Generalized Evidence Passing for Effect Handlers

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This paper studies compilation techniques for algebraic effect handlers. In particular, we present a sequence of refinements of algebraic effects, going via multi-prompt delimited control, *generalized evidence passing*, yield bubbling, and finally a monadic translation into plain lambda calculus which can be compiled efficiently to many target platforms. Along the way we explore various interesting points in the design space. We provide two implementations of our techniques, one as a library in Haskell, and one as a C backend for the Koka programming language. We show that our techniques are effective, by comparing against three other best-in-class implementations of effect handlers: multi-core OCaml, the *Ev.Eff* Haskell library, and the libhandler C library. We hope this work can serve as a basis for future designs and implementations of algebraic effects.

1 INTRODUCTION

Algebraic effects and handlers [Plotkin and Power 2003; Plotkin and Pretnar 2013] provide a powerful and flexible way to add structured control-flow abstraction to programming languages. Unfortunately, it is not straightforward to compile effect handlers into efficient code: effect operations are generally able to capture- and resume a delimited continuation, which usually requires special runtime support to do efficiently. For example, the effect handler implementation in multi-core OCaml [Dolan et al. 2017; Sivaramakrishnan et al. 2021] relies on a runtime system that uses segmented stacks which can be captured efficiently. Then, a natural question that arises is whether it is possible to compile effect handlers efficiently where the target platform does not directly support delimited continuations, for example, when compiling to C/LLVM, WASM [Haas et al. 2017], JavaScript, Java VM, .NET, etc.

In this paper we give a formalized translation and evaluation semantics from a typed effect handler calculus into a plain typed lambda calculus as a sequence of refinements:

(1) First we show how effect handler semantics can be expressed using standard *multi-prompt delimited control* semantics [Forster et al. 2019; Gunter et al. 1995] (Section 2.3).
(2) We refine this semantics further to *evidence passing semantics* (EPS) where the *evidence* for a handler prompt in the evaluation context is pushed down to each effect operation as an *evidence vector* (Section 2.4 and 3.1). This makes performing an operation a *local* transition that no longer needs to search through the evaluation context (Section 2.5 and 3.2).
(3) Next we also localize yielding to a handler prompt by *bubbling* each yield through the evaluation context instead of capturing in one step (Section 2.6 and 4.1). This closely follows the effect handler semantics as given by Pretnar [2015].
(4) With all evaluation transitions localized, we can now define a direct *monadic translation* of effect handlers into a plain typed lambda calculus using a multi-prompt monad (Section 2.7, 2.8, and 4). Such program can be directly compiled to any target platform (including C/LLVM, WASM, JavaScript, Java VM, .NET, etc) without requiring special runtime mechanisms .

Aside from the novel evidence passing semantics, many parts of the refinements are known compilation techniques for effect handlers – but we believe we are the first to formalize each within a single polymorphically typed framework (combined with evidence passing semantics). Specifically, we make the following contributions:

- We formalize each refinement and translation, and show they are sound and semantics preserving (Section 3 and 4). Along the way, we explore various interesting points in the design space:
  - The use of *segmented stacks* for implementing effect handlers in a *direct* way (as used by multi-core OCaml [Sivaramakrishnan et al. 2021]) versus translation into a multi-prompt monad (Section 2.3): segmented stacks need a dedicated runtime system but can capture and resume an operation in constant time (for *one-shot* resumptions), while a multi-prompt monad is linear in the continuation points.
  - Using *insertion-* versus *canonical* ordered evidence vectors (Section 2.4): the former is efficient to construct but needs a linear lookup for each operation, while a canonical vector is more expensive to construct upfront but can use constant time lookup for operations.
  - Using *short-cut resumptions* to minimize the stack usage of a resumption while increasing sharing of continuation points (Section 2.6.1); a similar technique is used in [Kiselyov and Ishii 2015] to compose monadic binds in an effect monad.
  - Using *bind-inlining* and *join-point sharing* for improved efficiency when translating into the multi-prompt monad (Section 2.7.1).
- Our evidence passing *semantics* (EPS) is a generalization of the work on evidence passing *translation* (EPT) [Xie et al. 2020]. In particular, EPT can only express a subset of full effect handlers that are restricted to *scoped* resumptions only, whereas EPS lifts the restriction and can fully express effect handlers (Section 2.9 and 3.1).
- We give the first formal account of optimized *tail-resumptive* operation semantics and show how this can evaluate an operation in-place and avoid performing an expensive yield-and-resume cycle in the majority of effect operations (Section 2.5 and 3.2). The tail-resumptive optimization is surprisingly subtle to get correct – in particular in combination with *unscoped resumptions* which we illustrate in Section 2.9.2. We prove the correctness of the tail-resumptive optimization by showing that an optimized program is contextually equivalent to the original one.
- We have implemented our techniques as a monadic library for effect handlers in Haskell, called *Mp.Eff* (for "multi-prompt effect"), generalizing the *Ev.Eff* library based on EPT provided by Xie and Leijen [2020]. Our implementation is based on insertion-ordered evidence vectors.
- We have also implemented our techniques in the Koka programming language [Koka 2019] compiling to standard C code (Section 2.8). The implementation uses canonical evidence vectors, short-cut resumptions, bind-inlining, and join-point sharing.
- We benchmarked the Koka implementation against four other implementations of effect handlers that compile to native code: the current state-of-the-art *direct* implementation of effect handlers in multi-core OCaml which uses a dedicated runtime system based on segmented stacks; our *Mp.Eff* Haskell library; the *Ev.Eff* Haskell library which has been shown by Xie and Leijen [2020] to perform very well compared to other Haskell effect handler libraries [Schrijvers et al. 2019; Wu and Schrijvers 2015; Wu et al. 2014]; and finally the *libhandler* C library which implements effect handlers directly in C by copying fragments of the stack [Leijen 2017a]. Comparing across systems and languages is always tricky but the results clearly indicate that our approach can have competitive performance (Section 5).

2 OVERVIEW

We start with a short discussion and examples of basic effect handlers and follow with an overview of each of our semantic refinements and translation techniques.

2.1 Algebraic Effects

With algebraic effect handlers, an effect l defines a set of operations op. For example, we can have a reader effect with an *ask* operation

$$\mathit{read}\ \{\ \mathit{ask}\ :\ ()\ \to\ \mathit{int}\ \}$$

and we can perform the *ask* operation writing perform *ask* (). A handler (handler h v) takes a list of operation clauses in h, and a computation v to be handled. Each operation clause in h takes the form $op \mapsto f$, providing the implementation f for the operation op from the handled effect where the implementation f is of form $\lambda x.\ \lambda k.\ e$: x binds the operation argument, and k binds the captured *resumption* that can be used to resume to the original call-site with the operation result. For example, we can handle the reader effect by always resuming with the constant 1:

$$h^{\mathit{read}}\ =\ \{\ \mathit{ask} \mapsto \lambda x.\ \lambda k.\ k\ 1\ \}$$

where the expression handler h^{read} (λ_. perform *ask* () + perform *ask* ()) evaluates to 2. The following evaluation rules give the essence of the untyped semantics for algebraic effect handlers [Xie and Leijen 2020]:

$$\begin{array}{lllll}
(\mathit{app}) & (\lambda x.\ e)\ v & \longrightarrow & e[x{:=}v] & \\
(\mathit{handler}) & \mathsf{handler}\ h\ f & \longrightarrow & \mathsf{handle}\ h\ (f\ ()) & \\
(\mathit{return}) & \mathsf{handle}\ h\ v & \longrightarrow & v & \\
(\mathit{perform}) & \mathsf{handle}\ h\ \mathsf{E}[\mathsf{perform}\ op\ v] & \longrightarrow & f\ v\ (\lambda x.\ \mathsf{handle}\ h\ \mathsf{E}[x]) & \\
 & & & \text{iff } op \notin \mathsf{bop}(\mathsf{E}) \wedge (op \mapsto f) \in h &
\end{array}$$

Rule (*app*) is standard β-reduction and applies a function to a value v by substituting x for the argument v in the function body. The (*handler*) takes a computation f, and applies the computation to a unit value under a new frame handle h. The computation to be handled (f) is always a unit-taking function as in [Xie et al. 2020], which essentially corresponds to a suspended computation as in the call-by-push-value approach [Levy 2006] used in several algebraic effect systems [Kammar and Pretnar 2017; Plotkin and Pretnar 2013].

The handle frame is only generated by handler, and treated as a strictly internal frame. When handling a computation under a handle h frame, there are two possible situations. In the first case, the computation evaluates to a value and the (*return*) transition discards the handle h frame and propagates the value. The second case captures the essence of algebraic effects handlers where an operation is *handled*. In rule (*perform*), perform op v calls an effect operation op by providing the operation argument v. The handle h frame handles the operation by applying the operation implementation f to the operation argument v, and the *resumption* ($\lambda x.$ handle h E[x]). The resumption captures the original handle, as well as the whole *evaluation context* E between handle and the operation call.

An evaluation context E is essentially an expression with a hole (□) in it, and the notation E[e] represents the expression obtained by plugging e into the hole of E (e.g., $(f\ (g\ \square))[x]\ =\ f\ (g\ x)$). In this rule, the condition $op \notin \mathsf{bop}(\mathsf{E})$ indicates that op is not in the bound operations of E, i.e. not handled by any handle frames in E, ensuring that it is always the *innermost* handle frame for the effect that handles the operation.

2.1.1 Examples. Here we consider some standard examples of algebraic effects, and we refer the reader to other work for more examples as well as practical uses of effect handlers [Bauer and Pretnar 2015; Hillerström and Lindley 2016; Kammar et al. 2013; Leijen 2017b; Pretnar 2015; Xie et al. 2020]. In the examples, we use $x \leftarrow e_1;\ e_2$ as a shorthand for $(\lambda x.\ e_2)\ e_1$, and use $e_1; e_2$ for $(\lambda\_.\ e_2)\ e_1$, where λ_ denotes a lambda whose binding is not used in the body.

Exceptions. The following definition defines an effect *exn* with one operation *throw*.

$$exn\ \{\ throw\ :\ \forall\alpha.\ ()\ \to\ \alpha\ \}$$

Given a datatype Maybe with two constructors Just and Nothing, we can define a handler for exceptions that reifies any exceptional computation with a Maybe result to return Nothing on an exception:

$$h^{exn}\ =\ \{\ throw \mapsto \lambda x.\ \lambda k.\ \mathsf{Nothing}\ \}$$

For example, suppose we define safe division as:

$$safediv\ =\ \lambda x\ y.\ \mathsf{if}\ (y == 0)\ \mathsf{then\ perform}\ throw\ ()\ \mathsf{else}\ x/y$$

then we have

$$\begin{array}{ll}
\quad \mathsf{handler}\ h^{exn}\ (\lambda_.\ \mathsf{Just}\ (safediv\ 42\ 2)) & \quad \mathsf{handler}\ h^{exn}\ (\lambda_.\ \mathsf{Just}\ (safediv\ 42\ 0)) \\
\longmapsto^* \mathsf{handle}\ h^{exn}\ (\mathsf{Just}\ (42/2)) & \longmapsto^* \mathsf{handle}\ h^{exn}\ (\mathsf{Just\ (perform}\ throw\ ())) \\
\longmapsto \mathsf{handle}\ h^{exn}\ (\mathsf{Just}\ 21) & \longmapsto (\lambda x.\ \lambda k.\ \mathsf{Nothing})\ ()\ (\lambda x.\ \mathsf{handle}\ h^{exn}\ (\mathsf{Just}\ x)) \\
\longmapsto \mathsf{Just}\ 21 & \longmapsto^* \mathsf{Nothing}
\end{array}$$

We use the notation $\longmapsto$ to allow expressions to take steps ($\longrightarrow$) inside evaluation contexts where $\longmapsto^*$ is the transitive closure of $\longmapsto$.

Reader. In the previous example we did not make use of the operation argument (x) or the resumption (k). Let's consider this time the evaluation of our first example with the *reader* effect:

$$\begin{array}{ll}
& \mathsf{handler}\ h^{read}\ (\lambda_.\ \mathsf{perform}\ ask\ ()\ +\ \mathsf{perform}\ ask\ ()) \\
\longmapsto^* & \mathsf{handle}\ h^{read}\ (\mathsf{perform}\ ask\ ()\ +\ \mathsf{perform}\ ask\ ()) \\
\longmapsto & (\lambda x.\ \lambda k.\ k\ 1)\ ()\ (\lambda x.\ \mathsf{handle}\ h^{read}\ (x\ +\ \mathsf{perform}\ ask\ ())) \\
\longmapsto^* & (\lambda x.\ \mathsf{handle}\ h^{read}\ (x\ +\ \mathsf{perform}\ ask\ ()))\ 1 \\
\longmapsto & \mathsf{handle}\ h^{read}\ (1\ +\ \mathsf{perform}\ ask\ ()) \quad \longmapsto^* (\lambda x.\ \mathsf{handle}\ h^{read}\ (1\ +\ x))\ 1 \quad \longmapsto^* 2
\end{array}$$

where both *ask* operations resume back to the original calling context with a result.

State. We can define a state handler using the monadic encoding [Kammar and Pretnar 2017], where performing an operation returns a function that takes in the current state.

$$\begin{array}{ll}
st\ \{\ get\colon\ ()\ \to\ \alpha, & h^{st}\ =\ \{get \mapsto \lambda x.\ \lambda k.\ (\lambda y.\ k\ y\ y), \\
\quad set\ :\ \alpha\ \to\ ()\ \} & \qquad\quad set \mapsto \lambda x.\ \lambda k.\ (\lambda y.\ k\ ()\ x)\ \}
\end{array}$$

The following program starts with an initial state 0.

$$\begin{array}{ll}
& (\mathsf{handler}\ h^{st}\ (\lambda_.\ \mathsf{perform}\ set\ 21;\ w \leftarrow \mathsf{perform}\ get\ ();\ (\lambda z.\ w\ +\ w))\ ))\ 0 \\
\longmapsto & (\mathsf{handle}\ h^{st}\ (\mathsf{perform}\ set\ 21;\ w \leftarrow \mathsf{perform}\ get\ ();\ (\lambda z.\ w\ +\ w))\ ))\ 0
\end{array}$$

In the following derivation, we make use of the *dot notation* [Xie and Leijen 2020]. Specifically, the notation $\mathsf{E}_1\ \bullet\ \mathsf{E}_2$ composes two evaluation contexts by plugging E_2 into the hole of E_1, resulting in a new evaluation context. The ($\bullet$) notation is right-associative and has the lowest precedence, so we often write $\mathsf{E}_1\ \bullet\ \mathsf{E}_2$ instead of $(\mathsf{E}_1)\ \bullet\ \mathsf{E}_2$. The notation $\mathsf{E}\ \bullet\ e$ has the same meaning as $\mathsf{E}[e]$, which plugs e into the hole of E, resulting in a new expression. Using the dot notation, the evaluation order of expressions becomes more apparent, and it is now easier to discuss one specific frame in the chain of evaluation contexts. We start by rewriting the last expression using the dot notation as:

$$=\quad \Box\ 0\ \bullet\ \mathsf{handle}\ h^{st}\ \Box\ \bullet\ (\Box;\ w \leftarrow \mathsf{perform}\ get\ ();\ (\lambda z.\ w\ +\ w))\ \bullet\ \mathsf{perform}\ set\ 21$$

For conciseness, we also often omit a trailing $\Box$ in an application context $e\ \Box\ \bullet\ \mathsf{E}$ and write instead $e\ \bullet\ \mathsf{E}$; this is usually the case for handle expressions:

$$=\quad \Box\ 0\ \bullet\ \mathsf{handle}\ h^{st}\ \bullet\ (\Box;\ w \leftarrow \mathsf{perform}\ get\ ();\ (\lambda z.\ w\ +\ w))\ \bullet\ \mathsf{perform}\ set\ 21$$

Writing contexts this way, it shows more clearly the stack of evaluation frames with the expression under evaluation at the end. We can now continue evaluating as:

$$\begin{array}{ll} \longmapsto^* \Box\ 0 \bullet (\lambda y.\ k\ ()\ 21) & \text{with } k = \lambda x.\ \mathsf{handle}\ h^{st} \bullet (\Box;\ w \leftarrow \mathsf{perform}\ get\ ();\ (\lambda z.\ w + w)) \bullet x \\ = \quad (\lambda y.\ k\ ()\ 21)\ 0 & \longmapsto k\ ()\ 21 \\ \longmapsto \Box\ 21 \bullet \mathsf{handle}\ h^{st} \bullet (\Box;\ w \leftarrow \mathsf{perform}\ get\ ();\ (\lambda z.\ w + w)) \bullet () & \\ = \quad \Box\ 21 \bullet \mathsf{handle}\ h^{st} \bullet ((\,);\ w \leftarrow \mathsf{perform}\ get\ ()\ ;\ (\lambda z.\ w + w)) & \\ \longmapsto \Box\ 21 \bullet \mathsf{handle}\ h^{st} \bullet (w \leftarrow \Box;\ (\lambda z.\ w + w)) \bullet \mathsf{perform}\ get\ () & \longmapsto^* 42 \end{array}$$

While this is a nice example of the expressiveness of effect handlers, it is clearly not the most efficient way to express mutable state. In practice, state can be implemented more efficiently using *parameterized handlers* [Plotkin and Pretnar 2009] or a primitive state handler [Xie and Leijen 2020].

Non-determinism. By having the resumption k available when handling, we can actually resume more than once. In the handler of *amb*, we implement non-determinism by collecting all possible results in a list by resuming the resumption twice, each time with one boolean result.

$$\begin{array}{ll} amb\ \{\ flip : () \to bool\ \} & \mathsf{handler}\ h^{amb}\ (\lambda\_.\ \ x \leftarrow \mathsf{perform}\ flip\ (); \\ h^{amb} = \{\ flip \mapsto \lambda\_.\ k.\ \ xs \leftarrow k\ \mathsf{True}; & \qquad\qquad\qquad\quad y \leftarrow \mathsf{perform}\ flip\ (); \\ \qquad\qquad\qquad\quad ys \leftarrow k\ \mathsf{False}; & \qquad\qquad\qquad\quad [x\ \&\&\ y]) \\ \qquad\qquad\qquad\quad xs\ ++\ ys\ \} & \longmapsto^* [\mathsf{True}, \mathsf{False}, \ \mathsf{False}, \mathsf{False}] \end{array}$$

2.2 Compiling Effect Handlers

As the examples show, algebraic effect handlers can be very expressive. Unfortunately, their expressive power also makes it not easy to compile them efficiently. The main culprit is the (*perform*) rule:

$$\mathsf{handle}\ h\ \mathsf{E}[\mathsf{perform}\ op\ v] \longrightarrow f\ v\ (\lambda x.\ \mathsf{handle}\ h\ \mathsf{E}[x]) \quad \text{iff } op \notin \mathsf{bop}(\mathsf{E})\ \wedge\ (op \mapsto f) \in h$$

This single rule combines two potentially expensive runtime operations:

(1) *Searching*: The innermost handler for *op* must be found which usually requires a linear search through the current handlers in the evaluation context (i.e. search up through the stack frames).
(2) *Capturing*: After finding the handler clause f, we need to capture the evaluation context (i.e. stack and registers) up to the found handler, and create a *resumption* function $(\lambda x.\ \mathsf{handle}\ h\ \mathsf{E}[x])$ which restores the captured context when invoked with a result. An added complication is that in the general case such resumption may never be called (as in h^{exn}), or invoked more than once (as in h^{amb}), which can present difficulties in the runtime (for scanning GC roots for example).

Capturing and restoring resumptions can be done relatively efficiently if the target runtime system implements segmented stacks (as in multi-core OCaml [Dolan et al. 2015] for example). However, many target platforms do not support directly capturing parts of the stack at all, like compilation to C (as in Koka), WASM, .NET, the Java VM, JavaScript, etc, and in these cases it is not even possible to implement (*perform*) in any direct way.

In this paper we address these compilation and runtime challenges by presenting various refinements of the operational semantics in combination with source translations. Each of these steps enables further optimizations and implementations, and we explore various interesting points in the design space along the way.

2.3 Multi-Prompt Semantics

As a first step, we are going to split the (*perform*) operation into two parts where we separate the searching for a handler from capturing and restoring a resumption. To capture and restore

a resumption we are going to use standard (typed) multi-prompt delimited control [Gunter et al. 1995]: instead of a handle h frame, we install a prompt $m\ h$ frame that is uniquely identified with a *marker* m, and performing an operation will use a yield $m\ f$ frame to yield to such prompt.

As an example, consider again the *reader* effect handler $h^{read} = \{\ ask \mapsto \lambda x.\ \lambda k.\ 1\ \}$, where we have the following evaluation (rewritten using the dot notation):

$$\begin{array}{ll} & \mathsf{handler}\ h^{read}\ (\lambda\_.\ \mathsf{perform}\ ask\ ()\ +\ \mathsf{perform}\ ask\ ()) \\ \longmapsto^* & \mathsf{handle}\ h^{read}\ \bullet\ (\square +\ \mathsf{perform}\ ask\ ())\ \bullet\ \mathsf{perform}\ ask\ () \\ \longmapsto & f\ ()\ (\lambda x.\ \mathsf{handle}\ h^{read}\ \bullet\ (\square +\ \mathsf{perform}\ ask\ ())\ \bullet\ x) \quad \text{where } f = \lambda x.\ \lambda k.\ 1 \\ \ldots & \end{array}$$

When using multi-prompt semantics, the first transition now installs a prompt $m\ h^{read}$ frame *instead* of a handle frame, where m is a unique *marker* identifying the prompt:

$$\begin{array}{ll} & \mathsf{handler}\ h^{read}\ (\lambda\_.\ \mathsf{perform}\ ask\ ()\ +\ \mathsf{perform}\ ask\ ()) \\ \longmapsto^* & \mathsf{prompt}\ m\ h^{read}\ (\mathsf{perform}\ ask\ +\ \mathsf{perform}\ ask\ ()) \\ = & \mathsf{prompt}\ m\ h^{read}\ \bullet\ (\square +\ \mathsf{perform}\ ask\ ())\ \bullet\ \mathsf{perform}\ ask\ () \end{array}$$

The next transition shows how we separate searching from capturing – perform $ask\ ()$ only finds the handler clause f but defers yielding to the prompt by using an explicit yield frame:

$$\longmapsto \mathsf{prompt}\ m\ h^{read}\ \bullet\ (\square +\ \mathsf{perform}\ ask\ ())\ \bullet\ \mathsf{yield}\ m\ (\lambda k.\ f\ ()\ k)$$

The yield $m\ g$ has two arguments: the marker m that uniquely identifies the prompt to yield to, and a function g that is applied to the resumption when reaching the prompt. Through the marker m, we can yield directly to the prompt which captures and applies the resumption:

$$\begin{array}{l} \longmapsto (\lambda k.\ f\ ()\ k)\ (\lambda x.\ \mathsf{prompt}\ m\ h^{read}\ \bullet\ (\square +\ \mathsf{perform}\ ask\ ())\ \bullet\ x) \\ \longmapsto f\ ()\ (\lambda x.\ \mathsf{prompt}\ m\ h^{read}\ \bullet\ (\square +\ \mathsf{perform}\ ask\ ())\ \bullet\ x) \\ \ldots \end{array}$$

This separation of concerns does not immediately buy us much but, as we will see, it opens up the way for optimizing each part individually by (1) using evidence passing semantics to avoid searching, and (2) using a monadic translation to enable capturing without requiring a special runtime system. Moreover, multi-prompt delimited control is one of the lowest level control operations that can be typed in the simply typed lambda calculus.

If one controls the target platform, it is possible to efficiently implement multi-prompt delimited control directly. This is done for example in multi-core OCaml using segmented stacks: here the call stack is split in segments where each prompt frame starts a fresh segment. The marker m can be implemented directly as the runtime pointer to that frame. Yielding up to a parent stack segment is now a constant time operation as only the stack segment pointer needs to be adjusted. Resuming once can also be done in constant time this way, but supporting multi-shot resumptions still requires a linear copy of the resumption stack segments (and this is one of the reasons why this is not directly supported in multi-core OCaml).

2.4 Evidence Passing Semantics

The (*perform*) operation is still a non-local transition as it searches through the evaluation context to find the innermost handler. We can make it local using *evidence passing semantics*, where we pass the current handlers in the evaluation context explicitly as an extra *evidence vector* w down to the perform operations. Instead of searching through the context, we can now look up the handler locally. Essentially, if the current evidence vector is w, then the (*perform*) rule becomes:

$$\mathsf{perform}\ op\ v \longrightarrow \mathsf{yield}\ m\ (\lambda k.\ f\ v\ k) \quad \text{where } (m, h) = w.l \wedge (op \mapsto f) \in h$$

The expression $w.l$ directly looks up the marker and handler (called *evidence*) for effect l from the evidence vector w. We apply the idea to our example, where we use the $\frown$ notation to indicate the current evidence vector and we sometimes omit the notation when it is irrelevant or obvious from the context. Evaluation always starts with an empty evidence vector $\langle\!\langle\rangle\!\rangle$:

$$\overbrace{\text{handler } h^{read}\ (\lambda\_.\ \text{perform } ask\ ()\ +\ \text{perform } ask\ ())}^{\langle\!\langle\rangle\!\rangle}$$

which evaluates into:

$$\longmapsto^* \overbrace{\text{prompt } m\ h^{read}}^{\langle\!\langle\rangle\!\rangle}\ \bullet\ \overbrace{(\square +\ \text{perform } ask\ ())\ \bullet\ \text{perform } ask\ ()}^{w\ =\ \langle\!\langle read : (m, h^{read})\rangle\!\rangle}$$

where the prompt frame modifies the evidence for rest of the evaluation context. At this point perform evaluates under an evidence vector $\langle\!\langle read : (m, h^{read})\rangle\!\rangle$, and we get:

$$\begin{aligned}
&\longmapsto \text{prompt } m\ h^{read}\ \bullet\ (\square +\ \text{perform } ask\ ())\ \bullet\ \text{yield } m\ (\lambda k.\ f\ ()\ k) \quad \text{where } (m,\ h^{read})\ =\ w.read\\
&\longmapsto (\lambda k.\ f\ ()\ k)\ (\lambda x.\ \text{prompt } m\ h^{read}\ \bullet\ (\square +\ \text{perform } ask\ ())\ \bullet\ x)\\
&\longmapsto f\ ()\ (\lambda x.\ \text{prompt } m\ h^{read}\ \bullet\ (\square +\ \text{perform } ask\ ())\ \bullet\ x)\\
&\ldots
\end{aligned}$$

Using evidence passing semantics makes the (*perform*) transition localized which can potentially be more efficient than searching through the evaluation context. When we treat the evidence vector as an abstract datatype there are two interesting variants depending on how the vectors are ordered:

(1) *Insertion order*: Insert handler evidence in the order of the actual handlers in the evaluation context. This is straightforward and also the approach we take in the associated Haskell library. However, it means that the lookup operation $w.l$ still needs to search linearly through the vector for the "innermost" handler. One way to implement such vector is as a linked list where each handler pushes itself on the list. Since evidence vectors are not first-class values, we can actually allocate this list on the evaluation stack directly and as such it becomes a linked list of handlers at runtime – this is exactly how many languages implement exception handlers where the w parameter is a pointer to the head of the exception handler list.

(2) *Canonical order*: Use a lexicographic order of the handler evidence based on their effect label. This requires a strongly typed calculus but it means that if the effect type is fully known at compile time, we can *statically* determine the index for a particular effect in the runtime evidence vector. For example, in systems that keep track of the effect type of expressions using *row types* [Hillerström and Lindley 2016; Leijen 2017b], the effect type of our example perform *ask* () is the singleton effect row $\langle read \rangle$, and we know statically that the dynamic runtime evidence vector will have the form $\langle\!\langle read : \_ \rangle\!\rangle$. We can thus replace the linear runtime lookup $w.read$ with a *constant-time* array access $w[0]$ instead. This is the approach used in the Koka compiler.

2.5 Tail-Resumptive Operations

With evidence semantics in place, the only expensive operation left is yielding and capturing a resumption. Fortunately, we can often avoid doing a full yield: almost all common operations in practice happen to be *tail resumptive* where the operation clause has the form:

$$op \mapsto \lambda x.\ \lambda k.\ k\ e \qquad \text{where } k \notin \text{fv}(e)$$

For example, the *ask* operation in our h^{read} handler is of this form[2]. It turns out we can perform such operations *in place*: instead of yielding up and eventually resuming with the final result, we can directly evaluate e on the current stack without doing an expensive yield followed by a resume. To this end, we extend each evidence in the evidence vector to store a triple (m, h, w) (instead of a tuple (m, h)), where the third component w is the *evidence context*: this is the evidence vector under which the handler h is defined and is used for the evaluated-in-place expression. We illustrate the use of this in our running example:

$$\overbrace{\text{handler } h^{read}\ (\lambda\_.\ \text{perform } ask\ ()\ +\ \text{perform } ask\ ())}^{\langle\rangle}$$
$$\longmapsto^* \overbrace{\text{prompt } m\ h^{read}}^{\langle\rangle}\ \bullet\ \overbrace{(\square +\ \text{perform } ask\ ())}^{\langle read : (m, h^{read}, \langle\rangle)\rangle}\ \bullet\ \text{perform } ask\ ()$$

Here, the evidence vector for perform is $\langle\!\langle read : (m, h^{read}, \langle\!\langle\rangle\!\rangle)\rangle\!\rangle$ and we can locally find the operation clause $ask \to \lambda x\ k.\ k\ 1 \in h^{read}$ and determine that it is tail-resumptive. Instead of generating yield as before, we instead evaluate e (as in $\lambda x.\ \lambda k.\ k\ e$, with e being 1 in this case) *in-place*:

$$\begin{aligned}
&\longmapsto \text{prompt } m\ h^{read}\ \bullet\ (\square +\ \text{perform } ask\ ())\ \bullet\ \text{under } read\ \bullet\ 1\\
&\longmapsto \text{prompt } m\ h^{read}\ \bullet\ (\square +\ \text{perform } ask\ ())\ \bullet\ 1\\
&\longmapsto \text{prompt } m\ h^{read}\ (1\ +\ \text{perform } ask\ ())\\
&\ldots
\end{aligned}$$

The operation clause is now evaluated in-place – but note it needs to be evaluated under an under l frame. Such frame ensures that if the operation clause e itself performs operations, these are resolved correctly with respect to the actual handler up in the evaluation context. Consider for example the following reader handler:

$$h_2\ =\ \{\ ask \mapsto \lambda x.\lambda k.\ k\ (\text{perform } ask\ ()\ +\ 1)\ \}$$

Here the operation clause is tail-resumptive, and itself performs an *ask* operation. Now consider:

$$\text{handler } h^{read}\ (\lambda\_.\ \text{handler } h_2\ (\lambda\_.\ \text{perform } ask()))$$
$$\longmapsto^* \overbrace{\text{prompt } m_1\ h^{read}}^{\langle\rangle}\ \bullet\ \overbrace{\text{prompt } m_2\ h_2}^{w_1 = \langle\!\langle read : (m_1, h^{read}, \langle\!\langle\rangle\!\rangle)\rangle\!\rangle}\ \bullet\ \overbrace{\text{perform } ask\ ()}^{w_2 = \langle\!\langle read : (m_2, h_2, w_1),\ read : (m_1, h^{read}, \langle\!\langle\rangle\!\rangle)\rangle\!\rangle}$$

At this point, the evidence vector at the second prompt is $w_1 = \langle\!\langle read : (m_1, h^{read}, \langle\!\langle\rangle\!\rangle)\rangle\!\rangle$, but the evidence vector at the perform contains two entries: $w_2 = \langle\!\langle read : (m_2, h_2, w_1),\ read : (m_1, h^{read}, \langle\!\langle\rangle\!\rangle)\rangle\!\rangle$. Here we see how the third member of the evidence always points to the "previous" evidence vector (e.g., w_1) under which a particular hander (e.g., h_2) is defined. If using insertion-ordered evidence vectors as a linked list, this is always just the tail of the list, but for canonical evidence vectors the previous vector must be kept explicitly. Since the operation clause is tail-resumptive, we get:

$$\begin{aligned}
&\longmapsto \overbrace{\text{prompt } m_1\ h^{read}}^{\langle\rangle}\ \bullet\ \overbrace{\text{prompt } m_2\ h_2}^{w_1}\ \bullet\ \overbrace{\text{under } read}^{w_2}\ \bullet\ \overbrace{\text{perform } ask\ ()\ +\ 1}^{w_1}\\
&=\ \ \text{prompt } m_1\ h^{read}\ \bullet\ \text{prompt } m_2\ h_2\ \bullet\ \text{under } read\ \bullet\ (\square +\ 1)\ \bullet\ \text{perform } ask\ ()
\end{aligned}$$

The evidence vector for the perform $ask\ ()\ +\ 1$ is now w_1 and *not* the unchanged w_2. Indeed, it would be incorrect to use w_2 or we would invoke the operation clause of h_2 again! The under *read* frame prevents this from happening and adjusts the evidence vector to the one under which the

[2]While h^{state} is not tail-resumptive, implementations of state based on parameterized handlers or primitive state are both tail-resumptive. h^{exn} and h^{amb} are not tail-resumptive because of their special nature (aborting the computation and non-determinism, respectively).

read handler is itself defined: this is exactly the third component of the evidence, $w_2.read.thd$, which is w_1 in our example. We now continue as:

$$\begin{array}{ll}
& \overbrace{\langle\rangle}\ \overbrace{\text{prompt } m_1\ h^{read}}\ \bullet\ \overbrace{\text{prompt } m_2\ h_2}^{w_1}\ \bullet\ \overbrace{\text{under } read}^{w_2}\ \bullet\ \overbrace{(\square + 1)\ \bullet\ \text{under } read}^{w_1}\ \bullet\ \overbrace{1}^{\langle\rangle} \\
\longmapsto & \text{prompt } m_1\ h^{read}\ \bullet\ \text{prompt } m_2\ h_2\ \bullet\ \text{under } read\ \bullet\ (\square + 1)\ \bullet\ \text{under } read\ \bullet\ 1 \\
\longmapsto & \text{prompt } m_1\ h^{read}\ \bullet\ \text{prompt } m_2\ h_2\ \bullet\ \text{under } read\ \bullet\ 2 \quad \longmapsto^* 2
\end{array}$$

Note how the second under *read* frame adjusts the evidence further to $\langle\!\langle\rangle\!\rangle$ (which is $w_1.read.thd$). The correct formalization of under is subtle, and we will come back to this in Section 2.9.

2.6 Bubbling Yields

Using evidence semantics, perform is a local transition which only leaves yields as a non-local transition for non-tail-resumptive operations. We can further make yield m local by *bubbling* it up until it meets its corresponding prompt m frame in the evaluation context. That is, instead of capturing the delimited evaluation context E wholesale, we are going to build a resumption function piecemeal while bubbling up. To this end, we extend yield $m\ v$ with an extra argument as yield $m\ v\ k$ where k is the current partially built up continuation (starting out as identity).

Consider our earlier exception effect in Section 2.1.1 where:

$$\begin{array}{ll}
& \text{handler } h^{exn}\ (\lambda_.\ safediv\ 42\ 0) \\
\longmapsto^* & \text{handle } h^{exn}\ \bullet\ \text{Just } \square\ \bullet\ \text{perform } throw\ () \\
\longmapsto & (\lambda x\ k.\ \text{Nothing})\ ()\ (\lambda x.\ \text{handle } h^{exn}\ (\text{Just } x))
\end{array}$$

When using yield bubbling we evaluate instead as (writing f for $\lambda x\ k.$ Nothing):

$$\begin{array}{ll}
& \text{handler } h^{exn}\ (\lambda_.\ safediv\ 42\ 0) \\
\longmapsto^* & \text{prompt } m\ h^{exn}\ \bullet\ \text{Just } \square\ \bullet\ \text{perform } throw\ () \\
\longmapsto & \text{prompt } m\ h^{exn}\ \bullet\ \text{Just } \square\ \bullet\ \text{yield } m\ (f\ ())\ id
\end{array}$$

At this point, the yield does a local transition and bubbles up only one step through the Just application, resulting in

$$\longmapsto \text{prompt } m\ h^{exn}\ \bullet\ \text{yield } m\ (f\ ())\ (\text{Just} \circ id)$$

Note how the resumption function changed from the initial identity id to the composition Just $\circ$ id. Generally, yields keep bubbling up this way extending their current resumption until they meet their target prompt:

$$\longmapsto f\ ()\ (\lambda x.\ \text{prompt } m\ h^{exn}\ ((\text{Just} \circ id)\ x)) \quad \longmapsto^* \text{Nothing}$$

Using bubbling removes any direct manipulation of the evaluation context E and only regular functions are used instead. The bubbling technique for implementing delimited continuations is well known and also used for example to give direct semantics to effect handlers [Kiselyov and Sivaramakrishnan 2017; Pretnar 2015].

2.6.1 Short-cut Resumptions. When bubbling up, a resumption is built up as a composition of continuations, $f_1 \circ \ldots \circ f_n \circ id$, and when resuming it is applied as $(f_1 \circ \ldots \circ f_n \circ id)\ x$ which will recreate all f_1 to f_n application frames on the evaluation stack which can be expensive. Instead, in an implementation we can represent the composition as a list $[f_1, \ldots, f_n]$, and resume instead as $resume\ [f_1, \ldots, f_n]\ x$ where *resume* folds through the list from the end:

$$resume\ []\ x\ =\ x \qquad\qquad resume\ (fs\ +\!\!+\ [f])\ x\ =\ resume\ fs\ (f\ x)$$

This can be done efficiently by using a queue or array representation (as done in Koka) and also uses minimal stack space by evaluating just one f continuation at a time. Moreover, any *further* yields in a frame f_i will bubble up directly through the current *resume* and thus capture all f_1 to f_{i-1}

continuations in one go (and will itself share those continuations through the various yields). We call these *short-cut resumptions* as these can be resumed by immediately starting at the deepest continuation point. This uses minimal stack space while increasing the use of shared continuations.

Note that while bubbling up we can also encounter prompt and under frames besides regular applications; for example, the final resumption may be of the form $f_1 \circ \ldots \circ f_i \circ \text{prompt } m\ h \circ f_{i+1} \circ \ldots \circ f_n$. When resuming, we need to ensure that such prompt and under frames are properly restored and cannot use short-cut's for those. Of course we can still use *resume* for the application fragments surrounding the prompt/under frames, e.g. *resume* $[f_1, \ldots, f_i] \circ \text{prompt } m\ h \circ$ *resume* $[f_{i+1}, \ldots, f_n]$.

2.7 Monadic Translation

At this point all transitions are local and no longer capture the evaluation context explicitly. This means we are now able to translate our core calculus into a pure lambda calculus together with a multi-prompt delimited control monad. This is a straightforward transformation where every (effectful) expression is sequenced through a monadic bind. Our running example:

$$\text{handler } h^{read}\ (\lambda\_.\ \text{perform } ask\ ()\ +\ \text{perform } ask\ ())$$

translates to the following monadic expression:

$$\text{handler } h^{read}\ (\lambda\_.\ \text{perform } ask\ ()\ \triangleright\ (\lambda x.\ \text{perform } ask\ ()\ \triangleright\ (\lambda y.\ \text{Pure}\ (x + y))))$$

where we write $(\triangleright)$ for monadic binding, and Pure for lifting pure expressions into the monad. Through the bind operation, the current continuation becomes explicit (as a function argument) and can be captured and resumed using regular function application, where bind is implemented essentially[3] as:

$$\begin{aligned} e \triangleright g\ =\ &\text{case } e \text{ of } \text{Pure } x \longrightarrow g\ x \\ &\quad\ \text{Yield } m\ f\ k \longrightarrow \text{Yield } m\ f\ (\lambda x.\ k\ x \triangleright g) \end{aligned}$$

Pure values are directly propagated while a yield bubbles up (upto its matching prompt) and appends each explicit continuation g to the built up resumption. Since all of this can be expressed in plain typed lambda calculus, this can be directly translated to almost any target platform – all control flow is now fully explicit.

2.7.1 Bind-Inlining and Join-Point Sharing. However, if done naively there may be a high cost to this translation: since every bind operation takes a lambda as its second argument this may lead to many closure allocations even for non-yielding code. Moreover, any direct tail-recursive calls are no longer directly tail-recursive as they occur under a lambda now!

To improve this we need two techniques: bind-inlining and join-point sharing. To avoid always allocating a lambda, we can use bind-inlining to simply inline every bind operation, expanding our example expression to:

$$\begin{aligned} &\text{handler } h^{read}\ (\lambda\_.\ \text{case perform } ask\ () \text{ of} \\ &\quad \text{Yield } m\ f\ k \longrightarrow \text{Yield } m\ f\ (\lambda z.\ k\ z \triangleright (\lambda x.\ \text{perform } ask\ () \triangleright (\lambda y.\ \text{Pure}\ (x + y)))) \\ &\quad \text{Pure } x \longrightarrow \text{case perform } ask\ () \text{ of } \text{Yield } m\ f\ k \longrightarrow \text{Yield } m\ f\ (\lambda z.\ k\ z \triangleright (\lambda y.\ \text{Pure}\ (x + y))) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{Pure } y \longrightarrow \text{Pure}\ (x + y)) \end{aligned}$$

For clarity, we did not inline the bind operations in an expansion itself. Nevertheless, we can already see that at every original bind operation, we duplicated the g argument in the expansion. This means that if we have a sequence of N statements, we may end up with $N!$ duplications.

[3] As shown in Section 4 the actual definition also propagates the current evidence vector as part of the monad.

To avoid such expansion, we need to use *join-point sharing*: we consider every g argument as a join point, and rewrite the initial translation to make this sharing explicit:

$$\mathit{join}_1 \;=\; \lambda x\; y.\; \mathsf{Pure}\;(x \;+\; y) \qquad\qquad \mathit{join}_2 \;=\; \lambda x.\; \mathsf{perform}\;\mathit{ask}\;() \rhd (\lambda y.\; \mathit{join}_1\;x\;y)$$
$$\mathsf{hander}\;h^{\mathit{read}}\;(\lambda\_.\;\mathsf{perform}\;\mathit{ask}\;() \rhd \mathit{join}_2)$$

From there, we perform bind-inlining only for non-*join* definitions, but also aggressively inline *join*-definitions for the Pure branches. This results effectively in a fully inlined *fast path* along the Pure branches:

$$\begin{array}{l}\mathsf{handler}\;h^{\mathit{read}}\;(\lambda\_.\;\mathsf{case}\;\mathsf{perform}\;\mathit{ask}\;()\;\mathsf{of}\\ \quad \mathsf{Yield}\;m\;f\;k \to \mathsf{Yield}\;m\;f\;(\lambda z.\;k\;z \rhd \mathit{join}_2)\\ \quad \mathsf{Pure}\;x \qquad \to \mathsf{case}\;\mathsf{perform}\;\mathit{ask}\;()\;\mathsf{of}\;\mathsf{Yield}\;m\;f\;k \to \mathsf{Yield}\;m\;f\;(\lambda z.\;k\;z \rhd \mathit{join}_1\;x)\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathsf{Pure}\;y \qquad \to \mathsf{Pure}\;(x+y))\end{array}$$

Note how the join_1 join point is shared by the join_2 definition as well, and the code expansion for N statements is now $2N$ which is much more manageable. In practice, the Koka compiler does a type-selective transformation and leaves out monadic binds for functions that are total (since those will never yield) which further reduces code expansion by a large factor.

This strategy ensures that we have a *fast path* along each Pure branch: if no operation performs a full yield, no allocation happens along this path and tail-recursive calls are preserved (and as such, this optimization works best when used together with tail-resumptive optimization). Only in the (hopefully rare) case that full yield is needed, the slow path along the Yield case is taken and a resumption is constructed on demand. When such a resumption is resumed, the execution is a bit slower as well as it takes the code path along the join_n definitions where the binds are *not* inlined – this is the price we pay for limiting the expansion. Note though that if the function is recursive, any further recursive calls will again start at the fast path.

Another possible approach to implementing delimited control primitives is by using continuation passing style (CPS) instead of the monadic approach. Using CPS, we would instead translate our example essentially as:

$$\mathsf{handler}\;h^{\mathit{read}}\;(\lambda\_\;k.\;\mathsf{perform}\;\mathit{ask}\;()\;(\lambda x.\;\mathsf{perform}\;\mathit{ask}\;()\;(\lambda y.\;k\;(x+y))))$$

This looks similar to the monadic approach, where instead of explicitly using a monadic bind ($\rhd$) we pass the current continuation as the last argument k to every effectful function. Unfortunately, since we now pass the continuation as an argument, we always need to allocate the lambda *in advance*. In contrast, in the monadic approach with bind-inlining we can immediately call the function and inspect its result without doing any allocation; only if it happens to return a Yield we actually need to allocate the continuation.

2.8 Compiling to C

At this point we can use regular compilation techniques to compile the plain lambda calculus to a target platform. As an example, we show here how Koka compiles to standard C. In our final calculus all effectful functions return a monadic result, either Pure or Yield. Since this monad is internal to the compiler we can optimize its representation: we always return results normally assuming Pure, and set a (thread-local) flag to indicate yielding (in which case the actual returned value is ignored). Moreover, every function has one extra parameter that holds the (thread-local) context `ctx` which contains the current evidence vector (`ctx→w`), and the yielding flag (`ctx→is_yielding`). For example, the expression $\lambda\_.\;\mathsf{perform}\;\mathit{ask}\;()\;+\;\mathsf{perform}\;\mathit{ask}\;()$ translates essentially as:

```
int expr( unit_t u, context_t* ctx) {
  int x = perform_ask( ctx→w[0], unit, ctx );
  if (ctx→is_yielding) { yield_extend(&join2,ctx); return 0; }
  int y = perform_ask( ctx→w[0], unit, ctx );
```

```
if (ctx→is_yielding) { yield_extend(alloc_closure_join1(x,ctx),ctx); return 0; }
return (x+y); }
```

Here we see how the evidence for the *read* handler is selected from the current evidence vector as `ctx→w[0]`. Here the offset 0 is known as the effect type is ⟨*read*⟩ and Koka uses canonical evidence vectors. If the effect row type was not fully known, e.g., a polymorphic row type ⟨*read* | μ⟩, the code would instead be `find_ev(ctx→w,tag_read)` to find the evidence dynamically. When yielding, the `yield_extend` calls are used to extend the currently build up resumption (as part of the `ctx`) with the current continuation (which is usually a join point).

There is still an overhead in always needing to check after every effectful call if we are yielding or not. Fortunately, this seems quite cheap on modern processors and the condition can be predicted well. In the future we would like to leverage C compiler primitives to implement the `is_yielding` flag in the processor *carry* flag as suggested by recent C++ proposals for error handling [Sutter 2019].

2.9 Generalized Evidence Passing

The closest related work to our approach is [Xie et al. 2020], which uses *evidence-passing translation* (EPT). Even though similar in its purpose, EPT differs fundamentally from our approach. First, while our evidence-passing semantics provides a set of direct evaluation rules for the algebraic effect calculus, EPT is defined via *elaboration* from the algebraic effect calculus into an *evidence calculus*. Second, our generalized evidence-passing semantics works for all algebraic effect handler programs, whereas in EPT resumptions are limited to *scoped* resumptions only – that is, resumptions can only be used under the same handler context as captured by the handler.

Specifically, in EPT, as the evidence vector is passed *statically* during elaboration, it is determined *before* running the program. However, the statically passed-in evidence vector may, as the program evaluates, no longer match the handlers in the current dynamic evaluation context (and in such case, EPT raises a runtime error). Scoped resumptions restrict the expressiveness of algebraic effects, including the use of *shallow handlers* [Hillerström and Lindley 2018] and code migration that resumes continuations on a different host [Kiselyov et al. 2006].

2.9.1 Non-Scoped Resumptions. We illustrate the problem of non-scoped resumptions using the following *evil* effect as shown by Xie et al. [2020]:

$$evil\ \{\ evil\ :\ ()\to()\ \}\qquad\qquad h^{evil} = \{\ evil \mapsto \lambda x\ k.\ k\ \}$$

The handler h^{evil} illustrates again the expressiveness of effect handlers: the captured resumption is a first-class value and thus can be returned directly, and in this example we are going to resume it later under a changed handler context. Suppose we have another reader handler that always returns 2:

$$h^{read_2} = \{\ ask \mapsto \lambda x\ k.\ k\ 2\ \}$$

Consider the following program, where $f = (\lambda k.\ \text{handler}\ h^{read_2}\ (\lambda\_.\ k\ ()))$, which takes a continuation and resumes it under a new handler (ignoring tail-resumptive optimization for now):

$$f\ (\text{handler}\ h^{read}\ (\lambda\_.\ \text{handler}\ h^{evil}\ (\lambda\_.\ \text{perform}\ ask\ ();\ \text{perform}\ evil\ ();\ \text{perform}\ ask\ ())))$$

$$\longmapsto \overbrace{f}^{\langle\rangle}\ \bullet\ \text{prompt}\ m_1\ h^{read}\ \bullet\ \overbrace{\text{prompt}\ m_2\ h^{evil}}^{w_1 = \langle read : (m_1,\ h^{read})\rangle}\ \bullet\ \overbrace{(\square\ ;\ \text{perform}\ evil\ ()\ ;\ \text{perform}\ ask\ ())\ \bullet\ \text{perform}\ ask\ ()}^{w_2 = \langle evil : (m_2,\ h^{evil}),\ read : (m_1, h^{read})\rangle}$$

It may seem that both *ask* operations will return 1 as they both have $read : (m_1, h^{read})$ in the evidence vector w_2 but, as we will see, that is not the case! The first *ask* returns 1 as expected though:

$$\longmapsto f\ \bullet\ \text{prompt}\ m_1\ h^{read}\ \bullet\ \text{prompt}\ m_2\ h^{evil}\ \bullet\ (\square\ ;\ \text{perform}\ evil\ ()\ ;\ \text{perform}\ ask\ ())\ \bullet\ 1$$

However, before we can handle the second *ask*, the operation *evil* is performed, which captures the second *ask* in the resumption k:

$$\begin{aligned}
&\longmapsto f \bullet \text{prompt } m_1\ h^{read} \bullet \text{prompt } m_2\ h^{evil} \bullet (\square\,;\ \text{perform } ask\ ()) \bullet \text{perform } evil\ () \\
&\longmapsto \overbrace{f \bullet \text{prompt } m_1\ h^{read}}^{\langle\rangle} \bullet \overbrace{k}^{w_1} \quad \text{where } k = \lambda x.\ \text{prompt } m_2\ h^{evil} \bullet (\square\,;\ \text{perform } ask\ ()) \bullet x \qquad (1)
\end{aligned}$$

As k is a value, it is propagated through the prompt m_1 frame:

$$\longmapsto f\ k \quad \longmapsto \text{handler } h^{read_2}\ (\lambda\_.\ k\ ())$$

At this point, the reader handler in the context is now changed to h^{read_2}:

$$\begin{aligned}
&\longmapsto \overbrace{\text{prompt } m_0\ h^{read_2}}^{\langle\rangle} \bullet \overbrace{k\ ()}^{w_3 = \langle read : (m_0,\ h^{read_2})\rangle} \qquad (2) \\
&\longmapsto \overbrace{\text{prompt } m_0\ h^{read_2}}^{\langle\rangle} \bullet \overbrace{\text{prompt } m_2\ h^{evil}}^{w_3} \bullet \overbrace{(\square\,;\ \text{perform } ask\ ()) \bullet ()}^{w_4 = \langle evil : (m_2,\ h^{evil}),\ read : (m_0,\ h^{read_2})\rangle}
\end{aligned}$$

and the *ask* operation is performed under w_4 using the new h^{read_2} and thus evaluates to 2 (and not 1)!

EPT rejects this program at runtime by detecting the *non-scoped* resumption k: k is captured under w_1 at (1), but is later applied under w_3 at (2). In particular, in EPT, both *ask* operations *statically* receive w_2 as the evidence vector during elaboration to the evidence calculus. As such resuming k under a changed evidence vector means the statically received evidence vector does no longer match the dynamic handler context anymore, and is thus not allowed in their system. In contrast, our generalized evidence passing semantics correctly models the dynamic behavior of the evidence vector, and can express the full semantics of algebraic effect handlers.

2.9.2 Non-Scoped Resumptions with Tail-Resumptive Optimization. Xie et al. [2020] also describe the tail-resumptive optimization, and argue that tail-resumptive operations are examples of scoped resumptions, but do not provide any formalization of the optimization.

It turns out that the tail-resumptive optimization is more challenging with generalized evidence passing semantics, and our formalization goes beyond what is sketched in [Xie et al. 2020]. In particular, the interaction between non-scoped resumptions and tail-resumptive operations is subtle and the formalization of under is tricky to get right. We illustrate this by performing the previous *evil* example from *inside* a tail-resumptive operation:

$$tl\ \{\ tl\ :\ ()\ \to\ Int\ \} \qquad h^{tl} = \ \{\ tl \mapsto \lambda x\ k.\ k\ (\text{perform } ask\ ();\ \text{perform } evil\ ();\ \text{perform } ask\ ())\ \}$$

Here we have the same sequence of operations as before, but this time these happen from inside an operation. Note that this operation is tail-resumptive, despite all effects performed before resuming. Now consider the following program, which performs tl under three handlers, and passes the result to f.

$$\begin{aligned}
&f\ (\text{handler } h^{read}\ (\lambda_.\ \text{handler } h^{evil}\ (\lambda_.\ \text{handler } h^{tl}\ (\lambda_.\ \text{perform } tl\ ()\ ))))\\
&\longrightarrow^* f \bullet \text{prompt } m_1\ h^{read} \bullet \text{prompt } m_2\ h^{evil} \bullet \text{prompt } m_3\ h^{tl} \bullet \text{perform } tl\ ()
\end{aligned}$$

We evaluate tl in-place under w_2, as h^{tl} is itself defined under w_2.

$$\begin{aligned}
\longrightarrow^* &\overbrace{f \bullet \text{prompt } m_1\ h^{read}}^{\langle\rangle} \bullet \overbrace{\text{prompt } m_2\ h^{evil}}^{w_1 = \langle read : (m_1, h^{read}, \langle\rangle)\rangle} \bullet \overbrace{\text{prompt } m_3\ h^{tl}}^{w_2 = \langle evil : (m_2,\ h^{evil}, w_1),\ read : (m_1, h^{read}, \langle\rangle)\rangle} \\
&\quad {\scriptstyle w_3 = \langle tl : (m_3,\ h^{tl}, w_2),\ evil : (m_2,\ h^{evil}, w_1)\ ,\ read : (m_1, h^{read}, \langle\rangle)\rangle} \\
&\bullet \overbrace{\text{under } tl}^{w_2} \bullet \overbrace{(\square\,;\ \text{perform } evil\ ();\ \text{perform } ask\ ()) \bullet \text{perform } ask\ ()}^{w_2}
\end{aligned}$$

ask is also tail-resumptive and gets evaluated in-place.

$$\begin{aligned}&\longrightarrow f \bullet \text{prompt } m_1\ h^{read} \bullet \text{prompt } m_2\ h^{evil} \bullet \text{prompt } m_3\ h^{tl}\\ &\quad \bullet \text{under } tl \bullet (\square\,;\ \text{perform } evil\ ();\ \text{perform } ask\ ()) \bullet \text{under } read \bullet 1 \qquad (3)\end{aligned}$$

We then perform *evil*, which again captures the resumption and passes it to f.

$$\begin{aligned}&\longrightarrow^* f \bullet \text{prompt } m_1\ h^{read} \bullet \text{prompt } m_2\ h^{evil} \bullet \text{prompt } m_3\ h^{tl}\\ &\quad \bullet \text{under } tl \bullet (\square;\ \text{perform } ask\ ()) \bullet \text{perform } evil\\ &\longrightarrow^* f\ k \quad \text{where } k = \lambda x.\ \text{prompt } m_2\ h^{evil} \bullet \text{prompt } m_3\ h^{tl} \bullet \text{under } tl \bullet (\square;\ \text{perform } ask\ ()) \bullet x \qquad (4)\end{aligned}$$

f applies k under a reader handler h^{read_2}:

$$\begin{aligned}&\longrightarrow^* \overbrace{\text{prompt } m_0\ h^{read_2}}^{\langle\rangle} \bullet \overbrace{\text{prompt } m_2\ h^{evil}}^{w_4=\langle read:(m_0,\ h^{read_2},\ \langle\rangle)\rangle} \bullet \overbrace{\text{prompt } m_3\ h^{tl}}^{w_5=\langle evil:(m_1,h^{evil},w_4),\ read:(m_0,\ h^{read_2},\ \langle\rangle)\rangle}\\ &\quad \bullet \overbrace{\text{under } tl}^{w_6=\langle tl:(m_3,\ h^{tl},w_5),\ evil:(m_2,\ h^{evil},w_4)\ ,\ read:(m_0,h^{read_2},\langle\rangle)\rangle} \bullet \overbrace{\text{perform } ask\ ()}^{w_5} \qquad (5)\\ &\longrightarrow^* 2\end{aligned}$$

The evaluation is quite subtle in several places. First, at (3) we introduced under *tl*. As shown at (4), the under frame can itself be captured by a resumption. This explains why we cannot directly apply the optimization but require an extra under frame: inside the resumption we still need to remember that operations happening after under *tl* can only reach handlers beyond h^{tl}.

However, at (3), it might be tempting to introduce the frame as under w_2 instead of *under tl*, as that would be enough to ensure that all operations afterwards are evaluated under w_2. By doing so, under could be formalized in a simpler way: under w_2 could simply ignore the current evidence vector and always pass w_2 to future operations. Our initial formalization did this but unfortunately this turns out to be unsound.

As shown at (5), the evidence vector for *under tl* itself has changed, from w_3 to w_6, and thus the evidence vector passed by *under tl* has also changed from w_2 to w_5, so that the last *ask* is handled by h^{read_2} and returns 2. The reader could check that 2 is indeed the desired result of the program if without tail-resumptive optimization. If we had used under w_2, the *ask* would wrongly return 1!

Proving the correctness of under is also challenging, as it essentially requires us to show that a program with tail-resumptive optimization will produce the same result as of the same program without the optimization. To this end, we show that the optimized program is *contextual equivalent* to the original program.

3 SEMANTICS

This section presents System F^{pw}, which features algebraic effects using multi-prompt and evidence passing semantics. The system is designed based on System F^{ϵ} [Xie et al. 2020], a plain polymorphic algebraic effect calculus.

3.1 Multi-Prompt with Evidence Passing Semantics

3.1.1 Syntax. Figure 1 defines the syntax. Expressions e include values v, applications $e\ e$, type applications $e\ \sigma$ and the internal frame prompt $m\ h$ and yield $m\ v$. Values include variables x, lambdas $\lambda^{\epsilon}\ x:\sigma.e$, which is annotated with the effect ϵ that may be performed when the lambda is applied, type lambdas $\Lambda\alpha^{k}.\ v$, and handler h and perform $op\ \epsilon\ \overline{\sigma}$. Since the calculus is explicitly typed and an operation signature can be polymorphic, performing an operation perform $op\ \epsilon\ \overline{\sigma}$ needs to indicate its context effect ϵ, as well as to explicitly pass the type arguments $\overline{\sigma}$. A handler h contains a list of operation clauses $op \mapsto f$, where f denotes a function expression. As we have

$$
\begin{array}{llll}
\text{Expression} & e & ::= & v \mid e\ e \mid e\ \sigma \\
 & & \mid & \text{prompt}\ m\ h\ e \\
 & & \mid & \text{yield}\ m\ v \\
\text{Value} & v & ::= & x \mid \lambda^{\epsilon} x\!:\!\sigma.\ e \mid \Lambda\alpha^{\kappa}.\ v \\
 & & \mid & \text{handler}\ h \\
 & & \mid & \text{perform}\ op\ \epsilon\ \overline{\sigma} \\
\text{Handler} & h & ::= & \{\ \overline{op_i \mapsto f_i}\ \} \\
\text{Evaluation ctx.} & \mathsf{E} & ::= & \Box \mid \mathsf{E}\ e \mid v\ \mathsf{E} \mid \mathsf{E}\ \sigma \\
 & & \mid & \text{prompt}\ m\ h\ \mathsf{E} \\
 & \mathsf{F} & ::= & \Box \mid \mathsf{F}\ e \mid v\ \mathsf{F} \mid \mathsf{F}\ \sigma
\end{array}
\qquad
\begin{array}{llll}
\text{Type} & \sigma & ::= & \alpha^{\kappa} \mid c^{\kappa}\ \overline{\sigma} \mid \sigma \to \epsilon\ \sigma \\
 & & \mid & \forall\alpha^{\kappa}.\ \sigma \\
\text{Effect row} & \epsilon & ::= & \langle\rangle \mid \langle l \mid \epsilon\rangle \\
\text{Kind} & \kappa & ::= & * \mid \kappa \to \kappa \mid \text{lab} \mid \text{eff} \\
\text{Type env.} & \Gamma & ::= & \varnothing \mid \Gamma,\ x\!:\!\sigma \\
\text{Effect ctx.} & \Sigma & ::= & \{\ \overline{l_i : sig_i}\ \} \\
\text{Effect sig.} & sig & ::= & \{\ \overline{op_i : \forall\overline{\alpha^{\kappa}}.\ \sigma_i \to \sigma_i'}\ \} \\
\text{Evidence} & ev & ::= & (m,\ h,\ w) \\
\text{Evidence vec.} & w & ::= & \langle\!\langle\rangle\!\rangle \mid \langle\!\langle l\!:\!ev \mid w\rangle\!\rangle
\end{array}
$$

$$
\begin{array}{lllll}
(app) & (\lambda^{\epsilon} x\!:\!\sigma.\ e)\ v & \longrightarrow & e[x{:=}v] & \\
(tapp) & (\Lambda\alpha^{\kappa}.\ v)\ \sigma & \longrightarrow & v[\alpha{:=}\sigma] & \\
(handler) & \text{handler}\ h\ v & \longrightarrow & \text{prompt}\ m\ h\ (v\ ()) & \text{with unique } m \\
(promptv) & \text{prompt}\ m\ h\ v & \longrightarrow & v & \\
(prompt) & \text{prompt}\ m\ h\ \mathsf{E}[\text{yield}\ m\ f] & \longrightarrow & f\ (\lambda^{\epsilon} x\!:\!\sigma_2.\ \text{prompt}\ m\ h\ \mathsf{E}[x]) & \\
 & & \text{with} & \varnothing \vdash_{\text{val}} f : (\sigma_2 \to \epsilon\ \sigma) \to \epsilon\ \sigma & \\
(perform) & w \vdash \text{perform}\ op\ \epsilon_0\ \overline{\sigma}\ v & \longrightarrow & \text{yield}\ m\ (\lambda^{\epsilon} k\!:\!\sigma_2[\overline{\alpha}{:=}\overline{\sigma}] \to \epsilon\ \sigma.\ f\ \overline{\sigma}\ v\ k) & \\
 & & \text{with} & (m, h, \_) = w.l \wedge (op \mapsto f) \in h & \\
 & & & (op\!:\!\forall\overline{\alpha}.\ \sigma_1 \to \sigma_2) \in \Sigma(l) \wedge \varnothing \vdash h : \sigma \mid l \mid \epsilon &
\end{array}
$$

$$
\frac{e \longrightarrow e'}{w \vdash \mathsf{F}[e] \longmapsto \mathsf{F}[e']}\,(step)
\qquad
\frac{w \vdash e \longrightarrow e'}{w \vdash \mathsf{F}[e] \longmapsto \mathsf{F}[e']}\,(stepw)
\qquad
\frac{}{w \vdash e \longmapsto^{*} e}
$$

$$
\frac{\langle\!\langle l : (m, h, w) \mid w\rangle\!\rangle \vdash e \longmapsto e'}{w \vdash \mathsf{F}[\text{prompt}\ m\ h\ e] \longmapsto \mathsf{F}[\text{prompt}\ m\ h\ e']}\,(promptw)
\qquad
\frac{w \vdash e \longmapsto^{*} e' \quad w \vdash e' \longmapsto e''}{w \vdash e \longmapsto^{*} e''}
$$

Fig. 1. F^{pw}: multi-prompt with evidence-passing semantics.

seen before, an evaluation context E is essentially an expression with a hole in it, which indicates explicitly the evaluation order of an expression. A *pure* evaluation context F has no prompt frame.

Types. Types σ include type variables α^{κ} of kind κ, type constructors $c^{\kappa}\ \overline{\sigma}$ where c^{κ} of kind κ is applied to the arguments $\overline{\sigma}$, function types $\sigma \to \epsilon\ \sigma$ annotated with the effect ϵ that may be performed when the function is applied, and polymorphic types $\forall\alpha^{\kappa}.\ \sigma$. An effect row ϵ can be either empty $\langle\rangle$ which denotes the *total* effect, or an extension $\langle l \mid \epsilon\rangle$ which extends ϵ with effect label l. We often leave out the effect annotation if it is total. For convenience, we use μ to denote effect variables (i.e., α^{eff}). An effect may end up with an effect variable (e.g., $\langle l \mid \mu\rangle$). We call those effects *open*, and effects that end in an empty effect (e.g., $\langle l \mid \langle\rangle\rangle$) *closed*.

Equivalence between row types ($\equiv$) is defined as follows. Row equivalence is reflexive, transitive, and can freely reorder distinct labels.

$$
\frac{}{\epsilon \equiv \epsilon}
\qquad
\frac{\epsilon_1 \equiv \epsilon_2 \quad \epsilon_2 \equiv \epsilon_3}{\epsilon_1 \equiv \epsilon_3}
\qquad
\frac{\epsilon_1 \equiv \epsilon_2}{\langle l \mid \epsilon_1\rangle \equiv \langle l \mid \epsilon_2\rangle}
\qquad
\frac{l_1 \neq l_2 \quad \epsilon_1 \equiv \epsilon_2}{\langle l_1, l_2 \mid \epsilon_1\rangle \equiv \langle l_2, l_1 \mid \epsilon_2\rangle}
$$

To distinguish among types, System F^{pw} uses a basic kind system. Kinds κ include the basic kind ($*$), functions ($\kappa \to \kappa$), the kind of labels (lab), and the kind of effects (eff). The judgment $\vdash_{\text{wf}}\ \sigma : \kappa$ checks the kind of types, whose definition is standard and is given in the appendix.

The term context Γ is standard. A global effect context Σ maps each effect l to its signature sig^l, which gives every operation its input and output types, i.e., $op_i : \forall \overline{\alpha}_i.\ \sigma_i \to \sigma'_i$. We assume each op is uniquely named, and we use the notation $op : \forall \overline{\alpha}.\ \sigma_1 \to \sigma_2 \in \Sigma(l)$ to denote the type of op that belongs to effect l.

Evidence Vectors. System F^{pw} incorporates *evidence passing semantics*. In particular, *evidence* ev is a triple consisting of a marker m, its corresponding handler h and the evidence vector w where h is defined. An *evidence vector* w is a map from effect labels to evidence. It can be either empty $\langle\!\rangle$, or an extension $\langle\!\langle ev \mid w \rangle\!\rangle$ which extends w with evidence ev. We use the notation $w.l$ to select evidence of label l from w. As we have discussed in Section 2.4, we treat the evidence vector as an abstract datatype, as it can be either canonical or insertion ordered, depending on how the extension operation $\langle\!\langle ev \mid w \rangle\!\rangle$ is implemented. Importantly though, for correctness of the evidence passing semantics, selection and extension should satisfy the following laws, so that $w.l$ always finds the most recent evidence of l, which corresponds to the dynamic semantics of algebraic effects where an operation is handled by its innermost handler.

$$\langle\!\langle l : ev \mid w \rangle\!\rangle.l = ev \qquad \langle\!\langle l' : ev \mid w \rangle\!\rangle.l = w.l \text{ iff } l \neq l'$$

3.1.2 Operational Semantics. The operational semantics rules of System F^{pw} (Figure 1) include three definitions: $\longrightarrow$ provides a primitive evaluation step, $\longmapsto$ evaluates expressions under evaluation contexts, and $\longmapsto^*$ defines the *transitive closure* of $\longmapsto$. In practice, evaluating an expression always start with an empty evidence vector. For clarify, we use a *lighter* color for all type information, which is needed for type soundness, but not directly for the dynamic semantics of algebraic effects.

($\longrightarrow$). During evaluation, we pass the current handlers down as an evidence vector. However, the evidence vector only matters when performing an operation, and many evaluation steps do not need to inspect the evidence vector. To make the difference clear, we separate the evaluation step into two categories: plain $e_1 \longrightarrow e_2$, and evaluation under an evidence vector $w \vdash e_1 \longrightarrow e_2$.

Rule (*app*) and (*tapp*) are standard. In rule (*handler*), handler installs a prompt m frame, with a fresh unique marker m, so that the marker can later be used to find the specific prompt. Values are propagated through the prompt frame (rule (*promptv*)).

As this system models the multi-prompt semantics, we split performing an operation into two parts: searching for a handler (rule (*perform*)), and capturing and restoring a resumption (rule (*prompt*)). Rule (*perform*) captures the essence of evidence passing semantics. Specifically, given the evidence vector w, performing an operation directly gets the handler h by selecting out the corresponding evidence by $w.l$. The operation implementation f from h is then used to handle the operation. As we will see shortly, the notation $\varnothing \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon$ says that h is a handler for effect l, which has result type σ and may itself perform effect ϵ. Notice the difference between ϵ_0 and $\epsilon - \epsilon_0$ is the effect where perform is defined, and ϵ is the effect where prompt h is defined. Finally, in rule (*prompt*), yield captures the resumption $(\lambda^{\epsilon} x : \sigma_2.\ \mathsf{prompt}\ m\ h\ \mathsf{E}[x])$, to which f is applied.

($\longmapsto$). When evaluating expressions under evaluation contexts, each rule is given the current evidence vector w. Rule (*step*) and (*stepw*) correspond respectively to a plain $\longrightarrow$ and a $w \vdash \longrightarrow$ step. Both rules evaluate under an F. That is because as shown in rule (*promptw*), the prompt $m\ h\ e$ frame modifies the evidence vector by inserting the new evidence $l : (m, h, w)$ and uses the evidence vector $\langle\!\langle l : (m, h, w) \mid w \rangle\!\rangle$ for evaluating e. Here the evaluation context is again an F ensuring that the evidence vectors always match the prompt frames in the context.

$$\Gamma \vdash e : \sigma \mid \epsilon \qquad \Gamma \vdash_{\mathsf{val}} v : \sigma \qquad \Gamma \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon$$

$$\frac{x{:}\sigma \in \Gamma}{\Gamma \vdash_{\mathsf{val}} x : \sigma}\ \textsc{var} \qquad \frac{\Gamma \vdash_{\mathsf{val}} v : \sigma}{\Gamma \vdash v : \sigma \mid \epsilon}\ \textsc{val}$$

$$\frac{\Gamma, x{:}\sigma_1 \vdash e : \sigma_2 \mid \epsilon}{\Gamma \vdash_{\mathsf{val}} \lambda^{\epsilon}\, x{:}\sigma_1.\, e : \sigma_1 \to \epsilon\ \sigma_2}\ \textsc{abs} \qquad \frac{\Gamma \vdash e_1 : \sigma_1 \to \epsilon\ \sigma \mid \epsilon \quad \Gamma \vdash e_2 : \sigma_1 \mid \epsilon}{\Gamma \vdash e_1\ e_2 : \sigma \mid \epsilon}\ \textsc{app}$$

$$\frac{\Gamma \vdash_{\mathsf{val}} v : \sigma \quad \kappa \neq \mathsf{lab}}{\Gamma \vdash_{\mathsf{val}} \Lambda\alpha^{\kappa}.\, v : \forall\alpha^{\kappa}.\, \sigma}\ \textsc{tabs} \qquad \frac{\Gamma \vdash e : \forall\alpha^{\kappa}.\, \sigma_1 \mid \epsilon \quad \vdash_{\mathsf{wf}} \sigma : \kappa}{\Gamma \vdash e\ \sigma : \sigma_1[\alpha{:=}\sigma] \mid \epsilon}\ \textsc{tapp}$$

$$\frac{op : \forall\overline{\alpha}.\, \sigma_1 \to \sigma_2 \in \Sigma(l)}{\Gamma \vdash_{\mathsf{val}} \mathsf{perform}\ op\ \epsilon\ \overline{\sigma} : \sigma_1[\overline{\alpha}{:=}\overline{\sigma}] \to \langle l \mid \epsilon\rangle\ \sigma_2[\overline{\alpha}{:=}\overline{\sigma}]}\ \textsc{perform}$$

$$\frac{\Gamma \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon}{\Gamma \vdash_{\mathsf{val}} \mathsf{handler}\ h : (() \to \langle l \mid \epsilon\rangle\ \sigma) \to \epsilon\ \sigma}\ \textsc{handler}$$

$$\frac{\Gamma \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon \quad \Gamma \vdash e : \sigma \mid \langle l \mid \epsilon\rangle}{\Gamma \vdash \mathsf{prompt}\ m\ h\ e : \sigma \mid \epsilon}\ \textsc{prompt} \qquad \frac{\Gamma \vdash_{\mathsf{val}} f : (\sigma \to \epsilon'\ \sigma') \to \epsilon'\ \sigma'}{\Gamma \vdash \mathsf{yield}\ m\ f : \sigma \mid \epsilon}\ \textsc{yield}$$

$$\frac{op_i : \forall\overline{\alpha}.\, \sigma_1 \to \sigma_2 \in \Sigma(l) \quad \overline{\alpha} \notin \mathsf{ftv}(\epsilon, \sigma) \quad \Gamma \vdash_{\mathsf{val}} f_i : \forall\overline{\alpha}.\, \sigma_1 \to \epsilon\ ((\sigma_2 \to \epsilon\ \sigma) \to \epsilon\ \sigma)}{\Gamma \vdash_{\mathsf{ops}} \{\ op_1 \mapsto f_1,\ \ldots,\ op_n \mapsto f_n\ \} : \sigma \mid l \mid \epsilon}\ \textsc{ops}$$

Fig. 2. Typing Rules for System F^{pw}.

Example. In Section 2, for better illustration, we have used the $\overset{w}{\frown}$ notation to indicate the current evidence vector. In the formal system, we always use $w \vdash$. The following example shows the evaluation derivation of handler $h^{read}\ (\lambda\_.\ \mathsf{perform}\ ask\ ()) \longmapsto^* 1$. We have omitted type annotations, and details regarding $\langle\!\langle\rangle\!\rangle \vdash e_1 \longmapsto e_3$ and $\langle\!\langle\rangle\!\rangle \vdash e_6 \longmapsto^* 1$.

(1) $e_1 = \mathsf{handler}\ h^{read}\ (\lambda\_.\ \mathsf{perform}\ ask\ ())$
(2) $e_2 = \mathsf{perform}\ ask\ ()$
(3) $e_3 = \mathsf{prompt}\ m\ h^{read}\ e_2$
(4) $e_4 = \mathsf{yield}\ m\ (\lambda k.\ (\lambda x\ k.\ 1)\ ()\ k)$
(5) $e_5 = \mathsf{prompt}\ m\ h^{read}\ e_4$
(6) $e_6 = (\lambda k.\ (\lambda x\ k.\ 1)\ ()\ k)\ (\lambda x.\ \mathsf{prompt}\ m\ h^{read}\ x)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\ldots}{\langle\!\langle\rangle\!\rangle \vdash e_1 \longmapsto^* e_3} \quad \dfrac{\langle\!\langle m,\ h^{read},\ \langle\!\langle\rangle\!\rangle\rangle \vdash e_2 \longrightarrow e_4}{\langle\!\langle\rangle\!\rangle \vdash e_3 \longmapsto e_5}}{\langle\!\langle\rangle\!\rangle \vdash e_1 \longmapsto^* e_5} \quad \dfrac{e_5 \longrightarrow e_6}{\langle\!\langle\rangle\!\rangle \vdash e_5 \longmapsto e_6}}{\langle\!\langle\rangle\!\rangle \vdash e_1 \longmapsto^* e_6} \quad \ldots}{\langle\!\langle\rangle\!\rangle \vdash e_1 \longmapsto^* 1}$$

3.1.3 Typing Rules. Figure 2 defines the typing rules for System F^{pw}. The judgment $\Gamma \vdash e : \sigma \mid \epsilon$ reads that, under the typing context Γ, the expression e has type σ and may perform effect ϵ. Values are not effectful and thus the typing judgment takes the form $\Gamma \vdash_{\mathsf{val}} v : \sigma$. The judgment $\Gamma \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon$ type-checks a handler h for effect l, with result type σ and effect ϵ.

Most rules are standard. Rule VAL can take in any effect. In rule ABS, the effect annotation from the lambda is passed to the body derivation, and the rule produces type $\sigma_1 \to \epsilon\ \sigma_2$, where ϵ indicates the effect that may be performed by the lambda body. In rule APP, we require three effects to match: the effect ϵ in the function $\sigma_1 \to \epsilon\ \sigma_2$, the effect ϵ of e_1 and of e_2.

Performing an operation introduces effects. In rule PERFORM, perform $op\ \epsilon\ \overline{\sigma}$ first gets the type of the operation from $\Sigma(l)$, and adds l to the context effect ϵ, generating $\langle l \mid \epsilon\rangle$. Dually, handling eliminates effects. In rule HANDLER, given a handler h for l, the rule takes an action with effect $\langle l \mid \epsilon\rangle$, and produces the result effect ϵ. Rule PROMPT is similar, but directly takes an expression e of effect $\langle l \mid \epsilon\rangle$. Rule OPS types a handler, where we assume $\{op_1, \ldots, op_n\} = \Sigma(l)$. Note that all operation implementations must have the same effect (ϵ) and type result (σ).

Rule YIELD is more subtle. Recall that the operational rule (*perform*) (in Figure 1) turns perform into yield, So we expect the result type of yield to match that of perform. Note that the result type of perform is the same as the argument type of the resumption k, and the type of the resumption k itself is the argument type of f in yield $m\ f$. Therefore, in rule YIELD, we directly get the result type from the type of f. To be more precise, we could also set the result effect of yield to match that of perform. But since yield is an internal frame, the current form is sufficient for type soundness.

3.1.4 Correctness, Preservation and Progress. In rule (*perform*), we refer to w as the current evidence vector, and we select out the handler from the evidence vector (instead of searching for it in the evaluation context). This means that for the correctness of evidence passing semantics, the current evidence vector *w must correspond exactly to the actual handlers in the dynamic evaluation context* – so that the handler selected from the evidence vector is indeed exactly the innermost handler that would be found with the original semantics of algebraic effects.

We use the notation $\lceil \mathsf{E} \rceil$ to extract all evidence from an evaluation context E. Specifically, if E is $\mathsf{F}_0 \bullet \text{prompt}\ m_1\ h_1 \bullet \mathsf{F}_1 \bullet \ldots \bullet \text{prompt}\ m_n\ h_n \bullet \mathsf{F}_n$, where each h_i is a handler for l_i, we have $\lceil \mathsf{E} \rceil = \langle\!\langle l_n : (m_n,\ h_n,\ \_) \mid \ldots \mid l_1 : (m_1,\ h_1, \_) \rangle\!\rangle$ (we ignore the third component as it is not used). In order to prove correctness, we show that a $\longmapsto$ step can be reasoned in terms of a $\longrightarrow$ step, where for a $w \vdash \longrightarrow$ step, the evidence vector is the original evidence vector extended by all evidence from the evaluation context:

Lemma 1. (*Inversion of* $\longmapsto$). If $w \vdash e_1 \longmapsto e_2$, then either

- $e_1 = \mathsf{E}[e_1']$, $e_2 = \mathsf{E}[e_2']$, and $e_1' \longrightarrow e_2'$; or
- $e_1 = \mathsf{E}[e_1']$, $e_2 = \mathsf{E}[e_2']$, and $\langle\!\langle \lceil \mathsf{E} \rceil \mid w \rangle\!\rangle \vdash e_1' \longrightarrow e_2'$.

Based on Lemma 1, we can now show that the marker m and the handler h found by evidence-passing semantics is indeed the innermost handler found dynamically from the evaluation context. The following theorem establishes the correctness of evidence passing semantics.

Theorem 1. (*Evidence corresponds to the evaluation context*).
If $\langle\!\langle \rangle\!\rangle \vdash \mathsf{E}[\text{perform}\ op^l\ \overline{\sigma}\ v] \longmapsto \mathsf{E}[\text{yield}\ m\ (\lambda k.\ f\ \overline{\sigma}\ v\ k)]$, then $\lceil \mathsf{E} \rceil . l = (m, h, \_)$, and $(op \mapsto f) \in h$.

Preservation and progress do not hold immediately for our system; instead we need to consider both prompt and yield as strictly *internal* frames that cannot be written directly by the programmer (and only occur during evaluation). For example, if we can write yield m ourselves, we can use an arbitrary m that does not match with any prompt in the context (and thus lose progress); similarly, we can write a yield $m\ f$ where the result type of f does not match the type expected by the prompt m in the context (and lose preservation).

By treating both prompt and yield as strictly internal frames we can ensure by construction that the previous problematic examples cannot occur, and can prove progress and preservation. In particular, we use a similar definition as the *handle-safe* definition from [Xie et al. 2020]:

Definition 1. (*Internal expressions*). An *internal-safe* expression is a well-typed closed expression that either (1) contains no internal construct; or (2) is itself reduced from an internal-safe expression.

Internal-safe expressions maintain two important invariants: (1) each prompt owns a unique m generated at rule (*handler*); and (2) when perform generates yield m in (*perform*), it has found the handler with the right type (and therefore, yield m will find the right prompt m in rule (*prompt*)). We prove that internal-safe System F^{pw} enjoys preservation and progress.

Theorem 2. (*Preservation of Internal-safe System* F^{pw}). If $\varnothing \vdash e_1 : \sigma \mid \langle\rangle$ where e_1 is internal-safe, and $\langle\!\langle \rangle\!\rangle \vdash e_1 \longmapsto e_2$, then $\varnothing \vdash e_2 : \sigma \mid \langle\rangle$.

$$
\begin{array}{lrcl}
\text{Expression} & e & ::= & \ldots \mid \mathsf{under}^{\epsilon,\epsilon}\ l\ e \\
\text{Evaluation context} & \mathsf{E} & ::= & \ldots \mid \mathsf{under}^{\epsilon,\epsilon}\ l\ \mathsf{E}
\end{array}
$$

$$
\begin{array}{lll}
(\mathit{performt}) \quad w \vdash \mathsf{perform}\ \mathit{op}\ \epsilon_0\ \overline{\sigma}\ v & \longrightarrow & (\Lambda\overline{\alpha}.\ \lambda^{\langle l \mid \epsilon_0\rangle} x : \sigma_1.\ \mathsf{under}^{\epsilon_0,\epsilon}\ l\ e)\ \overline{\sigma}\ v \\
& \text{with} & (m, h, w') = w.l \\
& & (\mathsf{op} \mapsto \Lambda\overline{\alpha}.\ \lambda^{\epsilon} x : \sigma_1.\ k : \sigma_2 \to \epsilon\ \sigma.\ k\ e) \in h \ \wedge\ k \notin \mathsf{fv}(e) \\
(\mathit{underv}) \quad \mathsf{under}^{\epsilon_0,\epsilon}\ l\ v & \longrightarrow & v
\end{array}
$$

$$
\frac{w' \vdash e \longmapsto e' \quad (m, h, w') = w.l}{w \vdash \mathsf{F}[\mathsf{under}^{\epsilon_0,\epsilon}\ l\ e] \longmapsto \mathsf{F}[\mathsf{under}^{\epsilon_0,\epsilon}\ l\ e']}\ (\mathit{underw}) \qquad \frac{\Gamma \vdash e : \sigma \mid \epsilon}{\Gamma \vdash \mathsf{under}^{\epsilon_0,\epsilon}\ l\ e : \sigma \mid \langle l \mid \epsilon_0\rangle}\ \textsc{under}
$$

Fig. 3. Tail resumptive operations

The progress theorem is more tricky, as perform does not find the handler from the evaluation context but instead from the evidence vector. Fortunately, from Lemma 1, we can show that the handler found from the evidence vector is always available in the evaluation context.

Theorem 3. (*Progress of Internal-safe System* F^{pw}). If $\varnothing \vdash e_1 : \sigma \mid \langle\rangle$ where e_1 is internal-safe, then either e_1 is a value, or $\langle\!\langle\rangle\!\rangle \vdash e_1 \longmapsto e_2$.

We can further prove that markers cannot be duplicated in the evaluation context.

Theorem 4. (*Uniqueness of Handlers for Internal-safe System* F^{pw}). For any internal-safe F^{pw} expression prompt $m_1\ h_1\ (\mathsf{E}_2[\mathsf{prompt}\ m_2\ h_2\ e])$, we have $m_1 \neq m_2$.

3.2 Tail-Resumptive Optimization

With evidence passing semantics, we are now ready to formalize the tail-resumptive optimization, which is given in Figure 3. We extend the definition of expressions with $\mathsf{under}^{\epsilon,\epsilon}\ l\ e$, and the definition of evaluation contexts with $\mathsf{under}^{\epsilon,\epsilon}\ l\ \mathsf{E}$.

3.2.1 Operational Semantics. Rule (*performt*) is the key to apply the tail-resumptive optimization. First, it gets the handler h from the evidence vector as before. But it then detects that the operation implementation $(\Lambda\overline{\alpha}.\ \lambda^{\epsilon} x : \sigma_1.\ \lambda k : \sigma_2 \to \epsilon\sigma.\ k\ e)$ is tail-resumptive (with $k \notin \mathsf{fv}(e)$), and so instead of yielding up, it generates $\mathsf{under}^{\epsilon_0,\epsilon}\ l\ e$, which directly evaluates e *in-place* with the type arguments $\overline{\sigma}$ and value argument v. When the expression evaluates to a value, the value is propagated through the under frame (rule (*underv*)).

Importantly, under needs to modify the evidence vector, so that operations happening after it can find the right handler. In rule (*underw*), given the current evidence vector w, under first finds the innermost evidence for l in the evidence vector, i.e., $(m,\ h,\ w')$, and then passes the evidence vector w', under which h is defined, to e. In other words, under skips the whole evidence fragment between h to itself, which should not be accessible to e.

3.2.2 Typing. Rule UNDER types an $\mathsf{under}^{\epsilon_0,\epsilon}\ l\ e$ expression. Note that the effect ϵ corresponds to the effect of e, while under itself produces $\langle l \mid \epsilon_0\rangle$. As with yield, in a more refined system, we can further state that ϵ_0 contains ϵ (as when generated in (*performt*)), but as under is internal, the current typing rule is sufficient for establishing soundness.

3.2.3 Correctness, Preservation and Progress. In what sense is the tail-resumptive optimization correct? If only the optimized expression could produce an equivalent result as of the original expression. However, the equivalence is not so obvious. To illustrate the subtlety, consider evaluating the expression $(\mathsf{prompt}\ m\ h \bullet \mathsf{E} \bullet \mathsf{perform}\ \mathit{op}\ \overline{\sigma}\ v)$ under the evidence vector w. Assume that

the *op* operation is handled by prompt $m\ h$, where $(\mathsf{op} \mapsto \Lambda\overline{\alpha}.\ \lambda x\ k.\ k\ e) \in h$ with $k \notin \mathrm{fv}(e)$, i.e., the implementation is tail-resumptive. If we evaluate the expression without tail-resumptive optimization, we get (for clarify we omit w in the derivation):

$$\begin{aligned}
&\mathsf{prompt}\ m\ h \bullet \mathsf{E} \bullet \mathsf{perform}\ op\ \epsilon_0\ \overline{\sigma}\ v \\
\longmapsto\ &\mathsf{prompt}\ m\ h \bullet \mathsf{E} \bullet \mathsf{yield}\ m\ (\lambda k.\ (\Lambda\overline{\alpha}.\ \lambda x\ k.\ k\ e)\ \overline{\sigma}\ v\ k) \\
\longmapsto^*\ &(\Lambda\overline{\alpha}.\ \lambda x\ k.\ k\ e)\ \overline{\sigma}\ v\ (\lambda x.\ \mathsf{prompt}\ m\ h\ \mathsf{E}[x]) \\
\longmapsto^*\ &(\lambda x.\ \mathsf{prompt}\ m\ h\ \mathsf{E}[x])\ e[\overline{\alpha}{:=}\overline{\sigma}, x{:=}v]
\end{aligned}$$

while with tail optimization we end up with:

$$\begin{aligned}
&\mathsf{prompt}\ m\ h \bullet \mathsf{E} \bullet \mathsf{perform}\ op^l\ \epsilon_0\ \overline{\sigma}\ v \\
\longmapsto\ &\mathsf{prompt}\ m\ h \bullet \mathsf{E} \bullet (\Lambda\overline{\alpha}.\ \lambda x.\ \mathsf{under}^{\epsilon_0,\epsilon}\ l \bullet e)\ [\overline{\sigma}]\ v \\
\longmapsto^*\ &\mathsf{prompt}\ m\ h \bullet \mathsf{E} \bullet \mathsf{under}^{\epsilon_0,\epsilon}\ l \bullet e[\overline{\alpha}{:=}\overline{\sigma},\ x{:=}v]
\end{aligned}$$

The two expressions are now quite different. Nevertheless, intuitively these two result expressions are *equivalent*: they both first evaluate $e[\overline{\alpha}{:=}\overline{\sigma},\ x{:=}v]$, and then pass the result to prompt $m\ h\ \mathsf{E}$, via beta-reduction and via propagation through under, respectively. The situation is a bit more tricky though as $e[\overline{\alpha}{:=}\overline{\sigma},\ x{:=}v]$ may perform an operation. However, even in that case, the operation will find the same handler: in the first case, it is obvious that the evidence vector passed to $e[\overline{\alpha}{:=}\overline{\sigma},\ x{:=}v]$ is w; in the second case, w is first extended by evidence from prompt $m\ h \bullet \mathsf{E}$, but then $\mathsf{under}^{\epsilon}\ l$ changes the evidence vector back to w! To capture the observation, we formalize an equivalent relation $e_1 \cong e_2$ between expressions (and evaluation contexts respectively) where e_1 has no under, and e_2 may have under. The relation $\cong$ is mostly structural, up to renaming of fresh markers, with the following rule:

$$\frac{e_1 \cong e_2 \quad \mathsf{E}_1 \cong \mathsf{E}_2 \quad l \notin \mathsf{bl}(\mathsf{E}_1) \quad \varnothing \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon}{(\lambda x.\ \mathsf{prompt}\ m\ h\ \mathsf{E}_1[x])\ e_1 \cong \mathsf{prompt}\ m\ h \bullet \mathsf{E}_2 \bullet \mathsf{under}^{\epsilon_0,\epsilon}\ l\ e_2}$$

We can prove that evaluation preserves the equivalent relation, except that expressions need to take several reduction steps to become equivalent again, as evaluating prompt under the two semantics takes different number of steps to reach the desired equivalent form.

Lemma 2. (*Evaluation Preserves* $\cong$). Given two closed internal-safe expressions $\varnothing \vdash e_1 : \sigma \mid \langle\rangle$ and $\varnothing \vdash e_2 : \sigma \mid \langle\rangle$, if $e_1 \cong e_2$, then either e_1 and e_2 are values, or there exist e_1', e_2' such that $\langle\!\langle\rangle\!\rangle \vdash e_1 \longmapsto^+ e_1'$, $\langle\!\langle\rangle\!\rangle \vdash e_2 \longmapsto^+ e_2'$, and $e_1' \cong e_2'$.

Based on Lemma 2, we show that the optimized and unoptimized expressions are *contextual equivalent*, with the intuition that we cannot tell them apart in any context.

Definition 2. (*Contextual Equivalence*).

$$\begin{aligned}
e_1 \cong_{ctx} e_2 \triangleq\ &\varnothing \vdash e_1 : \sigma \mid \epsilon \wedge \varnothing \vdash e_2 : \sigma \mid \epsilon \\
&\wedge \forall C.\ \varnothing \vdash C : (\sigma \mid \epsilon) \to (\mathit{Int} \mid \langle\rangle) \implies (\forall n.\ C[e_1] \longmapsto^* n \iff C[e_2] \longmapsto^* n)
\end{aligned}$$

where C is the standard definition of a *program context* that is under-free, and $C[e_1]$ is evaluated under the original semantics while $C[e_2]$ is with tail-resumptive optimization. The notation $\varnothing \vdash C : (\sigma_1 \mid \epsilon_1) \to (\sigma_2 \mid \epsilon_2)$ type-checks an program context, so that if $\varnothing \vdash e : \sigma_1 \mid \epsilon_1$, we have $\varnothing \vdash C[e] : \sigma_2 \mid \epsilon_2$.

Theorem 5. (*Tail-resumptive Optimization is Sound*). If $\varnothing \vdash e : \sigma \mid \epsilon$, then $e \cong_{ctx} e$.

The theorem may seem trivial, but given that $\cong_{ctx}$ uses different evaluation strategies for the left expression and the right one, the theorem states exactly what we want: starting from the same expression e, evaluating without and with tail-resumptive optimization produces the same result. We have also proved that Theorem 2 (Preservation) and Theorem 3 (Progress) remain valid for internal-safe System F^{pw} extended with under.

$$
\begin{array}{llll}
\text{Expressions} & e & ::= & v \mid e\ e \mid e\ \sigma \mid \mathsf{prompt}\ m\ h\ e \mid \mathsf{yield}\ m\ v\ v \\
(app_1) & v\ \square \bullet \mathsf{yield}\ m\ f\ k & \longrightarrow & \mathsf{yield}\ m\ f\ (\lambda x.\ v\ (k\ x)) \\
(app_2) & \square\ e \bullet \mathsf{yield}\ m\ f\ k & \longrightarrow & \mathsf{yield}\ m\ f\ (\lambda x.\ (k\ x)\ e) \\
(under) & \mathsf{under}\ l\ \square \bullet \mathsf{yield}\ m\ f\ k & \longrightarrow & \mathsf{yield}\ m\ f\ (\lambda x.\ \mathsf{under}\ l\ (k\ x)) \\
(prompt_1) & \mathsf{prompt}\ m\ h\ \square \bullet \mathsf{yield}\ m\ f\ k & \longrightarrow & f\ (\lambda x.\ \mathsf{prompt}\ m\ h\ (k\ x)) \\
(prompt_2) & \mathsf{prompt}\ n\ h\ \square \bullet \mathsf{yield}\ m\ f\ k & \longrightarrow & \mathsf{yield}\ m\ f\ (\lambda x.\ \mathsf{prompt}\ n\ h\ (k\ x)) \quad \text{iff } n \neq m \\
(perform) & w \vdash \mathsf{perform}\ op\ \epsilon_0\ \overline{\sigma}\ v & \longrightarrow & \mathsf{yield}\ m\ (\lambda k.\ f\ \overline{\sigma}\ v\ k)\ (\lambda x.\ x) \\
 & & & \text{with } (m, h, \_) = w.l \wedge (op \mapsto f) \in h
\end{array}
$$

Fig. 4. F^{pb}: Multi-prompt with bubble semantics.

4 TRANSLATION TO POLYMORPHIC LAMBDA CALCULUS

In order to compile to standard lambda calculus from our evidence passing effect handler calculus, we first need to ensure that all transitions are *local* and no longer manipulate evaluation contexts explicitly. The only operation that it is non-local with evidence passing semantics is the yield. As discussed in Section 2.6 we can make this local by bubbling up the yields step-by-step through the context while constructing a resumption.

4.1 Bubbling Yields

We briefly introduce System F^{pb} (Figure 4), which extends the semantics of F^{pw} where yield builds the resumption locally and bubbles up to its corresponding prompt frame. In this section, we focus on the dynamic semantics of System F^{pb}, with its full typed formalization and preservation and progress theorems given in Appendix C.

First, expressions now include a new form of yielding expression yield $m\ v\ v$ that takes an extra argument: first v is a continuation that waits for the resumption (like before), while the second v is the current resumption that is extended while bubbling up. We then replace the original rule (*prompt*) and (*perform*) in System F^{pw} (Figure 1) with rules in Figure 4. The new rule (*perform*) builds the continuation and the initial resumption, which is then bubbled up by the rest of the rules. In rule (*perform*) the yield now gets an extra argument $(\lambda x.\ x)$ which is the initial *partially built* resumption – at this time just an identity function. We can now gradually extend the resumption as yield bubbles up through every evaluation frame, as in rule (app_1), (app_2) and $(prompt_2)$. In rule (app_1), the frame $v\ \square$ is added to the current partially built resumption k, generating $(\lambda x.\ v\ (k\ x))$. Rule (app_2) is similar. Rule $(prompt_2)$ compares markers and finds that $n \neq m$ and adds the prompt frame to the resumption. The yield keeps bubbling up until it finds its matching handler in rule $(prompt_1)$, where we finally apply the continuation f to now completed resumption.

4.2 A Multi-prompt Delimited Control Monad

All transitions in the bubbling semantics are now *local* transitions, and we can implement these semantics using a multi-prompt delimited control monad, where each algebraic effect specific construct can be implemented directly as a regular function. In this section, we first establish the multi-prompt delimited control monad and then discuss the type directed translation from System F^{pb} into a polymorphic lambda calculus. We use standard techniques [Dyvbig et al. 2007] to implement delimited control as a monad. For better readability, throughout this section we use Haskell-like syntax. First, we define our monad Mon as:

$$\mathsf{type\ Mon}\ \mu\ \alpha\ =\ \mathsf{Evv}\ \mu \to \mathsf{Ctl}\ \mu\ \alpha$$

The evidence-passing semantics is established by taking an argument of type Evv μ, which corresponds to the current evidence vector for an effect μ, and returning in the control monad Ctl. The control monad is defined as[4]:

$$\begin{array}{l}\mathsf{data}\ \mathsf{Ctl}\ \mu\ \alpha\ =\\ \quad \mid \mathsf{Pure} : \alpha \to \mathsf{Ctl}\ \mu\ \alpha\\ \quad \mid \mathsf{Yield} : \forall \beta\ \mu'\ r.\ \mathsf{Marker}\ \mu'\ r \to ((\beta \to \mathsf{Mon}\ \mu'\ r) \to \mathsf{Mon}\ \mu'\ r) \to (\beta \to \mathsf{Mon}\ \mu\ \alpha) \to \mathsf{Ctl}\ \mu\ \alpha\end{array}$$

The Pure case returns a value result, while Yield implements yielding to a prompt (corresponding to yield $m\ f\ k$ in System F^{pb}). Markers carry explicit types and can later serve as the witness to type equality. When binding a yield, the resumption keeps being extended:

$$\begin{array}{lll}(f \circ g)\ x = f\ (g\ x) & & \text{(function composition)}\\ (f \star g)\ x = g\ x \rhd f & & \text{(Kleisli composition)}\\ e \rhd g = \lambda w.\ \mathsf{case}\ e\ w\ \mathsf{of}\ \mathsf{Pure}\ x \to g\ x\ w & & \text{(monadic bind)}\\ \qquad\qquad \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g \star k) & & \end{array}$$

With the multi-prompt monad, we can now define the monadic translation of types from System F^{pb}, where all effectful functions are made monadic:

$$\begin{array}{llll}\lfloor \forall \overline{\alpha^\kappa}.\ \sigma \rfloor = \forall \overline{\alpha^\kappa}.\ \lfloor \sigma \rfloor & & \lfloor \sigma_1 \to \epsilon\ \sigma_2 \rfloor = \lfloor \sigma_1 \rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor \sigma_2 \rfloor\\ \lfloor \alpha \rfloor = \alpha & & \lfloor c^\kappa\ \sigma_1 \ldots \sigma_n \rfloor = c^\kappa\ \lfloor \sigma_1 \rfloor \ldots \lfloor \sigma_n \rfloor\end{array}$$

We then implement prompt as a family of $prompt^l$ functions for each effect l:

$$\begin{array}{lll}prompt^l : \forall \mu\ \alpha.\ \mathsf{Marker}\ \mu\ \alpha \to \mathsf{Hnd}^l\ \mu\ \alpha \to \mathsf{Mon}\ \langle l \mid \mu \rangle\ \alpha \to \mathsf{Mon}\ \mu\ \alpha & & \\ prompt^l\ m\ h\ e = \lambda w.\ \mathsf{case}\ e\ \langle\!\langle l : (m, h, w) \mid w \rangle\!\rangle\ \mathsf{of} & & \\ \quad \mathsf{Pure}\ x \to \mathsf{Pure}\ x & & ((promptv) \text{ in Fig. 1})\\ \quad \mathsf{Yield}\ m'\ f\ k \mid m \neq m' \to \mathsf{Yield}\ m'\ f\ (prompt^l\ m\ h \circ k) & & ((prompt_2) \text{ in Fig. 4})\\ \quad \mathsf{Yield}\ m'\ f\ k \mid m = m' \to f\ (prompt^l\ m\ h \circ k)\ w & & ((prompt_1) \text{ in Fig. 4})\end{array}$$

Note how the evidence vector is passed as an explicit argument in the monad. The Pure case returns the value as is. For Yield, if it yields to another prompt, we keep building the resumption. In the third case, Yield meets the target prompt and we apply f to the built-up resumption (composed with $prompt^l\ m\ h$ as we use *deep* resumptions). Note that to type check this case, the equality of the markers $m = m'$ implies that $\mu = \mu'$ and $\alpha = r$ (as in the definition of Yield). For example, this can be encoded using explicit equality witnesses [Baars and Swierstra 2002] or equality constraints in Haskell [Sulzmann et al. 2007].

The *handler* function generates *prompt* with a fresh marker created by a utility function *freshm*.

$$\begin{array}{ll}handler^l : \forall \mu\ \alpha.\ \mathsf{Hnd}^l\ \mu\ \alpha \to ((\,) \to \mathsf{Mon}\ \langle l \mid \mu \rangle\ \alpha) \to \mathsf{Mon}\ \mu\ \alpha & \\ handler^l\ h\ f = freshm\ (\lambda m \to prompt^l\ m\ h\ (f\ (\,))) & ((handler) \text{ in Fig. 1})\end{array}$$

The type of a handler Hnd^l is generated for every effect signature $l : \{\ op_1 : \forall \overline{\alpha}_1.\ \sigma_1 \to \sigma_1',\ \ldots,\ op_n : \forall \overline{\alpha}_n.\ \sigma_n \to \sigma_n'\ \} \in \Sigma$ and is a record of operation definitions:

$$\mathsf{data}\ \mathsf{Hnd}^l\ \mu\ r = \mathsf{Hnd}^l\ (\forall \overline{\alpha}_1.\ \mathsf{Op}\ \lfloor \sigma_1 \rfloor\ \lfloor \sigma_1' \rfloor\ \mu\ r) \ldots (\forall \overline{\alpha}_n.\ \mathsf{Op}\ \lfloor \sigma_n \rfloor\ \lfloor \sigma_n' \rfloor\ \mu\ r)$$

together with a selector for each operation $op_i : \forall \overline{\alpha}.\ \sigma_1 \to \sigma_2 \in \Sigma(l)$:

$$\begin{array}{l}select^{op_i} : \forall \overline{\alpha}\ \mu\ r.\ \mathsf{Hnd}^l\ \mu\ r \to \mathsf{Op}\ \lfloor \sigma_1 \rfloor\ \lfloor \sigma_2 \rfloor\ \mu\ r\\ select^{op_i}\ (\mathsf{Hnd}^l\ op_1 \ldots op_n) = op_i\end{array}$$

[4]Our Ctl is different from that of [Xie et al. 2020] as the continuation and resumption in Yield both return in Mon, whereas they only need to return in Ctl (again because in their system the evidence vector is statically determined).

$$\textsc{val}\quad \frac{\Gamma \vdash_{\mathsf{val}} v : \sigma \rightsquigarrow v'}{\Gamma \vdash v : \sigma \mid \epsilon \rightsquigarrow \lambda w : \mathsf{Evv}\ \epsilon.\ \mathsf{Pure}\ \epsilon\ \lfloor\sigma\rfloor\ v'} \qquad \textsc{app}\quad \frac{\Gamma \vdash e_1 : \sigma_1 \to \epsilon\ \sigma \mid \epsilon \rightsquigarrow e_1' \quad \Gamma \vdash e_2 : \sigma_1 \mid \epsilon \rightsquigarrow e_2'}{\Gamma \vdash e_1\ e_2 : \sigma \mid \epsilon \rightsquigarrow e_1' \rhd (\lambda f : \lfloor \sigma_1 \to \epsilon\ \sigma \rfloor.\ e_2' \rhd f)}$$

$$\frac{op : \forall \overline{\alpha}.\ \sigma_1 \to \sigma_2 \in \Sigma(l)}{\begin{array}{l}\Gamma \vdash_{\mathsf{val}} \mathsf{perform}\ op\ \epsilon\ \overline{\sigma} : \sigma_1[\overline{\alpha}:=\overline{\sigma}] \to \langle l \mid \epsilon\rangle\ \sigma_2[\overline{\alpha}:=\overline{\sigma}] \\ \quad \rightsquigarrow perform^l\ \epsilon\ \lfloor\sigma_1[\overline{\alpha}:=\overline{\sigma}]\rfloor\ \lfloor\sigma_2[\overline{\alpha}:=\overline{\sigma}]\rfloor\ (\Lambda\mu\ r.\ select^{op}\ \lfloor\overline{\sigma}\rfloor\ \mu\ r)\end{array}}\ \textsc{perform}$$

Fig. 5. Monadic translation of F^{pb} (excerpt).

Where the $\mathsf{Op}\ \alpha\ \beta\ \mu\ r$ type represents operations from α to β defined in a handler with effect μ and result type r (the *answer* type). For example for a reader effect we have:

$$\begin{array}{l}\mathsf{data}\ \mathsf{Hnd}^{read}\ \mu\ r \quad = \mathsf{Hnd}^{read}\ (\mathsf{Op}\ ()\ \mathsf{int}\ \mu\ r) \\ select^{ask}\ (\mathsf{Hnd}^{read}\ ask) = ask\end{array}$$

For operations we distinguish between tail-resumptive operation implementations and normal implementations in order to do the tail-resumptive optimization:

$$\begin{array}{l}\mathsf{data}\ \mathsf{Op}\ \alpha\ \beta\ \mu\ r\ \ = \mathsf{Tail} \quad : (\alpha \to \mathsf{Mon}\ \mu\ \beta) \to \mathsf{Op}\ \alpha\ \beta\ \mu\ r \\ \qquad\qquad\quad \mid \mathsf{Normal} : (\alpha \to \mathsf{Mon}\ \mu\ ((\beta \to \mathsf{Mon}\ \mu\ r) \to \mathsf{Mon}\ \mu\ r)) \to \mathsf{Op}\ \alpha\ \beta\ \mu\ r\end{array}$$

We can now *perform* an operation by getting the handler from the evidence vector, and selecting the right operation from the handler record (e.g. ask). Depending on the operation constructor, we use $under^l$ for tail-resumptive operations or otherwise generate a Yield.

$$\begin{array}{l}perform^l : \forall \mu\ \alpha\ \beta.\ (\forall \mu'\ r.\ \mathsf{Hnd}^l\ \mu'\ r \to \mathsf{Op}\ \alpha\ \beta\ \mu'\ r) \to \alpha \to \mathsf{Mon}\ \langle l \mid \mu\rangle\ \beta \\ perform^l\ select\ x\ =\ \lambda w : \mathsf{Evv}\ \langle l \mid \mu\rangle.\ \mathsf{let}\ (m, h, w') = w.l\ \mathsf{in} \\ \quad \mathsf{case}\ select\ h\ \mathsf{of}\ \mathsf{Tail}\ f \quad \to under^l\ m\ w'\ (f\ x) \qquad \text{((}performt\text{) in Fig. 3)} \\ \qquad\qquad\qquad \mathsf{Normal}\ f \to \mathsf{Yield}\ m\ (\lambda y.\ f\ x \rhd (\lambda g.\ g\ y))\ (\lambda x.\ \lambda w.\ \mathsf{Pure}\ x) \quad \text{((}perform\text{) in Fig. 4)}\end{array}$$

Finally, *under* can be implemented with two mutually recursive definitions:

$$\begin{array}{l}under^l\ : \forall \mu\ \beta\ \mu'\ r.\ \mathsf{Marker}\ \mu'\ r \to \mathsf{Evv}\ \mu' \to \mathsf{Mon}\ \mu'\ \beta \to \mathsf{Mon}\ \mu\ \beta \\ underk^l : \forall \mu\ \beta\ \mu'\ r.\ \mathsf{Marker}\ \mu'\ r \to (\beta \to \mathsf{Mon}\ \mu'\ r) \to \beta \to \mathsf{Mon}\ \mu\ r\end{array}$$

The *under* function runs the action e under another evidence vector w', and ensures any resumption is itself continued under the right evidence through *underk*:

$$\begin{array}{lll}under^l\ m\ w'\ e\ =\ \lambda w : \mathsf{Evv}\ \mu.\ \mathsf{case}\ e\ w'\ \mathsf{of} & & \text{((}underw\text{) in Fig. 3)} \\ \qquad \mathsf{Pure}\ x \quad \to \mathsf{Pure}\ x & & \text{((}underv\text{) in Fig. 3)} \\ \qquad \mathsf{Yield}\ n\ f\ k \to \mathsf{Yield}\ n\ f\ (underk^l\ m\ k) & & \text{((}under\text{) in Fig. 4)}\end{array}$$

Note that it easy to make a mistake here: a well-typed (!) but semantically wrong implementation of $under^l$ is to return in the Yield case $\mathsf{Yield}\ n\ f\ (\lambda x.\ under^l\ m\ w'\ (k\ x))$ – as described in Section 2.9.2 this wrongly fixes the evidence vector to w'. Instead, we need to use the *underk* function which re-finds the correct evidence vector w' from the current evidence vector w to resume under:

$$\begin{array}{lll}underk^l\ m\ k\ x\ =\ \lambda w : \mathsf{evv}\ \mu.\ \mathsf{let}\ (m', h, w' : \mathsf{evv}\ \epsilon)\ =\ w.l\ \mathsf{in} & & \\ \qquad \mathsf{if}\ (m = m')\ \mathsf{then}\ under\ m\ w'\ (k\ x)\ w & & \text{((}underw\text{) in Fig. 3)}\end{array}$$

The marker is passed to under^l and underk^l in order to get the type equality from $m = m'$ (which should always hold for internal-safe expressions).

4.3 Monadic Translation

Using the multi-prompt monad definition, we can define a type-directed translation of System F^{pb} into a polymorphic lambda calculus. The translation takes the form $\Gamma \vdash e : \sigma \mid \epsilon \rightsquigarrow e'$, where

under Γ, the expression e with type σ and effect ϵ is translated to e'. Based on the multi-prompt monad, the translation is mostly straightforward where each construct translates directly to its corresponding function: prompt translates to *prompt*, handler translates to *handler*, etc. Figure 5 shows an excerpt of the translation rules, while the full translation is shown in Appendix C.

During translation, we have made type applications explicit. Rule VAL simply wraps the value translation inside Pure. Rule APP first evaluates e_1', binds the result to f, and then evaluates e_2' and passes the result to f. If any of the expressions evaluates to Yield, the monadic binding ($\triangleright$) will bubble it up (according to the rules (app_1) and (app_2) in Figure 4). Rule PERFORM shows how perform is translated using our monadic implementation of $perform^l$ and $select^{op}$.

We prove that our translation is sound, where we use the notation $\vdash_F$ for the typing judgment in the target polymorphic lambda calculus, whose full definition can be found in the appendix.

Theorem 6. (*Monadic Translation is Sound*). If $\varnothing \vdash e : \sigma \mid \langle\rangle \rightsquigarrow e'$, then $\varnothing \vdash_F e' : \mathsf{Mon}\ \langle\rangle\ \lfloor\sigma\rfloor$.

We can now show that our final monadic translation preserves the original semantics of algebraic effects in System F^ϵ [Xie et al. 2020]. In particular, we prove that given an user-provided expression e, we can apply the monadic translation to get an expression e' in the polymorphic lambda calculus, and evaluating e' preserves the original algebraic effects semantics. We use $e \Uparrow$ to denote that e diverges.

Theorem 7. (*Semantics Preserving*). Given $\varnothing \vdash e : int \mid \langle\rangle \rightsquigarrow e'$, if $e \longmapsto^* n$ in F^ϵ, then $e'\ \langle\!\rangle \longmapsto^*$ Pure $\langle\rangle$ *int* n in the polymorphic lambda calculus. If $e \Uparrow$ in F^ϵ, then $e'\ \langle\!\rangle \Uparrow$ in the polymorphic lambda calculus.

5 BENCHMARKS

In this section we benchmark five implementations of effect handlers.

(1) *Koka*: We have a full implementation of our techniques in the Koka compiler [Koka 2019] which compiles via standard C code. This uses generalized evidence passing with canonical evidence vectors, short-cut resumptions, bind-inlining and join-point sharing.
(2) *multi-core OCaml*: This is a fork of standard OCaml with the current state-of-the-art *direct* implementation of effect handlers based on segmented stacks [Sivaramakrishnan et al. 2021] (but without direct support for multi-shot resumptions).
(3) *Mp.Eff*: This is our implementation of generalized evidence passing effect handlers as a monadic library in Haskell. The library uses insertion-ordered evidence vectors and does not use short-cut resumptions. The implementation is closely based on the *Ev.Eff* library.
(4) *Ev.Eff*: A Haskell monadic effect handler library by Xie and Leijen [2020] based on evidence *translation* (and cannot handle non-scoped resumptions). They have shown that this library performs very well with respect to other effect handler implementations [Kiselyov and Ishii 2015; Schrijvers et al. 2019; Wu and Schrijvers 2015; Wu et al. 2014] and monad transformers.
(5) *libhandler*: a C library that implements effect handlers on top of the regular C stack and uses `longjmp` to transfer control [Leijen 2017a]. This is a *direct* implementation where capturing- and resuming is linear in the stack size as it copies and restores pieces of the C stack directly. It uses the tail-resumptive optimization and insertion ordered "evidence" where it searches through the handler frames on the stack.
(6) *libmprompt*: a C library that implements effect handlers using in-place growable stacks using virtual memory. This is a *direct* implementation with constant time stack switching without need to copy fragments of the stack. Implements tail-resumptive optimization and

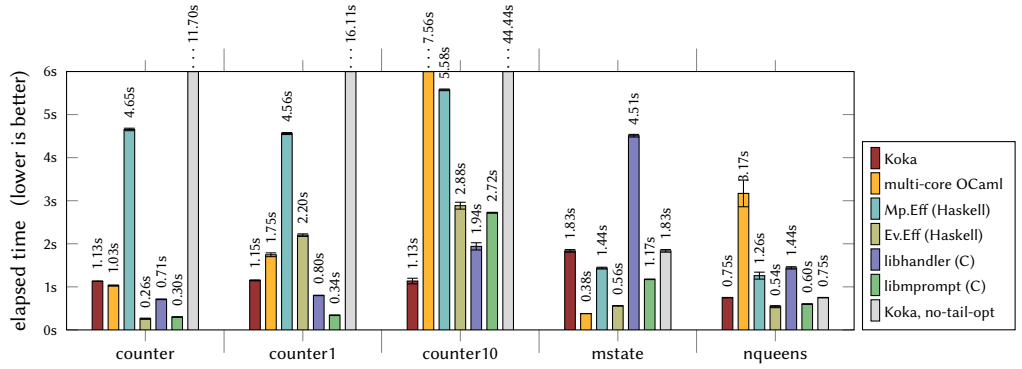

Fig. 6. Execution time averaged over 10 runs

uses insertion ordered "evidence" where it searches through the handler frames on the stack. [Leijen and Sivaramakrishnan 2021]

Comparing across systems is always difficult as many parts differ – for example, Koka uses compiler guided reference while multi-core OCaml and Haskell use a generational tracing collector, Koka has few standard optimizations while both OCaml and Haskell are sophisticated compilers with decades of development, etc. We selected current best-in-class implementations that compile to native code[5] so execution times are somewhat comparable, and we selected benchmarks that probe effect handling in particular, with minimal other computation and allocation overheads. As such, the results are meant to establish if the effect handler compilation strategies described in this paper are viable and can be competitive, but should not be interpreted as a measure of absolute performance between systems and languages. Execution times are shown in Figure 6. The execution times are averaged over 10 runs, on an AMD 5950X at 3.4Ghz with 32GiB memory running Ubuntu 20.04, with Koka v2.0.16, multi-core OCaml 4.10, libhandler v0.5, and GHC 8.6.5.

counter. This benchmark implements a state effect using a mutable reference such that both *get* and *set* operations are tail-resumptive. It then performs 200M *get* and *set* operations in a tight loop. The tail-resumptive optimization in Koka and the fast stack switching in OCaml seem to perform similarly and the execution times are very close. The libhandler C implementation is 1.5× faster than Koka – we believe this is because it does no allocation at all. In contrast, both Koka and OCaml allocate at each operation: OCaml allocates a continuation object per resumption [Sivaramakrishnan et al. 2021], and Koka allocates a resumption result function to support `finally` clauses (which are not used in any benchmark here) [Leijen 2018].

Mp.Eff is about 4× slower as Koka, but *Ev.Eff* is 4× faster! This is because GHC is able to fully inline the handler and operations and optimizes almost all effect handling code away. When we remove the inline pragma on the state handler definition, the benchmark takes about 2.02s which is more in line with the results seen in *counter1* and *counter10*. We also ran this benchmark with the tail-resumption optimization turned off; this causes Koka to always allocate a resumption and take the slow path through the monadic bindings making it 10× slower than the optimized version.

counter1. This is the same as *counter* but with one (unused) reader effect handler in between. This time Koka is 1.5× faster than OCaml: due to evidence passing, the execution times of the tail-resumptive *get* and *set* operations are independent of any other handlers that are in the context (as the handler is found at a constant offset in the canonical evidence vectors). In contrast, multi-core

[5] As opposed to using an interpreter, or using JavaScript as a target for example.

OCaml always yields up one handler stack segment at a time and thus each *get* and *set* operation needs to pass through the reader handler incurring a runtime cost.

counter10. Same as *counter1* but now with 10 reader handlers under the state handler. Again Koka execution is (almost) the same as for *counter1* but we can see that all implementations without tail-resumptive optimization or evidence-passing get slower with each added handler due to the linear search at each operation call.

The *counter10* benchmark may seem artificial but we believe this pattern to be common in practice. Many uses of effect handlers are to provide contextual state and environments; for example, a type checker may have a current substitution, the type environment, a unique identifier generator, etc. Such nested handlers may thus be quite common in general.

mstate. This is the same as *counter* but now implements the state effect in a monadic way as shown in Section 2.1.1. This means that the operations are no longer tail-resumptive since the resumption is captured under a lambda. To reduce the execution time, *mstate* only performs 20M *get* and *set* operations (versus 200M in the tail-resumptive *counter* benchmark). This is a worst-case for Koka as it needs to allocate a fresh resumption for each operation call, and it is about 5× slower than multi-core OCaml here. Surprisingly, both *Mp.Eff* and *Ev.Eff* are faster than Koka here – again, the small benchmark can be optimized impressively well by GHC.

nqueens. Calculates all solutions to the *queens* problem of size 12 using a *choice* effect to elegantly express backtracking similar to the non-determinism example in Section 2.1.1. Multiple resumptions are not directly supported in multi-core OCaml but we can use `Obj.clone_continuation` to manually copy resumptions[6]. Here OCaml is about 5× slower than Koka. We think that this is partly due to the *short-cut resumptions* (Section 2.6.1): in Koka when the call stack is 12 calls deep, each *choice* operation resumes 12 times, each time sharing all continuations; in contrast, OCaml and libhandler need to linearly copy the stack on each resumption. Similarly, the Haskell libraries also do very well here as these also share the resumption functions over multiple resumes.

Overall, the results look promising and show our compilation strategy can be competitive with specialized runtime implementations. With respect to evidence translation versus evidence passing, it seems evidence translation can have the advantage in performance: even though *Mp.Eff* and *Ev.Eff* have very similar implementations, the generalized evidence passing library is about twice as slow as the *Ev.Eff* library over our benchmarks. We believe this is partly caused by the more static nature of evidence in *Ev.Eff* and which makes it more amenable to compiler optimizations.

6 RELATED WORK

Throughout the paper, we compare with the most directly related work [Sivaramakrishnan et al. 2021; Xie et al. 2020; Xie and Leijen 2020] inline. Here we briefly discuss other related work.

In contrast to the monadic translation, Hillerström et al. [2017] describe a CPS based translation of effect handlers. Similar to bubbling and evidence passing, this avoids capturing the evaluation context by making all continuations explicit. Forster et al. [2019] show how delimited control, monads, and effect handlers can all be expressed in terms of each other in an untyped setting. However, their encoding of effect handlers in terms of shift-reset does not preserve typeability (due to the lack of answer type polymorphism). In our work typing is preserved by using multi-prompt delimited control with explicitly typed markers. Kiselyov and Sivaramakrishnan [2017] present a direct embedding of effect handlers in OCaml based on shift-reset (using the *delimcc* library), where they use an out-of-band technique [Kiselyov et al. 2006] to work around the lack of answer

[6]It works for our particular benchmark, but generally multiple resumptions do not work reliably (as currently implemented) for two reasons: the optimizer is not aware of multiple resumptions and may generate invalid code when optimizing across function calls (this is a problem for libhandler as well [Leijen 2017a]), and cloning a continuation does not compose with other operations that may not clone their own continuation (leading to a runtime crash).

type polymorphism. Kammar et al. [2013] also embed effect handlers in OCaml using shift-reset where they use a global mutable variable to hold the stack of handlers in the current dynamic scope (which can be considered as the current insertion-ordered evidence vector).

Brachthäuser et al. [2020] use capability passing to perform operations directly on a specific handler. This is also similar to the work of Zhang and Myers [2019] where handlers are passed by name as well. While they pass a capability individually for each handler, we uniformly pass a vector of handlers. Both of these approaches can be viewed as programming within an explicit evidence passing calculus.

7 CONCLUSION

Generalized evidence passing is a promising technique for compiling effect handlers to standard target platforms, and can offer competitive performance relative to specialized runtimes. Moreover, our formalization explores the design space of implementation techniques and their trade-offs. We hope our study will lead to further improvements of effect handlers implementations in the future.

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$$
\begin{array}{llll}
\text{Expression} & e & ::= & v \mid e\ e \mid e\ \sigma \mid \text{handle}\ h\ e \\
\text{Value} & v & ::= & x \mid \lambda^{\epsilon} x : \sigma.\ e \mid \Lambda \alpha^{\kappa}.\ v \\
 & & \mid & \text{handler}\ h \mid \text{perform}\ op\ \epsilon\ \overline{\sigma} \\
\text{Handler} & h & ::= & \{\ \overline{op_i \mapsto f_i}\ \}
\end{array}
$$

$$
\begin{array}{llll}
\text{Term context} & \Gamma & ::= & \varnothing \mid \Gamma,\ x : \sigma \\
\text{Effect context} & \Sigma & ::= & \{\ \overline{l_i : sig^{l_i}}\ \} \\
\text{Effect sig.} & sig^{l} & ::= & \{\ \overline{op_i : \forall \overline{\alpha_i^{\kappa_i}}.\ \sigma_i \to \sigma_i'}\ \}
\end{array}
$$

$$
\begin{array}{llll}
\text{Type} & \sigma & ::= & \alpha^{\kappa} \mid c^{\kappa}\ \overline{\sigma} \mid \sigma \to \epsilon\ \sigma \mid \forall \alpha^{\kappa}.\ \sigma \\
\text{Effect row} & \epsilon & ::= & \langle\rangle \mid \langle l \mid \epsilon \rangle \\
\text{Kind} & \kappa & ::= & * \mid \kappa \to \kappa \mid \text{lab} \mid \text{eff}
\end{array}
$$

$$
\begin{array}{llll}
\text{Evaluation ctx.} & \mathsf{E} & ::= & \square \mid \mathsf{E}\ e \mid v\ \mathsf{E} \mid \mathsf{E}\ \sigma \\
 & & \mid & \text{handle}\ h\ \mathsf{E} \\
 & \mathsf{F} & ::= & \square \mid \mathsf{F}\ e \mid v\ \mathsf{F} \mid \mathsf{F}\ \sigma
\end{array}
$$

$$
\begin{array}{lllll}
(app) & (\lambda^{\epsilon} x : \sigma.\ e)\ v & \longrightarrow & e[x := v] & \\
(tapp) & (\Lambda \alpha^{\kappa}.\ v)\ \sigma & \longrightarrow & v[\alpha := \sigma] & \\
(handler) & \text{handler}\ h\ v & \longrightarrow & \text{handle}\ h\ (v\ ()) & \\
(return) & \text{handle}\ h\ v & \longrightarrow & v & \\
(perform) & \text{handle}\ h\ \mathsf{E}[\text{perform}\ op\ \epsilon_0\ \overline{\sigma}\ v] & \longrightarrow & f\ \overline{\sigma}\ v\ (\lambda^{\epsilon} x : \sigma_2[\overline{\alpha} := \overline{\sigma}].\ \text{handle}\ h\ \mathsf{E}[x]) & \text{iff}\ op \notin \text{bop}(\mathsf{E}) \\
 & & & \wedge\ (op : (\forall \overline{\alpha}.\ \sigma_1 \to \sigma_2) \to f)\ \in h : \sigma \mid l \mid \epsilon &
\end{array}
$$

$$
\frac{e_1 \longrightarrow e_2}{\mathsf{E}[e_1] \longmapsto \mathsf{E}[e_2]}\ (step)
$$

Fig. 7. System F^{ϵ}: explicitly typed algebraic effect handlers.

APPENDICES

A BACKGROUND: POLYMORPHIC ALGEBRAIC EFFECT CALCULUS F^{ϵ}

This section briefly introduces System F^{ϵ} [Xie et al. 2020], a polymorphic algebraic effect calculus, which essentially extends the polymorphic lambda calculus with algebraic effects and row-based effect types. The system is used for later transformation and optimization.

A.1 Syntax

Figure 7 defines the syntax of System F^{ϵ}. Expressions include algebraic effects specific expressions handler h, perform $op\ \epsilon\ \overline{\sigma}$, and handle $h\ e$.

A.2 Operational Semantics

The operational semantics of System F^{ϵ} is given at the bottom of Figure 7. Rule (*app*) is the standard call-by-value beta reduction rule, and rule (*tapp*) is the standard type reduction rule. We have seen rule (*handler*), (*return*), and (*perform*) at the beginning of Section 2.1. The main difference here is that rule (*perform*) is explicitly typed. In particular, perform passes to the operation implementation f the type arguments $\overline{\sigma}$ along with the value argument v and the resumption $(\lambda^{\epsilon} x : \sigma_2[\overline{\alpha} := \overline{\sigma}].\ \text{handle}\ h\ \mathsf{E}[x])$, which is explicitly annotated with the type and effect annotation. The notation $(op : (\forall \overline{\alpha}.\ \sigma_1 \to \sigma_2) \to f)\ \in h : \sigma \mid l \mid \epsilon$ is a syntactic sugar for three conditions: (1) $(op \to f)\ \in h$, which gets the operation implementation f from h; (2) $op : (\forall \overline{\alpha}.\ \sigma_1 \to \sigma_2) \in \Sigma(l)$, which gets the type of the operation op from the global effect context Σ; and (3) $\varnothing \vdash_{\text{ops}} h : \sigma \mid l \mid \epsilon$.

A.3 Typing Rules

Figure 8 defines the typing rules for System F^{ϵ}.

Most rules are standard. In rule HANDLER, given a handler h for l, the rule takes an action with effect $\langle l \mid \epsilon \rangle$, and handles l, leaving effect ϵ. Rule HANDLE is similar, but directly takes an expression e of effect $\langle l \mid \epsilon \rangle$.

$$\Gamma \vdash e : \sigma \mid \epsilon \qquad \Gamma \vdash_{\mathsf{val}} v : \sigma \qquad \Gamma \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon$$

$$\frac{x : \sigma \in \Gamma}{\Gamma \vdash_{\mathsf{val}} x : \sigma}\ \textsc{var} \qquad \frac{\Gamma \vdash_{\mathsf{val}} v : \sigma}{\Gamma \vdash v : \sigma \mid \epsilon}\ \textsc{val}$$

$$\frac{\Gamma, x : \sigma_1 \vdash e : \sigma_2 \mid \epsilon}{\Gamma \vdash_{\mathsf{val}} \lambda^{\epsilon}\, x{:}\sigma_1.\, e : \sigma_1 \to \epsilon\ \sigma_2}\ \textsc{abs} \qquad \frac{\Gamma \vdash e_1 : \sigma_1 \to \epsilon\ \sigma \mid \epsilon \quad \Gamma \vdash e_2 : \sigma_1 \mid \epsilon}{\Gamma \vdash e_1\ e_2 : \sigma \mid \epsilon}\ \textsc{app}$$

$$\frac{\Gamma \vdash_{\mathsf{val}} v : \sigma \quad \kappa \neq \mathsf{lab}}{\Gamma \vdash_{\mathsf{val}} \Lambda\alpha^{\kappa}.\, v : \forall\alpha^{\kappa}.\, \sigma}\ \textsc{tabs} \qquad \frac{\Gamma \vdash e : \forall\alpha^{\kappa}.\, \sigma_1 \mid \epsilon \quad \vdash_{\mathsf{wf}} \sigma : \kappa}{\Gamma \vdash e\ \sigma : \sigma_1[\alpha{:=}\sigma] \mid \epsilon}\ \textsc{tapp}$$

$$\frac{op : \forall\overline{\alpha}.\, \sigma_1 \to \sigma_2 \in \Sigma(l)}{\Gamma \vdash_{\mathsf{val}} \mathsf{perform}\ op\ \epsilon\ \overline{\sigma} : \sigma_1[\overline{\alpha}{:=}\overline{\sigma}] \to \langle l \mid \epsilon \rangle\ \sigma_2[\overline{\alpha}{:=}\overline{\sigma}]}\ \textsc{perform}$$

$$\frac{\Gamma \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon}{\Gamma \vdash_{\mathsf{val}} \mathsf{handler}\ h : (() \to \langle l \mid \epsilon \rangle\ \sigma) \to \epsilon\ \sigma}\ \textsc{handler} \qquad \frac{\Gamma \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon \quad \Gamma \vdash e : \sigma \mid \langle l \mid \epsilon \rangle}{\Gamma \vdash \mathsf{handle}\ h\ e : \sigma \mid \epsilon}\ \textsc{handle}$$

$$\frac{op_i : \forall\overline{\alpha}.\, \sigma_1 \to \sigma_2 \in \Sigma(l) \quad \overline{\alpha} \notin \mathsf{ftv}(\epsilon, \sigma) \quad \Gamma \vdash_{\mathsf{val}} f_i : \forall\overline{\alpha}.\, \sigma_1 \to \epsilon\ ((\sigma_2 \to \epsilon\ \sigma) \to \epsilon\ \sigma)}{\Gamma \vdash_{\mathsf{ops}} \{\ op_1 \to f_1,\ \ldots,\ op_n \to f_n\ \} : \sigma \mid l \mid \epsilon}\ \textsc{ops}$$

Fig. 8. Typing Rules for System F^{ϵ}.

A.4 Preservation and Progress

Xie et al. [2020] have proved that System F^{ϵ} enjoys progress and preservation. Progress implies that in any well-typed total expression, all operations are handled properly.

Theorem 8. (*Progress*). If $\varnothing \vdash e_1 : \sigma \mid \langle\rangle$ then either e_1 is a value, or $e_1 \longmapsto e_2$.

Theorem 9. (*Preservation*). If $\varnothing \vdash e_1 : \sigma \mid \langle\rangle$ and $e_1 \longmapsto e_2$, then $\varnothing \vdash e_2 : \sigma \mid \langle\rangle$.

B MULTI-PROMPT

Figure 9 presents System F^{p}, which applies the multi-prompt semantics to algebraic effects.

B.1 Syntax

Expressions e include all expressions from System F^{ϵ}, except for handle, which is replaced by two internal multi-prompt constructs, prompt $m\ h\ e$ and yield $m\ v$. Similarly, the evaluation context frame handle is replaced by prompt. Notably, prompt and yield both carry a *marker*: a marker identifies a specific prompt, which can be used for searching as in yield. yield further carries a value v, which is a continuation that waits for the resumption, as we will see in the operational semantics rules.

B.2 Operational Semantics

Figure 9b defines the operational semantics rules. In rule (*handler*), handler this time installs a prompt m frame, with a fresh unique marker m, so that the marker can later be used to find the specific prompt. As before, values are propagated through the prompt frame (rule (*promptv*)).

As this system models the multi-prompt semantics, we split performing an operation into two parts: searching for a handler (rule (*perform*)), and capturing and restoring a resumption (rule (*prompt*)). In rule (*perform*), perform finds the corresponding prompt. But instead of directly

$$
\begin{array}{lll}
\text{Expression} & e & ::= \; v \mid e\ e \mid e\ \sigma \\
 & & \quad \mid \; \mathsf{prompt}\ m\ h\ e \mid \mathsf{yield}\ m\ v \\
\text{Evaluation context} & \mathsf{E} & ::= \; \Box \mid \mathsf{E}\ e \mid v\ \mathsf{E} \mid \mathsf{prompt}\ m\ h\ \mathsf{E}
\end{array}
$$

(a) Syntax

$$
\begin{array}{lllll}
(\mathit{handler}) & \mathsf{handler}\ h\ v & \longrightarrow & \mathsf{prompt}\ m\ h\ (v\ ()) & \text{with unique } m \\
(\mathit{promptv}) & \mathsf{prompt}\ m\ h\ v & \longrightarrow & v & \\
(\mathit{prompt}) & \mathsf{prompt}\ m\ h\ \mathsf{E}[\mathsf{yield}\ m\ f] & \longrightarrow & f\ (\lambda^{\epsilon} x : \sigma_2.\ \mathsf{prompt}\ m\ h\ \mathsf{E}[x]) & \\
 & & & \varnothing \vdash_{\mathsf{val}} f : (\sigma_2 \to \epsilon\ \sigma) \to \epsilon\ \sigma & \\
(\mathit{perform}) & \mathsf{prompt}\ m\ h\ \mathsf{E}[\mathsf{perform}\ \mathit{op}\ \epsilon_0\ \overline{\sigma}\ v] & \longrightarrow & \mathsf{prompt}\ m\ h\ \mathsf{E}[\mathsf{yield}\ m\ (\lambda^{\epsilon} k : \sigma_k.\ f\ \overline{\sigma}\ v\ k)] & \\
 & & \text{iff} & \mathit{op} \notin \mathsf{bop}(\mathsf{E}) & \\
 & & & (\mathit{op} : (\forall \overline{\alpha}.\ \sigma_1 \to \sigma_2) \to f) \in h : \sigma \mid l \mid \epsilon & \\
 & & & \sigma_k = \sigma_2[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ \sigma &
\end{array}
$$

(b) Operational semantics

$$
\frac{\Gamma \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon \qquad \Gamma \vdash e : \sigma \mid \langle l \mid \epsilon \rangle}{\Gamma \vdash \mathsf{prompt}\ m\ h\ e : \sigma \mid \epsilon}\ \textsc{prompt}
\qquad
\frac{\Gamma \vdash_{\mathsf{val}} f : (\sigma \to \epsilon'\ \sigma') \to \epsilon'\ \sigma'}{\Gamma \vdash \mathsf{yield}\ m\ f : \sigma \mid \epsilon}\ \textsc{yield}
$$

(c) Typing rules

Fig. 9. F^p: Explicitly typed with multi-prompt delimited control.

capturing the resumption, the rule produces yield m, with a continuation $(\lambda^{\epsilon} k : \sigma_k.\ f\ \overline{\sigma}\ v\ k)$, which, when applied to the resumption k, applies the operation implementation f to the type arguments $\overline{\sigma}$, value argument v and the resumption k. Then, in rule (*prompt*), yield captures the resumption $(\lambda^{\epsilon} x : \sigma_2.\ \mathsf{prompt}\ m\ h\mathsf{E}[x])$, to which f is applied.

B.3 Typing Rules

Figure 9c gives the typing rules for the two multi-prompt constructs. Rule PROMPT is similar as rule HANDLE, which handles l from the effect $\langle l \mid \epsilon \rangle$. Rule YIELD is more subtle. Recall that the operational rule (*perform*) (in Figure 9b) turns perform $\mathit{op}\ \epsilon_0 \overline{\sigma}\ v$ into yield, so the type of yield should match the result type of perform $\mathit{op}\ \epsilon_0\ \overline{\sigma}\ v$. According to the typing rule for perform, the result type of perform $\mathit{op}\ \epsilon_0\ \overline{\sigma}\ v$ is $\sigma_2[\overline{\alpha} := \overline{\sigma}]$ (as in rule (*perform*)), which is the argument type of the resumption k. Thus in YIELD, we get the result type from the argument type of f's argument. To be more precise, we could also set the result effect of yield to match that of perform $\mathit{op}\ \epsilon_0\ \overline{\sigma}\ v$. But since yield is an internal frame, as we will see in the next section, the current form is sufficient for type soundness.

B.4 Preservation and Progress

Theorem 10. (*Preservation of Internal-safe System* F^p). If $\varnothing \vdash e_1 : \sigma \mid \langle\rangle$ where e_1 is internal-safe, and $e_1 \longmapsto e_2$, then $\varnothing \vdash e_2 : \sigma \mid \langle\rangle$.

Theorem 11. (*Progress of Internal-safe System* F^p). If $\varnothing \vdash e_1 : \sigma \mid \langle\rangle$ where e_1 is internal-safe, then either e_1 is a value, or $e_1 \longmapsto e_2$.

C TYPED BUBBLE SEMANTICS

$$\begin{array}{lrcll} \text{Expressions} & e & ::= & \ldots & \text{(non-yield expressions from } \mathsf{F}^{p} \text{ (Fig. 1))} \\ & & | & \mathsf{yield}\ m\ v\ v & \text{(yield with bubbling)} \end{array}$$

(a) Expressions

$$\begin{array}{lll} (app_1) & v\,\square \bullet \mathsf{yield}\ m\ f\ k & \longrightarrow \mathsf{yield}\ m\ f\ (\lambda^{\epsilon} x\!:\!\sigma_2.\ v\ (k\ x)) \quad \varnothing \vdash_{\mathsf{val}} k : \sigma_2 \to \epsilon\ \sigma \\ (app_2) & \square\ e \bullet \mathsf{yield}\ m\ f\ k & \longrightarrow \mathsf{yield}\ m\ f\ (\lambda^{\epsilon} x\!:\!\sigma_2.\ (k\ x)\ e) \quad \varnothing \vdash_{\mathsf{val}} k : \sigma_2 \to \epsilon\ \sigma \\ (under) & \mathsf{under}^{\epsilon_0,\epsilon}\ l\ \square \bullet \mathsf{yield}\ m\ f\ k & \longrightarrow \mathsf{yield}\ m\ f\ (\lambda^{\epsilon_0} x\!:\!\sigma_2.\ \mathsf{under}^{\epsilon_0,\epsilon}\ l\ (k\ x)) \\ & & \qquad \varnothing \vdash_{\mathsf{val}} k : \sigma_2 \to \epsilon\ \sigma \\ (prompt_1) & \mathsf{prompt}\ m\ h\ \square \bullet \mathsf{yield}\ m\ f\ k & \longrightarrow f\ (\lambda^{\epsilon} x\!:\!\sigma_2.\ \mathsf{prompt}\ m\ h\ (k\ x)) \quad \varnothing \vdash_{\mathsf{val}} k : \sigma_2 \to \langle l \mid \epsilon \rangle\ \sigma \\ (prompt_2) & \mathsf{prompt}\ n\ h\ \square \bullet \mathsf{yield}\ m\ f\ k & \longrightarrow \mathsf{yield}\ m\ f\ (\lambda^{\epsilon} x\!:\!\sigma_2.\ \mathsf{prompt}\ n\ h\ (k\ x)) \quad \text{iff } n \neq m \\ & & \qquad \varnothing \vdash_{\mathsf{val}} k : \sigma_2 \to \langle l \mid \epsilon \rangle\ \sigma \end{array}$$

$$\begin{array}{lll} (perform) & w \vdash \mathsf{perform}\ op\ \epsilon_0\ \overline{\sigma}\ v & \longrightarrow \mathsf{yield}\ m\ (\lambda^{\epsilon} k\!:\!\sigma_k.\ f\ \overline{\sigma}\ v\ k)\ (\lambda^{\langle l \mid \epsilon_0 \rangle} x\!:\!\sigma_2[\overline{\alpha}\!:=\!\overline{\sigma}].\ x) \\ & & \text{with } (m, h, \_) = w.l \wedge (op \to f) \in h \\ & & \quad \sigma_k = \sigma_2[\overline{\alpha}\!:=\!\overline{\sigma}] \to \epsilon\ \sigma \\ & & \quad \varnothing \vdash h : \sigma \mid l \mid \epsilon \quad (op\!:\!\forall\overline{\alpha}.\ \sigma_1 \to \sigma_2) \in \Sigma(l) \end{array}$$

(b) Operational Semantics

$$\frac{\Gamma \vdash_{\mathsf{val}} f : (\sigma_2 \to \epsilon'\ \sigma') \to \epsilon'\ \sigma' \quad \Gamma \vdash_{\mathsf{val}} k : \sigma_2 \to \epsilon\ \sigma}{\Gamma \vdash \mathsf{yield}\ m\ f\ k : \sigma \mid \epsilon}\ \textsc{yieldb}$$

(c) Typing rule

Fig. 10. F^{pb}: Multi-prompt with bubble semantics.

Figure 10 gives the full typed formalization of the bubble semantics given in Section 4.1.

C.1 Typing

The rule YIELDB type-checks the new yield form where the argument type σ_2 of k always matches the expected operation result type σ_2 in f. The result type and effect of yield are the same as that of k. Note how the result type and effect of the partially built resumption change during each bubbling step in the operational semantics.

C.2 Preservation and Progress

Like before, we prove the preservation and the progress theorem of internal-safe System F^{pb}. The main challenge in the proof is to show that when the continuation f is applied to the complete resumption k, the type of k matches the argument type of f; that is, at that moment, we have $\sigma = \sigma'$ and $\epsilon = \epsilon'$ (as in rule YIELDB).

Theorem 12. (*Preservation of Internal-safe System* F^{pb}). If $\varnothing \vdash e_1 : \sigma \mid \langle\rangle$ where e_1 is internal-safe, and $\langle\!\langle\rangle\!\rangle \mid e_1 \longmapsto e_2$, then $\varnothing \vdash e_2 : \sigma \mid \langle\rangle$.

Theorem 13. (*Progress of Internal-safe System* F^{pb}). If $\varnothing \vdash e_1 : \sigma \mid \langle\rangle$ where e_1 is internal-safe, then either e_1 is a value, or $\langle\!\langle\rangle\!\rangle \vdash e_1 \longmapsto e_2$.

C.3 Monadic Translation

The full monadic translation is given in Figure 11.

$$\Gamma \vdash e : \sigma \mid \epsilon \rightsquigarrow e' \quad (\uparrow\ \uparrow_{F^{pb}}\ \downarrow\ \uparrow\ \downarrow_{F^{v}}) \qquad \Gamma \vdash_{val} v : \sigma \rightsquigarrow v' \quad (\uparrow\ \uparrow_{F^{pb}}\ \downarrow\ \downarrow_{F^{v}}) \qquad \Gamma \vdash_{ops} h : \sigma \mid l \mid \epsilon \rightsquigarrow h' \quad (\uparrow\ \uparrow_{F^{pb}}\ \downarrow\ \downarrow\ \downarrow\ \downarrow_{F^{v}})$$

$$\frac{x : \sigma \in \Gamma}{\Gamma \vdash_{val} x : \sigma \rightsquigarrow x}\ \text{VAR} \qquad \frac{\Gamma, x : \sigma_1 \vdash e : \sigma_2 \mid \epsilon \rightsquigarrow e'}{\Gamma \vdash_{val} \lambda^{\epsilon}\, x : \sigma_1.\, e : \sigma_1 \to^{\epsilon} \sigma_2 \rightsquigarrow \lambda x : \lfloor \sigma_1 \rfloor.\, e'}\ \text{ABS}$$

$$\frac{\Gamma \vdash_{val} v : \sigma \rightsquigarrow v'}{\Gamma \vdash v : \sigma \mid \epsilon \rightsquigarrow \mathsf{pure}\ \epsilon\ \lfloor \sigma \rfloor\ v'}\ \text{VAL} \qquad \frac{\Gamma \vdash_{val} v : \sigma \rightsquigarrow v' \quad \kappa \neq \mathsf{lab}}{\Gamma \vdash_{val} \Lambda \alpha^{\kappa}.\, v : \forall \alpha^{\kappa}.\, \sigma \rightsquigarrow \Lambda \alpha^{\kappa}.\, v'}\ \text{TABS}$$

$$\frac{\Gamma \vdash e_1 : \sigma_1 \to \epsilon\ \sigma \mid \epsilon \rightsquigarrow e_1' \quad \Gamma \vdash e_2 : \sigma_1 \mid \epsilon \rightsquigarrow e_2'}{\Gamma \vdash e_1\ e_2 : \sigma \mid \epsilon \rightsquigarrow e_1' \rhd (\lambda f : \lfloor \sigma_1 \to \epsilon\ \sigma \rfloor.\, e_2' \rhd f)}\ \text{APP}$$

$$\frac{\Gamma \vdash e : \forall \alpha^{\kappa}.\, \sigma_1 \mid \epsilon \rightsquigarrow e' \quad \vdash_{wf} \sigma : \kappa}{\Gamma \vdash e\ \sigma : \sigma_1[\alpha := \sigma] \mid \epsilon \rightsquigarrow e' \rhd (\lambda x : \lfloor \forall \alpha^{\kappa}.\, \sigma_1 \rfloor.\, pure\ \epsilon\ \lfloor \sigma_1[\alpha := \sigma] \rfloor\ (x\ \lfloor \sigma \rfloor))}\ \text{TAPP}$$

$$\frac{op : \forall \overline{\alpha}.\, \sigma_1 \to \sigma_2 \in \Sigma(l)}{\Gamma \vdash_{val} \mathsf{perform}\ op\ \epsilon\ \overline{\sigma} : \sigma_1[\overline{\alpha} := \overline{\sigma}] \to \langle l \mid \epsilon \rangle\ \sigma_2[\overline{\alpha} := \overline{\sigma}] \rightsquigarrow perform^{l}\ \epsilon\ \lfloor \sigma_1[\overline{\alpha} := \overline{\sigma}] \rfloor\ \lfloor \sigma_2[\overline{\alpha} := \overline{\sigma}] \rfloor\ (\Lambda \mu\ r.\ select^{op}\ \lfloor \overline{\sigma} \rfloor\ \mu\ r)}\ \text{PERFORM}$$

$$\frac{\begin{array}{c} op_i : \forall \overline{\alpha}.\, \sigma_1 \to \sigma_2 \in \Sigma(l) \quad \overline{\alpha} \not\cap \mathsf{ftv}(\epsilon\ \sigma) \\ \Gamma \vdash_{val} f_i : \forall \overline{\alpha}.\, \sigma_1 \to \epsilon\ ((\sigma_2 \to \epsilon\ \sigma) \to \epsilon\ \sigma) \rightsquigarrow f_i' \end{array}}{\Gamma \vdash_{ops} \{\ op_1 \to f_1,\ \ldots,\ op_n \to f_n\ \} : \sigma \mid l \mid \epsilon \rightsquigarrow \mathsf{Hnd}^{l}\ (\overline{\forall \overline{\alpha}.\, \mathsf{Normal}\ \lfloor \sigma_1 \rfloor\ \lfloor \sigma_2 \rfloor\ \epsilon\ \lfloor \sigma \rfloor\ f_i'})\ \}}\ \text{OPS}$$

$$\frac{\Gamma \vdash_{ops} h : \sigma \mid l \mid \epsilon \rightsquigarrow h'}{\Gamma \vdash_{val} \mathsf{handler}\ h : (() \to \langle l \mid \epsilon \rangle\ \sigma) \to \epsilon\ \sigma \rightsquigarrow handler^{l}\ \epsilon\ \lfloor \sigma \rfloor\ h'}\ \text{HANDLER}$$

$$\frac{\Gamma \vdash_{ops} h : \sigma \mid l \mid \epsilon \rightsquigarrow h' \quad \Gamma \vdash e : \sigma \mid \langle l \mid \epsilon \rangle \rightsquigarrow e'}{\Gamma \vdash \mathsf{prompt}\ m\ h\ e : \sigma \mid \epsilon \rightsquigarrow prompt^{l}\ \epsilon\ \lfloor \sigma \rfloor\ m\ h'\ e'}\ \text{PROMPT}$$

$$\frac{\Gamma \vdash_{val} f : (\sigma_2 \to \epsilon'\ \sigma') \to \epsilon'\ \sigma' \rightsquigarrow f' \quad \Gamma \vdash_{val} k : \sigma_2 \to \epsilon\ \sigma \rightsquigarrow k'}{\Gamma \vdash \mathsf{yield}\ m\ f\ k : \sigma \mid \epsilon \rightsquigarrow yield\ \epsilon\ \lfloor \sigma \rfloor\ \lfloor \sigma_2 \rfloor\ \epsilon'\ \lfloor \sigma' \rfloor\ m\ f'\ k'}\ \text{YIELDB}$$

$$\frac{\Gamma \vdash e : \sigma \mid \epsilon \rightsquigarrow e'}{\Gamma \vdash \mathsf{under}^{\epsilon_0, \epsilon}\ l\ e : \sigma \mid \langle l \mid \epsilon_0 \rangle \rightsquigarrow \lambda w : \mathsf{Evv}\ \langle l \mid \epsilon_0 \rangle.\ \mathsf{let}\ (m, \_, w') : \mathsf{Ev}\ \epsilon\ r\ =\ w.l\ \mathsf{in}\ under^{l}\ \langle l \mid \epsilon_0 \rangle\ \lfloor \sigma \rfloor\ \epsilon\ r\ m\ w'\ e'\ w}\ \text{UNDER}$$

Fig. 11. Monadic translation of F^{pb}.

The translation makes use of the following helper functions:

$pure : \forall \mu\ \alpha.\ \alpha \to \mathsf{Mon}\ \mu\ \alpha$
$pure\ x\ =\ \lambda w : \mathsf{Evv}\ \mu.\ \mathsf{Pure}\ \mu\ \alpha\ x$

$yield : \forall \mu\ \alpha\ \beta\ \mu'\ r.\ \mathsf{Marker}\ \mu'\ r \to ((\beta \to \mathsf{Mon}\ \mu'\ r) \to \mathsf{Mon}\ \mu'\ r) \to (\beta \to \mathsf{Mon}\ \mu\ \alpha) \to \mathsf{Mon}\ \mu\ \alpha$
$yield\ m\ clause\ k\ =\ \lambda w : \mathsf{Evv}\ \mu.\ \mathsf{Yield}\ \mu\ \alpha\ \beta\ \mu'\ r\ m\ clause\ k$

$$
\begin{array}{lll}
\text{Expression} & e & ::= \quad v \mid e\ e \mid e[\sigma] \\
\text{Values} & v & ::= \quad x \mid \lambda x\!:\!\sigma.\ e \mid \Lambda\alpha^{\kappa}.\ v \\
\\
\text{Evaluation context} & \mathsf{F} & ::= \quad \Box \mid \mathsf{F}\ e \mid v\ \mathsf{F} \mid \mathsf{F}\ [\sigma] \\
& \mathsf{E} & ::= \quad \mathsf{F}
\end{array}
$$

$$
\begin{array}{ll}
(\mathit{fapp}) & (\lambda^{\epsilon} x\!:\!\sigma.\ e)\ v \longrightarrow e[x{:=}v] \\
(\mathit{ftapp}) & (\Lambda\alpha^{\kappa}.\ v)\ [\sigma] \longrightarrow v[\alpha{:=}\sigma]
\end{array}
$$

$$
\frac{x\ :\ \sigma\ \in \Gamma}{\Gamma \vdash_{\mathsf{F}}\ x\ :\ \sigma}\ \textsc{fvar}
\qquad
\frac{\Gamma \vdash_{\mathsf{F}}\ v\ :\ \sigma}{\Gamma \vdash_{\mathsf{F}}\ \Lambda\alpha^{\kappa}.\ v\ :\ \forall\alpha^{\kappa}.\ \sigma}\ \textsc{ftabs}
\qquad
\frac{\Gamma, x\!:\!\sigma_1 \vdash_{\mathsf{F}}\ e\ :\ \sigma_2}{\Gamma \vdash_{\mathsf{F}}\ \lambda x\!:\!\sigma_1.\ e\ :\ \sigma_1 \to \sigma_2}\ \textsc{fabs}
$$

$$
\frac{\Gamma \vdash_{\mathsf{F}}\ e_1\ :\ \sigma_1 \to \sigma \qquad \Gamma \vdash\ e_2\ :\ \sigma}{\Gamma \vdash_{\mathsf{F}}\ e_1\ e_2\ :\ \sigma}\ \textsc{fapp}
\qquad
\frac{\Gamma \vdash_{\mathsf{F}}\ e\ :\ \forall\alpha^{\kappa}.\ \sigma_1 \qquad \vdash_{\mathsf{wf}}\ \sigma\ :\ \kappa}{\Gamma \vdash_{\mathsf{F}}\ e[\sigma]\ :\ \sigma_1[\alpha{:=}\sigma]}\ \textsc{ftapp}
$$

Fig. 12. System F^{v}: explicitly typed (higher kinded) polymorphic lambda calculus.

$$
\frac{}{\vdash_{\mathsf{wf}}\ \alpha^{\kappa}\ :\ \kappa}\ \textsc{kind-var}
\qquad
\frac{\vdash_{\mathsf{wf}}\ \sigma\ :\ *\qquad \kappa \neq \mathsf{lab}}{\vdash_{\mathsf{wf}}\ \forall\alpha^{\kappa}.\ \sigma\ :\ *}\ \textsc{kind-quant}
$$

$$
\frac{}{\vdash_{\mathsf{wf}}\ c^{\kappa}\ :\ \kappa}\ \textsc{kind-con}
\qquad
\frac{\vdash_{\mathsf{wf}}\ \sigma_1\ :\ \kappa_2 \to \kappa \qquad \vdash_{\mathsf{wf}}\ \sigma_2\ :\ \kappa_2}{\vdash_{\mathsf{wf}}\ \sigma_1\ \sigma_2\ :\ \kappa}\ \textsc{kind-app}
$$

$$
\frac{}{\vdash_{\mathsf{wf}}\ \langle\rangle\ :\ \mathsf{eff}}\ \textsc{kind-total}
\qquad
\frac{\vdash_{\mathsf{wf}}\ \epsilon\ :\ \mathsf{eff} \qquad \vdash_{\mathsf{wf}}\ l\ :\ \mathsf{lab}}{\vdash_{\mathsf{wf}}\ \langle l \mid \epsilon\rangle\ :\ \mathsf{eff}}\ \textsc{kind-row}
$$

$$
\frac{\vdash_{\mathsf{wf}}\ \sigma_1\ :\ *\qquad \vdash_{\mathsf{wf}}\ \sigma_2\ :\ *\qquad \vdash_{\mathsf{wf}}\ \epsilon\ :\ \mathsf{eff}}{\vdash_{\mathsf{wf}}\ \sigma_1 \to \epsilon\ \sigma_2}\ \textsc{kind-arrow}
$$

Fig. 13. Well-formedness of types.

We assume the type for evidence is Ev, and we use let $x\ :\ \sigma\ =\ e_1$ in e_2 as the syntactic sugar for $(\lambda x\ :\ \sigma.\ e_2)\ e_1$.

D POLYMORPHIC LAMBDA CALCULUS

Figure 12 presents System F^{v}, an explicitly typed (higher kinded) polymorphic lambda calculus with strict evaluation [Xie et al. 2020]. Types as in Figure 7 with no effects on the arrows.

E FULL RULES

E.1 Well-formed Types

The kinding rules for types are shown in Figure 13. The rules are standard mostly standard except we do not allow type abstraction over effect labels – or otherwise equivalence between types cannot be decided statically. The rules KIND-TOTAL, KIND-ROW, and KIND-ARROW are not strictly necessary and can be derived from KIND-APP.

$$\boxed{\Gamma \vdash_{\mathsf{ec}} \mathsf{E} : \sigma \to \sigma' \mid \epsilon}$$

$$\frac{}{\Gamma \vdash_{\mathsf{ec}} \square : \sigma \to \sigma \mid \epsilon \rightsquigarrow \lambda x{:}\mathsf{Mon}\ \epsilon\ \lfloor\sigma\rfloor.\ x}\ \textsc{cempty}$$

$$\frac{\Gamma \vdash e : \sigma_2 \mid \epsilon \rightsquigarrow e' \quad \Gamma \vdash_{\mathsf{ec}} \mathsf{E} : \sigma_1 \to (\sigma_2 \to\epsilon\ \sigma_3) \mid \epsilon \rightsquigarrow g}{\Gamma \vdash_{\mathsf{ec}} \mathsf{E}\ e : \sigma_1 \to \sigma_3 \mid \epsilon \rightsquigarrow (\lambda f.\ e' \rhd f) \star g}\ \textsc{capp1}$$

$$\frac{\Gamma \vdash_{\mathsf{val}} v : \sigma_2 \to\epsilon\ \sigma_3 \rightsquigarrow v' \quad \Gamma \vdash_{\mathsf{ec}} \mathsf{E} : \sigma_1 \to \sigma_2 \mid \epsilon \rightsquigarrow g}{\Gamma \vdash_{\mathsf{ec}} v\ \mathsf{E} : \sigma_1 \to \sigma_3 \mid \epsilon \rightsquigarrow v \star g}\ \textsc{capp2}$$

$$\frac{\Gamma \vdash_{\mathsf{ec}} \mathsf{E} : \sigma_1 \to \forall\alpha.\ \sigma_2 \mid \epsilon \rightsquigarrow g}{\Gamma \vdash_{\mathsf{ec}} \mathsf{E}\ \sigma : \sigma_1 \to \sigma_2\ [\alpha{:=}\sigma] \mid \epsilon \rightsquigarrow (\lambda x{:}\lfloor\forall\alpha.\ \sigma_2\rfloor.\ \mathit{pure}\ \epsilon\ \lfloor\sigma_2[\alpha{:=}\sigma]\rfloor\ (x\ \lfloor\sigma\rfloor)) \star g}\ \textsc{ctapp}$$

$$\frac{\Gamma \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon \quad \Gamma \vdash_{\mathsf{ec}} \mathsf{E} : \sigma_1 \to \sigma \mid \langle l \mid \epsilon\rangle}{\Gamma \vdash_{\mathsf{ec}} \mathsf{handle}\ h\ \mathsf{E} : \sigma_1 \to \sigma \mid \epsilon}\ \textsc{chandle}$$

For System F^{p},F^{pw},F^{pb}

$$\frac{\Gamma \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon \rightsquigarrow h' \quad \Gamma \vdash_{\mathsf{ec}} \mathsf{E} : \sigma_1 \to \sigma \mid \langle l \mid \epsilon\rangle \rightsquigarrow g}{\Gamma \vdash_{\mathsf{ec}} \mathsf{prompt}\ m\ h\ \mathsf{E} : \sigma_1 \to \sigma \mid \epsilon\ \mathit{prompt}^{l}\ \epsilon\ \lfloor\sigma\rfloor\ m\ h' \circ g}\ \textsc{cprompt}$$

$$\frac{\Gamma \vdash_{\mathsf{ec}} \mathsf{E} : \sigma_1 \to \sigma \mid \epsilon \rightsquigarrow g}{\begin{array}{l}\Gamma \vdash_{\mathsf{ec}} \mathsf{under}^{\langle\epsilon',\epsilon\rangle}\ l\ \mathsf{E} : \sigma_1 \to \sigma \mid \langle l \mid \epsilon'\rangle \rightsquigarrow \lambda x{:}\mathsf{Mon}\ \langle[\![\mathsf{E}]\!] \mid \epsilon\rangle\ \lfloor\sigma_1\rfloor.\ \lambda w{:}\mathsf{Evv}\ \langle l \mid \epsilon'\rangle. \\ \mathsf{let}\ (m, h', w'){:}\mathsf{Ev}\ \epsilon\ r\ =\ w.l\ \mathsf{in}\ \mathit{under}^{l}\ \langle l \mid \epsilon'\rangle\ \lfloor\sigma\rfloor\ \epsilon\ r\ m\ w'\ (g\ x)\ w\end{array}}\ \textsc{cunder}$$

Fig. 14. Evaluation context typing

E.2 Evaluation Context Typing

The evaluation context typing rules is given in Figure 14.

E.3 Program Context Typing

The definition of the program context in System F^{pw} is defined as follows, with its typing rules given in Figure 15. The notation $\varnothing \vdash C : (\sigma \mid \epsilon) \to (\mathit{Int} \mid \langle\rangle)$ used in the paper can be expressed as $\varnothing \vdash C : \sigma \to \mathit{Int} \mid \langle\rangle$ and $[\![\mathsf{E}]\!] = \epsilon$, where $[\![C]\!]$ gets all labels of the handlers in C.

$$\begin{array}{lrl}\text{Program ctx.} & C & ::= \square \mid C\ e \mid e\ C \mid C\ \sigma \\ & & \mid \mathsf{prompt}\ m\ h\ C \mid \mathsf{yield}\ m\ C \\ & & \mid \lambda^{\epsilon} x{:}\sigma.\ C \mid \Lambda\alpha^{\kappa}.\ C\end{array}$$

$\Gamma \vdash C : \sigma \to \sigma' \mid \epsilon$

$$\frac{}{\Gamma \vdash \square : \sigma \to \sigma \mid \epsilon}\ \textsc{pempty}$$

$$\frac{\Gamma \vdash e : \sigma_2 \mid \epsilon \quad \Gamma \vdash C : \sigma_1 \to (\sigma_2 \to\epsilon\ \sigma_3) \mid \epsilon}{\Gamma \vdash C\ e : \sigma_1 \to \sigma_3 \mid \epsilon}\ \textsc{papp1}$$

$$\frac{\Gamma \vdash_{\mathsf{val}} e : \sigma_2 \to\epsilon\ \sigma_3 \quad \Gamma \vdash C : \sigma_1 \to \sigma_2 \mid \epsilon}{\Gamma \vdash e\ C : \sigma_1 \to \sigma_3 \mid \epsilon}\ \textsc{papp2}$$

$$\frac{\Gamma \vdash C : \sigma_1 \to \forall\alpha.\ \sigma_2 \mid \epsilon}{\Gamma \vdash C\ \sigma : \sigma_1 \to \sigma_2\ [\alpha{:=}\sigma] \mid \epsilon}\ \textsc{ptapp}$$

$$\frac{\Gamma \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon \quad \Gamma \vdash C : \sigma_1 \to \sigma \mid \langle l \mid \epsilon \rangle}{\Gamma \vdash \mathsf{prompt}\ m\ h\ C : \sigma_1 \to \sigma \mid \epsilon}\ \textsc{pprompt}$$

$$\frac{\Gamma \vdash C : \sigma_1 \to (\sigma \to \epsilon'\ \sigma') \to \epsilon'\ \sigma'}{\Gamma \vdash \mathsf{yield}\ m\ C : \sigma_1 \to \sigma \mid \epsilon}\ \textsc{pyield}$$

$$\frac{\Gamma,\ x : \sigma \vdash C : \sigma_1 \to \sigma_2 \mid \epsilon}{\Gamma \vdash \lambda^{\epsilon}\ x{:}\sigma.\ C : \sigma_1 \to (\sigma \to \epsilon\ \sigma_2) \mid \epsilon'}\ \textsc{pabs}$$

$$\frac{\Gamma,\ x : \sigma \vdash C : \sigma_1 \to \sigma_2 \mid \epsilon \quad \kappa \neq \mathsf{lab}}{\Gamma \vdash \Lambda\alpha^{\kappa}.\ C : \sigma_1 \to \forall\alpha^{\kappa}.\ \sigma_2 \mid \epsilon'}\ \textsc{ptabs}$$

Fig. 15. Program context typing

F PROOFS

F.1 System F^{ϵ}

Here we show a list of lemmas of System F^{ϵ} from [Xie et al. 2020] that will be used in later proofs. Most of the lemmas can be extended trivially to support the new forms.

Lemma 3. (*Evaluation context typing*). If $\varnothing \vdash_{\mathsf{ec}} \mathsf{E} : \sigma_1 \to \sigma \mid \epsilon$ and $\varnothing \vdash e : \sigma_1 \mid \langle [\![\mathsf{E}]\!] \mid \epsilon \rangle$, then $\varnothing \vdash \mathsf{E}[e] : \sigma \mid \epsilon$.

Lemma 4. (*Effect corresponds to the evaluation context*). If $\varnothing \vdash \mathsf{E}[e] : \sigma \mid \epsilon$ then there exists σ_1 such that $\varnothing \vdash_{\mathsf{ec}} \mathsf{E} : \sigma_1 \to \sigma \mid \epsilon$, and $\varnothing \vdash e : \sigma_1 \mid \langle [\![\mathsf{E}]\!] \mid \epsilon \rangle$.

The following two lemmas are corollaries.

Lemma 5. (*Well typed operations are handled*). If $\varnothing \vdash \mathsf{E}[\mathsf{perform}\ op\ \overline{\sigma}\ v] : \sigma \mid \langle\rangle$ then E has the form $\mathsf{E}_1 \bullet \mathsf{handle}\ h \bullet \mathsf{E}_2$ with $op \notin \mathsf{bop}(\mathsf{E}_2)$ and $op \to f \in h$.

Lemma 6. (*Effects types are meaningful*). If $\varnothing \vdash \mathsf{E}[\mathsf{perform}\ op\ \overline{\sigma}\ v] : \sigma \mid \epsilon$ with $op \notin \mathrm{bop}(\mathsf{E})$, then $op \in \Sigma(l)$ and $l \in \epsilon$, i.e. effect types cannot be discarded without a handler.

F.2 System F^p: Multi-prompt Semantics

F.2.1 Preservation.

Lemma 7. (*Values can have any effect*). If $\Gamma \vdash v : \sigma \mid \epsilon_1$, then $\Gamma \vdash v : \sigma \mid \epsilon_2$.

Proof. (*Of Lemma 7*) Follows directly by VAL. □

Definition 3. (*m-mapping*). We say an expression e is m-mapping, if every m in e can uniquely determine its h.

Lemma 8. (*Internal-safe expressions are m-mapping*). Any internal-safe expression e is m-mapping.

Proof. (*Of Lemma 8*) For the base case, there is no internal construct in the expression, and so the goal is trivially true. In the inductive case, the expression is evaluated from an m-mapping internal-safe expressions. As every time a new prompt is generated, it owns a unique marker, and in any other reduction rules we cannot change existing markers, the goal holds. □.

Lemma 9. (*Small Step Preservation*). If $\varnothing \vdash e_1 : \sigma \mid \epsilon$ where e_1 is internal-safe, and $e_1 \longrightarrow e_2$, then $\varnothing \vdash e_2 : \sigma \mid \epsilon$.

Proof. (*Of Lemma 9*) By induction on $e_1 \longrightarrow e_2$. We only discuss new case.
case (*handler*) $\mathsf{handler}\ h\ v \longrightarrow \mathsf{prompt}\ m\ h\ (v\ ())$ with unique m with $\varnothing \vdash h : \sigma \mid l \mid \epsilon$.

$\varnothing \vdash \mathsf{handler}\ h\ v : \sigma \mid \langle\rangle$	given
$\varnothing \vdash_{\mathsf{val}} \mathsf{handler}\ h : (() \to \langle l \mid \epsilon\rangle\ \sigma) \to \epsilon\ \sigma$	(*app*) and (*val*)
$\varnothing \vdash v : (() \to \langle l \mid \epsilon\rangle\ \sigma)$	(*app*)
$\varnothing \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon$	(*handler*)
$\varnothing \vdash v\ () : \sigma \mid \langle l \mid \epsilon\rangle$	(*app*)
$\varnothing \vdash \mathsf{prompt}\ m\ h\ (v\ ()) : \sigma \mid \epsilon$	(*prompt*)

case (*promptv*) $\mathsf{prompt}\ m\ h\ v \longrightarrow v$.

$\varnothing \vdash \mathsf{prompt}\ m\ h\ v : \sigma \mid \epsilon$	given
$\varnothing \vdash v : \sigma \mid \langle l \mid \epsilon\rangle$	(*app*)
$\varnothing \vdash v : \sigma \mid \epsilon$	Lemma 7

case (*prompt*) $\mathsf{prompt}\ m\ h\ \mathsf{E}[\mathsf{yield}\ m\ f] \longrightarrow f\ (\lambda x^{\epsilon} : \sigma_2.\ \mathsf{prompt}\ m\ h\ \mathsf{E}[x])$
with $\varnothing \vdash_{\mathsf{val}} f : (\sigma_2 \to \epsilon\ \sigma) \to \epsilon\ \sigma$.

By Lemma 8, we know that each m maps to a unique handler, so the handler h is indeed what yield m is looking for. So h must be of the right type.

$\varnothing \vdash \text{prompt}\ m\ h\ \mathsf{E}[\text{yield}\ m\ f] : \sigma \mid \epsilon$	given
$\varnothing \vdash_{\text{ops}} h : \sigma \mid l \mid \epsilon$	(*prompt*)
$\varnothing \vdash \mathsf{E}[\text{yield}\ m\ f] : \sigma \mid \langle l \mid \epsilon \rangle$	above
$\varnothing \vdash_{\text{ec}} \mathsf{E} : \sigma_2 \mid \epsilon' \to \sigma \mid \langle l \mid \epsilon \rangle$	Lemma 4
$\varnothing \vdash_{\text{val}} f : (\sigma_2 \to \epsilon\ \sigma) \to \epsilon\ \sigma$	given
$x : \sigma_2 \vdash_{\text{ec}} \mathsf{E} : \sigma_2 \mid \epsilon' \to \sigma \mid \langle l \mid \epsilon \rangle$	weakening
$x : \sigma_2 \vdash x : \sigma_2 \mid \epsilon$	(*var*) and (*val*)
$x : \sigma_2 \vdash \mathsf{E}[x] : \sigma \mid \langle l \mid \epsilon \rangle$	weakening
$x : \sigma_2 \vdash \text{prompt}\ m\ h\ \mathsf{E}[x] : \sigma \mid \epsilon$	(*prompt*)
$\varnothing \vdash (\lambda x^{\epsilon} : \sigma_2.\ \text{prompt}\ m\ h\ \mathsf{E}[x]) : \sigma \mid \epsilon$	(*abs*)

case (*perform*) $\text{prompt}\ m\ h\ \mathsf{E}[\text{perform}\ op\ \epsilon'\ \overline{\sigma}\ v] \longrightarrow \text{prompt}\ m\ h\ \mathsf{E}[\text{yield}\ m\ (\lambda^{\epsilon}\ k : \sigma_k.\ f\ \overline{\sigma}\ v\ k)]$
iff $op \notin \text{bop}(\mathsf{E}) \wedge (op \to f \in h)$, with $\sigma_k = \sigma_2[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ \sigma, \varnothing \vdash h : \sigma \mid l \mid \epsilon, op : \forall \overline{\alpha}.\ \sigma_1 \to \sigma_2 \in \Sigma(l)$.

$\varnothing \vdash \text{prompt}\ m\ h\ \mathsf{E}[\text{perform}\ op\ \epsilon'\ \overline{\sigma}\ v] : \sigma \mid \epsilon$	given
$\varnothing \vdash_{\text{ops}} h : \sigma \mid l \mid \epsilon$	(*prompt*)
$\varnothing \vdash \mathsf{E}[\text{perform}\ op\ \epsilon'\ \overline{\sigma}\ v] : \sigma \mid \langle l \mid \epsilon \rangle$	above
$op : \forall \overline{\alpha}.\ \sigma_1 \to \sigma_2 \in \Sigma(l)$	given
$\varnothing \vdash_{\text{ec}} \mathsf{E} : \sigma_2[\overline{\alpha} := \overline{\sigma}] \to \sigma \mid \langle l \mid \epsilon \rangle$	Lemma 4
$\varnothing \vdash \text{perform}\ op\ \epsilon'\ \overline{\sigma}\ v : \sigma_2[\overline{\alpha} := \overline{\sigma}] \mid \langle [\![\mathsf{E}]\!] \mid l \mid \epsilon \rangle$	above and (*perform*)
$\varnothing \vdash \text{perform}\ op\ \epsilon'\ \overline{\sigma}\ v : \sigma_2[\overline{\alpha} := \overline{\sigma}] \mid \langle\ l \mid [\![\mathsf{E}]\!] \mid \epsilon \rangle$	$op \notin \text{bop}(\mathsf{E})$
$\varnothing \vdash \text{perform}\ op\ \epsilon'\ \overline{\sigma} : \sigma_1[\overline{\alpha} := \overline{\sigma}] \to \langle\ l \mid [\![\mathsf{E}]\!] \mid \epsilon \rangle\ \sigma_2[\overline{\alpha} := \overline{\sigma}] \mid \langle\ l \mid [\![\mathsf{E}]\!] \mid \epsilon \rangle$	(*app*)
$\varnothing \vdash_{\text{val}} v : \sigma_1[\overline{\alpha} := \overline{\sigma}]$	above and (*val*)
$(op \to f \in h)$	given
$\varnothing \vdash_{\text{val}} f : \forall \overline{\alpha}.\ \sigma_1 \to \epsilon\ ((\sigma_2 \to \epsilon\ \sigma) \to \epsilon\ \sigma)$	(*ops*)
$\overline{\alpha} \notin \text{ftv}(\sigma)$	above
$\varnothing \vdash f : \forall \overline{\alpha}.\ \sigma_1 \to \epsilon\ ((\sigma_2 \to \epsilon\ \sigma) \to \epsilon\ \sigma) \mid \epsilon$	(*val*)
$\varnothing \vdash f\ \overline{\sigma} : \sigma_1[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ ((\sigma_2[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ \sigma) \to \epsilon\ \sigma) \mid \epsilon$	(*tapp*)
$\varnothing \vdash f\ \overline{\sigma}\ v : ((\sigma_2[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ \sigma) \to \epsilon\ \sigma) \mid \epsilon$	(*app*)
$k : \sigma_k \vdash f\ \overline{\sigma}\ v : ((\sigma_2[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ \sigma) \to \epsilon\ \sigma) \mid \epsilon$	weakening
$k : \sigma_k \vdash f\ \overline{\sigma}\ v\ k : \sigma_k \to \epsilon\ \sigma \mid \epsilon$	(*app*)
$\varnothing \vdash_{\text{val}} (\lambda^{\epsilon}\ k : \sigma_k.\ f\ \overline{\sigma}\ v\ k) : \sigma_k \to \epsilon\ \sigma$	(*abs*)
$\varnothing \vdash_{\text{val}} (\lambda^{\epsilon}\ k : \sigma_k.\ f\ \overline{\sigma}\ v\ k) : (\sigma_2[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ \sigma) \to \epsilon\ \sigma$	$\sigma_k = \sigma_2[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ \sigma$
$\varnothing \vdash \text{yield}\ m\ (\lambda^{\epsilon}\ k : \sigma_k.\ f\ \overline{\sigma}\ v\ k) : \sigma_2[\overline{\alpha} := \overline{\sigma}] \mid \langle [\![\mathsf{E}]\!] \mid l \mid \epsilon \rangle$	(*yield*)
$\varnothing \vdash \mathsf{E}[\text{yield}\ m\ (\lambda^{\epsilon}\ k.\ f\ \overline{\sigma}\ v\ k)] : \sigma \mid \langle l \mid \epsilon \rangle$	Lemma 3
$\varnothing \vdash \text{prompt}\ m\ h\ \mathsf{E}[\text{yield}\ m\ (\lambda^{\epsilon}\ k.\ f\ \overline{\sigma}\ v\ k)\ ] : \sigma \mid \epsilon$	(*prompt*)

□

Proof. (*Of Theorem 10*)

$e_1 = \mathsf{E}[e_1']$	(*step*)
$e_1' \longrightarrow e_2'$	above
$e_2 = \mathsf{E}[e_2']$	above
$\varnothing \vdash \mathsf{E}[e_1'] : \sigma \mid \langle\rangle$	given
$\varnothing \vdash e_1' : \sigma_1 \mid [\![\mathsf{E}]\!]$	Lemma 4
$\varnothing \vdash \mathsf{E} : \sigma_1 \to \sigma \mid \langle\rangle$	above
$\varnothing \vdash e_2' : \sigma_1 \mid [\![\mathsf{E}]\!]$	Lemma 9
$\varnothing \vdash \mathsf{E}[e_2'] : \sigma \mid \langle\rangle$	Lemma 3

□

F.2.2 Progress.

Lemma 10. (*Progress with effects*). If $\varnothing \vdash e_1 : \sigma \mid \epsilon$ then either (1) e_1 is a value; or (2) $e_1 \longmapsto e_2$; or (3) $e_1 = \mathsf{E}[\mathsf{perform}\ op\ \epsilon\ \overline{\sigma}\ v]$, where $op \notin \mathsf{bop}(\mathsf{E})$; or (4) $e_1 = \mathsf{E}[\mathsf{yield}\ m\ f]$, where $m \notin \mathsf{bm}(\mathsf{E})$.

Proof. (*Of Lemma 10*) By induction on typing. Based on the progress theorem of System F^ϵ, here we only discuss the new cases.

case $\varnothing \vdash \mathsf{prompt}\ m\ h\ e : \sigma \mid \epsilon$. By I.H., we know that either e is a value, or $e \longmapsto e'$, or $e = \mathsf{E}[\mathsf{perform}\ op\ \epsilon\ \overline{\sigma}\ v]$, or $e = \mathsf{E}[\mathsf{yield}\ m'\ f]$ where $m' \notin \mathsf{bm}(\mathsf{E})$.

- If e is a value, then by (*promptv*) we have $\mathsf{prompt}\ m\ h\ e \longmapsto e$
- If $e \longmapsto e'$, then by (*step*) we have $\mathsf{prompt}\ m\ h\ e \longmapsto \mathsf{prompt}\ m\ h\ e'$.
- If $e = \mathsf{E}[\mathsf{perform}\ op\ \epsilon\ \overline{\sigma}\ v]$. We discuss whether op is bound in h.
  - $op \to f \in h$. Then by (*perform*) we have $\mathsf{prompt}\ m\ h\ \mathsf{E}[\mathsf{perform}\ op\ \epsilon\ \overline{\sigma}\ v] \longmapsto \mathsf{prompt}\ m\ h\ \mathsf{E}[\mathsf{yield}\ m\ (\lambda k.\ f\ \overline{\sigma}\ v\ k)]$.
  - $op \notin h$. Let $\mathsf{E}' = \mathsf{prompt}\ m\ h\ \mathsf{E}$, then we have $e_1 = \mathsf{E}'[\mathsf{perform}\ op\ \epsilon\ \overline{\sigma}\ v]$.
- If $e = \mathsf{E}[\mathsf{yield}\ m'\ f]$.
  - $m = m'$. Then by (*prompt*), we have $\mathsf{prompt}\ m\ h\ \mathsf{E}[\mathsf{yield}\ m\ f] \longmapsto f\ (\lambda x.\ \mathsf{prompt}\ m\ h\ \mathsf{E}[x])$.
  - $m \neq m'$. Let $\mathsf{E}' = \mathsf{prompt}\ m\ h\ \mathsf{E}$, then we have $e_1 = \mathsf{E}'[\mathsf{yield}\ m'\ f]$.

case $\varnothing \vdash \mathsf{yield}\ m\ f : \sigma \mid \epsilon$. The goal follows trivially. □.

Proof. (*Of Theorem 11*) Apply Lemma 10, then we know that either e_1 is a value, or $e_1 \longmapsto e_2$, or $e_1 = \mathsf{E}[\mathsf{perform}\ op\ \epsilon\ \overline{\sigma}\ v]$ where $op \notin \mathsf{bop}(\mathsf{E})$, or $e = \mathsf{E}[\mathsf{yield}\ m\ f]$, where $m \notin \mathsf{bm}(\mathsf{E})$.

For the first two cases, we have proved the goal. For the third case, we prove it by contradiction.

$\varnothing \vdash \mathsf{E}[\mathsf{perform}\ op\ \epsilon\ \overline{\sigma}\ v] : \sigma \mid \langle\rangle$	given
$l \notin \mathsf{bop}(\mathsf{E})$	given
$l \in \langle\rangle$	Lemma 6
Contradiction	

The last case is an impossible case as for internal-safe expressions, yield cannot appear without the corresponding prompt. This is because (1) initially yield only appears after applying rule (*perform*); (2) it is then directly followed by rule (*prompt*) so there is no way to pass it around. Thus there is no possible evaluation that can construct a standalone yield.

□

F.2.3 Simulation.

Definition 4. ($\lceil e \rceil^{\epsilon \Downarrow p}$ *and* $\lceil e \rceil^{\epsilon \Uparrow p}$). $\lceil e \rceil^{\epsilon \Downarrow p}$ turns an expression from F^ϵ to F^p by turning handle h into prompt $m\ h$ with fresh m of the correct type; and $\lceil e \rceil^{\epsilon \Uparrow p}$ turns a yield-free expression from F^p into F^ϵ by turning back prompt $m\ h$ into handle h. The definition can be lifted straightforward to handlers and evaluation contexts.

Lemma 11. (*Simulation (small step)*). If $e_1 \longrightarrow e_2$ in System F^ϵ, then there exists e_2' such that $\lceil e_1 \rceil^{\epsilon \Downarrow p} \longrightarrow^* e_2'$ in System F^p, and $\lceil e_2' \rceil^{\epsilon \Uparrow p} = e_2$.

Proof. (*Of Lemma 11*) By induction on $e_1 \longrightarrow e_2$. Most cases are straightforward. The only interesting case is

(*handler*) $\mathsf{handler}\ h\ \mathsf{E}[\mathsf{perform}\ op\ \epsilon_0\ \overline{\sigma}\ v] \longrightarrow f\ \overline{\sigma}\ v\ (\lambda x.\ \mathsf{handle}\ h\ \mathsf{E}[x])$ with $op \notin \mathsf{bop}(\mathsf{E}), (op \to f) \in h$.

In this case, we have

$\lceil \mathsf{handler}\ h\ \mathsf{E}[\mathsf{perform}\ op\ \epsilon_0\ \overline{\sigma}\ v] \rceil^{\epsilon \Downarrow p} = \mathsf{prompt}\ m\ \lceil h \rceil^{\epsilon \Downarrow p}\ (\lceil \mathsf{E} \rceil^{\epsilon \Downarrow p}[\mathsf{perform}\ op\ \epsilon_0\ \overline{\sigma}\ \lceil v \rceil^{\epsilon \Downarrow p}])$ with a

fresh m. Since $\lceil\rceil^{\epsilon \Downarrow p}$ does not change handlers in an evaluation context, obviously we have $op \notin \mathrm{bop}(\lceil \mathsf{E} \rceil^{\epsilon \Downarrow p})$.

$\mathsf{prompt}\ m\ \lceil h \rceil^{\epsilon \Downarrow p}\ (\lceil \mathsf{E} \rceil^{\epsilon \Downarrow p}[\mathsf{perform}\ op\ \epsilon_0\ \overline{\sigma}\ \lceil v \rceil^{\epsilon \Downarrow p}])$
$\longrightarrow$ (*perform*)
$\mathsf{prompt}\ m\ \lceil h \rceil^{\epsilon \Downarrow p}\ \lceil \mathsf{E} \rceil^{\epsilon \Downarrow p}[\mathsf{yield}\ m\ (\lambda k.\ \lceil f \rceil^{\epsilon \Downarrow p}\ \overline{\sigma}\ \lceil v \rceil^{\epsilon \Downarrow p}\ k)]$
$\longrightarrow$ (*prompt*)
$(\lambda k.\ \lceil f \rceil^{\epsilon \Downarrow p}\ \overline{\sigma}\ \lceil v \rceil^{\epsilon \Downarrow p}\ k)\ (\lambda x.\ \mathsf{prompt}\ m\ \lceil h \rceil^{\epsilon \Downarrow p}\ \lceil \mathsf{E} \rceil^{\epsilon \Downarrow p}[x])$
$\longrightarrow$ (*app*)
$\lceil f \rceil^{\epsilon \Downarrow p}\ \overline{\sigma}\ \lceil v \rceil^{\epsilon \Downarrow p}\ (\lambda x.\ \mathsf{prompt}\ m\ \lceil h \rceil^{\epsilon \Downarrow p}\ \lceil \mathsf{E} \rceil^{\epsilon \Downarrow p}[x])$

We have $\lceil \lceil f \rceil^{\epsilon \Downarrow p}\ \overline{\sigma}\ \lceil v \rceil^{\epsilon \Downarrow p}\ (\lambda x.\ \mathsf{prompt}\ m\ \lceil h \rceil^{\epsilon \Downarrow p}\ \lceil \mathsf{E} \rceil^{\epsilon \Downarrow p}[x]) \rceil^{\epsilon \Uparrow p} = f\ \overline{\sigma}\ v\ (\lambda x.\ \mathsf{handle}\ h\ \mathsf{E}[x])$.

□

Theorem 14. (*Simulation*). If $e_1 \longmapsto e_2$ in System F^{ϵ}, then there exists e_2' such that $\lceil e_1 \rceil^{\epsilon \Downarrow p} \longmapsto^* e_2'$ in System F^{p}, and $\lceil e_2' \rceil^{\epsilon \Uparrow p} = e_2$.

Proof. (*Of Theorem 14*) Follows directly by Lemma 11 and (*step*). □

F.3 System F^{pw}: Multi-prompt with Evidence Passing Semantics

F.3.1 Preservation.

Lemma 12. (*Small Step Preservation*). 1. If $\varnothing \vdash e_1 : \sigma \mid \epsilon$ where e_1 is internal-safe, and $e_1 \longrightarrow e_2$, then $\varnothing \vdash e_2 : \sigma \mid \epsilon$.
2. If $\varnothing \vdash e_1 : \sigma \mid \epsilon$ where e_1 is internal-safe, and $w \vdash e_1 \longrightarrow e_2$, where $w : \mathrm{evv}\ \epsilon$, then $\varnothing \vdash e_2 : \sigma \mid \epsilon$.

Proof. (*Of Theorem 9*) The $e_1 \longrightarrow e_2$ is the same as before (Lemma 9). So here we only discuss the case for $w \vdash e_1 \longrightarrow e_2$.

(*perform*) $w \vdash \mathsf{perform}\ op\ \epsilon_0\ \overline{\sigma}\ v \longrightarrow \mathsf{yield}\ m\ (\lambda^{\epsilon} k : \sigma_k.\ f\ \overline{\sigma}\ v\ k)$ with $(m, h, \_) = w.l$, and $(op \to f) \in h$, and $op : \forall \overline{\alpha}.\ \sigma_1 \to \sigma_2 \in \Sigma(l)$, and $\varnothing \vdash h : \sigma \mid l \mid \epsilon$, and $\sigma_k = \sigma_2[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ \sigma$.

$\varnothing \vdash h : \sigma \mid l \mid \epsilon$	given
$(op \to f) \in h$	given
$op : \forall \overline{\alpha}.\ \sigma_1 \to \sigma_2 \in \Sigma(l)$	given
$\varnothing \vdash_{\mathrm{val}} f : \forall \overline{\alpha}.\ \sigma_1 \to \epsilon\ (\sigma_2 \to \epsilon\ \sigma) \to \epsilon\ \sigma$	OPS
$\overline{\alpha} \not\setminus \mathrm{tv}(\epsilon, \sigma)$	above
$\varnothing \vdash f : \forall \overline{\alpha}.\ \sigma_1 \to \epsilon\ (\sigma_2 \to \epsilon\ \sigma) \to \epsilon\ \sigma \mid \epsilon$	VAL
$\varnothing \vdash f\ \overline{\sigma} : \sigma_1[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ (\sigma_2[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ \sigma) \to \epsilon\ \sigma \mid \epsilon$	TAPP
$\varnothing \vdash_{\mathrm{val}} \mathsf{perform}\ op\ \epsilon_0\ \overline{\sigma} : \sigma_1[\overline{\alpha} := \overline{\sigma}] \to \langle l \mid \epsilon_0 \rangle\ \sigma_2[\overline{\alpha} := \overline{\sigma}]$	PERFORM
$\varnothing \vdash \mathsf{perform}\ op\ \epsilon_0\ \overline{\sigma}\ v : \sigma_2[\overline{\alpha} := \overline{\sigma}] \mid \langle l \mid \epsilon_0 \rangle$	APP
$\varnothing \vdash_{\mathrm{val}} v : \sigma_1[\overline{\alpha} := \overline{\sigma}]$	above
$\varnothing \vdash f\ \overline{\sigma}\ v : (\sigma_2[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ \sigma) \to \epsilon\ \sigma \mid \epsilon$	APP
$k : \sigma_k \vdash f\ \overline{\sigma}\ v : (\sigma_2[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ \sigma) \to \epsilon\ \sigma \mid \epsilon$	weakening
$k : \sigma_k \vdash f\ \overline{\sigma}\ v\ k : \sigma \mid \epsilon$	APP
$\varnothing \vdash_{\mathrm{val}} \lambda k : \sigma_k.\ f\ \overline{\sigma}\ v\ k : \sigma_k \to \epsilon\ \sigma \mid \epsilon$	ABS
$\varnothing \vdash \mathsf{yield}\ m\ (\lambda k : \sigma_k.\ f\ \overline{\sigma}\ v\ k) : \sigma_2[\overline{\alpha} := \overline{\sigma}] \mid \langle l \mid \epsilon_0 \rangle$	YIELD

□

Lemma 13. (*Preservation*). If $\varnothing \vdash e_1 : \sigma \mid \epsilon$ where e_1 is internal-safe, and $w \vdash e_1 \longmapsto e_2$ where $w : \mathrm{evv}\ \epsilon$, then $\varnothing \vdash e_2 : \sigma \mid \epsilon$.

Proof. (*Of Lemma 13*) By induction on $w \vdash e_1 \longmapsto e_2$.
case (*step*)

$e_1 = \mathsf{F}[e_1']$	(*step*)
$e_1' \longrightarrow e_2'$	above
$e_2 = \mathsf{F}[e_2']$	above
$\varnothing \vdash \mathsf{F}[e_1'] : \sigma \mid \langle\rangle$	given
$\varnothing \vdash e_1' : \sigma_1 \mid [\![\mathsf{F}]\!]$	Lemma 4
$\varnothing \vdash \mathsf{F} : \sigma_1 \to \sigma \mid \langle\rangle$	above
$\varnothing \vdash e_2' : \sigma_1 \mid [\![\mathsf{F}]\!]$	Lemma 12
$\varnothing \vdash \mathsf{F}[e_2'] : \sigma \mid \langle\rangle$	Lemma 3

case (*stepw*)

$e_1 = \mathsf{F}[e_1']$	(*step*)
$w \vdash e_1' \longrightarrow e_2'$	above
$e_2 = \mathsf{F}[e_2']$	above
$\varnothing \vdash \mathsf{F}[e_1'] : \sigma \mid \langle\rangle$	given
$\varnothing \vdash e_1' : \sigma_1 \mid [\![\mathsf{F}]\!]$	Lemma 4
$\varnothing \vdash \mathsf{F} : \sigma_1 \to \sigma \mid \langle\rangle$	above
$\varnothing \vdash e_2' : \sigma_1 \mid [\![\mathsf{F}]\!]$	Lemma 12
$\varnothing \vdash \mathsf{F}[e_2'] : \sigma \mid \langle\rangle$	Lemma 3

case (*promptw*)

$e_1 = \mathsf{F}[\mathsf{prompt}\ m\ h\ e_1']$	(*promptw*)
$\langle\!\langle l : (m, h, w) \mid w \rangle\!\rangle \mid e_1' \longrightarrow e_2'$	above
$e_2 = \mathsf{F}[\mathsf{prompt}\ m\ h\ e_2']$	above
$\varnothing \vdash \mathsf{F}[\mathsf{prompt}\ m\ h\ e_1'] : \sigma \mid \langle\rangle$	given
$\varnothing \vdash e_1' : \sigma_1 \mid [\![\mathsf{F}\ (\mathsf{prompt}\ m\ h\ \square)]\!]$	Lemma 4
$\varnothing \vdash \mathsf{F}\ (\mathsf{prompt}\ m\ h\ \square]) : \sigma_1 \to \sigma \mid \langle\rangle$	above
$\varnothing \vdash e_2' : \sigma_1 \mid [\![\mathsf{F}\ (\mathsf{prompt}\ m\ h\ \square)]\!]$	I.H.
$\varnothing \vdash \mathsf{F}[\mathsf{prompt}\ m\ h\ e_2'] : \sigma \mid \langle\rangle$	Lemma 3

□

Proof. (*Of Theorem 2*) Follows directly by Lemma 13. □

F.3.2 Progress.

Lemma 14. (*Progress with effects*). If $\varnothing \vdash e_1 : \sigma \mid \epsilon$, then for any $w : \mathsf{evv}\ \epsilon$, we have either (1) e_1 is a value; or (2) $w \vdash e_1 \longmapsto e_2$; or (3) $e_1 = \mathsf{E}[\mathsf{yield}\ m\ f]$ where $m \notin \mathsf{bm}(\mathsf{E})$.

Proof. (*Of Lemma 14*) By induction on typing.

First, notice that the type of w is always correctly updates through (*step*), (*stepw*) and (*promptw*).

Most cases follow the progress theorem of System F^p, so here we only discuss the new cases.

case $\varnothing \vdash \mathsf{prompt}\ m\ h\ e : \sigma \mid \epsilon$. By I.H., we know that either e is a value, or $\langle\!\langle l : (m, h, w) \mid w \rangle\!\rangle \vdash e \longmapsto e'$, or $e = \mathsf{E}[\mathsf{yield}\ m'\ f]$, where $m' \notin \mathsf{bm}(\mathsf{E})$.

- If e is a value, then by (*promptv*) we have $\mathsf{prompt}\ m\ h\ e \longmapsto e$.
- If $\langle\!\langle l : (m, h, w) \mid w \rangle\!\rangle \vdash e \longmapsto e'$. Then by (*promptw*), $\mathsf{prompt}\ m\ h\ e \longmapsto \mathsf{prompt}\ m\ h\ e'$.
- If $e = \mathsf{E}[\mathsf{yield}\ m'\ f]$ where $m' \notin \mathsf{bm}(\mathsf{E})$.

- $m = m'$. Then by (*prompt*), we have prompt $m\ h\ \mathsf{E}[\text{yield}\ m\ f] \longmapsto f\ (\lambda x.\ \text{prompt}\ m\ h\ \mathsf{E}[x])$.
- $m \neq m'$. Let $\mathsf{E}' = \text{prompt}\ m\ h\ \mathsf{E}$, then we have $e_1 = \mathsf{E}'[\text{yield}\ m'\ f]$.

case $\varnothing \vdash \text{yield}\ m\ f : \sigma \mid \epsilon$. The goal follows trivially.

case $\varnothing \vdash \text{perform}\ op\ \epsilon\ \overline{\sigma}\ v : \sigma_2[\overline{\alpha}{:=}\overline{\sigma}] \mid \langle l \mid \epsilon \rangle$ where $op : \forall \overline{\alpha}.\ \sigma_1 \to \sigma_2 \in \Sigma(l), \varnothing \vdash_{\text{val}} v : \sigma_1[\overline{\alpha}{:=}\overline{\sigma}]$.

The main difference between the progress theorem for System F^{pw} from System F^{p} is that perform can always reduce in System F^{pw} under the evidence vector with the right type. The case when the operation argument is not a value can follow the existing standard proof steps for previous progress lemmas, so here we only discuss when it is a value.

Given $w : \text{evv}\ \langle l \mid \epsilon \rangle$, we can get $(m,\ h,\ \_) = w.l$ where h is a handler for effect l, and thus $(op \to f) \in h$.

So by (*perform*), we have

$w \vdash \text{perform}\ op\ \epsilon\ \overline{\sigma}\ v : \sigma_2[\overline{\alpha}{:=}\overline{\sigma}] \longrightarrow \text{yield}\ m\ (\lambda k.\ f\ \overline{\sigma}\ v\ k)$. □

Proof. (*Of Theorem 3*) Apply Lemma 14, then we know that either e_1 is a value, or $\langle\!\langle\rangle\!\rangle \vdash e_1 \longmapsto e_2$, or $e = \mathsf{E}[\text{yield}\ m\ f]$, where $m \notin \text{bm}(\mathsf{E})$.

For the first two cases, we have proved the goal.

The last case is an impossible case. For internal-safe expressions, yield can only appear under (*perform*). By Lemma 1, we know that (*perform*) is applied under the evidence vector $\langle\!\langle \lceil \mathsf{E} \rceil \mid \langle\!\langle\rangle\!\rangle \rangle\!\rangle$. And thus the corresponding prompt m must be in E.

□

F.3.3 Correspondence.

Proof. (*Proof for Lemma 1*) By induction on $\longmapsto$.

case (*step*).

$e_1 = \mathsf{F}[e_1']$ (*step*)
$e_2 = \mathsf{F}[e_2']$ above
$e_1' \longrightarrow e_2'$ above

case (*stepw*).

$e_1 = \mathsf{F}[e_1']$ (*step*)
$e_2 = \mathsf{F}[e_2']$ above
$w \vdash e_1' \longrightarrow e_2'$ above
$\langle\!\langle \lceil \mathsf{F} \rceil \mid w \rangle\!\rangle = w$ by definition

case (*promptw*).

$e_1 = \mathsf{F}[\mathit{prompt}\ m\ h\ e_1']$ (*step*)
$e_2 = \mathsf{F}[\mathit{prompt}\ m\ h\ e_2']$ above
$\langle\!\langle l : (m,\ h) \mid w \rangle\!\rangle \vdash e_1' \longmapsto e_2'$ above

By I.H., we have

subcase

$e_1' = \mathsf{E}'[e_1'']$ I.H.
$e_2' = \mathsf{E}'[e_2'']$
$e_1'' \longrightarrow e_2''$
$\mathsf{E} = \mathsf{F}\ (\mathit{prompt}\ m\ h\ \mathsf{E}')$ Let

subcase

$e_1' = \mathsf{E}'[e_1'']$ I.H.
$e_2' = \mathsf{E}'[e_2'']$
$\langle\!\langle \lceil \mathsf{E}' \rceil \mid l : (m,\ h,\ \_) \mid w \rangle\!\rangle \vdash e_1'' \longrightarrow e_2''$
$\langle\!\langle \lceil \mathsf{F}\ (\mathit{prompt}\ m\ h\ \mathsf{E}') \rceil \mid w \rangle\!\rangle = \langle\!\langle \lceil \mathsf{E}' \rceil \mid l : (m,\ h,\ \_) \mid w \rangle\!\rangle$ by definition
□

Proof. (*Of Theorem 1*) Follows directly by Lemma 1 and (*perform*). □

F.3.4 Simulation.

Lemma 15. (*Simulation (small step)*). Given $\varnothing \vdash e_1 : \sigma \mid \epsilon$, if $e_1 \longrightarrow e_2$ in System F^p, then for $w : \mathsf{evv}\ \epsilon$, we have $w \vdash e_1 \longmapsto e_2$ in System F^{pw}.

Proof. (*Of Lemma 15*) By induction on $e_1 \longrightarrow e_2$ in System F^p. Most cases are straightforward. The only interesting case is

(*perform*) $\mathsf{prompt}\ m\ h\ \mathsf{E}[\mathsf{perform}\ \mathit{op}\ \epsilon_0\ \overline{\sigma}\ v] \longrightarrow \mathsf{prompt}\ m\ h\ \mathsf{E}[\mathsf{yield}\ m\ (\lambda^{\epsilon} k : \sigma_k.\ f\ \overline{\sigma}\ v\ k)]$
iff $\mathit{op} \notin \mathsf{bop}(\mathsf{E}) \wedge (\mathit{op} \to f \in h), \mathit{op} : \forall \overline{\alpha}.\ \sigma_1 \to \sigma_2 \in \Sigma(l), \varnothing \vdash h : \sigma \mid l \mid \epsilon, \sigma_k = \sigma_2[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ \sigma$.

In System F^{pw} then, we know that

reducing

$w \vdash \mathsf{prompt}\ m\ h\ \mathsf{E}[\mathsf{perform}\ \mathit{op}\ \epsilon\ \overline{\sigma}\ v]$

means (by (*promptw*)) reducing

$\langle\!\langle l : (m, h, w) \mid w \rangle\!\rangle \vdash \mathsf{E}[\mathsf{perform}\ \mathit{op}\ \epsilon\ \overline{\sigma}\ v]$

which then means (by $\longmapsto$) reducing

$\langle\!\langle \lceil \mathsf{E} \rceil \mid \langle\!\langle l : (m, h, w) \mid w \rangle\!\rangle \rangle\!\rangle \vdash \mathsf{perform}\ \mathit{op}\ \epsilon\ \overline{\sigma}\ v$

Since $\mathit{op} \notin \mathsf{bop}(\mathsf{E})$, we know that $\langle\!\langle \lceil \mathsf{E} \rceil \mid \langle\!\langle l : (m, h, w) \mid w \rangle\!\rangle \rangle\!\rangle.l = (m,\ h,\ \_)$. We then have by (*perform*),

$\langle\!\langle \lceil \mathsf{E} \rceil \mid \langle\!\langle l : (m, h, w) \mid w \rangle\!\rangle \rangle\!\rangle \vdash \mathsf{perform}\ \mathit{op}\ \epsilon\ \overline{\sigma}\ v \longmapsto \mathsf{yield}\ m\ (\lambda k.\ f\ \overline{\sigma}\ v\ k)$

Therefore

$w \vdash \mathsf{prompt}\ m\ h\ \mathsf{E}[\mathsf{perform}\ \mathit{op}\ \epsilon\ \overline{\sigma}\ v] \longmapsto \mathsf{prompt}\ m\ h\ \mathsf{E}[\mathsf{yield}\ m\ (\lambda k.\ f\ \overline{\sigma}\ v\ k)]$

□

Lemma 16. (*Evaluation Step (II)*). 1. If $e_1 \longrightarrow e_2$, then $w \vdash \mathsf{E}[e_1] \longmapsto \mathsf{E}[e_2]$.
2. If $\langle\!\langle \lceil \mathsf{E} \rceil \mid w \rangle\!\rangle \vdash e_1 \longrightarrow e_2$, then $w \vdash \mathsf{E}[e_1] \longmapsto \mathsf{E}[e_2]$.

Proof. (*Of Lemma 16*) Both cases can be easily proved by induction on E. We take the first case as an example. **case** $\mathsf{E} = \square$. The goal follows directly by (*step*).
case $\mathsf{E} = \mathsf{E}'\ e$. By I.H., we have $w \vdash \mathsf{E}'[e_1] \longmapsto \mathsf{E}'[e_2]$. By (*stepw*), we have $(\mathsf{E}'\ e)[e_1] \longmapsto (\mathsf{E}'\ e)[e_2]$.
case The case for $v\ \mathsf{E}'$ and $\mathsf{E}'\ \sigma$ are similar as the case for $\mathsf{E}'\ e$, following directly by I.H. and (*stepw*).
case $\mathsf{E} = \mathsf{prompt}\ m\ h\ \mathsf{E}'$. By I.H., we have $\langle\!\langle l : (m, h, w) \mid w \rangle\!\rangle \vdash \mathsf{E}'[e_1] \longrightarrow \mathsf{E}[e_2]$. By (*promptw*), we have $(\mathsf{prompt}\ m\ h\ \mathsf{E}')[e_1] \longrightarrow (\mathsf{prompt}\ m\ h\ \mathsf{E}')[e_2]$. □

Lemma 17. (*Evaluation Step (III)*). If $\langle\!\langle \lceil \mathsf{E} \rceil \mid w \rangle\!\rangle \vdash e_1 \longmapsto e_2$, then $w \vdash \mathsf{E}[e_1] \longmapsto \mathsf{E}[e_2]$.

Proof. (*Of Lemma 17*) Given $\langle\!\langle \lceil \mathsf{E} \rceil \mid w \rangle\!\rangle \vdash e_1 \longmapsto e_2$, by Lemma 1, we know that there are two cases

- $e_1 = \mathsf{E}'[e_1']$, $e_2 = \mathsf{E}'[e_2']$, and $e_1' \longrightarrow e_2'$.

By Lemma 16 (1), we have that $w \vdash (\mathsf{E} \bullet \mathsf{E}')[e_1'] \longmapsto (\mathsf{E} \bullet \mathsf{E}')[e_2']$. That is, $w \vdash \mathsf{E}[e_1] \longmapsto \mathsf{E}[e_2]$.

- $e_1 = \mathsf{E}'[e_1']$, $e_2 = \mathsf{E}'[e_2']$, and $\langle\!\langle \lceil \mathsf{E}' \rceil \mid \langle\!\langle \lceil \mathsf{E} \rceil \mid w \rangle\!\rangle \rangle\!\rangle \vdash e_1' \longrightarrow e_2'$.

By Lemma 16 (2), we have that $w \vdash (\mathsf{E} \bullet \mathsf{E}')[e_1'] \longmapsto (\mathsf{E} \bullet \mathsf{E}')[e_2']$. That is, $\mathsf{E}[e_1] \longmapsto \mathsf{E}[e_2]$.
□

Theorem 15. (*Simulation*). Given $\varnothing \vdash e_1 : \sigma \mid \epsilon$, if $e_1 \longmapsto e_2$ in System F^p, then for $w : \mathsf{evv}\ \epsilon$, we have $w \vdash e_1 \longmapsto e_2$ in System F^{pw}.

Proof. (*Of Theorem 15*) In System F^p, $e_1 \longmapsto e_2$ means that, by (*step*), $e_1 = \mathsf{E}[e_1']$, and $e_2 = \mathsf{E}[e_2']$, and $e_1' \longrightarrow e_2'$.
From Lemma 15, we have that $\langle\!\langle \lceil \mathsf{E} \rceil \mid w \rangle\!\rangle \vdash e_1' \longmapsto e_2'$.
By Lemma 17, we have $w \vdash \mathsf{E}[e_1'] \longmapsto \mathsf{E}[e_2']$. That is, $w \vdash e_1 \longmapsto e_2$.
□

F.3.5 Uniqueness.

Proof. (*Of Theorem 4*) From Lemma 8, we know that internal-safe expressions are m-mapping.
Then suppose there is an internal-safe expression of form
$\mathsf{prompt}\ m\ h \bullet \mathsf{E} \bullet \mathsf{prompt}\ m\ h \bullet e$
where the marker m is duplicated, and these two ms, as the expression is m-mapping, have the same handler h, and h is a handler for effect l.
Since it is internal-safe, we know it is closed and well-typed. So we have

$\varnothing \vdash \mathsf{prompt}\ m\ h \bullet \mathsf{E} \bullet \mathsf{prompt}\ m\ h \bullet e : \sigma \mid \epsilon$	known
$\varnothing \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon$	PROMPT
$\varnothing \vdash \mathsf{E} \bullet \mathsf{prompt}\ m\ h \bullet e : \sigma \mid \langle l \mid \epsilon \rangle$	PROMPT
$\varnothing \vdash_{\mathsf{ec}} \mathsf{E} : \sigma_1 \to \sigma \mid \langle l \mid \epsilon \rangle$	Lemma 4
$\varnothing \vdash \mathsf{prompt}\ m\ h \bullet e : \sigma_1 \mid \langle \llbracket \mathsf{E} \rrbracket \mid l \mid \epsilon \rangle$	Lemma 4
$\varnothing \vdash_{\mathsf{ops}} h : \sigma_1 \mid l \mid \langle \llbracket \mathsf{E} \rrbracket \mid l \mid \epsilon \rangle$	PROMPT
$\sigma = \sigma_1$	
$\epsilon = \langle \llbracket \mathsf{E} \rrbracket \mid l \mid \epsilon \rangle$	
contradiction	

We have a contradiction because $\langle \llbracket \mathsf{E} \rrbracket \mid l \mid \epsilon \rangle$ contains at least one more label than ϵ, so they cannot be equivalent. □

F.4 Tail Resumptive Operation

F.4.1 Preservation.

Lemma 18. (*Small Step Preservation*). 1. If $\varnothing \vdash e_1 : \sigma \mid \epsilon$, where e_1 is internal-safe, and $e_1 \longrightarrow e_2$, then $\varnothing \vdash e_2 : \sigma \mid \epsilon$.
2. If $\varnothing \vdash e_1 : \sigma \mid \epsilon$, where e_1 is internal-safe, and $w \vdash e_1 \longrightarrow e_2$, where $w : \mathsf{evv}\ \epsilon$, then $\varnothing \vdash e_2 : \sigma \mid \epsilon$.

Proof. (*Of Theorem 9*) Based on the preservation lemma for System F^{pw} (Lemma 12), here we only discuss the new cases for under.
case (*performt*) $w \mid \mathsf{perform}\ op\ \epsilon_0\ \overline{\sigma}\ v \longrightarrow (\Lambda\overline{\alpha}.\lambda^{\langle l \mid \epsilon_0 \rangle} x : \sigma_1.\ \mathsf{under}^{\epsilon_0, \epsilon}\ l\ e)\ \overline{\sigma}\ v$ where $(m, h, w') = w.l$ and $\mathsf{op} : \forall\overline{\alpha}.\ \sigma_1 \to \sigma_2 \in \Sigma(l)$, and $(\mathsf{op} \to \Lambda\overline{\alpha}.\ \lambda^{\epsilon} x : \sigma_1\ k : \sigma_2 \to \sigma.\ k\ e) \in h$ with $k \notin \mathsf{fv}(e)$

$w : \mathsf{evv}\ \epsilon$	given
$(m, h, w') = w.l$	given
$\varnothing \vdash_{\mathsf{ops}} h : l \mid \sigma \mid \epsilon$	follows
$\mathsf{op} : \forall\overline{\alpha}.\ \sigma_1 \to \sigma_2 \in \Sigma(l)$	given
$(\mathsf{op} \to \Lambda\overline{\alpha}.\ \lambda^{\epsilon} x : \sigma_1\ k : \sigma_2 \to \sigma.\ k\ e) \in h$	given
$\varnothing \vdash (\Lambda\overline{\alpha}.\ \lambda^{\epsilon} x : \sigma_1\ k : \sigma_2 \to \sigma.\ k\ e) : \forall\overline{\alpha}.\ \sigma_1 \to \epsilon\ (\sigma_2 \to \epsilon\ \sigma) \to \epsilon\ \sigma$	OPS
$x : \sigma_1,\ k : \sigma_2 \to \sigma \vdash k\ e : \sigma \mid \epsilon$	TABS and ABS
$x : \sigma_1,\ k : \sigma_2 \to \sigma \vdash e : \sigma_2 \mid \epsilon$	APP
$k \notin \mathsf{fv}(e)$	given
$x : \sigma_1 \vdash e : \sigma_2 \mid \epsilon$	strengthening
$x : \sigma_1 \vdash \mathsf{under}^{\epsilon_0,\ \epsilon}\ l\ e : \sigma_2 \mid \langle l \mid \epsilon_0 \rangle$	UNDER
$\varnothing \vdash \Lambda\overline{\alpha}.\ \lambda x : \sigma_1.\ \mathsf{under}^{\epsilon_0,\ \epsilon}\ l\ e : \forall\overline{\alpha}.\ \sigma_1 \to \langle l \mid \epsilon_0 \rangle\ \sigma_2 \mid \langle l \mid \epsilon_0 \rangle$	TABS and ABS
$\varnothing \vdash \mathsf{perform}\ op\ \epsilon_0\ \overline{\sigma}\ v : \sigma_2[\overline{\alpha} := \overline{\sigma}]$	given and PERFORM
$\varnothing \vdash v : \sigma_1[\overline{\alpha} := \overline{\sigma}] \mid \langle l \mid \epsilon_0 \rangle$	
$\varnothing \vdash (\Lambda\overline{\alpha}.\ \lambda x : \sigma_1.\ \mathsf{under}^{\epsilon_0,\ \epsilon}\ l\ e)\ \overline{\sigma}\ v : \sigma_2[\overline{\alpha} := \overline{\sigma}] \mid \langle l \mid \epsilon_0 \rangle$	TAPP and APP

case (*under*) $\mathsf{under}^{\epsilon_0,\ \epsilon}\ l\ v \longrightarrow v$.

Given that $\varnothing \vdash \mathsf{under}^{\epsilon_0,\ \epsilon}\ l\ v : \sigma \mid \langle l \mid \epsilon_0 \rangle$, by UNDER we have $\varnothing \vdash v : \sigma \mid \epsilon$. As Lemma 7, we have $\varnothing \vdash v : \sigma \mid \langle l \mid \epsilon_0 \rangle$. □

Lemma 19. (*Preservation*). If $\varnothing \vdash e_1 : \sigma \mid \epsilon$, where e_1 is internal-safe, and $w \vdash e_1 \longmapsto e_2$ where $w : \mathsf{evv}\ \epsilon$, then $\varnothing \vdash e_2 : \sigma \mid \langle\rangle$.

Proof. (*Of Theorem 13*) By induction on $w \vdash e_1 \longmapsto e_2$. We discuss the only new case (*underw*). That is, $e_1 = \mathsf{F}[\mathsf{under}^{\epsilon_0,\ \epsilon}\ l\ e]$.

$$\frac{w' \vdash e \longmapsto e' \quad (m, h, w') = w.l}{w \vdash \mathsf{F}[\mathsf{under}^{\epsilon_0,\epsilon}\ l\ e] \longmapsto \mathsf{F}[\mathsf{under}^{\epsilon_0,\epsilon}\ l\ e']}\,(underw)$$

Because e_1 is internal safe, so $\mathsf{under}^{\epsilon_0,\ \epsilon}$ was initially generated by (*performt*).

$w \vdash \mathsf{perform}\ op\ \epsilon_0\ \overline{\sigma}\ v \mid \longrightarrow \mid (\Lambda\overline{\alpha}.\ \lambda^{\langle l \mid \epsilon_0 \rangle} x : \sigma_1.\ \mathsf{under}^{\epsilon_0,\epsilon}\ l\ e)\ \overline{\sigma}\ v$, where $(m, h, w') = w.l$, and $(\mathsf{op} \to \Lambda\overline{\alpha}.\ \lambda^{\epsilon} x : \sigma_1.\ k : \sigma_2 \to \epsilon\ \sigma.\ k\ e) \in h$ with $k \notin \mathsf{fv}(e)$.

At that point, we know that the expression is of effect $\langle l \mid \epsilon_0 \rangle$, so $w : \mathsf{evv}\ \langle l \mid \epsilon_0 \rangle$. Given $(m, h, w') = w.l$ and $(\mathsf{op} \to \Lambda\overline{\alpha}.\ \lambda^{\epsilon} x : \sigma_1.\ k : \sigma_2 \to \epsilon\ \sigma.\ k\ e) \in h$, we know that $w' : \mathsf{evv}\ \epsilon$. That is, effect context of label l from effect $\langle l \mid \epsilon_0 \rangle$ is ϵ.

Back to (*underw*), we know that $w' : w\ \epsilon$. Now we have

$\varnothing \vdash \mathsf{F}[\mathsf{under}^{\epsilon_0,\ \epsilon}\ l\ e] : \sigma \mid \langle l \mid \epsilon_0 \rangle$	given
$\varnothing \vdash_{\mathsf{ec}} \mathsf{F} : \sigma_1 \to \sigma \mid \langle l \mid \epsilon_0 \rangle$	above
$\varnothing \vdash \mathsf{under}^{\epsilon_0,\ \epsilon}\ l\ e : \sigma_1 \mid \langle l \mid \epsilon_0 \rangle$	Lemma 4
$\varnothing \vdash e : \sigma_1 \mid \epsilon$	UNDER
$\varnothing \vdash e' : \sigma_1 \mid \epsilon$	I.H.
$\varnothing \vdash \mathsf{under}^{\epsilon_0,\ \epsilon}\ l\ e' : \sigma_1 \mid \langle l \mid \epsilon_0 \rangle$	UNDER
$\varnothing \vdash \mathsf{F}[\mathsf{under}^{\epsilon_0,\ \epsilon}\ l\ e] : \sigma \mid \langle l \mid \epsilon_0 \rangle$	Lemma 3

□

Theorem 16. (*Preservation*). If $\varnothing \vdash e_1 : \sigma \mid \langle\rangle$, where e_1 is internal-safe, and $\langle\!\langle\rangle\!\rangle \vdash e_1 \longmapsto e_2$, then $\varnothing \vdash e_2 : \sigma \mid \langle\rangle$.

Proof. (*Of Theorem 16*) Follows directly by Lemma 19 with $w = \langle\!\langle\rangle\!\rangle$. □

F.4.2 Progress.

Definition 5. (*Well-formed evaluation contexts*). We say that an evaluation context is well-formed, if it is of the following form:

$$
\begin{array}{lcl}
\mathsf{F} & ::= & \Box \mid \mathsf{F}\ e \mid v\ \mathsf{F} \\
& \mid & \mathsf{prompt}\ m\ h \bullet \mathsf{E} \bullet \mathsf{under}^{\epsilon_0,\epsilon}\ l\ \mathsf{F} \quad l \notin [\![\mathsf{E}]\!], \varnothing \vdash h : \sigma \mid l \mid \epsilon \\
\mathsf{E} & ::= & \Box \mid \mathsf{E}\ e \mid v\ \mathsf{E} \\
& \mid & \mathsf{prompt}\ m\ h \bullet \mathsf{E}' \bullet \mathsf{under}^{\epsilon_0,\epsilon}\ l\ \mathsf{E} \quad l \notin [\![\mathsf{E}']\!], \varnothing \vdash h : \sigma \mid l \mid \epsilon \\
& \mid & \mathsf{prompt}\ m\ h\ \mathsf{E}
\end{array}
$$

Lemma 20. (*Internal-safe expressions have well-formed evaluation contexts*). If an internal-safe F^p expressions e, which was initially reduced ($\longmapsto$) under $\langle\!\langle\rangle\!\rangle$, can be written as $\mathsf{E}[e']$, then E is a well-formed evaluation context.

Proof. (*Of Lemma 20*) This lemma is to rule out expressions like

$\mathsf{prompt}\ m\ h^{l_1} \bullet \mathsf{prompt}\ m\ h^{l_2} \bullet \mathsf{under}\ l_1 \bullet \mathsf{under}\ l_2\ e$

The term type-checks because l_1 and l_2 are handled. But it does not evaluate as (*skipw*) does not apply: because of under l_1, (*underw*) would remove the prompt $m\ h^{l_2}$ in the evidence vector, and thus under l_2 would fail to find any l_2 evidence in the evidence vector. So skipw for under l_2 does not apply.

The lemma is restricted to internal-safe expressions that are evaluated ($\longmapsto$) under $\langle\!\langle\rangle\!\rangle$, as we want to rule out stand-alone under $l\ e$, which itself can be an internal-safe expressions as it can be reduced from perform with a proper non-empty evidence vector, but its evaluation context under $l\ \Box$ is not well-formed.

We prove our goal by case-analyzing the definition of internal-safe expressions:

- When e contains no under at all, then the goal is trivially true.
- If e is reduced from an internal-safe expression e_1, then we need to show that at every step, the property is preserved.
  The key observation is that whenever an under is introduced ((*performt*)), its l is chosen from the current available evidence vector w. According to (*underw*), all existing unders have already removed all the evidence from their prompt until the corresponding under. Thus the only possible prompt that the newly introduced under is paired to can only wrap well-formed evaluation contexts and the new evaluation context is thus also well-formed.

□

Lemma 21. (*Progress with effects*). If $\varnothing \vdash e_1 : \sigma \mid \epsilon$, then for any $w : \mathsf{evv}\ \epsilon$, we have either (1) e_1 is a value; or (2) $w \vdash e_1 \longmapsto e_2$; or (3) $e_1 = \mathsf{E}[\mathsf{yield}\ m\ f]$, where $m \notin \mathsf{bm}(\mathsf{E})$; or (4) $e_1 = \mathsf{E}[\mathsf{under}^{\epsilon}\ l\ e]$, where E is well-formed, and there is no $h \in \mathsf{E}$ such that $\varnothing \vdash_{\mathsf{ops}} h : l \mid \_ \mid \epsilon$, which we denote using $l^{\epsilon} \notin \lceil \mathsf{E} \rceil$.

Proof. (*Of Lemma 21*) In the fourth case, it is possible that an expression contains several unders that cannot be correctly reduced, but here we only need to know the first under that causes the trouble. That's why we can still show E is well-formed. For example, under l_1 (under $l_2\ e$) can be represented as $\Box[\mathsf{under}\ l_1\ (\mathsf{under}\ l_2\ e)]$.

We first do induction on the size of e_1, and then do induction on typing.

Based on the progress theorem of System F^p (Lemma 14), here we only discuss the new cases for under.

case $\varnothing \vdash \mathsf{prompt}\ m\ h\ e : \sigma \mid \epsilon$. By I.H., we know that either e is a value, or $\langle\!\langle l : (m, h, w) \mid w \rangle\!\rangle \vdash e \longmapsto e'$, or $e = \mathsf{E}[\mathsf{yield}\ m'\ f]$, where $m' \notin \mathsf{bm}(\mathsf{E})$, or $e = \mathsf{E}[\mathsf{under}^{\epsilon}\ l\ e_1']$, where E is well-formed and $l^{\epsilon} \notin \lceil \mathsf{E} \rceil$.

We have already talked about the first three cases in Lemma 14. In the last case, we have $e = \mathsf{E}[\mathsf{under}^{\epsilon}\ l\ e_1']$, where $l^{\epsilon} \notin [\![\mathsf{E}]\!]$, and E is well-formed.

- $\varnothing \vdash_{\mathsf{ops}} h : l \mid \_ \mid \epsilon$. That means under l is paired with the current prompt. Because E is well-formed, we now have both prompt $m\ h\ \mathsf{E}$ and prompt $m\ h\ \mathsf{E}[\mathsf{under}\ l\ \square]$ well-formed. By I.H. on e_1' (the size of e_1' is smaller than prompt $m\ h\ \mathsf{E}[\mathsf{under}\ l\ e_1']$), we have
  - e_1' is a value. Then we know that under $l\ v \longrightarrow v$ by (*under*). Because prompt $m\ h\ \mathsf{E}$ is well-formed, we have prompt $m\ h\ \mathsf{E}[\mathsf{under}\ l\ v] \longmapsto$ prompt $m\ h\ \mathsf{E}[v]$.
  - $w \vdash e_1' \longmapsto e_2'$. Because of (*underw*), under l removes all the evidence vector between the prompt m and under l, we know that evaluating $w \vdash$ prompt $m\ h\ \mathsf{E}[\mathsf{under}\ l\ e_1']$ reduces to evaluating $w \vdash e_1'$. The type of w is of the right type because of preservation. So we have prompt $m\ h\ \mathsf{E}[\mathsf{under}\ l\ e_1'] \longmapsto$ prompt $m\ h\ \mathsf{E}[\mathsf{under}\ l\ e_2']$.
  - $e_1' = \mathsf{E}'[\mathsf{yield}\ m'\ f]$ where $m' \notin \mathsf{bm}(\mathsf{E}')$. Then again since under l removes all the evidence vector between the prompt m and under l, we have $e_1' = (\text{prompt } m\ h\ \mathsf{E} \bullet \mathsf{under}\ l \bullet \mathsf{E}')[\mathsf{yield}\ m_1\ f]$, and $m' \notin \mathsf{bm}(\text{prompt } m\ h\ \mathsf{E} \bullet \mathsf{under}\ l \bullet \mathsf{E}')$.
  - $e_1' = \mathsf{E}'[\mathsf{under}^{\epsilon'}\ l'\ e_2']$ where $l'^{\epsilon'} \notin [\![\mathsf{E}']\!]$ and E' is well-formed. Again since under l removes all the evidence vector between prompt m and under l, we have $e_1' = (\text{prompt } m\ h\ \mathsf{E} \bullet \mathsf{under}\ l \bullet \mathsf{E}')[\mathsf{under}\ l'\ e_2']$, and $l'^{\epsilon'} \notin \lceil \text{prompt } m\ h\ \mathsf{E} \bullet \mathsf{under}\ l \bullet \mathsf{E}' \rceil$.
- h is not a handler for l. Then we have $(\text{prompt } m\ h\ \mathsf{E})[\mathsf{under}\ l\ e_1']$ and $l^{\epsilon} \notin [\![(\text{prompt } m\ h\ \mathsf{E})]\!]$.

□

Theorem 17. (*Progress of Internal-safe System* F^{pw} *with* under). If $\varnothing \vdash e_1 : \sigma \mid \langle\rangle$ where e_1 is an internal-safe expression, we have either e_1 is a value, or $\langle\!\langle\rangle\!\rangle \vdash e_1 \longmapsto e_2$.

Proof. (*Of Theorem 17*) Apply Lemma 21, then we know that either e_1 is a value, or $\langle\!\langle\rangle\!\rangle \vdash e_1 \longmapsto e_2$, or $e = \mathsf{E}[\mathsf{yield}\ m\ f]$ where $m \notin \mathsf{bm}(\mathsf{E})$, or $e_1 = \mathsf{E}[\mathsf{under}^{\epsilon}\ l\ e]$ where E is well-formed, and $l^{\epsilon} \notin \lceil \mathsf{E} \rceil$.

For the first two cases, we have proved the goal.

For the third case, we can prove it by contradiction, following the proof for System F^{p} (Theorem 3).

The last case is new. However it is an impossible case. By Lemma 20, we know that internal-safe expressions can only have well-formed evaluation context. That means $\mathsf{E} \bullet \mathsf{under}^{\epsilon}\ l\ \square$ must be well-formed. However we already know that $l^{\epsilon} \notin \lceil \mathsf{E} \rceil$, and thus $\mathsf{E} \bullet \mathsf{under}^{\epsilon}\ l\ \square$ cannot be well-formed. So we have a contradiction.

□

F.4.3 Coherence.

Lemma 22. ($\cong$ *preserves the handler context*). Given $\mathsf{E}_1 \cong \mathsf{E}_2$, and $\Gamma \vdash \mathsf{E}_1 : \sigma_1 \to \sigma_2 \mid \langle\rangle$, then $\lceil \mathsf{E}_1 \rceil = \lceil \mathsf{E}_2 \rceil$.

Proof. (*Of Lemma 22*) Most cases follow directly. The only interesting case is the equivalence of evaluation contexts lifted by EQ-UNDER. In this case we have $(\lambda x.\ \text{prompt } m\ h\ \mathsf{E}_1[x])\ \mathsf{E}_1 \cong \text{prompt } m\ h \bullet \mathsf{E}_2 \bullet \mathsf{under}^{\epsilon}\ l\ \mathsf{E}_2$, with $\mathsf{E}_1 \cong \mathsf{E}_2$.

We then have $\lceil (\lambda x.\ \text{prompt } m\ h\ \mathsf{E}_1[x])\ \mathsf{E}_1 \rceil = \lceil \mathsf{E}_1 \rceil$.

Also, since under removes the evidence vector between prompt $m\ h$ and under, we have $\lceil \text{prompt } m\ h \bullet \mathsf{E}_2 \bullet \mathsf{under}^{\epsilon}\ l\ \mathsf{E}_2 \rceil = \lceil \mathsf{E}_2 \rceil$.

By I.H., we have $\lceil \mathsf{E}_1 \rceil = \lceil \mathsf{E}_2 \rceil$.

□

Lemma 23. ($\cong$ *preserves evaluation (general)*). Given $e_1 \cong e_2$ where e_1 and e_2 are well-typed non-values, and $\mathsf{E}_1 \cong \mathsf{E}_2$ where $\mathsf{E}_1[e_1]$ and $\mathsf{E}_2[e_2]$ are well-typed internal-expressions with effect $\langle\rangle$, there exist e_1', e_2' such that $\langle\!\rangle \vdash \mathsf{E}_1[e_1] \longmapsto^+ e_1'$, and $\langle\!\rangle \vdash \mathsf{E}_2[e_2] \longmapsto^+ e_2'$, and $e_1' \cong e_2'$.

Proof. (*Of Lemma 23*) By induction on e_1. We write $e_1 \longmapsto e_2$ for $\langle\!\rangle \vdash e_1 \longmapsto e_2$.

case $e_1 = e_3\ e_4$, where e_3 is not a value.

$e_2 = e_5\ e_6$	by $\cong$
$e_3 \cong e_5$	above
$e_4 \cong e_6$	above
$\mathsf{E}_1 \bullet \Box\ e_4 \bullet e_3 \longmapsto^+ e_3'$	I.H.
$\mathsf{E}_2 \bullet \Box\ e_5 \bullet e_5 \longmapsto^+ e_5'$	I.H.
$e_3' \cong e_5'$	I.H.

case $e_1 = v_1\ e_3$, where e_3 is not a value.

We discuss the shape of e_2.

subcase $e_2 = v_2\ e_4$.

$v_1 \cong v_2$	by $\cong$
$e_3 \cong e_4$	above
$\mathsf{E}_1 \bullet v_1\ \Box \bullet e_3 \longmapsto^+ e_3'$	I.H.
$\mathsf{E}_2 \bullet v_2\ \Box \bullet e_4 \longmapsto^+ e_4'$	I.H.
$e_3' \cong e_4'$	I.H.

subcase $e_1 = (\lambda x.\ \mathsf{prompt}\ m\ h\ \mathsf{E}_1'[x])\ e_3$ and $e_2 = \mathsf{prompt}\ m\ h \bullet \mathsf{E}_2' \bullet \mathsf{under}\ l\ e_4$ with $l \notin [\![\mathsf{E}]\!]$.

$e_3 \cong e_4$	by $\cong$
$\mathsf{E}_1' \cong \mathsf{E}_2'$	by $\cong$
$\mathsf{E}_1 \bullet (\lambda x.\ \mathsf{prompt}\ m\ h\ \mathsf{E}'[x])\ \Box \bullet e_3 \longmapsto^+ e_3'$	I.H.
$\mathsf{E}_2 \bullet \mathsf{prompt}\ m\ h \bullet \mathsf{E}_2'\ .\mathsf{under}\ l\ \Box \bullet e_4 \longmapsto^+_u e_4'$	I.H.
$e_3' \cong e_4'$	I.H.

case $e_1 = (\lambda x.\ e_3)\ v_1$.

subcase $e_2 = (\lambda x.\ e_4)\ v_2$.

$e_3 \cong e_4$	by $\cong$
$v_1 \cong v_2$	above
$\mathsf{E}_1[e_1] \longmapsto \mathsf{E}_1[e_3[x{:=}v_1]\ ]$	(*app*)
$\mathsf{E}_2[e_2] \longmapsto \mathsf{E}_2[e_4[x{:=}v_2]\ ]$	(*app*)
$e_3[x{:=}v_1] \cong e_4[x{:=}v_2]$	by substitution

subcase $e_1 = (\lambda x.\ \mathsf{prompt}\ m\ h\ \mathsf{E}_1'[x])\ v_1$ and $e_2 = \mathsf{prompt}\ m\ h \bullet \mathsf{E}_2' \bullet \mathsf{under}^\epsilon\ l\ v_2$ with $l \notin [\![\mathsf{E}]\!]$.

$e_3 \cong e_4$	by $\cong$
$\mathsf{E}_1' \cong \mathsf{E}_2'$	by $\cong$
$v_1 \cong v_2$	above
$\mathsf{E}_1[e_1] \longmapsto \mathsf{E}_1[\mathsf{prompt}\ m\ h\ \mathsf{E}_1'[v_1]\ ]$	(*app*)
$\mathsf{E}_2[e_2] \longmapsto \mathsf{E}_2[\mathsf{prompt}\ m\ h\ \mathsf{E}_2'[v_2]\ ]$	(*under*)
$\mathsf{E}_1[\mathsf{prompt}\ m\ h\ \mathsf{E}_1'[v_1]\ ] \cong \mathsf{E}_2[\mathsf{prompt}\ m\ h\ \mathsf{E}_2'[v_2]\ ]$	congruence

case $e_1 = \mathsf{handler}\ h\ v_1$.

$e_2 = \mathsf{handler}\ h\ v_2$ by $\cong$
$v_1 \cong v_2$ above
$\mathsf{E}_1[\mathsf{handler}\ h\ v_1] \longmapsto \mathsf{E}_1[\mathsf{prompt}\ m\ h\ (v_1\ ())]$ (*handler*)
$\mathsf{E}_2[\mathsf{handler}\ h\ v_2] \longmapsto \mathsf{E}_2[\mathsf{prompt}\ m\ h\ (v_2\ ())]$ (*handler*)
$\mathsf{E}_1[\mathsf{prompt}\ m\ h\ (v_1\ ())] \cong \mathsf{E}_1[\mathsf{prompt}\ m\ h\ (v_2\ ())]$ congruence

case $e_1 = \mathsf{perform}\ op\ \overline{\sigma}\ v_1$.

$\mathsf{E}_1[\mathsf{perform}\ op\ \overline{\sigma}\ v_1] \longmapsto \mathsf{E}_1[\mathsf{yield}\ m\ (\lambda k.\ f[\overline{\sigma}]\ v_1\ k)]$ (*perform*)
$(m,\ h,\ \_) = \lceil \mathsf{E}_1 \rceil.l$ and $(op \to f) \in h$ above
$e_2 = \mathsf{perform}\ op\ \overline{\sigma}\ v_2$ by $\cong$
$v_1 \cong v_2$ above

subcase f is not a tail-resumptive operation.

$\lceil \mathsf{E}_1 \rceil = \lceil \mathsf{E}_2 \rceil$ Lemma 22
$\mathsf{E}_2[\mathsf{perform}\ op\ \overline{\sigma}\ v_2] \longmapsto \mathsf{E}_2[\mathsf{yield}\ m\ (\lambda k.\ f[\overline{\sigma}]\ v_2\ k)]$ (*perform*)
$\mathsf{E}_1[\mathsf{yield}\ m\ (\lambda k.\ f[\overline{\sigma}]\ v_1\ k)] = \mathsf{E}_2[\mathsf{yield}\ m\ (\lambda k.\ f[\overline{\sigma}]\ v_2\ k)]$ congruence

subcase $f = \Lambda\overline{\alpha}.\ \lambda x.\ k\ e$ is a tail-resumptive operation.

As $\mathsf{E}_1[e_1]$ is internal-safe. According to progress, yield m is going to be handled.

$\mathsf{E}_1 = \mathsf{E}_1' \bullet \mathsf{prompt}\ m\ h \bullet \mathsf{E}_1''$ suppose
$\mathsf{E}_1[\mathsf{yield}\ m\ (\lambda k.\ f[\overline{\sigma}]\ v_1\ k)] \longmapsto \mathsf{E}_1'[(\lambda k.\ f[\overline{\sigma}]\ v_1\ k)\ (\lambda x.\ prompt\ m\ h\ \mathsf{E}_1''[x])]$ (*prompt*)
$\mathsf{E}_1[\mathsf{yield}\ m\ (\lambda k.\ f[\overline{\sigma}]\ v_1\ k)] \longmapsto \mathsf{E}_1'[(\lambda x.\ prompt\ m\ h\ \mathsf{E}_1''[x])\ e[\overline{\alpha}:=\overline{\sigma},\ x:=v_1]\ ]$ (*tapp*) and (*ap*
$\mathsf{E}_2 = \mathsf{E}_2' \bullet \mathsf{prompt}\ m\ h \bullet \mathsf{E}_2''$ by $\cong$
$\mathsf{E}_1' \cong \mathsf{E}_2'$ by $\cong$
$\mathsf{E}_1'' \cong \mathsf{E}_2''$ by $\cong$
$\mathsf{E}_2[\mathsf{perform}\ op\ \overline{\sigma}\ v_2] \longmapsto \mathsf{E}_2[(\Lambda\overline{\alpha}.\ \lambda x.\ \mathsf{under}\ l \bullet e)\ [\overline{\sigma}]\ v_2\ ]$ (*performt*)
$\mathsf{E}_2[\mathsf{perform}\ op\ \overline{\sigma}\ v_2] \longmapsto \mathsf{E}_2[\mathsf{under}\ l \bullet e[\overline{\alpha}:=\overline{\sigma},\ x:=v_2]\ ]$ (*tapp*) and (*ap*
$(\lambda x.\ prompt\ m\ h\ \mathsf{E}_1''[x])\ e[\overline{\alpha}:=\overline{\sigma},\ x:=v_1] = \mathsf{prompt}\ m\ h \bullet \mathsf{E}_2'' \bullet \mathsf{under}\ l \bullet e[\overline{\alpha}:=\overline{\sigma},\ x:=v_2]$ EQ-UNDER
$\mathsf{E}_1'[(\lambda x.\ prompt\ m\ h\ \mathsf{E}_1''[x])\ e[\overline{\alpha}:=\overline{\sigma},\ x:=v_1]\ ]$
$= \mathsf{E}_2' \bullet \mathsf{prompt}\ m\ h.\ \mathsf{E}_2'' \bullet \mathsf{under}\ l \bullet e[\overline{\alpha}:=\overline{\sigma}, x:=v_2]$ congruence
$= \mathsf{E}_2[\mathsf{under}\ l \bullet e[\overline{\alpha}:=\overline{\sigma}, x:=v_2]\ ]$ by substitution

case $e_1 = \mathsf{prompt}\ m\ h\ e_3$.

subcase $e_3 = v_1$ is a value.

$e_2 = \mathsf{prompt}\ m\ h\ v_2$ by $\cong$
$v_1 \cong v_2$ above
$\mathsf{E}_1[\mathsf{prompt}\ m\ h\ v_1] \longmapsto \mathsf{E}_1[v_1]$ (*promptv*)
$\mathsf{E}_2[\mathsf{prompt}\ m\ h\ v_2] \longmapsto \mathsf{E}_2[v_2]$ (*promptv*)
$v_1 \cong v_2$ known

subcase e_3 is not a value.

$e_2 = \mathsf{prompt}\ m\ h\ e_4$ by $\cong$
$e_3 \cong e_4$ above
$\mathsf{E}_1.\ \mathsf{prompt}\ m\ h \bullet e_3 \longmapsto e_1'$ I.H.
$\mathsf{E}_2.\ \mathsf{prompt}\ m\ h \bullet e_4 \longmapsto e_2'$ I.H.
$e_1' \cong e_2'$ I.H.

case $e_1 = \mathsf{yield}\ m\ f_1$. According to progress, $\mathsf{E}_1[e_1]$ is going to be handled.

$\mathsf{E}_1 = \mathsf{E}_1' \bullet \text{prompt } m\ h \bullet \mathsf{E}_1''$	suppose
$\mathsf{E}_1[e_1] \longmapsto \mathsf{E}_1'[f_1\ (\lambda x.\ \textit{prompt}\ m\ h\ \mathsf{E}_1''[x])]$	(*prompt*)
$e_2 = \text{yield } m\ f_2$	by $\cong$
$f_1 \cong f_2$	above
$\mathsf{E}_1 \cong \mathsf{E}_2$	given
$\mathsf{E}_2 = \mathsf{E}_2' \bullet \text{prompt } m\ h \bullet \mathsf{E}_2''$	by $\cong$
$\mathsf{E}_1' \cong \mathsf{E}_2'$	by $\cong$
$\mathsf{E}_1'' \cong \mathsf{E}_2''$	by $\cong$
$\mathsf{E}_2[e_2] \longmapsto \mathsf{E}_2'[f_2\ (\lambda x.\ \textit{prompt}\ m\ h\ \mathsf{E}_2''[x])]$	(*prompt*)
$\mathsf{E}_1'[f_1\ (\lambda x.\ \textit{prompt}\ m\ h\ \mathsf{E}_1''[x])] = \mathsf{E}_2'[f_2\ (\lambda x.\ \textit{prompt}\ m\ h\ \mathsf{E}_2''[x])]$	congruence

□

Proof. (*Of Lemma 2*) If they are both values then we are done.

Otherwise, according to the definition of $\cong$, it's impossible that one is a value and the other is an expression, so they are both expressions. We then apply Lemma 23 with $\epsilon = \langle\rangle$ and, $\mathsf{E}_1 = \mathsf{E}_2 = \square$, and we are done. □

Proof. (*Of Theorem 5*) If $C[e] \longmapsto^* n$ under the unoptimized semantics, then as $C[e] \cong C[e]$ and by Lemma 2, we must have $C[e] \longmapsto^* e'$ under the tail-resumptive optimization semantics, with $e' \cong n$.
According to the definition of $\cong$, we must have $e' = n$.
That means $C[e] \longmapsto^* n$ under the tail-resumptive optimization semantics.
The case from right to left is the same. □

F.5 System F^{pb}: Bubbling Semantics

F.5.1 Preservation.

Lemma 24. (*Small Step Preservation*). 1. If $\varnothing \vdash e_1 : \sigma \mid \epsilon$ where e_1 is internal-safe, and $e_1 \longrightarrow e_2$, then $\varnothing \vdash e_2 : \sigma \mid \epsilon$.
2. If $\varnothing \vdash e_1 : \sigma \mid \epsilon$ where e_1 is internal-safe, and $w \vdash e_1 \longrightarrow e_2$, where $w : \text{evv}\ \epsilon$, then $\varnothing \vdash e_2 : \sigma \mid \epsilon$.

Proof. (*Of Lemma 24*) We case analyze the $\longrightarrow$ relation, and only discuss all new cases.
case (app_1) $v\ \square \bullet \text{yield } m\ f\ k \longrightarrow \text{yield } m\ f\ (\lambda^{\epsilon'} x : \sigma_2.\ v\ (k\ x))$, where $\varnothing \vdash_{\text{val}} k : \sigma_2 \to \epsilon'\ \sigma'$.

$\varnothing \vdash v\ \square \bullet \text{yield } m\ f\ k : \sigma_0 \mid \epsilon'$	given
$\varnothing \vdash_{\text{val}} k : \sigma_2 \to \epsilon'\ \sigma'$	given
$x : \sigma_2 \vdash k : \sigma_2 \to \epsilon'\ \sigma' \mid \epsilon'$	VAL and weakening
$x : \sigma_2 \vdash k\ x : \sigma' \mid \epsilon'$	APP
$\varnothing \vdash \text{yield } m\ f\ k : \sigma' \mid \epsilon'$	APP
$\varnothing \vdash v : \sigma' \to \epsilon'\ \sigma_0 \mid \epsilon'$	APP
$\varnothing \vdash_{\text{val}} f : (\sigma_2 \to \epsilon\ \sigma) \to \epsilon\ \sigma$	YIELD
$x : \sigma_2 \vdash v : \sigma' \to \epsilon'\ \sigma_0 \mid \epsilon'$	weakening
$x : \sigma_2 \vdash v\ (k\ x) : \sigma_0 \mid \epsilon'$	APP
$\varnothing \vdash_{\text{val}} \lambda^{\epsilon'} x : \sigma_2.\ v\ (k\ x) : \sigma_2 \to \epsilon'\ \sigma_0$	ABS
$\text{yield } m\ f\ (\lambda^{\epsilon'} x : \sigma_2.\ v\ (k\ x)) : \sigma_0 \mid \epsilon'$	YIELD

case (app_2) $\square\ e \bullet \text{yield } m\ f\ k \longrightarrow \text{yield } m\ f\ (\lambda^{\epsilon'} x : \sigma_2.\ (k\ x)\ e)$, where $\varnothing \vdash_{\text{val}} k : \sigma_2 \to \epsilon'\ \sigma'$

$\varnothing \vdash \Box e \bullet \text{yield}\ m\ f\ k : \sigma_2' \mid \epsilon'$	given
$\varnothing \vdash_{\text{val}} k : \sigma_2 \to \epsilon'\ \sigma'$	given
$x : \sigma_2 \vdash k : \sigma_2 \to \epsilon'\ \sigma' \mid \epsilon'$	VAL and weakening
$x : \sigma_2 \vdash k\ x : \sigma' \mid \epsilon'$	APP
$\sigma' = \sigma_1' \to \epsilon'\ \sigma_2'$	APP
$\varnothing \vdash \text{yield}\ m\ f\ k : \sigma_1' \to \epsilon'\ \sigma_2' \mid \epsilon'$	APP
$\varnothing \vdash e : \sigma_1' \mid \epsilon'$	APP
$\varnothing \vdash_{\text{val}} f : (\sigma_2 \to \epsilon\ \sigma) \to \epsilon\ \sigma$	YIELD
$x : \sigma_2 \vdash e : \sigma_1' \mid \epsilon'$	weakening
$x : \sigma_2 \vdash (k\ x)\ e : \sigma_2' \mid \epsilon'$	APP
$\varnothing \vdash_{\text{val}} \lambda^{\epsilon'} x : \sigma_2.\ (k\ x)\ e : \sigma_2 \to \epsilon'\ \sigma_2'$	ABS
$\text{yield}\ m\ f\ (\lambda^{\epsilon'} x : \sigma_2.\ (k\ x)\ e) : \sigma_2' \mid \epsilon'$	YIELD

case (*under*) $\text{under}^{\epsilon_0,\epsilon}\ l\ \Box \bullet \text{yield}\ m\ f\ k \longrightarrow \text{yield}\ m\ f\ (\lambda^{\langle l \mid \epsilon_0 \rangle} x : \sigma_2.\ \text{under}^{\epsilon_0,\epsilon}\ l\ (k\ x))$ $\varnothing \vdash_{\text{val}} k : \sigma_2 \to \epsilon\ \sigma$

$\varnothing \vdash \text{under}^{\epsilon_0,\epsilon}\ l\ \Box \bullet \text{yield}\ m\ f\ k : \sigma \mid \langle l \mid \epsilon_0 \rangle$	given
$\varnothing \vdash \text{yield}\ m\ f\ k : \sigma \mid \epsilon$	UNDER
$\varnothing \vdash_{\text{val}} f : (\sigma_2 \to \epsilon'\ \sigma') \to \epsilon'\ \sigma'$	YIELD
$\varnothing \vdash_{\text{val}} k : \sigma_2 \to \epsilon\ \sigma$	given
$x : \sigma_2 \vdash k : \sigma_2 \to \epsilon\ \sigma \mid \epsilon$	VAL and weakening
$x : \sigma_2 \vdash k\ x : \sigma \mid \epsilon$	APP
$x : \sigma_2 \vdash \text{under}^{\epsilon_0,\epsilon}\ l\ (k\ x) : \sigma \mid \langle l \mid \epsilon_0 \rangle$	UNDER
$\varnothing \vdash \lambda^{\langle l \mid \epsilon_0 \rangle} x : \sigma_2.\ \text{under}^{\epsilon_0,\epsilon}\ l\ (k\ x) : \sigma_2 \to \langle l \mid \epsilon_0 \rangle\ \sigma$	ABS
$\varnothing \vdash \text{yield}\ m\ f\ (\lambda^{\epsilon_0} x : \sigma_2.\ \text{under}^{\epsilon_0,\epsilon}\ l\ (k\ x)) : \sigma \mid \langle l \mid \epsilon_0 \rangle$	YIELD

case ($prompt_1$) $\text{prompt}\ m\ h\ \Box \bullet \text{yield}\ m\ f\ k \longrightarrow f\ (\lambda^{\epsilon} x : \sigma_2.\ \text{prompt}\ m\ h\ (k\ x))$, where $\varnothing \vdash_{\text{val}} k : \sigma_2 \to \langle l \mid \epsilon \rangle\ \sigma$.

Because it's internal safe, we know that the type of h indeed matches that of f.

$\varnothing \vdash \text{prompt}\ m\ h\ \Box \bullet \text{yield}\ m\ f\ k : \sigma \mid \epsilon$	given
$\varnothing \vdash_{\text{val}} k : \sigma_2 \to \langle l \mid \epsilon \rangle\ \sigma$	given
$x : \sigma_2 \vdash k : \sigma_2 \to \langle l \mid \epsilon \rangle\ \sigma \mid \langle l \mid \epsilon \rangle$	VAL and weakening
$x : \sigma_2 \vdash k\ x : \sigma \mid \langle l \mid \epsilon \rangle$	APP
$\varnothing \vdash \text{yield}\ m\ f\ k : \sigma \mid \langle l \mid \epsilon \rangle$	PROMPT
$\varnothing \vdash_{\text{ops}} h : \sigma \mid l \mid \epsilon$	PROMPT
$\varnothing \vdash_{\text{val}} f : (\sigma_2 \to \epsilon\ \sigma) \to \epsilon\ \sigma$	YIELD
$x : \sigma_2 \vdash \text{prompt}\ m\ h\ (k\ x) : \sigma \mid \epsilon$	PROMPT
$\varnothing \vdash_{\text{val}} \lambda^{\epsilon'} x : \sigma_2.\ \text{prompt}\ m\ h\ (k\ x) : \sigma_2 \to \epsilon\ \sigma$	ABS
$\varnothing \vdash f\ (\lambda^{\epsilon'} x : \sigma_2.\ \text{prompt}\ m\ h\ (k\ x)) : \sigma \mid \epsilon$	APP

case ($prompt_2$) $\text{prompt}\ n\ h\ \Box \bullet \text{yield}\ m\ f\ k \longrightarrow \text{yield}\ m\ f\ (\lambda^{\epsilon'} x : \sigma_2.\ \text{prompt}\ n\ h\ (k\ x))$, iff $n \neq m$, $\varnothing \vdash_{\text{val}} k : \sigma_2 \to \langle l \mid \epsilon' \rangle\ \sigma'$.

$\varnothing \vdash \text{prompt}\ n\ h\ \square \bullet \text{yield}\ m\ f\ k : \sigma' \mid \epsilon'$	given
$\varnothing \vdash_{\text{val}} k : \sigma_2 \to \langle l \mid \epsilon' \rangle\ \sigma'$	given
$x : \sigma_2 \vdash k : \sigma_2 \to \langle l \mid \epsilon' \rangle\ \sigma' \mid \langle l \mid \epsilon' \rangle$	VAL and weakening
$x : \sigma_2 \vdash k\ x : \sigma' \mid \langle l \mid \epsilon' \rangle$	APP
$\varnothing \vdash \text{yield}\ m\ f\ k : \sigma' \mid \langle l \mid \epsilon' \rangle$	PROMPT
$\varnothing \vdash_{\text{ops}} h : \sigma' \mid l \mid \epsilon'$	PROMPT
$\varnothing \vdash_{\text{val}} f : (\sigma_2 \to \epsilon\ \sigma) \to \epsilon\ \sigma$	YIELD
$x : \sigma_2 \vdash \text{prompt}\ m\ h\ (k\ x) : \sigma' \mid \epsilon'$	PROMPT
$\varnothing \vdash_{\text{val}} \lambda^{\epsilon'} x : \sigma_2.\ \text{prompt}\ m\ h\ (k\ x) : \sigma_2 \to \epsilon'\ \sigma'$	ABS
$\text{yield}\ m\ f\ (\lambda^{\epsilon'} x : \sigma_2.\ \text{prompt}\ n\ h\ (k\ x)) : \sigma' \mid \epsilon'$	YIELD

case (*perform*) $w \vdash \text{perform}\ op\ \epsilon'\ \overline{\sigma}\ v \longrightarrow \text{yield}\ m\ (\lambda^{\epsilon} k : \sigma_k.\ f\ \overline{\sigma}\ v\ k)\ (\lambda^{\langle l \mid \epsilon' \rangle} x : \sigma_2[\overline{\alpha} := \overline{\sigma}].\ x)$, with $(m, h, \_) = w.l, (op : (\forall \overline{\alpha}.\ \sigma_1 \to \sigma_2) \to f) \in h : \sigma \mid l \mid \epsilon, \sigma_k = \sigma_2[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ \sigma$.

$\varnothing \vdash \text{perform}\ op\ \epsilon'\ \overline{\sigma}\ v : \sigma_2[\overline{\alpha} := \overline{\sigma}] \mid \langle l \mid \epsilon' \rangle$	given
$(op : (\forall \overline{\alpha}.\ \sigma_1 \to \sigma_2) \to f) \in h : \sigma \mid l \mid \epsilon$	given
$\varnothing \vdash_{\text{val}} f : \forall \overline{\alpha}.\ \sigma_1 \to \epsilon\ (\sigma_2 \to \epsilon\ \sigma) \to \epsilon\ \sigma$	OPS
$\varnothing \vdash_{\text{val}} f\ \overline{\sigma} : \sigma_1[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ (\sigma_2[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ \sigma) \to \epsilon\ \sigma$	TAPP
$\varnothing \vdash_{\text{val}} v : \sigma_1[\overline{\alpha} := \overline{\sigma}]$	PERFORM and VAL
$\varnothing \vdash f\ \overline{\sigma}\ v : (\sigma_2[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ \sigma) \to \epsilon\ \sigma \mid \epsilon$	APP
$k : \sigma_k \vdash f\ \overline{\sigma}\ v : (\sigma_2[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ \sigma) \to \epsilon\ \sigma \mid \epsilon$	weakening
$k : \sigma_k \vdash f\ \overline{\sigma}\ v\ k : \sigma \mid \epsilon$	APP
$\varnothing \vdash_{\text{val}} \lambda^{\epsilon} k : \sigma_k.\ f\ \overline{\sigma}\ v\ k : \sigma_k \to \sigma$	ABS
$\varnothing \vdash_{\text{val}} \lambda^{\langle l \mid \epsilon' \rangle} x : \sigma_2[\overline{\alpha} := \overline{\sigma}].\ x : \sigma_2[\overline{\alpha} := \overline{\sigma}] \to \langle l \mid \epsilon' \rangle\ \sigma_2[\overline{\alpha} := \overline{\sigma}]$	ABS
$\text{yield}\ m\ (\lambda^{\epsilon} k : \sigma_k.\ f\ \overline{\sigma}\ v\ k)\ (\lambda^{\langle l \mid \epsilon' \rangle} x : \sigma_2[\overline{\alpha} := \overline{\sigma}].\ x) : \sigma_2[\overline{\alpha} := \overline{\sigma}] \mid \langle l \mid \epsilon' \rangle$	YIELD

□

Proof. (*Of Theorem 12*) The same as preservation for System F^{pw} (Theorem 2), with Lemma 24. □

F.5.2 Progress.

Lemma 25. (*Progress with effects*). If $\varnothing \vdash e_1 : \sigma \mid \epsilon$, then for any $w : \text{evv}\ \epsilon$, we have either (1) e_1 is a value; or (2) $w \vdash e_1 \longmapsto e_2$; or (3) $e_1 = \text{yield}\ m\ f\ k$.

Proof. (*Of Lemma 25*) The proof is essentially the same as progress with effects for System F^{pw} (Lemma 14). The main difference between progress for System F^{pb} from System F^{pw} is that notice in option (3) we have $\text{yield}\ m\ f\ k$ as the outermost instead of $\text{E}[\text{yield}\ m\ f]$ as in Lemma 14. That is because under the bubble semantics, yield can always evaluate under (or, bubble out of) the outer evaluation context.

We take prompt as an example.

case $\varnothing \vdash \text{prompt}\ m\ h\ e : \sigma \mid \epsilon$. By I.H., we know that either e is a value, or $\langle\!\langle l : (m, h, w) \mid w \rangle\!\rangle \vdash e \longmapsto e'$, or $e = \text{yield}\ m\ f\ e$.

- If e is a value, then by (*promptv*) we have $\text{prompt}\ m\ h\ e \longmapsto e$.
- If $\langle\!\langle l : (m, h, w) \mid w \rangle\!\rangle \vdash e \longmapsto e'$. Then by (*promptw*), $\text{prompt}\ m\ h\ e \longmapsto \text{prompt}\ m\ h\ e'$.
- If $e = \text{yield}\ m'\ f\ k$.
  - $m = m'$. Then by ($prompt_1$), we have $\text{prompt}\ m\ h\ \text{yield}\ m\ f\ k \longmapsto f\ (\lambda x.\ \text{prompt}\ m\ h\ (k\ x))$.
  - $m \neq m'$. Then by ($prompt_2$), we have $\text{prompt}\ m\ h\ \text{yield}\ m'\ f\ k \longmapsto \text{yield}\ m'\ f\ (\lambda x.\ \text{prompt}\ m\ h\ (k\ x))$.

□

Proof. (*Of Theorem 13*) The same as the progress theorem for System F^{pw} (Theorem 3), with Lemma 25. □

F.5.3 Simulation.

Definition 6. ($\lceil e \rceil^{pw \Downarrow pb}$ *and* $\lceil e \rceil^{pw \Uparrow pb}$). $\lceil e \rceil^{pw \Downarrow pb}$ turns an expression from F^{pw} to F^{pb} by turning yield $m\ f$ into yield $m\ f\ (\lambda x.\ x)$; and $\lceil e \rceil^{pw \Uparrow pb}$ turns an expression from F^{pb} into F^{pw} by turning back yield $m\ f\ k$ into yield $m\ f$. The definition can be lifted straightforward to handlers and evaluation contexts.

Definition 7. (*Eta-expansion of evaluation contexts*). We define $=_\eta$ as a congruent relation between expressions with $\mathsf{E}[v] \ =_\eta \ (\lambda x.\ \mathsf{E}[x])\ v$.

Lemma 26. (*Simulation (small step)*). Given $\varnothing \ \vdash \ e_1 \ : \ \sigma \mid \epsilon$, and $w \ : \ \mathsf{evv}\ \epsilon$,
(1) if $e_1 \longrightarrow e_2$ in internal-safe System F^{pw}, we have $\lceil e_1 \rceil^{pw \Downarrow pb} \longmapsto^* e_2'$ in System F^{pb}, and $e_2' \ =_\eta \ e_2$;
(2) if $w \ \vdash \ e_1 \longrightarrow e_2$ in internal-safe System F^{pw}, we have $w \ \vdash \ \lceil e_1 \rceil^{pw \Downarrow pb} \longmapsto e_2'$ in System F^{pb}, and $e_2' \ =_\eta \ e_2$.

Proof. (*Of Lemma 26*) This Lemma is defined for internal-safe System F^{pw} because when yielding, in the (*prompt*) rule in System F^{pw}, yield non-deterministically finds a marker, while in System F^{pb} in terms of the bubble semantics, yield always finds the closest corresponding marker. Therefore we restrict the lemma to internal-safe System F^{pw} so that we know markers in the evaluation context are unique and two semantics coincide.

By induction on $e_1 \longrightarrow e_2$ and $w \ \vdash \ e_1 \longrightarrow e_2$ in System F^{pb}. Most cases are straightforward. Here we discuss only interesting cases.

case (*perform*) $w \ \vdash \ \mathsf{perform}\ op\ \epsilon_0\ \overline{\sigma}\ v \longrightarrow \mathsf{yield}\ m\ (\lambda^{\epsilon} k : \sigma_k.\ f\ \overline{\sigma}\ v\ k)$.

Then by (*perform*) from System F^{pb}, we have

$w \ \vdash \ \mathsf{perform}\ op\ \epsilon'\ \overline{\sigma}\ \lceil v \rceil^{pw \Downarrow pb} \longrightarrow \mathsf{yield}\ m\ (\lambda^{\epsilon} k : \sigma_k.\ \lceil f \rceil^{pw \Downarrow pb}\ \overline{\sigma}\ v\ \lceil k \rceil^{pw \Downarrow pb})\ (\lambda^{\langle l \mid \epsilon' \rangle} x : \sigma_2[\overline{\alpha} := \overline{\sigma}].\ x)$

Now

$\lceil \mathsf{yield}\ m\ (\lambda^{\epsilon} k : \sigma_k.\ \lceil f \rceil^{pw \Downarrow pb}\ \overline{\sigma}\ v\ \lceil k \rceil^{pw \Downarrow pb})\ (\lambda^{\langle l \mid \epsilon' \rangle} x : \sigma_2[\overline{\alpha} := \overline{\sigma}].\ x) \rceil^{pw \Uparrow pb}$
$= \ \mathsf{yield}\ m\ (\lambda^{\epsilon} k : \sigma_k.\ f\ \overline{\sigma}\ v\ k)$

case (*prompt*) $\mathsf{prompt}\ m\ h\ \mathsf{E}[\mathsf{yield}\ m\ f] \longrightarrow f\ (\lambda^{\epsilon} y : \sigma_2.\ \mathsf{prompt}\ m\ h\ \mathsf{E}[y])$

We have

$\lceil \mathsf{prompt}\ m\ h\ \mathsf{E}[\mathsf{yield}\ m\ f] \rceil^{pw \Downarrow pb} = \ \mathsf{prompt}\ m\ \lceil h \rceil^{pw \Downarrow pb}\ \lceil \mathsf{E} \rceil^{pw \Downarrow pb} [\mathsf{yield}\ m\ \lceil f \rceil^{pw \Downarrow pb}\ (\lambda x.\ x)]$

Now by the operational semantics rules, we know that yield $m\ f\ (\lambda x.\ x)$ will bubble up until it finds prompt. During this process, suppose $\lceil \mathsf{E} \rceil^{pw \Downarrow pb}$ consists of multiple "minimal" evaluation contexts $(\mathsf{E}_1 \bullet \mathsf{E}_2 \ldots \mathsf{E}_n)$. By "minimal" evaluation contexts we mean that each E_i cannot be destruct anymore to non-empty $\mathsf{E}_i' \bullet \mathsf{E}_i''$.

Note that initially, we have $k \ = \ (\lambda x.\ x)$ which is equivalent to $\lambda x.\ \square[x]$. For each bubble rules, we have

$\mathsf{E}_i \bullet \mathsf{yield}\ m\ f\ k \longrightarrow \mathsf{yield}\ m\ f\ (\lambda x.\ (\mathsf{E} \bullet k\ \square)\ [x])$

So bubbling through all the evaluation contexts until prompt, we have the final k' built up as

$k' \ = \ (\lambda x_1.\ (\mathsf{E}_1 \bullet (\lambda x_2.\ (\mathsf{E}_2 \bullet (\lambda x_3.\ (\mathsf{E}_3 \bullet (\ldots (\lambda x.\ (\square)\ [x]))\ \square)\ [x_3])\ \square)\ [x_2])\ \square)\ [x_1])$

We then have

$\mathsf{prompt}\ m\ \lceil h \rceil^{pw \Downarrow pb}\ (\mathsf{yield}\ m\ f\ k') \longrightarrow \lceil f \rceil^{pw \Downarrow pb}\ (\lambda y.\ \mathsf{prompt}\ m\ h\ (k'\ y))$

We have
$k'\ y$
$=$
$(\lambda x_1.\ (\mathsf{E}_1 \bullet (\lambda x_2.\ (\mathsf{E}_2 \bullet (\lambda x_3.\ (\mathsf{E}_3 \bullet (\ldots (\lambda x.\ (\square)\ [x]))\ \square)\ [x_3])\ \square)\ [x_2])\ \square)\ [x_1])\ y$
$=_\eta$
$(\mathsf{E}_1 \bullet (\lambda x_2.\ (\mathsf{E} \bullet (\lambda x_3.\ (\mathsf{E} \bullet (\ldots (\lambda x.\ (\square)\ [x]))\ \square)\ [x_3])\ \square)\ [x_2])\ \square)\ [y]$
$=$
$\mathsf{E}\ [(\lambda x_2.\ (\mathsf{E} \bullet ((\lambda x_3.\ (\mathsf{E}_3 \bullet (\ldots (\lambda x.\ (\square)\ [x]))\ \square)\ [x_3]))\ \square)\ [x_2])\ y]$
$=_\eta$
$\mathsf{E}_1\ [(\mathsf{E}_2 \bullet (\lambda x_3.\ (\mathsf{E}_3 \bullet (\ldots (\lambda x.\ (\square)\ [x]))\ \square)\ [x_3])\ \square)[y]\ ]$
$=$
$\mathsf{E}_1\ [\mathsf{E}_2\ [(\lambda x_3.\ (\mathsf{E}_3 \bullet (\ldots (\lambda x.\ (\square)\ [x]))\ \square)\ [x_3])\ y]\ ]$
$=_\eta$
$\mathsf{E}_1\ [\mathsf{E}_2\ [\ (\mathsf{E}_3 \bullet (\ldots (\lambda x.\ (\square)\ [x]))\ \square)\ [y]\ ]\ ]$
$=$
$\mathsf{E}_1\ [\mathsf{E}_2\ [\ \mathsf{E}_3\ [(\ldots (\lambda x.\ (\square)\ [x]))\ y]\ ]\ ]$
$=$
$\ldots$
$\mathsf{E}_1\ [\mathsf{E}_2\ [\ \mathsf{E}_3\ [\ \ldots[\ \mathsf{E}_n\ [(\lambda x.\ (\square)\ [x])\ y]\ ]\ ]\ ]$
$=_\eta$
$\mathsf{E}_1\ [\mathsf{E}_2\ [\ \mathsf{E}_3\ [\ \ldots[\ \mathsf{E}_n\ [y]\ ]\ ]\ ]$
That means that
$\lceil k'\ y \rceil^{p w \Uparrow p b} = \lceil \mathsf{E}_1\ [\mathsf{E}_2\ [\ \mathsf{E}_3\ [\ \ldots[\ \mathsf{E}_n\ [y]\ ]\ ]\ ] \rceil^{p w \Uparrow p b} = \mathsf{E}[y]$
Therefore
$\lceil \lceil f \rceil^{p w \Downarrow p b}\ (\lambda y.\ \mathsf{prompt}\ m\ h\ (k'\ y)) \rceil^{p w \Uparrow p b} = f\ (\lambda y.\ \mathsf{prompt}\ m\ h\ \mathsf{E}[y])$ □

Theorem 18. (*Simulation*). Given $\varnothing \vdash e_1 : \sigma \mid \epsilon$, if $w \vdash e_1 \longmapsto e_2$ in internal-safe System F^{pw}, we have $w \vdash \lceil e_1 \rceil^{p w \Downarrow p b} \longmapsto^* e_2'$ in System F^{pb}, and $e_2' =_\eta e_2$.

Proof. (*Of Theorem 18*) Follows directly based on Lemma 26. □

F.6 Monadic Translation

F.6.1 Translation Soundness.

Lemma 27. (*Monadic Translation is Sound*). 1. If $\Gamma \vdash e : \sigma \mid \epsilon \rightsquigarrow e'$, then $\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} e' : \mathsf{Mon}\ \epsilon\ \lfloor \sigma \rfloor$.
2. If $\Gamma \vdash_{\mathsf{val}} v : \sigma \rightsquigarrow v'$, then $\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} v' : \lfloor \sigma \rfloor$.
3. If $\Gamma \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon \rightsquigarrow h'$, then $\lfloor \Gamma \rfloor \vdash h' : \mathsf{Hnd}^l\ \epsilon\ \lfloor \sigma \rfloor$.
4. If $\Gamma \vdash_{\mathsf{ec}} \mathsf{E} : \sigma_1 \to \sigma_2 \mid \epsilon \rightsquigarrow g$, then $\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} g : \mathsf{Mon}\ \langle [\![\mathsf{E}]\!] \mid \epsilon \rangle\ \lfloor \sigma_1 \rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor \sigma_2 \rfloor$.

Proof. (*Of Lemma 27*) **Part 1** By induction on the translation.
case $e = v$.

$\Gamma \vdash v : \sigma \mid \epsilon \rightsquigarrow \mathsf{pure}\ \epsilon\ \lfloor \sigma \rfloor\ v'$	given
$\Gamma \vdash_{\mathsf{val}} v : \sigma \mid \epsilon \rightsquigarrow v'$	VAL
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} v' : \lfloor \sigma \rfloor$	Part 2
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} \mathsf{Pure}\ \epsilon\ \lfloor \sigma \rfloor\ v' : \mathsf{Mon}\ \epsilon\ \lfloor \sigma \rfloor$	Pure, FTAPP and FAPP

case $e = e\ \sigma$.

$\Gamma \vdash e\,\sigma : \sigma_1[\alpha{:=}\sigma] \mid \epsilon \rightsquigarrow e' \triangleright (\lambda x{:}\lfloor \forall \alpha^{\kappa}.\ \sigma_1 \rfloor.\ \mathsf{pure}\ \epsilon\ \lfloor \sigma_1[\alpha{:=}\sigma] \rfloor\ (x\ \lfloor \sigma \rfloor))$ given
$\Gamma \vdash e : \forall \alpha.\ \sigma_1 \mid \epsilon \rightsquigarrow e'$ TAPP
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} e' : \mathsf{Mon}\ \epsilon\ (\forall \alpha.\ \lfloor \sigma_1 \rfloor)$ I.H.
$\lfloor \sigma_1[\alpha{:=}\sigma] \rfloor = \lfloor \sigma_1 \rfloor[\alpha{:=}\lfloor \sigma \rfloor]$ by substitution
$\lfloor \Gamma \rfloor,\ x : \forall \alpha.\ \lfloor \sigma_1 \rfloor \vdash_{\mathsf{F}} \mathsf{pure}\ \epsilon\ \lfloor \sigma_1[\alpha{:=}\sigma] \rfloor\ (x\ \lfloor \sigma \rfloor) : \mathsf{Mon}\ \epsilon\ \lfloor \sigma_1[\alpha{:=}\sigma] \rfloor$ *pure*, FTAPP and FAPP
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} \lambda x{:}\ \lfloor \forall \alpha.\ \sigma_1 \rfloor].\ \mathit{pure}\ \epsilon\ \lfloor \sigma_1[\alpha{:=}\sigma] \rfloor\ (x\ \lfloor \sigma \rfloor) : (\forall \alpha.\ \lfloor \sigma_1 \rfloor) \to \mathsf{Mon}\ \epsilon\ \lfloor \sigma_1[\alpha{:=}\sigma] \rfloor$ FABS
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} e' \triangleright (\lambda x{:}\ \lfloor \forall \alpha.\ \sigma_1 \rfloor.\ \mathit{pure}\ \epsilon\ \lfloor \sigma_1[\alpha{:=}\sigma] \rfloor\ (x\ \lfloor \sigma \rfloor)) : \mathsf{Mon}\ \epsilon\ \lfloor \sigma_1[\alpha{:=}\sigma] \rfloor$ $\triangleright$

case $e = e_1\ e_2$.

$\Gamma \vdash e_1\ e_2 : \sigma \mid \epsilon \rightsquigarrow e_1' \triangleright (\lambda f{:}\lfloor \sigma_1 \to \epsilon\ \sigma \rfloor.\ e_2' \triangleright f)$ given
$\Gamma \vdash e_1 : \sigma_1 \to \epsilon\ \sigma \mid \epsilon \rightsquigarrow e_1'$ APP
$\Gamma \vdash e_2 : \sigma_1 \mid \epsilon \rightsquigarrow e_2'$ above
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} e_1' : \mathsf{Mon}\ \epsilon\ (\lfloor \sigma_1 \rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor \sigma \rfloor)$ I.H
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} e_2' : \mathsf{Mon}\ \epsilon\ \lfloor \sigma_1 \rfloor$ I.H
$\lfloor \Gamma \rfloor,\ f{:}\ (\lfloor \sigma_1 \rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor \sigma \rfloor) \vdash_{\mathsf{F}} f : \lfloor \sigma_1 \rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor \sigma \rfloor$ FVAR
$\lfloor \Gamma \rfloor,\ f{:}\ (\lfloor \sigma_1 \rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor \sigma \rfloor) \vdash_{\mathsf{F}} e_2' : \mathsf{Mon}\ \epsilon\ \lfloor \sigma_1 \rfloor$ weakening
$\lfloor \Gamma \rfloor,\ f{:}\ (\lfloor \sigma_1 \rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor \sigma \rfloor) \vdash_{\mathsf{F}} e_2' \triangleright f : \mathsf{Mon}\ \epsilon\ \lfloor \sigma \rfloor$ $\triangleright$
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} (\lambda f{:}\lfloor \sigma_1 \to \epsilon\ \sigma \rfloor.\ e_2' \triangleright f) : (\lfloor \sigma_1 \rfloor \to \mathsf{Mon}\ \lfloor \sigma \rfloor) \to \mathsf{Mon}\ \epsilon\ \lfloor \sigma \rfloor$ FABS
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} e_1' \triangleright (\lambda f{:}\lfloor \sigma_1 \to \epsilon\ \sigma \rfloor.\ e_2' \triangleright f) : \mathsf{Mon}\ \epsilon\ \lfloor \sigma \rfloor$ $\triangleright$

case $e = \mathsf{prompt}\ m\ h\ e$.

$\Gamma \vdash \mathsf{prompt}\ m\ h\ e : \sigma \mid \epsilon \rightsquigarrow \mathit{prompt}^{l}\ \epsilon\ \lfloor \sigma \rfloor\ m\ h'\ e'$ given
$\Gamma \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon \rightsquigarrow h'$ PROMPT
$\Gamma \vdash e : \sigma \mid \langle l \mid \epsilon \rangle \rightsquigarrow e'$ above
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} h' : \mathsf{Hnd}^{l}\ \epsilon\ \lfloor \sigma \rfloor$ Part 3
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} e' : \mathsf{Mon}\ \langle l \mid \epsilon \rangle\ \sigma$ I.H.
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} \mathit{prompt}^{l}\ \epsilon\ \lfloor \sigma \rfloor\ m\ h'\ e' : \mathsf{Mon}\ \epsilon\ \sigma$ *prompt*, FTAPP and FAPP

case $e = \mathsf{yield}\ m\ f\ k$.

$\Gamma \vdash \mathsf{yield}\ m\ h\ e_0 : \sigma \mid \epsilon \rightsquigarrow \mathit{yield}\ \epsilon\ \lfloor \sigma \rfloor\ \lfloor \sigma_2 \rfloor\ \epsilon'\ \lfloor \sigma' \rfloor\ m\ f'\ k'$ given
$\Gamma \vdash_{\mathsf{val}} f : (\sigma_2 \to \epsilon'\ \sigma') \to \epsilon'\ \sigma' \rightsquigarrow f'$ YIELDB
$\Gamma \vdash_{\mathsf{val}} k : \sigma_2 \to \epsilon\ \sigma \rightsquigarrow k'$ above
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} f' : (\lfloor \sigma_2 \rfloor \to \mathsf{Mon}\ \epsilon'\ \lfloor \sigma' \rfloor) \to \mathsf{Mon}\ \epsilon'\ \lfloor \sigma' \rfloor$ Part 3
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} k' : \lfloor \sigma_2\ \rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor \sigma \rfloor$ Part 3
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} \mathit{yield}\ \epsilon\ \lfloor \sigma \rfloor\ \lfloor \sigma_2 \rfloor\ \epsilon'\ \lfloor \sigma' \rfloor\ m\ f'\ k' : \mathsf{Mon}\ \epsilon\ \lfloor \sigma \rfloor$ yield, FTAPP and FAPP

case $e = \mathsf{under}^{\epsilon_0,\epsilon}\ l\ e$.

$\Gamma \vdash \mathsf{under}^{\epsilon_0,\epsilon}\ l\ e : \sigma \mid \langle l \mid \epsilon_0 \rangle \rightsquigarrow \lambda w{:}\mathsf{Evv}\ \langle l \mid \epsilon_0 \rangle.\ \mathsf{let}\ (m, \_,\ w'){:}\mathsf{Ev}\ \epsilon\ r = w.l$
$\quad \mathsf{in}\ \mathit{under}^{l}\ \langle l \mid \epsilon_0 \rangle\ \lfloor \sigma \rfloor\ \epsilon\ r\ m\ w'\ e'\ w$ given
$\Gamma \vdash e : \sigma \mid \epsilon \rightsquigarrow e'$ UNDER
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} e' : \mathsf{Mon}\ \epsilon\ \lfloor \sigma \rfloor$ I.H.
$\lfloor \Gamma \rfloor,\ w{:}\mathsf{Evv}\ \langle l \mid \epsilon_0 \rangle \vdash_{\mathsf{F}} \mathsf{let}\ (m, \_,\ w'){:}\mathsf{Ev}\ \epsilon\ r = w.l$
$\quad \mathsf{in}\ \mathit{under}^{l}\ \langle l \mid \epsilon_0 \rangle\ \lfloor \sigma \rfloor\ \epsilon\ r\ m\ w'\ e'\ w : \mathsf{Ctl}\ \langle l \mid \epsilon_0 \rangle\ \lfloor \sigma \rfloor$ under^{l}
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} \lambda w{:}\mathsf{Evv}\ \langle l \mid \epsilon_0 \rangle.\ \mathsf{let}\ (m, \_,\ w'){:}\mathsf{Ev}\ \epsilon\ r = w.l$
$\quad \mathsf{in}\ \mathit{under}^{l}\ \langle l \mid \epsilon_0 \rangle\ \lfloor \sigma \rfloor\ \epsilon\ r\ m\ w'\ e'\ w : \mathsf{Mon}\ \langle l \mid \epsilon_0 \rangle\ \lfloor \sigma \rfloor$ FABS

Part 2

By induction on the translation.

case $v = x$.

$\Gamma \vdash_{\mathsf{val}} x : \sigma \rightsquigarrow x$ given
$x : \sigma \in \Gamma$ VAR
$x : \lfloor\sigma\rfloor \in \lfloor\Gamma\rfloor$ follows
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} x : \lfloor\sigma\rfloor$ FVAR

case $v = \lambda^{\epsilon} x : \sigma.\ e$.

$\Gamma \vdash_{\mathsf{val}} \lambda^{\epsilon} x : \sigma_1 . e : \sigma_1 \to \epsilon\ \sigma_2 \rightsquigarrow \lambda x : \lfloor\sigma_1\rfloor .\ e'$ given
$\Gamma, x : \sigma_1 \vdash e : \sigma_2 \mid \epsilon \rightsquigarrow e'$ ABS
$\lfloor\Gamma\rfloor, x : \lfloor\sigma_1\rfloor \vdash_{\mathsf{F}} e' : \mathsf{Mon}\ \epsilon\ \lfloor\sigma_2\rfloor$ Part 1
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} \lambda x : \lfloor\sigma_1\rfloor .\ e' : \lfloor\sigma_1\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma_2\rfloor$ FABS

case $v = \Lambda\alpha^{\kappa} .\ v_0$.

$\Gamma \vdash_{\mathsf{val}} \Lambda\alpha^{\kappa} .\ v : \forall\alpha .\ \sigma \rightsquigarrow \Lambda\alpha^{\kappa} .\ v'$ given
$\Gamma \vdash_{\mathsf{val}} v : \sigma \rightsquigarrow v'$ TABS
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} v' : \lfloor\sigma\rfloor$ I.H.
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} \Lambda\alpha .\ v' : \forall\alpha .\ \lfloor\sigma\rfloor$ FTABS

case $v = \mathsf{handler}\ h$.

$\Gamma \vdash_{\mathsf{val}} \mathsf{handler}^{\epsilon}\ h : (() \to \langle l \mid \epsilon\rangle\ \sigma) \to \epsilon\ \sigma \rightsquigarrow \mathit{handler}^{l}\ \epsilon\ \lfloor\sigma\rfloor\ h'$ given
$\Gamma \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon \rightsquigarrow h'$ MHANDLE
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} h' : \mathsf{Hnd}^{l}\ \epsilon\ \lfloor\sigma\rfloor$ Part 3
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} \mathit{handler}^{l}\ \epsilon\ \lfloor\sigma\rfloor\ h' : (() \to \mathsf{Mon}\ \langle l \mid \epsilon\rangle\ \sigma) \to \mathsf{Mon}\ \epsilon\ \sigma$ *handler*, FTAPP. FAPP

case $v = \mathsf{perform}\ \mathit{op}\ \epsilon\ \overline{\sigma}$.

$\Gamma \vdash_{\mathsf{val}} \mathsf{perform}\ \mathit{op}\ \epsilon\ \overline{\sigma} : \sigma_1[\overline{\alpha} := \overline{\sigma}] \to \langle l \mid \epsilon\rangle\ \sigma_2[\overline{\alpha} := \overline{\sigma}]$
$\quad \rightsquigarrow \mathit{perform}^{l}\ \epsilon\ \lfloor\sigma_1[\overline{\alpha} := \overline{\sigma}]\rfloor\ \lfloor\sigma_2[\overline{\alpha} := \overline{\sigma}]\rfloor\ (\Lambda\mu\ r .\ \mathit{select}^{\mathit{op}}\ \lfloor\overline{\sigma}\rfloor\ \mu\ r)$ given
$\lfloor\sigma_1[\overline{\alpha} := \overline{\sigma}]\rfloor = \lfloor\sigma_1\rfloor[\overline{\alpha} := \lfloor\overline{\sigma}\rfloor]$ by substitution
$\lfloor\sigma_2[\overline{\alpha} := \overline{\sigma}]\rfloor = \lfloor\sigma_2\rfloor[\overline{\alpha} := \lfloor\overline{\sigma}\rfloor]$ by substitution
$\mathit{op} : \forall\overline{\alpha} .\ \sigma_1 \to \sigma_2 \in \Sigma(l)$ MPERFORM
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} (\Lambda\mu\ r .\ \mathit{select}^{\mathit{op}}\ \lfloor\overline{\sigma}\rfloor\ \mu\ r) : \forall\mu\ r .\ \mathsf{Hnd}^{l}\ \mu\ r \to \mathsf{Op}\ \lfloor\sigma_1\rfloor[\overline{\alpha} := \lfloor\overline{\sigma}\rfloor]\ \lfloor\sigma_2\rfloor[\overline{\alpha} := \lfloor\overline{\sigma}\rfloor]\ \mu\ r$ FTABS, $\mathit{select}^{\mathit{op}}$
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} \mathit{perform}^{l}\ \epsilon\ \lfloor\sigma_1[\overline{\alpha} := \overline{\sigma}]\rfloor\ \lfloor\sigma_2[\overline{\alpha} := \overline{\sigma}]\rfloor\ (\Lambda\mu\ r .\ \mathit{select}^{\mathit{op}}\ \lfloor\overline{\sigma}\rfloor\ \mu\ r) :$
$\quad \lfloor\sigma_1[\overline{\alpha} := \overline{\sigma}]\rfloor \to \mathsf{Mon}\ \langle l \mid \epsilon\rangle\ \lfloor\sigma_2[\overline{\alpha} := \overline{\sigma}]\rfloor$ *perform*, FTAPP, FAPP

Part 3

$\Gamma \vdash_{\mathsf{ops}} \{\mathit{op}_1 \to f_1, \ldots, \mathit{op}_n \to f_n\} : \sigma \mid l \mid \epsilon \rightsquigarrow \mathsf{Hnd}^{l}\ (\overline{\forall\overline{\alpha} .\ \mathsf{Normal}\ \lfloor\sigma_1\rfloor\ \lfloor\sigma_2\rfloor\ \epsilon\ \lfloor\sigma\rfloor\ f_i'})$ given
$\Gamma \vdash_{\mathsf{val}} f_i : \forall\overline{\alpha} .\ \sigma_1 \to \epsilon\ (\sigma_2 \to \epsilon\ \sigma) \to \epsilon\ \sigma \rightsquigarrow f_i'$ OPS
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} f_i' : \forall\overline{\alpha} .\ \lfloor\sigma_1\rfloor \to \mathsf{Mon}\ \epsilon\ ((\lfloor\sigma_2\rfloor \to \mathsf{Mon}\ \lfloor\sigma\rfloor) \to \mathsf{Mon}\ \lfloor\sigma\rfloor)$ Part 2
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} \mathsf{Normal}\ \lfloor\sigma_1\rfloor\ \lfloor\sigma_2\rfloor\ \epsilon\ \lfloor\sigma\rfloor\ f_i' : \forall\overline{\alpha} .\ \mathsf{Op}\ \lfloor\sigma_1\rfloor\ \lfloor\sigma_2\rfloor\ \epsilon\ \lfloor\sigma\rfloor$ Op
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} \mathsf{Hnd}^{l}\ (\overline{\mathsf{Normal}\ \lfloor\sigma_1\rfloor\ \lfloor\sigma_2\rfloor\ \epsilon\ \lfloor\sigma\rfloor\ f_i'}) : \mathsf{Hnd}^{l}\ \epsilon\ \lfloor\sigma\rfloor$ Hnd and FAPP

Part 4 case CEMPTY.

$\lfloor\Gamma\rfloor \vdash \lambda x : \mathsf{Mon}\ \epsilon\ \lfloor\sigma\rfloor .\ x : \mathsf{Mon}\ \epsilon\ \lfloor\sigma\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma\rfloor$

case CAPP1.

$\Gamma \vdash_{\mathsf{ec}} \mathsf{E}\ e : \sigma_1 \to \sigma_3 \mid \epsilon \rightsquigarrow (\lambda f : \lfloor\sigma_2\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma_3\rfloor.\ e' \rhd f) \star g$ given
$\Gamma \vdash e : \sigma_2 \mid \epsilon \rightsquigarrow e'$
$\Gamma \vdash_{\mathsf{ec}} \mathsf{E} : \sigma_1 \to (\sigma_2 \to \epsilon\ \sigma_3) \mid \epsilon \rightsquigarrow g$
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} e' : \mathsf{Mon}\ \epsilon\ \lfloor\sigma_2\rfloor$ Part 1
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} g : \mathsf{Mon}\ \langle [\![\mathsf{E}]\!] \mid \epsilon\rangle\ \sigma_1 \to \mathsf{Mon}\ \epsilon\ (\lfloor\sigma_2\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma_3\rfloor)$ I.H.
$\lfloor\Gamma\rfloor, f : \lfloor\sigma_2\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma_3\rfloor \vdash_{\mathsf{F}} e' : \mathsf{Mon}\ \epsilon\ \lfloor\sigma_2\rfloor$ weakening
$\lfloor\Gamma\rfloor, f : \lfloor\sigma_2\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma_3\rfloor \vdash_{\mathsf{F}} (e' \rhd f) : \mathsf{Mon}\ \epsilon\ \lfloor\sigma_3\rfloor$ $(\rhd)$
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} (\lambda f : \lfloor\sigma_2\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma_3\rfloor.\ e' \rhd f) : (\lfloor\sigma_2\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma_3\rfloor) \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma_3\rfloor$ FABS
$\lfloor\Gamma\rfloor, f : \lfloor\sigma_2\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma_3\rfloor \vdash_{\mathsf{F}} g : \mathsf{Mon}\ \langle [\![\mathsf{E}]\!] \mid \epsilon\rangle\ \sigma_1 \to \mathsf{Mon}\ \epsilon\ (\lfloor\sigma_2\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma_3\rfloor)$ weakening
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} (\lambda f : \lfloor\sigma_2\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma_3\rfloor.\ e' \rhd f) \star g : \mathsf{Mon}\ \langle [\![\mathsf{E}]\!] \mid \epsilon\rangle\ \lfloor\sigma_1\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma_3\rfloor$ $(\star)$

case CAPP2.

$\Gamma \vdash_{\mathsf{ec}} v\ \mathsf{E} : \sigma_1 \to \sigma_3 \mid \epsilon \rightsquigarrow v \star g$ given
$\Gamma \vdash_{\mathsf{val}} v : \sigma_2 \to \epsilon\ \sigma_3 \rightsquigarrow v'$
$\Gamma \vdash_{\mathsf{ec}} \mathsf{E} : \sigma_1 \to \sigma_2 \mid \epsilon \rightsquigarrow g$
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} v' : \lfloor\sigma_2\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma_3\rfloor$ Part 2
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} g : \mathsf{Mon}\ \langle [\![\mathsf{E}]\!] \mid \epsilon\rangle\ \lfloor\sigma_1\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma_2\rfloor$ I.H.
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} v \star g : \mathsf{Mon}\ \langle [\![\mathsf{E}]\!] \mid \epsilon\rangle\ \lfloor\sigma_1\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma_3\rfloor$ $(\star)$

case CTAPP.

$\Gamma \vdash_{\mathsf{ec}} \mathsf{E}\ \sigma : \sigma_1 \to \sigma_2[\alpha{:=}\sigma] \mid \epsilon \rightsquigarrow (\lambda x : \lfloor\forall\alpha.\ \sigma_2\rfloor.\ \mathit{pure}\ \epsilon\ \lfloor\sigma_2[\alpha{:=}\sigma]\rfloor\ (x\ \lfloor\sigma\rfloor)) \star g$ given
$\Gamma \vdash_{\mathsf{ec}} \mathsf{E} : \sigma_1 \to \forall\alpha.\ \sigma_2 \mid \epsilon \rightsquigarrow g$
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} g : \mathsf{Mon}\ \langle [\![\mathsf{E}]\!] \mid \epsilon\rangle\ \lfloor\sigma_1\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\forall\alpha.\ \sigma_2\rfloor$ I.H.
$\lfloor\Gamma\rfloor, x : \lfloor\forall\alpha.\ \sigma_2\rfloor \vdash_{\mathsf{F}} (x\ \lfloor\sigma\rfloor)) : \lfloor\sigma_2\rfloor[\alpha{:=}\lfloor\sigma\rfloor]$ FTAPP
$\lfloor\Gamma\rfloor, x : \lfloor\forall\alpha.\ \sigma_2\rfloor \vdash_{\mathsf{F}} (x\ \lfloor\sigma\rfloor)) : \lfloor\sigma_2[\alpha{:=}\sigma]\rfloor$ by subst
$\lfloor\Gamma\rfloor, x : \lfloor\forall\alpha.\ \sigma_2\rfloor \vdash_{\mathsf{F}} \mathit{pure}\ \epsilon\ \lfloor\sigma_2[\alpha{:=}\sigma]\rfloor\ (x\ \lfloor\sigma\rfloor)) : \mathsf{Mon}\ \epsilon\ \lfloor\sigma_2[\alpha{:=}\sigma]\rfloor$ *pure*
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} (\lambda x : \lfloor\forall\alpha.\ \sigma_2\rfloor.\ \mathit{pure}\ \epsilon\ \lfloor\sigma_2[\alpha{:=}\sigma]\rfloor\ (x\ \lfloor\sigma\rfloor)) : \lfloor\forall\alpha.\ \sigma_2\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma_2[\alpha{:=}\sigma]\rfloor$ FABS
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} (\lambda x : \lfloor\forall\alpha.\ \sigma_2\rfloor.\ \mathit{pure}\ \epsilon\ \lfloor\sigma_2[\alpha{:=}\sigma]\rfloor\ (x\ \lfloor\sigma\rfloor)) \star g : \mathsf{Mon}\ \langle [\![\mathsf{E}]\!] \mid \epsilon\rangle\ \lfloor\sigma_1\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma_2[\alpha{:=}\sigma]\rfloor$ $(\star)$

case CPROMPT.

$\Gamma \vdash_{\mathsf{ec}} \Gamma \vdash_{\mathsf{ec}} \mathsf{prompt}\ m\ h\ \mathsf{E} : \sigma_1 \to \sigma \mid \epsilon\ \mathit{prompt}^l\ \epsilon\ \lfloor\sigma\rfloor\ m\ h' \circ g$ given
$\Gamma \vdash_{\mathsf{ops}} h : \sigma \mid l \mid \epsilon \rightsquigarrow h'$
$\Gamma \vdash_{\mathsf{ec}} \mathsf{E} : \sigma_1 \to \sigma \mid \langle l \mid \epsilon\rangle \rightsquigarrow g$
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} h' : \mathsf{Hnd}^l\ \epsilon\ \lfloor\sigma\rfloor$ Part 3
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} g : \mathsf{Mon}\ \langle [\![\mathsf{E}]\!] \mid l \mid \epsilon\rangle\ \lfloor\sigma_1\rfloor \to \mathsf{Mon}\ \langle l \mid \epsilon\rangle\ \lfloor\sigma\rfloor$ I.H.
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} \mathit{prompt}^l\ \epsilon\ \lfloor\sigma\rfloor\ m\ h : \mathsf{Mon}\ \langle l \mid \epsilon\rangle\ \lfloor\sigma\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma\rfloor$ prompt^l
$\lfloor\Gamma\rfloor \vdash_{\mathsf{F}} (\mathit{prompt}^l\ \epsilon\ \lfloor\sigma\rfloor\ m\ h) \circ g : \mathsf{Mon}\ \langle [\![\mathsf{E}]\!] \mid l \mid \epsilon\rangle\ \lfloor\sigma_1\rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor\sigma\rfloor$ $(\circ)$

case CUNDER.

$\Gamma \vdash_{\mathsf{ec}} \mathsf{under}^{\epsilon',\epsilon}\ l\ \mathsf{E} : \sigma_1 \to \sigma \mid \langle l \mid \epsilon' \rangle \rightsquigarrow \lambda x{:}\mathsf{Mon}\ \langle [\![\mathsf{E}]\!] \mid \epsilon \rangle\ \lfloor \sigma_1 \rfloor.\ \lambda w{:}\mathsf{Evv}\ \langle l \mid \epsilon' \rangle.$ given
$\quad$ let $(m, h', w') : \mathsf{Ev}\ \epsilon\ r = w.l$ in $under^l\ \langle l \mid \epsilon' \rangle\ \lfloor \sigma \rfloor\ \epsilon\ r\ m\ w'\ (g\ x)\ w$
$\Gamma \vdash_{\mathsf{ec}} \mathsf{E} : \sigma_1 \to \sigma \mid \epsilon \rightsquigarrow g$
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} g : \mathsf{Mon}\ \langle [\![\mathsf{E}]\!] \mid \epsilon \rangle\ \lfloor \sigma_1 \rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor \sigma \rfloor$ I.H.
$\lfloor \Gamma \rfloor, x : \mathsf{Mon}\ \langle [\![\mathsf{E}]\!] \mid \epsilon \rangle\ \lfloor \sigma_1 \rfloor \vdash_{\mathsf{F}} g\ x : \mathsf{Mon}\ \langle [\![\mathsf{E}]\!] \mid \epsilon \rangle\ \lfloor \sigma_1 \rfloor \to \mathsf{Mon}\ \epsilon\ \lfloor \sigma \rfloor$ I.H.
$\lfloor \Gamma \rfloor, x : \mathsf{Mon}\ \langle [\![\mathsf{E}]\!] \mid \epsilon \rangle\ \lfloor \sigma_1 \rfloor, w : \mathsf{Evv}\ \langle l \mid \epsilon' \rangle \vdash_{\mathsf{F}}$
$\quad$ let $(m, h', w') : \mathsf{Ev}\ \epsilon\ r = w.l$ in $under^l\ \langle l \mid \epsilon' \rangle\ \lfloor \sigma \rfloor\ \epsilon\ r\ m\ w'\ (g\ x)\ w : \mathsf{Ctl}\ \langle l \mid \epsilon' \rangle\ \lfloor \sigma \rfloor$ $under^l$
$\lfloor \Gamma \rfloor \vdash_{\mathsf{F}} \lambda x{:}\mathsf{Mon}\ \langle [\![\mathsf{E}]\!] \mid \epsilon \rangle\ \lfloor \sigma_1 \rfloor.\ \lambda w{:}\mathsf{Evv}\ \langle l \mid \epsilon' \rangle.$
$\quad$ let $(m, h', w') : \mathsf{Ev}\ \epsilon\ r = w.l$
$\quad\quad$ in $under^l\ \langle l \mid \epsilon' \rangle\ \lfloor \sigma \rfloor\ \epsilon\ r\ m\ w'\ (g\ x)\ w : \mathsf{Mon}\ \langle [\![\mathsf{E}]\!] \mid \epsilon \rangle\ \lfloor \sigma_1 \rfloor \to \mathsf{Mon}\ \langle l \mid \epsilon' \rangle\ \lfloor \sigma \rfloor$
$(m, h', w') : \mathsf{Ev}\ \epsilon\ r,\ h',\ w' = w.l$ given
Since under removes labels from the context effect, we consider
$\langle [\![\mathsf{under}^{\epsilon',\epsilon}\ l\ \mathsf{E}]\!] \mid l \mid \epsilon' \rangle$
$= \langle [\![\mathsf{E}]\!] \mid [\![\mathsf{under}^{\epsilon',\epsilon}\ l]\!] \mid l \mid \epsilon' \rangle$
$= \langle [\![\mathsf{E}]\!] \mid \epsilon \rangle$
□

Proof. (*Proof for Theorem 6*) Applying Lemma 27 Part 1 with $\Gamma = \varnothing$ and $\epsilon = \langle\rangle$. □

Lemma 28. . If $\Gamma \vdash_{\mathsf{ec}} \mathsf{F} : \sigma_1 \to \sigma_2 \mid \epsilon \rightsquigarrow g$, then $g = g_1 \star (g_2 \star (\ .. \star g_n))$, where $g_n = id$.

Proof. (*Of Lemma 28*) By straightforward induction on the translation. Note the g_n comes from the translation of □, which is always *id*. □.

F.6.2 Simulation.

Lemma 29. (*Simulation ($\longrightarrow$)*). 1. If $\varnothing \vdash e_1 : \sigma \mid \epsilon \rightsquigarrow e_1'$ and $\varnothing \vdash e_2 : \sigma \mid \epsilon \rightsquigarrow e_2'$, and $e_1 \longrightarrow e_2$ in internal-safe System F^{pb}, and $w' : \mathsf{Evv}\ \epsilon$ then $e_1'\ w' \longmapsto^* e_2'\ w'$;
2. If $\varnothing \vdash e_1 : \sigma \mid \epsilon \rightsquigarrow e_1'$ and $\varnothing \vdash e_2 : \sigma \mid \epsilon \rightsquigarrow e_2'$, and $w \vdash e_1 \longrightarrow e_2$ in internal-safe System F^{pb}, where w elaborates to $w' : \mathsf{Evv}\ \epsilon$, then $e_1'\ \lfloor w \rfloor \longmapsto^* e_2'\ \lfloor w \rfloor$;

Proof. (*Of Theorem 29*) To prove simulation, we apply many kinds of functional laws and equivalence throughout the proof.

Also, to make the proof concise, we sometimes omit the type arguments to regular functions.

Part 1. Induction on the operational rules.
case (*app*) $(\lambda^{\epsilon}\ x{:}\sigma_1.\ e)\ v \longrightarrow e[x{:=}v]$.
$\varnothing \vdash (\lambda^{\epsilon}\ x{:}\sigma_1.\ e)\ v : \sigma \mid \epsilon \rightsquigarrow (pure\ \epsilon\ \lfloor \sigma_1 \to \epsilon\ \sigma_2 \rfloor\ (\lambda^{\epsilon}\ x{:}\sigma_1.\ e')) \rhd (\lambda f : \lfloor \sigma_1 \to \epsilon\ \sigma \rfloor.\ pure\ \epsilon\ \lfloor \sigma_1 \rfloor\ v' \rhd f)$
$((pure\ \epsilon\ \lfloor \sigma_1 \to \epsilon\ \sigma_2 \rfloor\ (\lambda^{\epsilon}\ x{:}\sigma_1.\ e')) \rhd (\lambda f : \lfloor \sigma_1 \to \epsilon\ \sigma \rfloor.\ pure\ \epsilon\ \lfloor \sigma_1 \rfloor\ v' \rhd f))\ w'$
$\longmapsto^* (\lambda w.\ (\lambda f : \lfloor \sigma_1 \to \epsilon\ \sigma \rfloor.\ pure\ \epsilon\ \lfloor \sigma_1 \rfloor\ v' \rhd f)\ (\lambda^{\epsilon}\ x{:}\sigma_1.\ e')\ w)\ w'$
$\longmapsto^* (\lambda f : \lfloor \sigma_1 \to \epsilon\ \sigma \rfloor.\ pure\ \epsilon\ \lfloor \sigma_1 \rfloor\ v' \rhd f)\ (\lambda^{\epsilon}\ x{:}\sigma_1.\ e')\ w'$
$\longmapsto^* (pure\ \epsilon\ \lfloor \sigma_1 \rfloor\ v' \rhd (\lambda^{\epsilon}\ x{:}\sigma_1.\ e'))\ w'$
$\longmapsto^* (\lambda w.\ (\lambda^{\epsilon}\ x{:}\sigma_1.\ e')\ v'\ w)\ w'$
$\longmapsto^* (\lambda^{\epsilon}\ x{:}\sigma_1.\ e')\ v'\ w'$
$\longmapsto^* e'[x{:=}v']\ w'$
case (*tapp*) $(\Lambda\alpha^{\kappa}.\ v)\ [\sigma] \longrightarrow v[\alpha{:=}\sigma]$.

$\varnothing \vdash (\Lambda\alpha^{\kappa}.\ v)\ [\sigma] : \sigma_1[\alpha{:=}\sigma] \mid \epsilon$
$\quad \rightsquigarrow (\mathit{pure}\ \epsilon\ \lfloor\forall\alpha^{\kappa}.\ \sigma_1\rfloor\ (\Lambda\alpha^{\kappa}.\ v')) \rhd (\lambda x{:}\lfloor\forall\alpha^{\kappa}.\ \sigma_1\rfloor.\ \mathit{pure}\ \epsilon\ \lfloor\sigma_1[\alpha{:=}\sigma]\rfloor\ (x\ \lfloor\sigma\rfloor))$
$((\mathit{pure}\ \epsilon\ \lfloor\forall\alpha^{\kappa}.\ \sigma_1\rfloor\ (\Lambda\alpha^{\kappa}.\ v')) \rhd (\lambda x{:}\lfloor\forall\alpha^{\kappa}.\ \sigma_1\rfloor.\ \mathit{pure}\ \epsilon\ \lfloor\sigma_1[\alpha{:=}\sigma]\rfloor\ (x\ \lfloor\sigma\rfloor)))\ w'$
$\longmapsto^* (\lambda w.\ (\lambda x{:}\lfloor\forall\alpha^{\kappa}.\ \sigma_1\rfloor.\ \mathit{pure}\ \epsilon\ \lfloor\sigma_1[\alpha{:=}\sigma]\rfloor\ (x\ \lfloor\sigma\rfloor))\ (\Lambda\alpha^{\kappa}.\ v')\ w)\ w'$
$\longmapsto^* (\lambda x{:}\lfloor\forall\alpha^{\kappa}.\ \sigma_1\rfloor.\ \mathit{pure}\ \epsilon\ \lfloor\sigma_1[\alpha{:=}\sigma]\rfloor\ (x\ \lfloor\sigma\rfloor))\ (\Lambda\alpha^{\kappa}.\ v')\ w'$
$\longmapsto^* \mathit{pure}\ \epsilon\ \lfloor\sigma_1[\alpha{:=}\sigma]\rfloor\ ((\Lambda\alpha^{\kappa}.\ v')\ \lfloor\sigma\rfloor)\ w'$
$\longmapsto^* \mathit{pure}\ \epsilon\ \lfloor\sigma_1[\alpha{:=}\sigma]\rfloor\ v'[\alpha{:=}\sigma]\ w'$
$\varnothing \vdash v : \sigma_1 \mid \epsilon \rightsquigarrow \mathit{pure}\ \epsilon\ \lfloor\sigma_1\rfloor\ v'$
$\varnothing \vdash v[\alpha{:=}\sigma] : \sigma_1[\alpha{:=}\sigma] \mid \epsilon \rightsquigarrow \mathit{pure}\ \epsilon\ \lfloor\sigma_1[\alpha{:=}\sigma]\rfloor\ (v'[\alpha{:=}\sigma])$ By substitution

case (*handler*) (handler h) $v \longrightarrow$ prompt $m\ h\ (v\ ())$ with m unique.

$\varnothing \vdash$ (handler h) $v : \sigma \mid \epsilon$
$\quad \rightsquigarrow \mathit{pure}\ \epsilon\ \lfloor(() \to \langle l \mid \epsilon\rangle\ \sigma) \to \epsilon\ \sigma\rfloor\ (\mathit{handler}^l\ \epsilon\ \lfloor\sigma\rfloor\ h') \rhd (\lambda f{:}\lfloor() \to \epsilon\ \sigma\rfloor.\ \mathit{pure}\ \langle l \mid \epsilon\rangle\ ()\ v \rhd f)$
$(\mathit{pure}\ \epsilon\ \lfloor(() \to \langle l \mid \epsilon\rangle\ \sigma) \to \epsilon\ \sigma\rfloor\ (\mathit{handler}^l\ \epsilon\ \lfloor\sigma\rfloor\ h') \rhd (\lambda f{:}\lfloor() \to \epsilon\ \sigma\rfloor.\ \mathit{pure}\ \langle l \mid \epsilon\rangle\ ()\ v \rhd f))\ w'$
$\longmapsto^* (\lambda w.\ (\lambda f{:}\lfloor() \to \epsilon\ \sigma\rfloor.\ \mathit{pure}\ \langle l \mid \epsilon\rangle\ ()\ v \rhd f)\ (\mathit{handler}^l\ \epsilon\ \lfloor\sigma\rfloor\ h')\ w)\ w'$
$\longmapsto^* (\lambda f{:}\lfloor() \to \epsilon\ \sigma\rfloor.\ \mathit{pure}\ \langle l \mid \epsilon\rangle\ ()\ v \rhd f)\ (\mathit{handler}^l\ \epsilon\ \lfloor\sigma\rfloor\ h')\ w'$
$\longmapsto^* (\mathit{pure}\ \langle l \mid \epsilon\rangle\ ()\ v \rhd (\mathit{handler}^l\ \epsilon\ \lfloor\sigma\rfloor\ h'))\ w'$
$\longmapsto^* (\lambda w.\ (\mathit{handler}^l\ \epsilon\ \lfloor\sigma\rfloor\ h')\ v\ w)\ w'$
$\longmapsto^* (\mathit{handler}^l\ \epsilon\ \lfloor\sigma\rfloor\ h')\ v\ w'$
$\longmapsto^* \mathit{freshm}\ (\lambda m \to \mathrm{prompt}^l\ \epsilon\ \lfloor\sigma\rfloor\ m\ h'\ (v'\ ()))\ w'$
$\longmapsto^* \mathrm{prompt}^l\ \epsilon\ \lfloor\sigma\rfloor\ m\ h'\ (v'\ ())\ w'$ *m* fr
$\varnothing \vdash \mathrm{prompt}^l\ m\ h\ (v\ ()) : \sigma \mid \epsilon$
$\quad \rightsquigarrow \mathit{prompt}^l\ \epsilon\ \lfloor\sigma\rfloor\ m\ h'\ (\mathit{pure}\ \epsilon\ \lfloor() \to \langle l \mid \epsilon\rangle\ \sigma\rfloor\ v \rhd (\lambda f{:}\lfloor() \to \langle l \mid \epsilon\rangle\ \sigma\rfloor.\ \mathit{pure}\ \epsilon\ \lfloor()\rfloor\ () \rhd f))$
$\mathit{prompt}^l\ \epsilon\ \lfloor\sigma\rfloor\ m\ h'\ (\mathit{pure}\ \epsilon\ \lfloor() \to \langle l \mid \epsilon\rangle\ \sigma\rfloor\ v \rhd (\lambda f{:}\lfloor() \to \langle l \mid \epsilon\rangle\ \sigma\rfloor.\ \mathit{pure}\ \epsilon\ \lfloor()\rfloor\ () \rhd f))\ w'$
$\longmapsto^* \mathit{prompt}^l\ \epsilon\ \lfloor\sigma\rfloor\ m\ h'\ (\lambda w.\ (\lambda f{:}\lfloor() \to \langle l \mid \epsilon\rangle\ \sigma\rfloor.\ \mathit{pure}\ \epsilon\ \lfloor()\rfloor\ () \rhd f)\ v\ w)\ w'$
$\longmapsto^* (\lambda w.\ \mathrm{case}\ (\lambda w.\ (\lambda f{:}\lfloor() \to \langle l \mid \epsilon\rangle\ \sigma\rfloor.\ \mathit{pure}\ \epsilon\ \lfloor()\rfloor\ () \rhd f)\ v\ w)\ \langle\!\langle l{:}(m,\ h', w \mid\ w\rangle\!\rangle\ \mathrm{of}\ \ldots)\ w'$
$\longmapsto^* \mathrm{case}\ (\lambda w.\ (\lambda f{:}\lfloor() \to \langle l \mid \epsilon\rangle\ \sigma\rfloor.\ \mathit{pure}\ \epsilon\ \lfloor()\rfloor\ () \rhd f)\ v\ w)\ \langle\!\langle l{:}m,\ h', w' \mid\ w'\rangle\!\rangle\ \mathrm{of}\ \ldots$
$\longmapsto^* \mathrm{case}\ (\lambda f{:}\lfloor() \to \langle l \mid \epsilon\rangle\ \sigma\rfloor.\ \mathit{pure}\ \epsilon\ \lfloor()\rfloor\ () \rhd f)\ v\ \langle\!\langle l{:}m,\ h', w' \mid\ w'\rangle\!\rangle\ \mathrm{of}\ \ldots$
$\longmapsto^* \mathrm{case}\ (\mathit{pure}\ \epsilon\ \lfloor()\rfloor\ () \rhd\ v)\ \langle\!\langle l{:}m,\ h', w' \mid\ w'\rangle\!\rangle\ \mathrm{of}\ \ldots$
$\longmapsto^* \mathrm{case}\ (\lambda w.\ v\ ()\ w)\ \langle\!\langle l{:}m,\ h', w' \mid w'\rangle\!\rangle\ \mathrm{of}\ \ldots$
$\longmapsto^* \mathrm{case}\ v\ ()\ \langle\!\langle l{:}m,\ h', w' \mid w'\rangle\!\rangle\ \mathrm{of}\ \ldots$
$= \mathrm{prompt}^l\ \epsilon\ \lfloor\sigma\rfloor\ m\ h'\ (v'\ ())\ w'$

case (*promptv*) prompt $m\ h\ v \longrightarrow v$.

$\varnothing \vdash \mathrm{prompt}\ m\ h\ v : \sigma \mid \epsilon \rightsquigarrow \mathit{prompt}^l\ \epsilon\ \lfloor\sigma\rfloor\ m\ h'\ (\mathit{pure}\ \langle l \mid \epsilon\rangle\ \lfloor\sigma\rfloor\ v)$
$\mathit{prompt}^l\ \epsilon\ \lfloor\sigma\rfloor\ m\ h'\ (\mathit{pure}\ \langle l \mid \epsilon\rangle\ \lfloor\sigma\rfloor\ v)\ w'$
$\longmapsto^* \mathrm{Pure}\ \langle l \mid \epsilon\rangle\ \lfloor\sigma\rfloor\ v$
$= \mathit{pure}\ \langle l \mid \epsilon\rangle\ \lfloor\sigma\rfloor\ v\ w'$

case (prompt_1) prompt $m\ h$ (yield $m\ f\ k$) $\longrightarrow f\ (\lambda^{\epsilon}\ x{:}\ \sigma_2.\ \mathrm{prompt}\ m\ h\ (k\ x))$
with $\varnothing \vdash_{\mathrm{val}} k : \sigma_2 \to \langle l \mid \epsilon\rangle\ \sigma$.

$\varnothing \vdash \text{prompt}\ m\ h\ (\text{yield}\ m\ f) \rightsquigarrow \mathit{prompt}^l\ m\ h'\ (\mathit{yield}\ \epsilon'\ m\ f'\ k')$
$\mathit{prompt}^l\ m\ h'\ (\mathit{yield}\ m\ f'\ k')\ w'$
$\longmapsto^* f'\ (\mathit{prompt}^l\ m\ h' \circ k')\ w'$
$= f'\ (\lambda x.\ \mathit{prompt}^l\ m\ h'\ (k'\ x))\ w'$
$\varnothing \vdash f\ (\lambda^\epsilon\ x\colon \sigma_2.\ \text{prompt}\ m\ h\ (k'\ x))$
$\quad \rightsquigarrow (\mathit{pure}\ f') \rhd (\lambda g.\ \mathit{pure}\ (\lambda x.\ \mathit{prompt}^l\ m\ h'\ (\mathit{pure}\ k' \rhd (\lambda h.\ \mathit{pure}\ x \rhd h))) \rhd g)$
$\longmapsto^* ((\mathit{pure}\ f') \rhd (\lambda g.\ \mathit{pure}\ (\lambda x.\ \mathit{prompt}^l\ m\ h'\ (\mathit{pure}\ k' \rhd (\lambda h.\ \mathit{pure}\ x \rhd h))) \rhd g))\ w'$
$\longmapsto^* (\lambda w.\ (\lambda g.\ \mathit{pure}\ ((\lambda x.\ \mathit{prompt}^l\ m\ h'\ (\mathit{pure}\ k' \rhd (\lambda h.\ \mathit{pure}\ x \rhd h))) \rhd g)\ f)\ w)\ w'$
$\longmapsto^* (\lambda g.\ \mathit{pure}\ ((\lambda x.\ \mathit{prompt}^l\ m\ h'\ (\mathit{pure}\ k' \rhd (\lambda h.\ \mathit{pure}\ x \rhd h))) \rhd g)\ f)\ w'$
$\longmapsto^* (\mathit{pure}\ (\lambda x.\ \mathit{prompt}^l\ m\ h'\ (\mathit{pure}\ k' \rhd (\lambda h.\ \mathit{pure}\ x \rhd h))) \rhd f')\ w'$
$\longmapsto^* (\lambda w.\ f'\ (\lambda x.\ \mathit{prompt}^l\ m\ h'\ (\mathit{pure}\ k' \rhd (\lambda h.\ \mathit{pure}\ x \rhd h)))\ w)\ w'$
$\longmapsto^* f'\ (\lambda x.\ \mathit{prompt}^l\ m\ h'\ (\mathit{pure}\ k' \rhd (\lambda h.\ \mathit{pure}\ x \rhd h)))\ w'$
$\longmapsto^* f'\ (\lambda x.\ \mathit{prompt}^l\ m\ h'\ (\lambda w.\ (\lambda h.\ \mathit{pure}\ x \rhd h)\ k'\ w))\ w'$
$= f'\ (\lambda x.\ \mathit{prompt}^l\ m\ h'\ ((\lambda h.\ \mathit{pure}\ x \rhd h)\ k'))\ w'$
$\longmapsto^* f'\ (\lambda x.\ \mathit{prompt}^l\ m\ h'\ (\mathit{pure}\ x \rhd k')\ w'$
$\longmapsto^* f'\ (\lambda x.\ \mathit{prompt}^l\ m\ h'\ (\lambda w.\ k'\ x\ w)\ w'$
$= f'\ (\lambda x.\ \mathit{prompt}^l\ m\ h'\ (k'\ x)\ w'$

case (prompt_2) $\text{prompt}\ n\ h\ (\text{yield}\ m\ f\ k) \longrightarrow \text{yield}\ m\ f\ (\lambda^\epsilon x : \sigma_2.\ \text{prompt}\ n\ h\ (k\ x))$ iff $n \neq m$

$\varnothing \vdash \text{prompt}\ n\ h\ (\text{yield}\ m\ f) \rightsquigarrow \mathit{prompt}^l\ n\ h'\ (\mathit{yield}\ \epsilon'\ m\ f'\ k')$
$\mathit{prompt}^l\ n\ h'\ (\mathit{yield}\ m\ f'\ k')\ w'$
$\longmapsto^* \text{Yield}\ m\ f'\ (\mathit{prompt}\ n\ h' \circ k')$
$= \text{Yield}\ m\ f'\ (\lambda x.\ \mathit{prompt}\ n\ h'\ (k'\ x))$
$\varnothing \vdash \text{yield}\ m\ f\ (\lambda^\epsilon x : \sigma_2.\ \text{prompt}\ n\ h\ (k\ x)) \rightsquigarrow \mathit{yield}\ m\ f'\ (\lambda x.\ \text{prompt}\ n\ h'\ (k'\ x))$
$\mathit{yield}\ m\ f'\ (\lambda x.\ \text{prompt}\ n\ h'\ (k'\ x)\ w'$
$= \mathit{Yield}\ m\ f'\ (\lambda x.\ \text{prompt}\ n\ h'\ (k'\ x)$

case (app_1) $v\ (\text{yield}\ m\ f\ k) \longrightarrow \text{yield}\ m\ f\ (\lambda^\epsilon x : \sigma_2.\ v\ (k\ x))$

$\varnothing \vdash v\ (\text{yield}\ m\ f\ k) \rightsquigarrow \mathit{pure}\ v' \rhd (\lambda f.\ (\mathit{yield}\ m\ f'\ k') \rhd f)$
$(\mathit{pure}\ v' \rhd (\lambda f.\ (\mathit{yield}\ m\ f'\ k') \rhd f))\ w'$
$\longmapsto^* (\lambda w.\ (\lambda f.\ (\mathit{yield}\ m\ f'\ k') \rhd f)\ v'\ w)\ w'$
$\longmapsto^* (\lambda f.\ (\mathit{yield}\ m\ f'\ k') \rhd f)\ v'\ w'$
$\longmapsto^* ((\mathit{yield}\ m\ f'\ k') \rhd v')\ w'$
$\longmapsto^* (\lambda w.\ \text{Yield}\ m\ f'\ (v \star k'))\ w'$
$\longmapsto^* \text{Yield}\ m\ f'\ (v' \star k')$
$= \text{Yield}\ m\ f'\ (\lambda x.\ k'\ x \rhd v')$
$\varnothing \vdash \text{yield}\ m\ f\ (\lambda^\epsilon x : \sigma_2.\ v\ (k\ x))$
$\quad \rightsquigarrow \mathit{yield}\ m\ f'\ (\lambda x.\ \mathit{pure}\ v' \rhd (\lambda f.\ (\mathit{pure}\ k' \rhd (\lambda g.\ \mathit{pure}\ x \rhd g)) \rhd f))$
$\mathit{yield}\ m\ f'\ (\lambda x.\ \mathit{pure}\ v' \rhd (\lambda f.\ (\mathit{pure}\ k' \rhd (\lambda g.\ \mathit{pure}\ x \rhd g)) \rhd f))\ w'$
$\longmapsto^* \text{Yield}\ m\ f'\ (\lambda x.\ (\mathit{pure}\ k' \rhd (\lambda g.\ \mathit{pure}\ x \rhd g)) \rhd v')$
$\longmapsto^* \text{Yield}\ m\ f'\ (\lambda x.\ (\mathit{pure}\ x \rhd k') \rhd v')$
$\longmapsto^* \text{Yield}\ m\ f'\ (\lambda x.\ k'\ x \rhd v')$

case (app_2) $\text{yield}\ m\ f\ k\ e \longrightarrow \text{yield}\ m\ f\ (\lambda^\epsilon x : \sigma_2.\ (k\ x)\ e) \mid \varnothing \vdash_{\text{val}} k : \sigma_2 \to \epsilon\ \sigma$

$\varnothing \vdash (\text{yield } m\, f\, k)\ e \rightsquigarrow (\mathit{yield}\ m\, f'\, k') \rhd (\lambda f.\ e' \rhd f'))$
$((\mathit{yield}\ m\, f'\, k') \rhd (\lambda f.\ e' \rhd f'))\ w'$
$\longmapsto^* (\lambda w.\ (\text{Yield } m\, f'\, ((\lambda f.\ e' \rhd f') \star k)))\ w'$
$\longmapsto^* \text{Yield } m\, f'\, ((\lambda f.\ e' \rhd f') \star k')$
$= \text{Yield } m\, f'\, (\lambda x.\ k'\ x \rhd (\lambda f.\ e' \rhd f'))$
$\varnothing \vdash \text{yield } m\, f\, (\lambda^{\epsilon} x : \sigma_2.\ (k\ x)\ e)$
$\rightsquigarrow \mathit{yield}\ m\, f'\, (\lambda x.\ (\mathit{pure}\ k' \rhd (\lambda g.\ \mathit{pure}\ x \rhd g)) \rhd (\lambda f.\ e' \rhd f)))$
$\mathit{yield}\ m\, f'\, (\lambda x.\ (\mathit{pure}\ k' \rhd (\lambda g.\ \mathit{pure}\ x \rhd g)) \rhd (\lambda f.\ e' \rhd f)))\ w'$
$\longmapsto^* \text{Yield } m\, f'\, (\lambda x.\ (\mathit{pure}\ x \rhd k') \rhd (\lambda f.\ e' \rhd f)))$
$\longmapsto^* \text{Yield } m\, f'\, (\lambda x.\ k'\ x \rhd (\lambda f.\ e' \rhd f))$

case (*under*) $\text{under}^{\epsilon_0,\epsilon}\ l\ (\text{yield } n\, f\, k) \longrightarrow \text{yield } n\, f\, (\lambda^{\epsilon_0} x : \sigma_2.\ \text{under}^{\epsilon_0,\epsilon}\ l\ (k\ x))$

$\varnothing \vdash \text{under } l\ (\text{yield } n\, f\, k) \rightsquigarrow \lambda w.\ \text{let } (m, \_, w') = w.l \text{ in } \mathit{under}^l\ m\ w'\ (\text{yield } n\, f'\, k')\ w$
$(\lambda w.\ \text{let } (m, \_, w_1) = w.l \text{ in } \mathit{under}^l\ m\ w_1\ (\text{yield } n\, f'\, k')\ w)\ w'$
$\longmapsto^* \text{let } (m, \_, w_1) = w'.l \text{ in } \mathit{under}^l\ m\ w_1\ (\text{yield } n\, f'\, k')\ w'$
$\longmapsto^* \text{let } (m, \_, w_1) = w'.l \text{ in } (\text{Yield } n\, f'\, (\mathit{underk}^l\ m\ k'))$
$= \text{let } (m, \_, w_1) = w'.l \text{ in Yield } n\, f'\, (\lambda x.\ \lambda w_2.\ \mathit{underk}^l\ m\ k'\ x\ w_2)$
$= \text{let } (m, \_, w_1) = w'.l \text{ in}$
$\quad \text{Yield } n\, f'\, (\lambda x.\ \lambda w_2.\ \text{let } (m_2, \_, w_2') = w_2.l \text{ in } \mathit{if}\ (m = m_2)\ \mathit{then}\ \mathit{under}^l\ m\ w_2'\ (k'\ x)\ w_2)$
$\varnothing \vdash \text{yield } n\, f\, (\lambda^{\epsilon} x : \sigma_2.\ \text{under } l\ (k\ x))$
$\rightsquigarrow \text{yield } n\, f'\, (\lambda x.\ \lambda w_2.\ \text{let } (m, \_, w_2') = w_2.l \text{ in } \mathit{under}\ m\ w_2'\ (\mathit{pure}\ k' \rhd (\lambda f.\ \mathit{pure}\ x \rhd f))\ w_2)$
$\text{yield } n\, f'\, (\lambda x.\ \lambda w_2.\ \text{let } (m, \_, w_2') = w_2.l \text{ in } \mathit{under}\ m\ w_2'\ (\mathit{pure}\ k' \rhd (\lambda f.\ \mathit{pure}\ x \rhd f))\ w_2)\ w'$
$\longmapsto^* \text{Yield } n\, f'\, (\lambda x.\ \lambda w_2.\ \text{let } (m, \_, w_2') = w_2.l \text{ in } \mathit{under}\ m\ w_2'\ (k'\ x)\ w_2)$
$= \text{Yield } n\, f'\, (\lambda x.\ \lambda w_2.\ \text{let } (m, \_, w_2') = w_2.l \text{ in } \mathit{under}\ m\ w_2'\ (k'\ x)\ w_2)$

For internal-safe expressions, we know that $m = m_2$ is always true, so the expressions are equivalent.

Part 2. **case** (*perform*) $w \vdash \text{perform } \mathit{op}\ \epsilon_0\ \overline{\sigma}\ v \longrightarrow \text{yield } m\ (\lambda^{\epsilon} k : \sigma_k.\ f\ \overline{\sigma}\ v\ k)\ (\lambda^{\langle l | \epsilon_0 \rangle} x : \sigma_2[\overline{\alpha} := \overline{\sigma}].\ x)$
with $(m, h, \_) = w.l, (\mathit{op} : (\forall \overline{\alpha}.\ \sigma_1 \to \sigma_2) \to f) \in h : \sigma \mid l \mid \epsilon, \sigma_k = \sigma_2[\overline{\alpha} := \overline{\sigma}] \to \epsilon\ \sigma.$

$\varnothing \vdash \text{perform } \mathit{op}\ \epsilon_0\ \overline{\sigma}\ v : \sigma_2[\overline{\alpha} := \overline{\sigma}] \mid \langle l \mid \epsilon_0 \rangle$
$\rightsquigarrow (\mathit{pure}\ (\mathit{perform}^l\ \epsilon\ \lfloor \sigma_1[\overline{\alpha} := \overline{\sigma}] \rfloor\ \lfloor \sigma_2[\overline{\alpha} := \overline{\sigma}] \rfloor\ (\Lambda \mu\ r.\ \mathit{select}^{\mathit{op}}\ \lfloor \overline{\sigma} \rfloor\ \mu\ r))) \rhd (\lambda f.\ \mathit{pure}\ v' \rhd f)$
$(\mathit{pure}\ (\mathit{perform}^l\ \epsilon\ \lfloor \sigma_1[\overline{\alpha} := \overline{\sigma}] \rfloor\ \lfloor \sigma_2[\overline{\alpha} := \overline{\sigma}] \rfloor\ (\Lambda \mu\ r.\ \mathit{select}^{\mathit{op}}\ \lfloor \overline{\sigma} \rfloor\ \mu\ r))) \rhd (\lambda f.\ \mathit{pure}\ v' \rhd f)\ w'$
$\longmapsto^* \mathit{perform}^l\ \epsilon\ \lfloor \sigma_1[\overline{\alpha} := \overline{\sigma}] \rfloor\ \lfloor \sigma_2[\overline{\alpha} := \overline{\sigma}] \rfloor\ (\Lambda \mu\ r.\ \mathit{select}^{\mathit{op}}\ \lfloor \overline{\sigma} \rfloor\ \mu\ r)\ v'\ w'$
$\longmapsto^* \mathit{perform}^l\ \epsilon\ \lfloor \sigma_1[\overline{\alpha} := \overline{\sigma}] \rfloor\ \lfloor \sigma_2[\overline{\alpha} := \overline{\sigma}] \rfloor\ (\Lambda \mu\ r.\ \mathit{select}^{\mathit{op}}\ \lfloor \overline{\sigma} \rfloor\ \mu\ r)\ v'\ w'$
$\longmapsto^* \text{Yield } m\ (\lambda k.\ f'\ v' \rhd (\lambda g.\ g\ k))\ \mathit{pure}$
$\varnothing \vdash \text{yield } m\ (\lambda^{\epsilon} k : \sigma_k.\ f\ \overline{\sigma}\ v\ k)\ (\lambda^{\langle l | \epsilon_0 \rangle} x : \sigma_2[\overline{\alpha} := \overline{\sigma}].\ x) : \sigma_2[\overline{\alpha} := \overline{\sigma}] \mid \langle l \mid \epsilon_0 \rangle$
$\rightsquigarrow \mathit{yield}\ m\ (\lambda^{\epsilon} k : \sigma_k.\ (\mathit{pure}\ f')\ \overline{\sigma} \rhd (\lambda g_1.\ \mathit{pure}\ v' \rhd g_1) \rhd (\lambda g.\ \mathit{pure}\ k \rhd g))\ (\lambda^{\langle l\ | \epsilon_0 \rangle} x : \sigma_2[\overline{\alpha} := \overline{\sigma}].\ \mathit{pure}\ x)$
$\mathit{yield}\ m\ (\lambda k.\ (\mathit{pure}\ f')\ \overline{\sigma} \rhd (\lambda g_1.\ \mathit{pure}\ v' \rhd g_1) \rhd (\lambda g.\ \mathit{pure}\ k \rhd g))\ (\lambda x.\ \mathit{pure}\ x)\ w'$
$= \mathit{yield}\ m\ (\lambda k.\ (\mathit{pure}\ f')\ \overline{\sigma} \rhd (\lambda g_1.\ \mathit{pure}\ v' \rhd g_1) \rhd (\lambda g.\ \mathit{pure}\ k \rhd g))\ \mathit{pure}\ w'$
$\longmapsto^* \mathit{yield}\ m\ (\lambda k.\ f'\ v' \rhd (\lambda g.\ \mathit{pure}\ k \rhd g))\ \mathit{pure}\ w'$
$\longmapsto^* \mathit{yield}\ m\ (\lambda k.\ f'\ v' \rhd (\lambda g.\ g\ k))\ \mathit{pure}\ w'$
$\longmapsto^* \text{Yield } m\ (\lambda k.\ f'\ v' \rhd (\lambda g.\ g\ k))\ \mathit{pure}$

□

Lemma 30. (*Simulation* ($\longmapsto$)). If $\varnothing \vdash e_1 : \sigma \mid \epsilon \rightsquigarrow e_1'$ and $0 \vdash e_2 : \sigma \mid \epsilon \rightsquigarrow e_2'$, and $w : \text{evv } \epsilon$, and $w \vdash e_1 \longmapsto e_2$ in internal-safe System F^{pb}, then $e_1' \lfloor w \rfloor \longmapsto^* e_2' \lfloor w \rfloor$;

Proof. **case**

$$\frac{e_1 \longrightarrow e_2}{w \vdash \mathsf{F}[e_1] \longmapsto \mathsf{F}[e_2]}\,(step)$$

$\varnothing \vdash \mathsf{F}[e_1] : \sigma \mid \epsilon \rightsquigarrow g\ e_1'$ given
$\varnothing \vdash \mathsf{F} : \sigma_1 \to \sigma \mid \epsilon \rightsquigarrow g$
$\varnothing \vdash e_1 : \sigma_1 \mid \epsilon \rightsquigarrow e_1'$
$g = g_1 \star (g_2 \star (\ldots \star g_n))$ Lemma 28
$g_n = id$ above
$e_1'\ w' \longrightarrow e_2'\ w'$ Lemma 30
$g\ e_1'$
$= (g_1 \star (g_2 \star (\ldots \star g_n)))\ e_1'$
$= (g_2 \star (\ldots \star g_n))\ e_1' \rhd g_1$
$= ((g_n\ e_1') \rhd \ldots) \rhd g_2) \rhd g_1$
$g\ e_1'\ w'$
$= (((g_n\ e_1') \rhd \ldots) \rhd g_2) \rhd g_1)\ w'$
$= \mathsf{case}\ (((g_n\ e_1') \rhd \ldots) \rhd g_2)\ w'\ \mathsf{of}\ \mathsf{Pure}\ x \to g_1\ x\ w';\ \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g_1 \star k)$
$= \mathsf{case}$
$\quad (\mathsf{case}\ (((g_n\ e_1') \rhd \ldots))\ w'\ \mathsf{of}\ \mathsf{Pure}\ x \to g_2\ x\ w';\ \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g_2 \star k))$
$\quad \mathsf{of}\ \mathsf{Pure}\ x \to g_1\ x\ w';\ \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g_1 \star k)$
$= \ldots$
$= \mathsf{case}$
$\quad \mathsf{case}$
$\quad\quad \ldots$
$\quad\quad (\mathsf{case}\ g_n\ e_1'\ w')\ \mathsf{of}\ \mathsf{Pure}\ x \to g_{n-1}\ x\ w';\ \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g_{n-1} \star k))$
$\quad\quad \ldots$
$\quad \mathsf{of}\ \mathsf{Pure}\ x \to g_2\ x\ w';\ \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g_2 \star k))$
$\mathsf{of}\ \mathsf{Pure}\ x \to g_1\ x\ w';\ \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g_1 \star k)$
$= \mathsf{case}$ $g_n = id$
$\quad \mathsf{case}$
$\quad\quad \ldots$
$\quad\quad \mathsf{case}\ (id\ e_1'\ w')\ \mathsf{of}\ \mathsf{Pure}\ x \to g_{n-1}\ x\ w';\ \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g_{n-1} \star k))$
$\quad\quad \ldots$
$\quad \mathsf{of}\ \mathsf{Pure}\ x \to g_2\ x\ w';\ \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g_2 \star k))$
$\mathsf{of}\ \mathsf{Pure}\ x \to g_1\ x\ w';\ \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g_1 \star k)$
$\longmapsto^* \mathsf{case}$
$\quad \mathsf{case}$
$\quad\quad \ldots$
$\quad\quad \mathsf{case}\ (e_1'\ w')\ \mathsf{of}\ \mathsf{Pure}\ x \to g_{n-1}\ x\ w';\ \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g_{n-1} \star k))$
$\quad\quad \ldots$
$\quad \mathsf{of}\ \mathsf{Pure}\ x \to g_2\ x\ w';\ \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g_2 \star k))$
$\mathsf{of}\ \mathsf{Pure}\ x \to g_1\ x\ w';\ \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g_1 \star k)$
$\longmapsto^* \mathsf{case}$
$\quad \mathsf{case}$
$\quad\quad \ldots$
$\quad\quad \mathsf{case}\ (e_2'\ w')\ \mathsf{of}\ \mathsf{Pure}\ x \to g_{n-1}\ x\ w';\ \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g_{n-1} \star k))$
$\quad\quad \ldots$
$\quad \mathsf{of}\ \mathsf{Pure}\ x \to g_2\ x\ w';\ \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g_2 \star k))$
$\mathsf{of}\ \mathsf{Pure}\ x \to g_1\ x\ w';\ \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g_1 \star k)$
$= \mathsf{case}$ $g_n = id$
$\quad \mathsf{case}$
$\quad\quad \ldots$
$\quad\quad (g_n\ e_2'\ w')\ \mathsf{of}\ \mathsf{Pure}\ x \to g_{n-1}\ x\ w';\ \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g_{n-1} \star k))$
$\quad\quad \ldots$
$\quad \mathsf{of}\ \mathsf{Pure}\ x \to g_2\ x\ w';\ \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g_2 \star k))$
$\mathsf{of}\ \mathsf{Pure}\ x \to g_1\ x\ w';\ \mathsf{Yield}\ m\ f\ k \to \mathsf{Yield}\ m\ f\ (g_1 \star k)$

case

$$\frac{w \vdash e \longrightarrow e'}{w \vdash \mathsf{F}[e] \longmapsto \mathsf{F}[e']}(stepw)$$

The same as the previous case with part 2 of Lemma 29.

case

$$\frac{\langle\!\langle l : (m, h, w) \mid w \rangle\!\rangle \vdash e_1 \longmapsto e_2}{w \vdash \mathsf{F}[\mathsf{prompt}\ m\ h\ e_1] \longmapsto \mathsf{F}[\mathsf{prompt}\ m\ h\ e_2]}(promptw)$$

Similar as previous cases, except that the final expression in the case of the last ($\triangleright$) is $id\ (prompt\ m\ h'\ e')\ w'$.

$e_1' \langle\!\langle l : (m, h', w') \mid w\ of\ \dots \rangle\!\rangle \longmapsto^* e_2' \langle\!\langle l : (m, h', w') \mid w\ of\ \dots \rangle\!\rangle$ Lemma 29
$id\ (prompt\ m\ h'\ e_1')\ w'$
$= (prompt\ m\ h'\ e_1')\ w'$
$= \mathsf{case}\ (e_1' \langle\!\langle l : (m, h', w') \mid w)\ of\ \dots \rangle\!\rangle$
$\longmapsto^* \mathsf{case}\ (e_2' \langle\!\langle l : (m, h', w') \mid w)\ of\ \dots \rangle\!\rangle$
$= (prompt\ m\ h'\ e_2')\ w'$

case

$$\frac{w' \vdash e \longmapsto e' \quad (m, h, w') = w.l}{w \vdash \mathsf{F}[\mathsf{under}^{\epsilon_0,\epsilon}\ l\ e] \longmapsto \mathsf{F}[\mathsf{under}^{\epsilon_0,\epsilon}\ l\ e']}(underw)$$

Similar as previous cases, except that the final expression in the case of the last ($\triangleright$) is $id\ (\lambda w.\ \mathsf{let}\ (m,\ \_,\ w_1) = w$

$e_1'\ w' \longmapsto^* e_2'\ w'$ Lemma 29
$id\ (\lambda w.\ \mathsf{let}\ (m,\ \_,\ w_1) = w.l\ \mathsf{in}\ under^l\ m\ w_1\ e_1'\ w)\ w'$
$= (\lambda w.\ \mathsf{let}\ (m,\ \_,\ w_1) = w.l\ \mathsf{in}\ under^l\ m\ w_1\ e_1'\ w)\ w'$
$= \mathsf{let}\ (m,\ \_,\ w_1) = w'.l\ \mathsf{in}\ under^l\ m\ w_1\ e_1'\ w'$
$= \mathsf{let}\ (m,\ \_,\ w_1) = w'.l\ \mathsf{in}\ \mathsf{case}\ e_1'\ w'\ \mathsf{of}\ \dots$
$\longmapsto \mathsf{let}\ (m,\ \_,\ w_1) = w'.l\ \mathsf{in}\ \mathsf{case}\ e_2'\ w'\ \mathsf{of}\ \dots$
$= id\ (\lambda w.\ \mathsf{let}\ (m,\ \_,\ w_1) = w.l\ \mathsf{in}\ under^l\ m\ w_1\ e_2'\ w)\ w'$

Theorem 19. (*Simulation*). If $\varnothing \vdash e_1 : \sigma \mid \langle\rangle \rightsquigarrow e_1'$ and $0 \vdash e_2 : \sigma \mid \langle\rangle \rightsquigarrow e_2'$, and $\langle\!\langle\rangle\!\rangle \vdash e_1 \longmapsto e_2$ in internal-safe System F^{pb}, then $e_1' \langle\!\langle\rangle\!\rangle \longmapsto^* e_2' \langle\!\langle\rangle\!\rangle$;

Proof. (*Of Theorem 19*) Apply Lemma 30 with $\epsilon = \langle\rangle$ and $w = \langle\!\langle\rangle\!\rangle$. □

F.7 Semantics Preserving

Proof. (*Of Theorem 7*) We have established the simulation theorems for all refinements: Theorem 14, Theorem 15, Theorem 18 and Theorem 19.

If $e \longmapsto^* n$ in System F^{ϵ}, we know that from Theorem 14, there exists e_2, such that $\lceil e \rceil^{\epsilon \Downarrow p} \longmapsto^* e_2$ and $\lceil e_2 \rceil^{\epsilon \Uparrow p} = n$. Since e is user-provided which contains no internal frames, so $\lceil e \rceil^{\epsilon \Downarrow p} = e$. Also according to the definition of $\lceil \rceil^{\epsilon \Uparrow p}$, we know $e_2 = n$. So we have $e \longmapsto^* n$ in System F^{p}.

Then by Theorem 15, we know that $e \longmapsto^* n$ in System F^{pw}.

Theorem 18 does not directly lead to $e \longmapsto^* n$ in System F^{pb}, because of the $=_\eta$ relation. However, since the relation only occurs when we build up the resumption, it is easy to show that evaluation preserves the $=_\eta$ relation, as if the resumption is not used that the equivalence is preserved, and when the resumption is used we immediately have $(\lambda x.\ \mathsf{E}[x])\ v \longrightarrow \mathsf{E}[v]$.

Then we get $e \longmapsto^* e_3$ in System F^{pb}, where $n =_\eta e_3$. By progress, we can further evaluate e_3 to n, i.e., $e \longmapsto^* n$. Note e_3 cannot loop as we can only add a finite number of the $=_\eta$ sequence during the evaluation of $e \longmapsto^* e_3$.

Finally, given $\varnothing \vdash e : \mathit{int} \mid \langle\rangle \rightsquigarrow e'$, and $\varnothing \vdash n : \mathit{int} \mid \langle\rangle \rightsquigarrow \mathit{pure}\ \langle\rangle\ \mathit{int}\ n$, by Theorem 19, we know that $e'\ \langle\!\rangle \longrightarrow^* \mathit{pure}\ \langle\rangle\ n\ \langle\!\rangle \longrightarrow^* \mathsf{Pure}\ \langle\rangle\ \mathit{int}\ n$.

On the other hand, if e diverges (i.e., $e \Uparrow$), then following the same reasoning as above, we can show that $e'\ \langle\!\rangle$ diverges.

□