoretical centrality to measures of influence-based centrality, and axiomatic characterization of influence-based centrality.

- Efficient Approximation of Influence-based Centrality: We further provide an algorithmic framework for efficient approximation of these influence-based centrality measures.

3 An Algebraic Characterization of Network Influences

In this section, we present our main technical result — a surprising discovery during our research — which provides a graph-theoretical characterization of the space of influence cascading profiles. Specifically, we identify a simple set of classical graphs, and prove that when treated as BFS propagation instances, they form a linear basis in the space of all stochastic cascading-sequence profiles. This graph-theoretical characterization of influence models is instrumental to our systematic characterization of a family of influence-based network centralities. Moreover, we believe that this result is also important on its own right, and is potentially useful in other settings studying general influence propagation models.

3.1 Profiles of Influence Processes

Both in theory and in practice, different influence models may differ in many details， particularly regarding their underlying influence logic and network structures. To analyze influence models and formulate influence efficiency, researchers usually focus on certain profiles of the models' influence processes rather than all details of influence logic. For example, in their celebrated formulation of influence maximization Kempe, Kleinberg, and Tardos [18] essentially focus on the *influence-spread profile* of the influence model. For an influence model $\mathcal{I}$ over a groundset V, its *influence-spread profile* — $[\ \sigma_{\mathcal{I}}(S)\ ]_{S\in 2^V-\emptyset}$ — consisting of $2^{|V|}-1$ non-negative numbers, one for each non-empty *seed set* in V: For each $S \in 2^V - \emptyset$, the *influence spread* $\sigma_{\mathcal{I}}(S)$ is equal to the expected number of nodes this seed set can activate as the initial active set in the influence process. The *influence-spread profile* can be viewed as the summarization another richer stochastic profile — the *seed-target (probabilistic) profile* — of the underlying influence process. This profile of the influence model can be specified by a family of $2^{|V|}-1$ distributions, one for each seed set in V. For each $S \in 2^V - \emptyset$, the support of the distribution, $[\ P_{\mathcal{I}}(S \to T)\ ]_{\{T\subseteq V|S\subseteq T\}}$, with seed set S consists of all sets $\{T \subseteq V|S \subseteq T\}$. Then the influence spread is:

$$\sigma_{\mathcal{I}}(S) = \sum_{T\subseteq V|S\subseteq T} P_{\mathcal{I}}(S \to T)\cdot |T|.$$

In [7], the axiomatic characterization of influence-based centrality is based on the stochastic seed-target profile of the influence process. As we discussed in introduction, the compressed nature of stochastic seed-target profiles seriously limits the scope of influence-based centrality formulations to those independent of details of the influence propagation. In other words, the seed-target profile is only suitable for centrality measures addressing the final influence spread, but is not detailed enough to study other centrality extensions including extensions to degree, closeness, harmonic centralities, etc.

Stochastic Cascading Profiles: In this paper, we will focus on the following stochastic cascading profiles — also considered in [8] for studying the equivalence between high-order network-influence frameworks — which records basic influence propagation information. The *stochastic cascading profile* (SCP) captures (the time series of) activation sequences — rather than activation logic — of influence processes: We say that a sequence of sets $S_0, S_1, S_2, \ldots$ ($S_t \subseteq V$ for all $t \geq 0$) is *progressive* if

(a) for all $t \geq 0$, $S_t \subseteq S_{t+1}$;

(b) for any $t \geq 0$, if $S_t = S_{t+1}$, then for all $t' > t$, $S_{t'} = S_t$; and

(c) $S_0 \neq \emptyset$.

In the above definition, S_t represents the set of network nodes that become *active* by step t during the propagation, and $\Delta_t = S_t - S_{t-1}$ denotes the set of nodes newly *activated* at step t. Thus, the cascading sequence $(S_0, S_1, \ldots, S_{n-1})$ corresponds to the *progress* diffusion model in the literature and provides a layered structure starting from seed set S_0, similar to network broadcasting. However, unlike broadcasting, the layered cascading sequences are formed *stochastically* in network influence. At this level of abstraction, an influence model can be viewed as a probabilistic mechanism to generate cascading sequences. In each time step, already activated nodes stochastically activate more nodes in the next step, and this stochastic propagation ends when no new nodes are activated in a step. Therefore, each influence model generates a *stochastic cascading profile* (SCP), summarizing the probabilistic distribution over the possible cascading sequences from each seed set. Let $n = |V|$. For such progressive sequences, it is clear that the influence propagation stops within at most $n - 1$ steps, and thus we only need a sequence $S_0, S_1, \ldots, S_{n-1}$ to represent it.

Definition 3.1 (Stochastic Cascading Profile). *Each influence model $\mathcal{I}$ over a node set V (of size n) generates a* stochastic cascading profile *$P_{\mathcal{I}}$, which is a mapping from every nonempty seed set $S_0 \in 2^V \setminus \{\emptyset\}$ to a distribution $P_{\mathcal{I},S_0}$ over all progressive sequences $(S_0, S_1, \ldots, S_{n-1})$ starting from S_0. That is, for every nonempty seed set S_0, $P_{\mathcal{I},S_0}(S_0, S_1, \ldots, S_{n-1})$ gives the probability that a progressive sequence $(S_0, S_1, \ldots, S_{n-1})$ is generated from the influence process starting from seed set S_0.*

Note that if $(S_0, S_1, \ldots, S_{n-1})$ is not a valid cascading sequence, then $P_{\mathcal{I},S_0}(S_0, S_1, \ldots, S_{n-1}) = 0$. Thus, $P_{\mathcal{I},S_0}$ can be viewed as a distribution over all n-step set sequences, i.e., for every S_0:

$$\sum_{S_1,\ldots,S_{n-1}} P_{\mathcal{I},S_0}(S_0, S_1, \ldots, S_{n-1}) = 1.$$

We will drop the index S_0 from $P_{\mathcal{I},S_0}$ when it is clear from context.

Note also that in influence propagation models, the *influence-spread profile* can be summarized from the stochastic cascading profile as:

$$\sigma_{\mathcal{I}}(S) = \sum_{S_1,\ldots,S_{n-1}} P_{\mathcal{I},S}(S, S_1, \ldots, S_{n-1}) \cdot |S_{n-1}|, \quad \forall S \in 2^V \setminus \{\emptyset\}.$$

The more coarse-grained seed-target profile discussed above is then:

$$P_{\mathcal{I}}(S \to T) = \sum_{S_1,\ldots,S_{n-2},T} P_{\mathcal{I},S}(S, S_1, \ldots, S_{n-2}, T), \quad \forall S, T \in 2^V \setminus \{\emptyset\}.$$

3.2 A Graph-Theoretical Basis of Influence Profiles

Note that for any directed graph $G = (V, E)$, we can equivalently interpret it as a (deterministic) diffusion model, where the diffusion is carried out by the breadth-first search (BFS). In particular, for any seed set S_0, we have a deterministic cascading sequence $(S_0, S_1, \ldots, S_{n-1})$, where S_t is all the nodes that can be reached within t steps of BFS. Thus, the stochastic cascading profile of this "deterministic influence instance" is such that this and only this BFS sequence has probability 1; all other sequences starting from S_0 have probability 0. We call this instance the *BFS influence instance* corresponding to graph G, and denote it as $\mathcal{I}_G^{(\mathrm{BFS})}$.

Mathematically, each stochastic cascading profile (as defined in Definition 3.1) can be represented as a vector of probabilities over monotonic cascading sequences. In other words, the vector contains entries for each monotonic cascading sequences. Because for the profile of each set S_0, $P_{\mathcal{I},S_0}$, the probability of all valid cascading sequences add up to 1, so one entry is redundant. As our "canonical choice," we remove the entry $P_{\mathcal{I},S_0}(S_0, S_0, \ldots, S_0)$ from the "vector" of stochastic cascading profiles, and express it implicitly. Note that the resulting vector has an exponential number of dimensions, as there are exponential number of monotonic set sequences, and we will use M_n to denote its dimension (we will drop the index n when clear from context).

The set of "basis" graphs are the layered graphs, as depicted in Figure 1. Formally, for a vertex set V, for an integer $t \geq 0$, and $t+1$ disjoint nonempty subsets $R_0, R_1, \ldots, R_t \subseteq V$, a *layered graph* $L_V(R_0, \ldots, R_t)$ is a directed graph in which every node in R_{i-1} has a directed edge pointing to every node in R_i, for $i \in [t]$, and the rest nodes in $V \setminus \cup_{i=1}^{t} R_i$ are isolated nodes with no connections to and from any other nodes. We say that the BFS influence instance of the layered

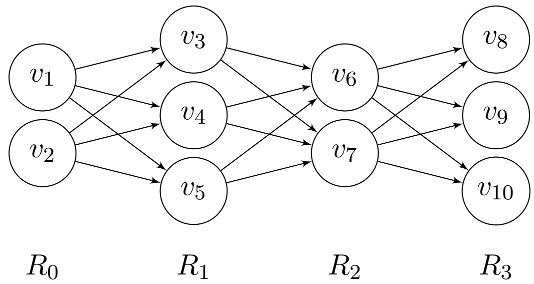

图 1: an example of a layered-graph instance with 4 layers (R_0, R_1, R_2, R_3) where $R_0 = \{v_1, v_2\}$, $R_1 = \{v_3, v_4, v_5\}$, $R_2 = \{v_6, v_7\}$, $R_3 = \{v_8, v_9, v_{10}\}$.

graph $L_V(R_0, \ldots, R_t)$, namely $\mathcal{I}^{(\text{BFS})}_{L_V(R_0,\ldots,R_t)}$, is a *layered-graph instance*, and for convenience we also use $\mathcal{I}_V(R_0, \ldots, R_t)$ to denote this instance. When the context is clear, we ignore V in the subscript. A trivial layered graph instance is when $t = 0$, in which case all nodes are isolated and there is no edge in the graph. We call this the *null influence instance*, and denote it as $\mathcal{I}^N$ (or $\mathcal{I}_V(V)$ to make it consistent with the layered-graph notation). Technically, in $\mathcal{I}^N$, only $P_{\mathcal{I}^N}(S_0, S_0, \ldots, S_0) = 1$, and all other probability values are 0, which means its corresponding vector form is the all-zero vector.

As a fundamental characterization of the mathematical space of influence profiles, we prove the following theorem, which states that all nontrivial layered-graph instances form a linear basis in the space of stochastic cascading profiles:

Theorem 3.1 (Graph-Theoretical Basis)**.** *Let $\mathcal{L}$ be the set of all nontrivial layered-graph instances under node set V, i.e.,*

$$\mathcal{L} = \{\mathcal{I}_V(R_0, \ldots, R_t) \mid t = 1, \ldots, n-1, \emptyset \neq R_i \subseteq V, \textit{all } R_i\textit{'s are disjoint}\}.$$

Then, the set of vectors corresponding to the nontrivial layered-graph instances in $\mathcal{L}$ forms a basis in $\mathbb{R}^M$.

Proof: Although the proof of this theorem is quite technical, its underlying principle is quite basic. We first provide some intuitions. Note that $M = |\mathcal{L}|$. Thus, the central argument in the proof is to show that elements in $\mathcal{L}$ are independent, which we will establish using proof-by-contradiction: Suppose the profiles corresponding to layered graphs are not independent. That is, there are not-all-zero coefficients $\lambda_{\mathcal{I}}$ such that a linear combination (denoted by $P = \sum_{\mathcal{I} \in \mathcal{L}} \lambda_{\mathcal{I}} P_{\mathcal{I}}$) of these profiles is zero. We consider a carefully-designed inclusion-exclusion form linear combination of the entries of P and show that this combination is exactly some $\lambda_{\mathcal{I}} \neq 0$, which means $P \neq \mathbf{0}$.

Now the formal proof. Because $|\mathcal{L}| = M$, to prove the theorem, it is sufficent for us to show vectors in $\mathcal{L}$ are independent. Suppose not, i.e., there is a nontrivial group of $\{\lambda(R_0, \ldots, R_t)\}$ such that

$$\sum_{R_0,\ldots,R_t:\,\mathcal{I}(R_0,\ldots,R_t)\in\mathcal{L}} \lambda(R_0,\ldots,R_t)P_{\mathcal{I}(R_0,\ldots,R_t)} = \mathbf{0}.$$

Let $\mathcal{I}(R_0^*,\ldots,R_{t^*}^*)$ be a layered-graph instance

- Such that $\lambda(R_0^*,\ldots,R_{t^*}^*) \neq 0$;
- Among those satisfying the condition above, with the largest number of layers (i.e. t^*);
- Among those satisfying the conditions above, with the largest number of vertices in the first layer (i.e. $|R_0^*|$).

Note that fixing the seed set, the propagation on a layered-graph instance is deterministic. That is, there is exactly one cascading sequence with the fixed seed set, which happens with probability 1. Let $Seq_{\mathcal{I}(R_0,\ldots,R_t)}(S_0)$ be the unique BFS sequence which happens on $\mathcal{I}(R_0,\ldots,R_t)$ with seed set S_0. We show that

$$\sum_{\emptyset\neq S_0\subseteq R_0^*}(-1)^{1+|S_0|}\sum_{R_0,\ldots,R_t:\,\mathcal{I}(R_0,\ldots,R_t)\in\mathcal{L}} \lambda(R_0,\ldots,R_t)P_{\mathcal{I}(R_0,\ldots,R_t)}(Seq_{\mathcal{I}(R_0^*,\ldots,R_{t^*}^*)}(S_0)) \neq 0,$$

which contradicts the assumption of non-independence and thereby concludes the proof.

We now compute the left hand side of the above formula.

$$\begin{aligned}
&\sum_{\emptyset\neq S_0\subseteq R_0^*}(-1)^{1+|S_0|}\sum_{R_0,\ldots,R_t:\,\mathcal{I}(R_0,\ldots,R_t)\in\mathcal{L}} \lambda(R_0,\ldots,R_t)P_{\mathcal{I}(R_0,\ldots,R_t)}(Seq_{\mathcal{I}(R_0^*,\ldots,R_{t^*}^*)}(S_0))\\
=&\sum_{R_0,\ldots,R_t:\,\mathcal{I}(R_0,\ldots,R_t)\in\mathcal{L}} \lambda(R_0,\ldots,R_t)\sum_{\emptyset\neq S_0\subseteq R_0^*}(-1)^{1+|S_0|}P_{\mathcal{I}(R_0,\ldots,R_t)}(Seq_{\mathcal{I}(R_0^*,\ldots,R_{t^*}^*)}(S_0))\\
=&\sum_{t<t^*,R_0,\ldots,R_t:\,\mathcal{I}(R_0,\ldots,R_t)\in\mathcal{L}} \lambda(R_0,\ldots,R_t)\sum_{\emptyset\neq S_0\subseteq R_0^*}(-1)^{1+|S_0|}P_{\mathcal{I}(R_0,\ldots,R_t)}(Seq_{\mathcal{I}(R_0^*,\ldots,R_{t^*}^*)}(S_0))\\
+&\sum_{t>t^*,R_0,\ldots,R_t:\,\mathcal{I}(R_0,\ldots,R_t)\in\mathcal{L}} \lambda(R_0,\ldots,R_t)\sum_{\emptyset\neq S_0\subseteq R_0^*}(-1)^{1+|S_0|}P_{\mathcal{I}(R_0,\ldots,R_t)}(Seq_{\mathcal{I}(R_0^*,\ldots,R_{t^*}^*)}(S_0))\\
+&\sum_{R_0,\ldots,R_{t^*}:\,\mathcal{I}(R_0,\ldots,R_{t^*})\in\mathcal{L}} \lambda(R_0,\ldots,R_{t^*})\sum_{\emptyset\neq S_0\subseteq R_0^*}(-1)^{1+|S_0|}P_{\mathcal{I}(R_0,\ldots,R_{t^*})}(Seq_{\mathcal{I}(R_0^*,\ldots,R_{t^*}^*)}(S_0))\\
=&\sum_{R_0,\ldots,R_{t^*}:\,\mathcal{I}(R_0,\ldots,R_{t^*})\in\mathcal{L}} \lambda(R_0,\ldots,R_{t^*})\sum_{\emptyset\neq S_0\subseteq R_0^*}(-1)^{1+|S_0|}P_{\mathcal{I}(R_0,\ldots,R_{t^*})}(Seq_{\mathcal{I}(R_0^*,\ldots,R_{t^*}^*)}(S_0))\\
=&\sum_{R_0:\,R_0\cap R_0^*\neq\emptyset,\,\mathcal{I}(R_0,R_1^*,\ldots,R_{t^*}^*)\in\mathcal{L}} \lambda(R_0,R_1^*,\ldots,R_{t^*}^*)\\
\times&\sum_{\emptyset\neq S_0\subseteq R_0^*}(-1)^{1+|S_0|}P_{\mathcal{I}(R_0,R_1^*,\ldots,R_{t^*}^*)}(Seq_{\mathcal{I}(R_0^*,\ldots,R_{t^*}^*)}(S_0)).
\end{aligned}$$

Now consider the summand in the last line of the above equation. For $R_0 \neq R_0^*$,

$$
\begin{aligned}
&\sum_{\emptyset \neq S_0 \subseteq R_0^*} (-1)^{1+|S_0|} P_{\mathcal{I}(R_0, R_1^*, \ldots, R_{t^*}^*)}(Seq_{\mathcal{I}(R_0^*, \ldots, R_{t^*}^*)}(S_0)) \\
= &\sum_{\emptyset \neq X \subseteq R_0^* \cap R_0} \sum_{Y \subseteq R_0^* \setminus R_0} (-1)^{1+|X|+|Y|} P_{\mathcal{I}(R_0, R_1^*, \ldots, R_{t^*}^*)}(Seq_{\mathcal{I}(R_0^*, \ldots, R_{t^*}^*)}(X \cup Y)) \\
+ &\sum_{\emptyset \neq Y \subseteq R_0^* \setminus R_0} (-1)^{1+|Y|} P_{\mathcal{I}(R_0, R_1^*, \ldots, R_{t^*}^*)}(Seq_{\mathcal{I}(R_0^*, \ldots, R_{t^*}^*)}(Y)) \\
= &\sum_{\emptyset \neq X \subseteq R_0^* \cap R_0} (-1)^{1+|X|} \sum_{Y \subseteq R_0^* \setminus R_0} (-1)^{|Y|} \\
= &\sum_{\emptyset \neq X \subseteq R_0^* \setminus R_0} (-1)^{1+|Y|} \times 0 \\
= &\; 0.
\end{aligned}
$$

And for $R_0 = R_0^*$,

$$
\sum_{\emptyset \neq S_0 \subseteq R_0^*} (-1)^{1+|S_0|} P_{\mathcal{I}(R_0^*, R_1^*, \ldots, R_{t^*}^*)}(Seq_{\mathcal{I}(R_0^*, \ldots, R_{t^*}^*)}(S_0)) = \sum_{\emptyset \neq S_0 \subseteq R_0^*} (-1)^{1+|S_0|} = 1.
$$

Plugging these in, we get

$$
\begin{aligned}
&\sum_{\emptyset \neq S_0 \subseteq R_0^*} (-1)^{1+|S_0|} \sum_{R_0, \ldots, R_t \colon \mathcal{I}(R_0, \ldots, R_t) \in \mathcal{L}} \lambda(R_0, \ldots, R_t) P_{\mathcal{I}(R_0, \ldots, R_t)}(Seq_{\mathcal{I}(R_0^*, \ldots, R_{t^*}^*)}(S_0)) \\
= &\sum_{R_0 \colon R_0 \cap R_0^* \neq \emptyset, \mathcal{I}(R_0, R_1^*, \ldots, R_{t^*}^*) \in \mathcal{L}} \lambda(R_0, R_1^*, \ldots, R_{t^*}^*) \sum_{\emptyset \neq S_0 \subseteq R_0^*} (-1)^{1+|S_0|} P_{\mathcal{I}(R_0, R_1^*, \ldots, R_{t^*}^*)}(Seq_{\mathcal{I}(R_0^*, \ldots, R_{t^*}^*)}(S_0)) \\
= &\sum_{R_0 \colon R_0 \cap R_0^* \neq \emptyset, R_0 \neq R_0^*, \mathcal{I}(R_0, R_1^*, \ldots, R_{t^*}^*) \in \mathcal{L}} \lambda(R_0, R_1^*, \ldots, R_{t^*}^*) \times 0 + \lambda(R_0^*, R_1^*, \ldots, R_{t^*}^*) \times 1 \\
= &\; \lambda(R_0^*, \ldots, R_{t^*}^*) \\
\neq &\; 0.
\end{aligned}
$$

□

4 Influence-based Centrality Characterization: A Unified Framework

Recall that a graph-theoretical centrality, such as degree, distance, PageRank, and betweenness, summarizes network data to measure the importance of each node in a network structure. Likewise, the objective of influence-based centrality formulations is to summarize the network-influence data in order to measure the importance of every node in influence diffusion processes.

The stochastic cascading profiles and their graph-theoretical basis (Theorem 3.1) provides a unified theoretical framework for extending graph-theoretical-centrality formulation to influence-based centrality formulation and axiomatic characterization.

1. First, summarize a given influence model by its stochastic cascading profile.

2. Continue the summarization from the stochastic cascading profile to a centrality measure.

In the rest of this paper, we focus on centrality formulation as the summarization of stochastic cascading profiles. Formally:

Definition 4.1 (Influence-based Centrality Measure)**.** *For influence processes over an n node groundset V, an* influence-based centrality measure *ψ is a mapping from a stochastic cascading profile $P_{\mathcal{I}}$ to a real-valued vector $(\psi_v(\mathcal{I}))_{v \in V} \in \mathbb{R}^n$.*

The objective is to formulate network centrality measures that reflect dynamic influence propagation. Theorem 3.1 lays the foundation for a systematic framework to generalize graph-theoretical centrality formulation to network centrality measures. It allows us to connect stochastic cascading profiles of influence-propagation with static graphs. Applying this algebraic structure of network influences, we can systematically extend a large family of graph-theoretical centralities — such as degree, closeness, harmonic, reachability centralities — to influence-based centrality formulations. In the process, by applying Theorem 3.1, we establish a complete *axiomatic characterization*, in Theorem 4.2, for this family of *stochastic sphere-of-influence centralities.* This axiomatic characterization illustrates that (a) our graph-theory-to-influence-dynamics extension is not only reasonable but also the only feasible mathematical choice, and (b) layered graphs are the key family of graphs for comparing different influence measures, since a graph-theoretical centrality measure on layered graphs fully determines the conforming influence-based centrality measure.

In Section 4.1, we first examine a unified centrality family that are natural for layered graphs. In Section 4.2, we then systematically lift these graph-theoretical centrality formulations to influence models. In Section 4.3, using Theorem 3.1, we prove a representation theorem, providing an axiomatic characterization for these influence-based centrality formulation.

4.1 A Unified Family of Sphere-of-Influence Centralities

In this subsection, we discuss a family of graph-theoretical centrality measures that contains various forms of "sphere-of-influence" and closeness centralities. These centrality measures have a common feature: the centrality of node v is fully determined by the distances from v to all nodes.

Consider a directed graph $G = (V, E)$. Let $N_G^+(v)$ and $N_G^-(v)$ denote the set of out-neighbors and in-neighbors, respectively, of a node v. Let $d_G(u, v)$ be the graph distance from u to v in G.

If v is not reachable from u in G then we set $d(u,v) = \infty$. Let $d_G(S,v) = \min_{u \in S} d_G(u,v)$ be the distance from a subset $S \subseteq V$ to node v. Let $\Gamma_G(S)$ be the set of nodes reachable from $S \subseteq V$ in G. When the context is clear, we would remove G from the subscripts in the above notations.

Recall that a graph-theoretical centrality measure is a mapping μ from a graph G to a real-valued vector $(\mu_v(G))_{v \in V} \in \mathbb{R}^n$, where $\mu_v(G)$ denote the centrality of v in G.

For every $S \subseteq V$ and $v \in V$, let $\vec{d}_G(S)$ be the vector in $\mathbb{R}^n$ consisting of the distance from S to every node u, i.e. $\vec{d}_G(S) = (d(S,u))_{u \in V}$. Let $\vec{d}_G(v) = \vec{d}_G(\{v\})$. We use $\mathbb{R}_\infty$ to denote $\mathbb{R} \cup \{\infty\}$. For each $f : \mathbb{R}^n_\infty \to \mathbb{R}$, we can define:

Definition 4.2 (Distance-based Centrality)**.** *A distance-based centrality $\mu^{\text{ind}}[f]$ with function $f : \mathbb{R}^n_\infty \to \mathbb{R}$ is defined as $\mu^{\text{ind}}[f]_v(G) = f(\vec{d}_G(v))$.*

Definition 4.2 is a general formulation. It includes several classical graph-theoretical centrality formulations as special cases: (a) The *degree centrality* (or immediate sphere of influence), $\mu^{\text{ind-deg}}$, is defined as the out-degree of a node v in graph G, that is, $\mu_v^{\text{ind-deg}}(G) = |N_G^+(v)|$. It is defined by $f^{\text{deg}}(\vec{d}) = |\{u \in V \mid d_u = 1\}|$. (b) The *closeness centrality*, $\mu^{\text{ind-cls}}$, is defined as the reciprocal of the average distance to other nodes, $\mu_v^{\text{ind-cls}}(G) = \frac{1}{\sum_{u \neq v} d_G(v,u)}$. It is defined by $f^{\text{cls}}(\vec{d}) = \frac{1}{\sum_{u \in V} d_u}$. If G is not strongly connected, then $\mu_v^{\text{ind-cls}}(G) = 0$ for any v that cannot reach all other nodes, and thus closeness centrality is not expressive enough for such graphs. (c) *harmonic centrality*, $\mu^{\text{ind-har}}$, is defined: $\mu_v^{\text{ind-har}}(G) = \sum_{u \neq v} \frac{1}{d_G(v,u)}$. It is defined by $f^{\text{har}}(\vec{d}) = \sum_{u \in V, d_u > 0} \frac{1}{d_u}$. Note that harmonic centrality is closely related with closeness centrality, and is applicable to network with disjointed components. (d) The *reachability centrality* measure $\mu^{\text{ind-rch}}$: $\mu_v^{\text{ind-rch}}(G) = |\Gamma_G(\{v\})|$, which means the reachability centrality of v is the number of nodes v could reach in G. It is defined by $f^{\text{rch}}(\vec{d}) = |\{u \in V \mid d_u < \infty\}|$. (e) The *sphere-of-influence centrality* measure $\mu^{\text{ind-SoI}(\delta)}$: For a threshold parameter δ, $\mu_v^{\text{ind-SoI}(\delta)} = |\{u : d_u \leq \delta\}|$. It is defined by $f^{\text{SoI}(\delta)}(\vec{d}) = |\{u \in V \mid d_u \leq \delta\}|$. Clearly, as δ varies from 1 to $n-1$ (or ∞), the sphere-of-influence centrality interpolates the degree centrality and the reachability centrality.

4.2 Stochastic Sphere-of-Influence: Lifting from Graph to Influence Models

Thus, Definition 4.2 represents a unified family of sphere-of-influence centralities for graphs. The function f — which is usually a non-increasing function of distance profiles — captures the *scale* of the impact, based on the distance of nodes from the *source*. By unifying these centralities under one general centrality class, we are able to systematically derive and study their generalization in the network-influence models. The key step is to transfer the graph distance in directed graph to *cascading distance* in cascading sequences. For any cascading sequence $(S_0, S_1, \ldots, S_{n-1})$ starting from seed set S_0, let $d_u(S_0, S_1, \ldots, S_{n-1}) = t$ if $u \in \Delta_t = S_t \setminus S_{t-1}$ ($\Delta_0 = S_0$), and

$d_u(S_0, S_1, \ldots, S_{n-1}) = \infty$ if $u \notin S_{n-1}$. We call $d_u(S_0, S_1, \ldots, S_{n-1})$ the *cascading distance* from seed set S_0 to node u, since it represents the number of steps needed for S_0 to activate u in the cascading sequence. Then, we define the cascading distance vector $\vec{d}(S_0, S_1, \ldots, S_{n-1})$ as $(d_u(S_0, S_1, \ldots, S_{n-1}))_{u \in V} \in \mathbb{R}^n$.

In particular, when we consider a cascading sequence $(\{v\}, S_1, \ldots, S_{n-1})$ starting from a single node v, set $\Delta_1 = S_1 \setminus \{v\}$ can be viewed as the out-neighbor of v, set S_{n-1} can be viewed as the all nodes reachable from v, and for every node $u \in \Delta_t = S_t \setminus S_{t-1}$, the distance from v to u is t.

Definition 4.3 (Individual Stochastic Sphere-of-Influence Centrality)**.** *For each function $f : \mathbb{R}^n_\infty \to \mathbb{R}$, the influence-based individual stochastic sphere-of-influence centrality $\psi^{\text{ind}}[f]$ is defined as:*

$$\psi^{\text{ind}}[f]_v(\mathcal{I}) = \mathbb{E}_{(S_1,\ldots,S_{n-1}) \sim P_{\mathcal{I},\{v\}}}[f(\vec{d}(\{v\}, S_1, \ldots, S_{n-1}))].$$

Definition 4.3 systematically extends the family of graph-theoretical centralities of Definition 4.2 to influence models. Natural influence-based centralities, e.g., the single-node influence (SNI) centrality defined in [7] (using each node v's influence spread $\sigma_{\mathcal{I}}(\{v\})$ as the measure of its influence-based centrality), can be expressed by this extension:

Proposition 4.1. $\forall$ *influence profile $\mathcal{I} = (V, E, P_{\mathcal{I}})$:*

$$\text{SNI}(\mathcal{I}) = \psi^{\text{ind}}[f^{\text{rch}}](\mathcal{I}).$$

Proof: For any $v \in V$,

$$\begin{aligned}
\text{SNI}_v(\mathcal{I}) &= \sigma_{\mathcal{I}}(v) \\
&= \mathbb{E}_{(S_1,\ldots,S_{n-1}) \sim P_{\mathcal{I}}} \sum_{u \in V} \mathbb{I}[d_u \leq \infty] \\
&= \mathbb{E}_{(S_1,\ldots,S_{n-1}) \sim P_{\mathcal{I}}} f^{\text{rch}}(\vec{d}(\{v\}, S_1, \ldots, S_{n-1})) \\
&= \psi^{\text{SNSSoI}}[f^{\text{rch}}](\mathcal{I}).
\end{aligned}$$

□

The influence-based centrality formulations of Definition 4.3 enjoy the following graph-theoretical conformity property.

Definition 4.4 (Graph-Theoretical Conformity)**.** *An influence-based centrality measure ψ conforms with a graph-theoretical centrality measure μ if for every directed graph G, $\psi(\mathcal{I}_G^{(\text{BFS})}) = \mu(G)$.*

Proposition 4.2. *For any function $f : \mathbb{R}^n_\infty \to \mathbb{R}$, $\psi^{\text{ind}}[f]$ conforms with $\mu^{\text{ind}}[f]$.*

Proof: For any directed graph $G = (V, E)$ and any set $v \in V$, let $(\{v\}, S_1^v, \ldots, S_{n-1}^v)$ be the BFS sequence starting from v in G. Then we have

$$\begin{aligned}\psi[f]_v(\mathcal{I}_G) &= \mathbb{E}_{(S_1,\ldots,S_{n-1})\sim P_{\mathcal{I}_G}(\{v\})}[f(\vec{d}(\{v\}, S_1, \ldots, S_{n-1}))] \\ &= f(\vec{d}(\{v\}, S_1^v, \ldots, S_{n-1}^v)) \\ &= f(\vec{d}_G(v)) = \mu[f]_v(G).\end{aligned}$$

Thus the lemma holds. □

4.3 Axiomatic Characterization of Influence-based Centrality Formulations

Given the multiplicity of the (potential) centrality formulations, "how should we characterize each formulation?" and "how should we compare different formulations?" are fundamental questions in network analysis. Inspired by the pioneering work of Arrow [2] on social choice, Shapley [41] on cooperation games and coalition, Palacios-Huerta & Volij [37] on measures of intellectual influence, and Altman & Tennenholtz [1] on PageRank, Chen and Teng [7] proposed an axiomatic framework for characterizing and analyzing influence-based network centrality. They identify two principle axioms that all desirable influenced-based centrality formulations should satisfy.

4.3.1 Principle Axioms for Influenced-Based Centrality

The first axiom — ubiquitous axiom for centrality characterization, e.g. [40] — states that labels on the nodes should have no effect on centrality measures. We state the axiom in the framework of stochastic cascading profiles.

Axiom 4.1 (Anonymity)**.** *For any influence instance $\mathcal{I}$ over a groundset V, and any permutation π of V, let $\pi(\mathcal{I})$ denote the isomorphic instance: for any cascading sequence $(S_0, S_1, \ldots, S_{n-1})$, $P_{\pi(\mathcal{I}),\Pi(S_0)}(\pi(S_0), \pi(S_1), \ldots, \pi(S_{n-1})) = P_{\mathcal{I},S_0}(S_0, S_1, \ldots, S_{n-1})$. Then:*

$$\psi_v(\mathcal{I}) = \psi_{\pi(v)}(\pi(\mathcal{I})), \quad \forall v \in V. \tag{1}$$

The second axiom concerns *Bayesian social influence* [7] through a given network: For any three influence profiles $\mathcal{I}$, $\mathcal{I}_1$, $\mathcal{I}_2$ over the same vertex set V, we say $\mathcal{I}$ is a *Bayesian* of $\mathcal{I}_1$ and $\mathcal{I}_2$ if there exists $\alpha \in [0, 1]$ such that $P_{\mathcal{I}} = \alpha P_{\mathcal{I}_1} + (1 - \alpha)P_{\mathcal{I}_2}$. In other words, $\mathcal{I}$ can be interpreted as a stochastic diffusion model where we first make a random selection — with probability α of model $\mathcal{I}_1$ and with probability $(1 - \alpha)$ of model $\mathcal{I}_2$ — and then carry out the diffusion process according to the selected model. We also say that $\mathcal{I}$ is a *convex combination* of $\mathcal{I}_1$ and $\mathcal{I}_2$. The axiom reflects the linearity-of-expectation principle. If an influence instance is a convex combination of two other influence instances, the centrality value of a vertex is the same convex combination of the corresponding centrality values in the two other instances.

Axiom 4.2 (Bayesian)**.** *For any $\alpha \in [0, 1]$, for any influence profiles $\mathcal{I}$, $\mathcal{I}_1$ and $\mathcal{I}_2$ over common vertex set V such that $P_{\mathcal{I}} = \alpha P_{\mathcal{I}_1} + (1 - \alpha) P_{\mathcal{I}_2}$,*

$$\psi_v(\mathcal{I}) = \alpha \psi_v(\mathcal{I}_1) + (1 - \alpha) \psi_v(\mathcal{I}_2), \quad \forall v \in V. \tag{2}$$

4.3.2 Characterization of Influence-Based Centrality: Single Node Perspective

Theorem 3.1 shows that all influence profiles can be represented as a linear combination of non-trivial layered-graph instances. The result enables us to study and compare centrality measures by looking at their instantiation in the simple layered graph instances. The Bayesian property together with the linear basis of nontrivial layered-graph instances leads to the following characterization theorem.

Theorem 4.1 (Uniquess)**.** *A Bayesian influence-based centrality measure is uniquely determined by its values on layered-graph instances (including the null instance).*

The proof of Theorem 4.1 relies on a general lemma about linear mapping as given in [7], as restated below.

Lemma 4.1 (Lemma 11 of [7])**.** *Let ψ be a mapping from a convex set $D \subseteq \mathbb{R}^M$ to $\mathbb{R}^n$ satisfying that for any vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_s \in D$, for any $\alpha_1, \alpha_2, \ldots, \alpha_s \geq 0$ and $\sum_{i=1}^{s} \alpha_i = 1$, $\psi(\sum_{i=1}^{s} \alpha_i \cdot \vec{v}_i) = \sum_{i=1}^{s} \alpha_i \cdot \psi(\vec{v}_i)$. Suppose that D contains a set of linearly independent basis vectors of $\mathbb{R}^M$, $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_M\}$ and also vector $\vec{0}$. Then for any $\vec{v} \in D$, which can be represented as $\vec{v} = \sum_{i=1}^{M} \lambda_i \cdot \vec{b}_i$ for some $\lambda_1, \lambda_2, \ldots, \lambda_M \in \mathbb{R}$, we have*

$$\psi(\vec{v}) = \psi\left(\sum_{i=1}^{M} \lambda_i \cdot \vec{b}_i\right) = \sum_{i=1}^{M} \lambda_i \cdot \psi(\vec{b}_i) + \left(1 - \sum_{i=1}^{M} \lambda_i\right) \cdot \psi(\vec{0}).$$

Proof: [Proof of Theorem 4.1] By the definition of the null influence instance $\mathcal{I}^N$ (same as the trivial layered-graph instance $\mathcal{I}_V(V)$), we can see that the vector representation of the null influence instance is the all 0 vector, because the entries corresponding to $P_{\mathcal{I}^N}(S_0, S_0, \ldots, S_0)$ are not included in the vector by definition. Then by Theorem 3.1 and Lemma 4.1, we know that for any Bayesian centrality measure ψ, its value on any influence instance $\mathcal{I}$, $\psi(\mathcal{I})$, can be represented as a linear combination of the $\mathcal{I}$'s values on layered-graph instances (including the null instance). Thus, the theorem holds. □

We now use the consequence of Theorem 3.1 and Axioms Anonymity and Bayesian to establish a complete characterization of the family of stochastic sphere-of-influence centralities formulated in Definition 4.3. We prove the following axiomatic representation theorem:

Theorem 4.2 (Characterization of Individual Centrality)**.** *For any anonymous function $f : \mathbb{R}^n_\infty \to \mathbb{R}$, $\psi^{\text{ind}}[f]$ (defined in Definition 4.3) is the unique influence-based centrality that conforms with $\mu^{\text{ind}}[f]$ (defined in Definition 4.2) that satisfies both Axiom* Anonymity *and Axiom* Bayesian*.*

This characterization illustrates that:

(a) our extension of graph-theoretical centralities to influence-based centralities is not only reasonable but the only feasible mathematical choice, and

(b) layered graphs are the key family of graphs comparing different influence measures, since a graph-theoretical centrality measure on layered graphs fully determines the conforming influence-based centrality measure.

Proof: [Proof of Theorem 4.2] Since layered-graph instances are all BFS instances derived from the special class of directed graphs, by Theorem 4.1, we have that any Bayesian centrality conforming with a classical graph-theoretical centrality is unique. The theorem then follows directly from Proposition 4.2 and, the next two propositions (Propositions 4.3 and 4.4). □

Proposition 4.3. *If a function $f : \mathbb{R}^n_\infty \to \mathbb{R}$ is* anonymous *— i.e., $f(\vec{d})$ is* permutation-invariant *— then $\psi^{\text{ind}}[f]$ (as defined in Definition 4.3) satisfies Axiom* Anonymity *and $\mu^{\text{ind}}[f]$ (as defined in Definition 4.2) satisfies the graph-theoretical counterpart of Axiom* Anonymity*.*

Proof:

$$
\begin{aligned}
&\psi^{\text{ind}}[f]_{\pi(v)}(\pi(\mathcal{I})) \\
&= \mathbb{E}_{(S_1,\ldots,S_{n-1})\sim P_{\pi(\mathcal{I})}(\{\pi(v)\})}[f(\vec{d}(\{\pi(v)\},\pi(S_1),\ldots,\pi(S_{n-1})))] \\
&= \mathbb{E}_{(S_1,\ldots,S_{n-1})\sim P_{\mathcal{I}}(\{v\})}[f(\vec{d}(\{v\},S_1,\ldots,S_{n-1}))] \\
&= \psi^{\text{ind}}[f]_v(\mathcal{I}).
\end{aligned}
$$

□

Proposition 4.4. *For any function $f : \mathbb{R}^n_\infty \to \mathbb{R}$, $\psi^{\text{ind}}[f]$ satisfies Axiom* Bayesian*.*

Proof: Suppose $\psi[f]$ is an influence-based distance-function centrality measure. Let instances $\mathcal{I}$, $\mathcal{I}_1$, $\mathcal{I}_2$ be that $P_{\mathcal{I}} = \alpha P_{\mathcal{I}_1} + (1-\alpha)P_{\mathcal{I}_2}$. Then for every sequence $(S_1,\ldots,S_{n-1})$ drawn from distribution $P_{\mathcal{I}}(\{v\})$, it is equivalently drawn with probability α from $P_{\mathcal{I}_1}(\{v\})$, and with probability $1-\alpha$ from $P_{\mathcal{I}_2}(\{v\})$. Therefore,

$$
\begin{aligned}
\psi[f]_v(\mathcal{I}) &= \mathbb{E}_{(S_1,\ldots,S_{n-1})\sim P_{\mathcal{I}}(\{v\})}[f(\vec{d}(\{v\},S_1,\ldots,S_{n-1}))] \\
&= \alpha\cdot\mathbb{E}_{(S_1,\ldots,S_{n-1})\sim P_{\mathcal{I}_1}(\{v\})}[f(\vec{d}(\{v\},S_1,\ldots,S_{n-1}))]
\end{aligned}
$$

$$+ (1-\alpha) \cdot \mathbb{E}_{(S_1,\ldots,S_{n-1})\sim P_{\mathcal{I}_2}(\{v\})}[f(\vec{d}(\{v\}, S_1, \ldots, S_{n-1}))]$$
$$= \alpha \cdot \psi[f]_v(\mathcal{I}_1) + (1-\alpha) \cdot \psi[f]_v(\mathcal{I}_2).$$

Thus the proposition holds. □

4.3.3 Characterization of Influence-Based Centrality: Group Perspective

As highlighted in Domingos-Richardson [38, 15] and Kempe-Kleinberg-Tardos [18], social influence propagation and viral marketing are largely group-based phenomena. Besides characterizing individuals' influence-based centralities, perhaps the more important task is to characterize the influence-based centrality of groups, and individuals' roles in group cooperation. This is the group centrality and Shapley centrality introduced in this section. When distinction is necessary, we refer to the centrality defined in Section 4.2 as *individual centrality*. Similar to individual centrality, we provide a characterization theorem showing that influence-based group centrality is also uniquely characterized by their values on layered-graph instances, as long as they satisfy the group version of Axioms Anonymity and Bayesian.

Group centrality measures the importance of each group in a network. Formally,

Definition 4.5 (Influence-based Group Centrality)**.** *An* influence-based group centrality measure *ψ^{grp} is a mapping from an influence profile $\mathcal{I} = (V, E, P_{\mathcal{I}})$ to a real-valued vector $(\psi^{\mathrm{grp}}_S(\mathcal{I}))_{S\in 2^V} \in \mathbb{R}^{2^n}$.*

For any function $f : \mathbb{R}^n_\infty \to \mathbb{R}$, both Definition 4.2 and Definition 4.3 have a natural extension: For $S \subseteq V$,

$$\mu^{\mathrm{grp}}[f]_S(G) = f(\vec{d}_G(S)),$$

$$\psi^{\mathrm{grp}}[f]_S(\mathcal{I}) = \mathbb{E}_{(S_1,\ldots,S_{n-1})\sim P_{\mathcal{I}}(\{v\})}[f(\vec{d}(S, S_1, \ldots, S_{n-1}))].$$

Axioms Anonymity and Bayesian extend naturally as well as the following characterization based on Theorem 3.1.

Theorem 4.3 (Characterization of Group Centrality)**.** *For any anonymous function $f : \mathbb{R}^n_\infty \to \mathbb{R}$, $\psi^{\mathrm{grp}}[f]$ is the unique influence-based group centrality that conforms with $\mu^{\mathrm{grp}}[f]$ and satisfies both Axiom* Anonymity *and Axiom* Bayesian*.*

Proof: The Bayesian part of the proof is essentially the same as the proof of Proposition 4.4, with subset S replacing node v. The Anonymity part of the proof is essentially the same as the proof of Proposition 4.3, with subset S replacing node v. The conformity part of the proof is essentially the same as the proof of Proposition 4.2, with subset S replacing node v. $\square$

Therefore, we can again reduce the analysis of an influence-based group centrality to the analysis of the measure on the particular layered-graph instances.

4.4 Shapley Centrality: Individuals' Roles in Social Influence

A group centrality measure has 2^n dimensions, so we further use the Shapley value [41] in cooperative game theory to reduce it to n dimensions, and refer to it as the influence-based Shapley centrality. The Shapley centrality of a node measures its importance when the node collaborate with other nodes in groups. Due to the linearity of the Shapley value, we obtain the same characterization for the Shapley centrality: it is the unique one conforming with the graph-theoretical Shapley centrality and satisfying Axioms Anonymity and Bayesian, and the uniqueness is fully determined by its values on layered-graph instances.

4.4.1 Cooperative Games and Shapley Value

A *cooperative game* [41] is defined by tuple (V, τ), where V is a set of n players, and $\tau : 2^V \to \mathbb{R}$ is called *characteristic function* specifying the cooperative utility of any subset of players. In cooperative game theory, a ranking function ϕ is a mapping from a characteristic function τ to a vector $(\phi_v(\tau))_{v \in V} \in \mathbb{R}^n$, indicating the importance of each individual in the cooperation. One famous ranking function is the Shapley value ϕ^{Shapley} [41], as defined below. Let Π be the set of all permutations of V, and $\pi \sim \Pi$ denote a random permutation π drawn uniformly from set Π. For any $v \in V$ and $\pi \in \Pi$, let $S_{\pi,v}$ denote the set of nodes in V preceding v in permutation π. Then, $\forall v \in V$:

$$\begin{aligned}
\phi_v^{\text{Shapley}}(\tau) =& \frac{1}{n!} \sum_{\pi \in \Pi} \left(\tau(S_{\pi,v} \cup \{v\}) - \tau(S_{\pi,v}) \right) \\
=& \sum_{S \subseteq V \setminus \{v\}} \frac{|S|!(n - |S| - 1)!}{n!} \left(\tau(S \cup \{v\}) - \tau(S) \right) \\
=& \mathbb{E}_{\pi \sim \Pi}[\tau(S_{\pi,v} \cup \{v\}) - \tau(S_{\pi,v})].
\end{aligned}$$

The Shapley value of a player v measures the expected marginal contribution of v on the set of players ordered before v in a random order. Shapley [41] proved a remarkable representation theorem: The Shapley value is the unique ranking function that satisfies all the following four conditions:

- Efficiency: $\sum_{v \in V} \phi_v(\tau) = \tau(V)$.

- Symmetry: For any $u, v \in V$, if $\tau(S \cup \{u\}) = \tau(S \cup \{v\})$, $\forall S \subseteq V \setminus \{u, v\}$, then $\phi_u(\tau) = \phi_v(\tau)$.
- Linearity: For any two characteristic functions τ and ω, for any $\alpha, \beta > 0$, $\phi(\alpha\tau + \beta\omega) = \alpha\phi(\tau) + \beta\phi(\omega)$.
- Null Player: For any $v \in V$, if $\tau(S \cup \{v\}) - \tau(S) = 0$, $\forall S \subseteq V \setminus \{v\}$, then $\phi_v(\tau) = 0$.

Efficiency states that the total utility is fully distributed. Symmetry states that two players' ranking values should be the same if they have the identical marginal utility profile. Linearity states that the ranking values of the weighted sum of two coalition games is the same as the weighted sum of their ranking values. Null Player states that a player's ranking value should be zero if the player has zero marginal utility to every subset.

4.4.2 Shapley Centrality of Influence Models

Shapley's celebrated concept — as highlighted in [7] — offers a formulation for assessing individuals' performance in group influence settings. It can be used to systematically compress exponential-dimensional group centrality measures into n-dimensional individual centrality measures.

Definition 4.6 (Influence-based Shapley Centrality)**.** *An* influence-based *Shapley centrality ψ^{Shapley} is an individual centrality measure corresponding to a group centrality ψ^{grp}:*

$$\begin{aligned}\psi_v^{\text{Shapley}}(\mathcal{I}) &= \phi_v^{\text{Shapley}}(\psi^{\text{grp}}(\mathcal{I})) \\ &= \mathbb{E}_{\pi\sim\Pi}[\psi^{\text{grp}}_{S_{\pi,v}\cup\{v\}}(\mathcal{I}) - \psi^{\text{grp}}_{S_{\pi,v}}(\mathcal{I})].\end{aligned}$$

We also denote it as $\psi^{\text{Shapley}} = \phi^{\text{Shapley}} \circ \psi^{\text{grp}}$.

In [7], Chen and Teng analyze the Shapley value of the influence-spread function, which is a special case of the following "Shapley extension" of Definition 4.3.

Definition 4.7 (Shapley Centrality of Stochastic Sphere-of-Influence)**.** *For each $f : \mathbb{R}^n_\infty \to \mathbb{R}$, the* Shapley centrality of Stochastic Sphere-of-Influence *$\psi^{\text{Shapley}}[f]$ is defined as:*

$$\psi^{\text{Shapley}}[f]_v(\mathcal{I}) = \phi_v^{\text{Shapley}}(\psi^{\text{grp}}[f](\mathcal{I})).$$

Shapley centrality μ^{Shapley} can also be defined similarly based on graph-theoretical group centrality (see, for example, [34]). We will refer to the extension of Definition 4.2 as $\mu^{\text{Shapley}}[f]$. Using Theorem 3.1, we can establish the following characterization.

Theorem 4.4 (Characterization of Shapley Centrality)**.** *For any anonymous function $f : \mathbb{R}^n_\infty \to \mathbb{R}$, $\psi^{\text{Shapley}}[f]$ is the unique influence-based centrality that conforms with $\mu^{\text{Shapley}}[f]$ and satisfies both Axiom* Anonymity *and Axiom* Bayesian*.*

Proof: We first show that $\psi^{\text{Shapley}}[f]$ is Bayesian. Let $\mathcal{I}$, $\mathcal{I}_1$, $\mathcal{I}_2$ be three influence instances with the same vertex set V, and $\alpha \in [0, 1]$, where $P_{\mathcal{I}} = \alpha P_{\mathcal{I}_1} + (1 - \alpha) P_{\mathcal{I}_2}$. Then for every node $v \in V$, we have

$$\begin{aligned}
\psi_v^{\text{Shapley}}(\mathcal{I}) &= \phi_v^{\text{Shapley}}(\psi^{\text{grp}}(\mathcal{I})) \\
&= \phi_v^{\text{Shapley}}(\psi^{\text{grp}}(\alpha P_{\mathcal{I}_1} + (1 - \alpha) P_{\mathcal{I}_2})) \\
&= \phi_v^{\text{Shapley}}(\alpha \psi^{\text{grp}}(\mathcal{I}_1) + (1 - \alpha) \psi^{\text{grp}}(\mathcal{I}_2)) \\
&= \alpha \phi_v^{\text{Shapley}}(\psi^{\text{grp}}(\mathcal{I}_1)) + (1 - \alpha) \phi_v^{\text{Shapley}}(\psi^{\text{grp}}(\mathcal{I}_2)) \qquad (3) \\
&= \alpha \psi_v^{\text{Shapley}}(\mathcal{I}_1) + (1 - \alpha) \psi_v^{\text{Shapley}}(\mathcal{I}_2),
\end{aligned}$$

where Eq. (3) is due to the linearity of the Shapley value, which is easy to verify by the following derivations:

$$\begin{aligned}
\phi_v^{\text{Shapley}}(\alpha \tau_1 + \beta \tau_2) &= \mathbb{E}_{\pi \sim \Pi}[(\alpha \tau_1 + \beta \tau_2)(S_{\pi,v} \cup \{v\}) - (\alpha \tau_1 + \beta \tau_2)(S_{\pi,v})] \\
&= \alpha \mathbb{E}_{\pi \sim \Pi}[\tau_1(S_{\pi,v} \cup \{v\}) - \tau_1(S_{\pi,v})] + \beta \mathbb{E}_{\pi \sim \Pi}[\tau_2(S_{\pi,v} \cup \{v\}) - \tau_2(S_{\pi,v})] \\
&= \alpha \phi_v^{\text{Shapley}}(\tau_1) + \beta \phi_v^{\text{Shapley}}(\tau_2).
\end{aligned}$$

Now we show that $\psi^{\text{Shapley}}[f]$ conforms with $\mu^{\text{Shapley}}[f]$. For any directed graph $G = (V, E)$ and any node $v \in V$, let $(\{v\}, S_1^v, \ldots, S_{n-1}^v)$ be the BFS sequence starting from v in graph G. We have

$$\begin{aligned}
\psi^{\text{Shapley}}[f]_v(\mathcal{I}_G) &= \phi_v^{\text{Shapley}}(\psi^{\text{grp}}[f](\mathcal{I}_G)) \\
&= \mathbb{E}_{\pi \sim \Pi}[\psi^{\text{grp}}[f]_{S_{\pi,v} \cup \{v\}}(\mathcal{I}_G) - \psi^{\text{grp}}[f]_{S_{\pi,v}}(\mathcal{I}_G)] \\
&= \mathbb{E}_{\pi \sim \Pi}[\mu^{\text{grp}}[f]_{S_{\pi,v} \cup \{v\}}(G) - \mu^{\text{grp}}[f]_{S_{\pi,v}}(G)] \qquad (4) \\
&= \phi_v^{\text{Shapley}}(\mu^{\text{grp}}[f](G)) \\
&= \mu^{\text{Shapley}}[f](G),
\end{aligned}$$

where Eq. (4) is because influence-based group centrality $\psi^{\text{grp}}[f]$ conforms with the structure-based group centrality $\mu^{\text{grp}}[f]$ (Theorem 4.3). Anonymity follows from anonymity of group centralities (Theorem 4.3). Uniqueness then follows from Theorem 4.1. □

Theorem 4.4 systematically extends the work of [7] to all sphere-of-influence formulations. The SNI and Shapley centrality analyzed in [7] are $\psi^{\text{ind}}[f^{\text{rch}}]$ and $\psi^{\text{Shapley}}[f^{\text{rch}}]$, respectively. In our process of generalizing the work of [7], we also *resolve an open question* central to the axiomatic characterization of [7], which is based on a family of *critical set instances* that do not correspond to a graph-theoretical interpretation. In fact, the influence-spread functions of these "axiomatic" critical set instances used in [7] are not submodular. In contrast, influence-spread functions of the popular independent cascade (IC) and linear threshold (LT) models, as well as, the trigger models of Kempe-Kleinberg-Tardos, are submodular. The submodularity of these influence-spread functions plays an instrumental role in influence maximization algorithms [18, 6]. Thus, it is a fundamental and mathematical question whether influence profiles can be characterized by "simpler" influence instances. Our layered-graph characterization (Theorem 3.1) resolves this open question by connecting all influence profiles with simple BFS cascading sequence in the layered graphs, which is a special case of the IC model and possess the submodularity property. In summary, our layered-graph characterization is instrumental to the series of characterizations we could provide in this paper for influence-based individual, group, and Shapley centralities (Theorem 4.2, 4.3, 4.4).

4.5 A Road Map

Figure 2 summarizes our systematic extension of graph-theoretical centralities (the lower three boxes) to influence-based centralities (the upper three boxes). Starting from the classical graph-theoretical distance-based individual centrality (e.g. harmonic centrality), by transferring the concept of graph distance to cascading distance, we could lift it to the stochastic sphere-of-influence individual centrality (e.g. influence-based harmonic centrality). From individual centralities (either graph-theoretical or influence-based), we could use group distance to extend them to group centralities. From group centralities, we could apply Shapley value to obtain Shapley centralities. Therefore, Figure 2 provides a road map on how to extend many classical graph-theoretical centralities to influence-based centralities.

5 Efficient Algorithm for Approximating Influence-based Centrality

Besides studying the characterization of influence-based centralities, we also want to compute these centrality measures efficiently. Accurate computation is in general infeasible (e.g. it is #P-hard to compute influence-based reachability centrality $\psi^{\text{ind}}[f^{\text{rch}}]$ in the triggering model [44, 11]). Thus, we are looking into approximating centrality values. Instead of designing one algorithm for each centrality, we borrow the algorithmic framework from [7] and show how to adapt the framework to approximate different centralities. Same as in [7], the algorithmic framework applies to the triggering model of influence propagation. For efficient computation, we further assume that the distance function f is *additive*, i.e. $f(\vec{d}) = \sum_{u \in V} g(d_u)$ for some scalar function $g : \mathbb{R}_\infty \to \mathbb{R}$ satisfying $g(\infty) = 0$. The degree, harmonic, and reachability centralities all satisfy this condition.

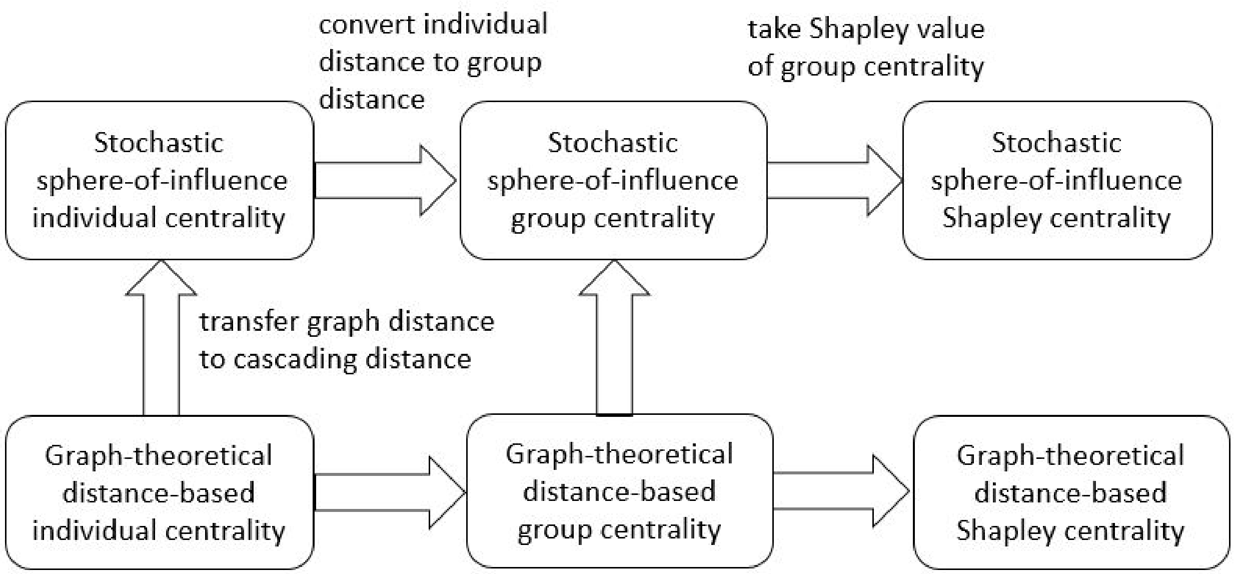

图 2: Road map for the systematic extension of graph-theoretical distance-based centralities to influence-based centralities.

In particular, we have $f^{\text{deg}}(\vec{d}) = \sum_{u \in V} g^{\text{deg}}(d_u)$, with $g^{\text{deg}}(d_u) = 1$ if $d_u = 1$ and $g^{\text{deg}}(d_u) = 0$ otherwise; $f^{\text{har}}(\vec{d}) = \sum_{u \in V} g^{\text{har}}(d_u)$, with $g^{\text{har}}(d_u) = 1/d_u$ if $d_u > 0$ and $g^{\text{har}}(d_u) = 0$ otherwise; and $f^{\text{rch}}(\vec{d}) = \sum_{u \in V} g^{\text{rch}}(d_u)$, where $g^{\text{rch}}(d_u) = 1$ if $d_u < \infty$ and $g^{\text{rch}}(d_u) = 0$ otherwise.

The algorithmic framework for estimating individual and Shapley forms of sphere-of-influence centrality is given in Algorithm 1, and is denoted ICE-RR (for Influence-based Centrality Estimate via RR set). The algorithm uses the approach of reverse-reachable sets (RR sets) [5, 43, 42]. An RR set R_v is generated by randomly selecting a node v (called the *root* of R_v) with equal probability, and then reverse simulating the influence propagation starting from v. In the triggering model, it is simply sampling a random triggering set $T(v)$ for v, putting all nodes in $T(v)$ into R_v, and then recursively sampling triggering sets for all nodes in $T(v)$, until no new nodes are generated.

The algorithm has two phases. In the first phase (lines 1–22), the number θ of RR sets needed for the estimation is computed. The mechanism for obtaining θ follows the IMM algorithm in [42] and is also the same as in [7]. In the second phase (lines 23–34), θ RR sets are generated, and for each RR set R, the centrality estimate of $u \in R$, est_u, is updated properly depending on the centrality type.

Comparing to the algorithm in [7], our change is in lines 8–14 and lines 26–31. First, when generating an RR set R_v, we not only stores the nodes, but for each $u \in R_v$, we also store the distance from u to root v in the reverse simulation paths $d_{R_v}(u, v)$. Technically, $d_{R_v}(u, v)$ is the graph distance from u to v in the subgraph G_{R_v}, where $G_{R_v} = (V, E_{R_v})$ with $E_{R_v} = \{(w, u) \mid$

Input: Network: $G = (V, E)$; Parameters: random triggering set distribution $\{\boldsymbol{T}(v)\}_{v \in V}$, $\varepsilon > 0$, $\ell > 0$, $k \in [n]$, node-wise distance function g

Output: $\hat{\psi}_v$, $\forall v \in V$: estimated centrality value

{Phase 1. Estimate the number of RR sets needed }
LB $= 1$; $\varepsilon' = \sqrt{2} \cdot \varepsilon$; $\theta_0 = 0$
$\text{est}_v = 0$ for every $v \in V$
for $i = 1$ to $\lfloor \log_2 n \rfloor - 1$ **do**
  $x = n/2^i$
  $\theta_i = \left\lceil \frac{n \cdot ((\ell+1) \ln n + \ln \log_2 n + \ln 2) \cdot (2 + \frac{2}{3}\varepsilon')}{\varepsilon'^2 \cdot x} \right\rceil$
  for $j = 1$ to $\theta_i - \theta_{i-1}$ **do**
    generate a random RR set R_v rooted at v, and for each $u \in R_v$, record the distance $d_{R_v}(u, v)$ from u to v in this reverse simulation.
    if estimating individual centrality **then**
      for every $u \in R_v$, $\text{est}_u = \text{est}_u + g(d_{R_v}(u, v))$
    else
      {estimating Shapley centrality}
      for every $u \in R_v$, $\text{est}_u = \text{est}_u + \phi_u^{\text{Shapley}}(g(d_{R_v}(\cdot, v)))$
    end if
  end for
  $\text{est}^{(k)} =$ the k-th largest value in $\{\text{est}_v\}_{v \in V}$
  if $n \cdot \text{est}^{(k)}/\theta_i \geq (1 + \varepsilon') \cdot x$ **then**
    LB $= n \cdot \text{est}^{(k)}/(\theta_i \cdot (1 + \varepsilon'))$
    break
  end if
end for
$\theta = \left\lceil \frac{n((\ell+1) \ln n + \ln 4)(2 + \frac{2}{3}\varepsilon)}{\varepsilon^2 \cdot \text{LB}} \right\rceil$
{Phase 2. Estimate the centrality value}
$\text{est}_v = 0$ for every $v \in V$
for $j = 1$ to θ **do**
  generate a random RR set R_v rooted at v, and for each $u \in R_v$, record the distance $d_{R_v}(u, v)$ from u to v in this reverse simulation.
  if estimating individual centrality **then**
    for every $u \in R_v$, $\text{est}_u = \text{est}_u + g(d_{R_v}(u, v))$
  else
    for every $u \in R_v$, $\text{est}_u = \text{est}_u + \phi_u^{\text{Shapley}}(g(d_{R_v}(\cdot, v)))$
  end if
end for
for every $v \in V$, $\hat{\psi}_v = n \cdot \text{est}_v/\theta$
return $\hat{\psi}_v$, $v \in V$

Algorithm 1: ICE-RR: Efficient estimation of sphere-of-influence centralities via RR-sets, for the triggering model and additive distance function $f(\vec{d}) = \sum_{u \in V} g(d_u)$.

$u \in R_v, w \in T(u)\}$ is the subgraph generated by the triggering sets sampled during the reverse simulation. Note that with this definition, for $u \notin R_v$, we have $d_{R_v}(u,v) = \infty$. Next, if we are estimating individual centrality, we simply update the estimate est_u by adding $g(d_{R_v}(u,v))$. If we are estimating Shapley centrality, we need to update est_u by adding $\phi_u^{\text{Shapley}}(g(d_{R_v}(\cdot,v)))$, the Shapley value of u on the set function $g(d_{R_v}(\cdot,v)) : S \in 2^V \mapsto g(d_{R_v}(S,v)) \in \mathbb{R}$. We will show below that the computation of $\phi_u^{\text{Shapley}}(g(d_{R_v}(\cdot,v)))$ for all $u \in R_v$ together is linear to $|R_v|$, so it is in the same order of generating R_v and does not incur significant extra cost. Note that the algorithm in [7] corresponds to our algorithm with $g = g^{\text{rch}}$. The correctness of the algorithm replies on the following crucial lemma.

Lemma 5.1. *Let R_v be a random RR set with root v generated in a triggering model instance $\mathcal{I}$. Then, $\forall u \in V$, u's stochastic sphere-of-influence individual centrality with function $f(\vec{d}) = \sum_{u \in V} g(d_u)$ is $\psi[f]_u(\mathcal{I}) = n \cdot \mathbb{E}[g(d_{R_v}(u,v))]$, where the expectation is taking over the distribution of RR set R_v. Similarly, u's influence-based Shapley centrality with f is $\psi^{\text{Shapley}}[f]_u(\mathcal{I}) = n \cdot \mathbb{E}[\phi_u^{\text{Shapley}}(g(d_{R_v}(\cdot,v)))]$.*

Proof: Consider first the influence-based distance-function centrality. We have

$$
\begin{aligned}
n \cdot \mathbb{E}[g(d_{R_v}(u,v))] &= n \cdot \sum_{w \in V} \Pr\{w = v\} \mathbb{E}[g(d_{R_v}(u,v)) \mid w = v] \\
&= n \cdot \sum_{w \in V} \frac{1}{n} \mathbb{E}[g(d_{R_w}(u,w))] \\
&= \sum_{w \in V} \mathbb{E}_{(S_1,\ldots,S_{n-1}) \sim P_{\mathcal{I}}(\{u\})}[g(d_w(\{u\}, S_1, \ldots, S_{n-1}))] \qquad (5) \\
&= \mathbb{E}_{(S_1,\ldots,S_{n-1}) \sim P_{\mathcal{I}}(\{u\})}\left[\sum_{w \in V} g(d_w(\{u\}, S_1, \ldots, S_{n-1}))\right] \\
&= \mathbb{E}_{(S_1,\ldots,S_{n-1}) \sim P_{\mathcal{I}}(\{u\})}[f(\vec{d}(\{u\}, S_1, \ldots, S_{n-1}))] \\
&= \psi[f]_u(\mathcal{I}),
\end{aligned}
$$

where Eq. (5) is by Lemma 5.2 below.

Next consider the influence-based distance-function Shapley centrality.

$$
\begin{aligned}
n \cdot \mathbb{E}[\phi_u^{\text{Shapley}}(g(d_{R_v}(\cdot,v)))] &= n \cdot \sum_{w \in V} \Pr\{w = v\} \mathbb{E}[\phi_u^{\text{Shapley}}(g(d_{R_v}(\cdot,v))) \mid w = v] \\
&= n \cdot \sum_{w \in V} \frac{1}{n} \mathbb{E}[\phi_u^{\text{Shapley}}(g(d_{R_w}(\cdot,w)))] \\
&= \phi_u^{\text{Shapley}}\left(\sum_{w \in V} \mathbb{E}[g(d_{R_w}(\cdot,w))]\right) \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
&= \phi_u^{\text{Shapley}}(\psi^{\text{grp}}[f](\mathcal{I})) &(7)\\
&= \psi^{\text{Shapley}}[f]_u(\mathcal{I}),
\end{aligned}
$$

where Eq. (6) is by the linearity of the Shapley value, as already argued in the proof of Theorem 4.3, and Eq. (7) follows the similar derivation step as in the case of individual centrality above. □

Lemma 5.2. *For fixed nodes $u, w \in V$, suppose we generate a random RR set R_w rooted at w, according to a triggering model instance $\mathcal{I}$. Then we have*

$$\mathbb{E}[g(d_{R_w}(u,w))] = \mathbb{E}_{(S_1,\ldots,S_{n-1})\sim P_{\mathcal{I}}(\{u\})}[g(d_w(\{u\}, S_1, \ldots, S_{n-1}))].$$

Proof: We know that the triggering model is equivalent to the following live-edge graph model [18]: for every node $v \in V$, sample its triggering set $T(v)$ and add edges (u, v) to a live-edge graph L for all $u \in T(v)$ (these edges are called live edges). Then the diffusion from a seed set S is the same as the BFS propagation in L from S. Since reverse simulation for generating RR set R_w also do the same sampling of the triggering sets, we can couple the reverse simulation process with the forward propagation by fixing a live-edge graph L. For a fixed live-edge graph L, the subgraph G_{R_w} generated by reverse simulation from the fixed root w is simply the induced subgraph of L induced by all nodes that can reach w in L. Thus $d_{R_w}(u, w)$ is the fixed distance from u to w in L, namely $d_L(u, w)$. On the other hand, with the fixed L, the cascading sequence starting from u is the fixed BFS sequence starting from u in L. Then in this BFS sequence $d_w(\{u\}, S_1, \ldots, S_{n-1})$ is the distance from u to w in this sequence, which is the same as the distance from u to w in L. Therefore, $d_w(\{u\}, S_1, \ldots, S_{n-1}) = d_{R_w}(u, w)$ for fixed live-edge graph L. We can then vary L according to its distribution, and obtain

$$\mathbb{E}[g(d_{R_w}(u,w))] = \mathbb{E}_{(S_1,\ldots,S_{n-1})\sim P_{\mathcal{I}}(\{u\})}[g(d_w(\{u\}, S_1, \ldots, S_{n-1}))].$$

□

From Lemma 5.1, we can understand that lines 28 and 30 are simply accumulating empirical values of $g(d_{R_v}(u, v))$ and $\phi_u^{\text{Shapley}}(g(d_{R_v}(\cdot, v)))$ for individual centrality and Shapley centrality, respectively, and line 33 averages this cumulative value and then multiply it by n to obtain the final centrality estimate. With the above lemma, the correctness of the algorithm ICE-RR is shown by the following theorem.

Theorem 5.1. *Let $(\psi_v)_{v\in V}$ be the true centrality value for an influence-based individual or Shapley centrality with additive function f, and let $\psi^{(k)}$ be the k-th largest value in $(\psi_v)_{v\in V}$. For any $\epsilon > 0$, $\ell > 0$, and $k \in [n]$, Algorithm ICE-RR returns the estimated centrality $(\hat{\psi}_v)_{v\in V}$ that satisfies (a) unbiasedness: $\mathbb{E}[\hat{\psi}_v] = \psi_v, \forall v \in V$; and (b) robustness: under the condition that $\psi^{(k)} \geq 1$, with probability at least $1 - \frac{1}{n^\ell}$:*

$$\begin{cases} |\hat{\psi}_v - \psi_v| \le \varepsilon \psi_v & \forall v \in V \ \textit{with}\ \psi_v > \psi^{(k)}, \\ |\hat{\psi}_v - \psi_v| \le \varepsilon \psi^{(k)} & \forall v \in V \ \textit{with}\ \psi_v \le \psi^{(k)}. \end{cases} \tag{8}$$

Proof: The proof follows exactly the same proof structure as the proof in [7]. All we need to change is to use our crucial lemma connecting RR sets with centrality measures (Lemma 5.1) to replace the corresponding lemma (Lemma 23) in [7]. □

In terms of time complexity, for individual centrality, lines 10 and 28 take constant time for each $u \in R_v$, so it has the same complexity as the algorithm in [7]. For Shapley centrality, the following lemma shows that the computation of $\phi_u^{\text{Shapley}}(g(d_{R_v}(\cdot, v)))$ for all v is $O(|R_v|)$, same as the complexity of generating R_v, so it will not add complexity to the overall running time. Suppose R_v has Δ levels in total (i.e., $\Delta = \max\{d_{R_v}(u', v) \mid u' \in R_v\}$), and let $s_i = |\{u' \mid u' \in R_v,\, d_{R_v}(u', v) \ge i\}|$.

Lemma 5.3. *For any function $g : \mathbb{R}_\infty \to \mathbb{R}$ with $g(\infty) = 0$,*

$$\phi_u^{\text{Shapley}}(g(d_{R_v}(\cdot, v))) = \frac{1}{|R_v|} g(k) + \frac{1}{|R_v|!} \sum_{k < i \le \Delta} (g(k) - g(i)) \left(\sum_{0 \le j \le s_i} \frac{s_i!}{j!} - \sum_{0 \le j \le s_{i+1}} \frac{s_{i+1}!}{j!} \right).$$

In $O(|R_v|)$ time we can compute this value for all nodes in R_v (assuming infinite precision). For degree centrality, $\phi_u^{\text{Shapley}}(g^{\text{deg}}(d_{R_v}(\cdot, v))) = 1/|\{w \mid d_{R_v}(w, v) = 1\}|$ if $d_{R_v}(u, v) = 1$, and otherwise it is 0. For reachability centrality, $\phi_u^{\text{Shapley}}(g^{\text{rch}}(d_{R_v}(\cdot, v))) = 1/|R_v|$.

Proof: We prove only for general g. For u at the k-th level,

$$\begin{aligned}
&\phi_u^{\text{Shapley}}(g(d_{R_v}(\cdot, v))) \\
&= \frac{1}{|R_v|!} \sum_{\pi} (g(d_{R_v}(S_{\pi,u} \cup \{u\}, v)) - g(d_{R_v}(S_{\pi,u}, v))) \\
&= \frac{1}{|R_v|} g(k) + \frac{1}{|R_v|!} \sum_{\pi} \sum_{k < i \le \Delta} \mathbb{I}[d_{R_v}(S_{\pi,u}, v) = i](g(k) - g(i)) \\
&= \frac{1}{|R_v|} g(k) + \frac{1}{|R_v|!} \sum_{k < i \le \Delta} (g(k) - g(i)) \sum_{1 \le j \le |R_v|} \left[\binom{s_i}{j-1} (j-1)! - \binom{s_{i+1}}{j-1} (j-1)! \right] \\
&= \frac{1}{|R_v|} g(k) + \frac{1}{|R_v|!} \sum_{k < i \le \Delta} (g(k) - g(i)) \left(\sum_{1 \le j \le s_i + 1} \frac{s_i!}{(s_i - j + 1)!} - \sum_{1 \le j \le s_{i+1} + 1} \frac{s_{i+1}!}{(s_{i+1} - j + 1)!} \right) \\
&= \frac{1}{|R_v|} g(k) + \frac{1}{|R_v|!} \sum_{k < i \le \Delta} (g(k) - g(i)) \left(\sum_{0 \le j \le s_i} \frac{s_i!}{j!} - \sum_{0 \le j \le s_{i+1}} \frac{s_{i+1}!}{j!} \right).
\end{aligned}$$

One possible way to compute the value above is:

1. Compute $x!$ and $\sum_{0 \le i \le x} \frac{1}{i!}$ for all $x \in [|R_v|]$ in time $O(|R_v|)$.

2. Now the term $\sum_{0 \le j \le s_i} \frac{s_i!}{j!}$ can be computed in additional constant time for any i. We can compute

$$\frac{1}{|R_v|!} \sum_{k < i \le \Delta} (g(k) - g(i)) \left(\sum_{0 \le j \le s_i} \frac{s_i!}{j!} - \sum_{0 \le j \le s_{i+1}} \frac{s_{i+1}!}{j!} \right)$$

for all $0 \le k \le \Delta$ in additional total time $O(\Delta)$ by first computing a suffix sum of

$$g(i) \left(\sum_{0 \le j \le s_i} \frac{s_i!}{j!} - \sum_{0 \le j \le s_{i+1}} \frac{s_{i+1}!}{j!} \right).$$

That is,

$$\sum_{k < i \le \Delta} g(i) \left(\sum_{0 \le j \le s_i} \frac{s_i!}{j!} - \sum_{0 \le j \le s_{i+1}} \frac{s_{i+1}!}{j!} \right).$$

for all $k \in \{0, \ldots, \Delta\}$.

3. Assign the values to vertices in each level in total time $O(|R_v|)$.

□

Therefore, the time complexity follows [7]:

Theorem 5.2. *Under the assumption that sampling a triggering set $T(v)$ takes time at most $O(|N^-(v)|)$ time, and the condition $\ell \ge (\log_2 k - \log_2 \log_2 n)/\log_2 n$, the expected running time of* ICE-RR *is $O(\ell(m+n)\log n \cdot \mathbb{E}[\sigma(\tilde{\boldsymbol{v}})]/(\psi^{(k)}\varepsilon^2))$, where $\mathbb{E}[\sigma(\tilde{\boldsymbol{v}})]$ is the expected influence spread of a random node $\tilde{\boldsymbol{v}}$ drawn from V with probability proportional to the in-degree of $\tilde{\boldsymbol{v}}$.*

Theorems 5.1 and 5.2 together show that our algorithm ICE-RR provides a framework to efficiently estimate all individual and Shapley centralities in the family of influence-based stochastic sphere-of-influence centralities. We further remark that, although algorithm ICE-RR is shown for computing individual centralities and Shapley centralities, it can be easily adapted to computing group centralities as well. Of course, a group centrality has 2^n values, so it is not feasible to list all of them. But if we consider that the algorithm is to estimate n group centrality values for n given sets, then we only need to replace est_u with est_S for every S in the input, and change the lines corresponding to individual centrality (lines 10 and 28) to "for each S in the input, $\text{est}_S = \text{est}_S + g(d_{R_v}(S, v))$". This change is enough for estimating n group centrality values.

6 Future Work

Many topics concerning the interaction between network centralities and influence dynamics can be further explored. One open question is how to extend other centralities that are not covered by sphere-of-influence to influence-based centralities. For example, betweenness centrality of a node v is determined not only by the distance from v to other nodes, but by all-pair distances, while PageRank and other eigenvalue centralities are determined by the entire graph structure. Therefore, one may need to capture further aspects of the influence propagation to provide natural extensions to these graph-theoretical centralities. Another open question is how to characterize centrality for a class of influence profiles, e.g. all submodular influence profiles, all triggering models, etc. Empirical comparisons of different influence-based centralities, as well as studying the applications that could utilize influence-based centralities, are all interesting and important topics worth further investigation. In spirit of Ker-I Ko's last project, how should we formulate and characterize centrality in competitive influence model?

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