Probabilistic Inference Based Message-Passing For Resource Constrained DCOPs

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Decision Making in Multiagent Systems

Examples

+ Sensor networks, Distributed meeting scheduling, multi-robot coordination

Goal: How do a set of agents decide the best alternative using only local coordination?

Challenges

+ No central control/knowledge
+ Communication overhead
+ Shared resources
Motivating Example

+ Meeting scheduling with budget constraint
  + Two branch each has a limited travel budget
  + Two meetings have to be scheduled, each having two options
+ **Goal:** Schedule meeting such that total utility is maximised

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**Loc-1**

1. Meeting-1: 3rd Aug
   - Utility: $400
   - Cost: $200

2. Meeting-2: 1st Aug
   - Utility: $1000
   - Cost: $250

   - Utility: $500
   - Cost: $250

4. Meeting-1: 2nd Aug
   - Utility: $450
   - Cost: $200

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**Loc-2**

1. Meeting-2: 4th Aug
   - Utility: $500
   - Cost: $250

2. Meeting-1: 2nd Aug
   - Utility: $450
   - Cost: $200

   - Utility: $500
   - Cost: $250

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Budget Summary:

- Loc-1: Budget = $500
- Loc-2: Budget = $600
Resource Constrained DCOPs

+ variables: $X = (x_1, \ldots, x_n)$, finite domains
+ Constraints: $\theta_{ij}(x_i, x_j) : x_i \times x_j \rightarrow \mathbb{R}$
+ Goal: Find joint assignment s.t. $\max_x \sum_{(i,j)} \theta_{ij}(x_i, x_j)$

+ A set $R = \{r_1, \ldots, r_m\}$ of shared resources
+ Utilisation: $u_i(r, x_i) : R \times x_i \rightarrow \mathbb{R}^+$
+ Resource constraint:

$$\forall r \in R : \sum_{i \in V} u_i(r, x_i) \leq C(r)$$
Related Work and Challenges

+ **Exact Algorithm:**
  - [Bowring et al., AAMAS-06]- Extends ADOPT with multiply constraints
  - [Matsui et al., AAAI-08]- Introduces resources as a virtual variable.
  + Limited Scalability

+ **Approximate Algorithm:**
  - No dedicated approximate solver for RC-DCOP
  - For tight resource constraints, approximate DCOP solvers fail to find even one feasible solution
  - Empirically true for max-sum

+ **Probabilistic Inference**
  - [Kumar and Zilberstein, NIPS-10]- MAP estimation in graphical model
  - Extend the above to handle resource constraints
Our Contribution

Formulate QP for continuous relaxation of RC-DCOP

QP formulation \equiv \text{Likelihood Maximization in a Bayes Net Mixture model}

Maximise the likelihood in the mixture of BN using Expectation Maximisation

Develop Message-Passing based on EM
Continuous Relaxation of RC-DCOP

Maximize overall utility

\[
\max_{p=\{p_1, \ldots, p_n\}} \sum_{(i,j) \in E} \sum_{x_i, x_j} p_i(x_i)p_j(x_j)\theta_{ij}(x_i, x_j)
\]

expected contribution by edge (i,j)

s.t.

\[
\sum_{x_i} p_i(x_i) = 1 \quad \forall i \in V
\]

Normalisation Constraint

\[
\sum_{i \in Nb(r)} \sum_{x_i} p_i(x_i)u_i(r, x_i) \leq C(r) \quad \forall r \in R
\]

Resource Constraint

\[
\sum_{i,J \in X} p_i(x_i)\theta_{ij}(x_i, x_j)\tilde{\theta}(x_i, x_j)
\]

expected contribution by edge (i,j)

\[
\sum_{i \in Nb(r)} \sum_{x_i} p_i(x_i)u_i(r, x_i)
\]

consumption by agent i

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**Graphical Representation**

The graphical representation illustrates the relationships between variables and decisions in the continuous relaxation of RC-DCOP. Nodes represent variables or states, and edges denote dependencies or contributions. The graph includes constraints and objectives as described in the mathematical formulation.

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Our Contribution

Formulate QP for continuous relaxation of RC-DCOP

QP formulation \(\equiv\) Likelihood Maximization in a Bayes Net Mixture model

Maximise the likelihood in the mixture of BN using Expectation Maximisation

Develop Message-Passing based on EM
Conversion to mixture of Bayes Net

+ CPT for Bayes Net \( l = 1 \):

\[
P(\hat{\theta} = 1 | x_1, x_2) = \frac{\theta_{12}(x_1, x_2) - \theta_{\min}}{\theta_{\max} - \theta_{\min}}
\]

**Theorem:** Maximising the likelihood \( P(\hat{\theta} = 1; p) \) of observing the reward variable subject to resource constraint is equivalent to solving the QP relaxation of RC-DCOP.

Our Contribution

- Formulate QP for continuous relaxation of RC-DCOP
- QP formulation $\equiv$ Likelihood Maximization in a Bayes Net Mixture model
- Maximise the likelihood in the mixture of BN using Expectation Maximisation
- Develop Message-Passing based on EM
EM for Mixture of BN

- Use EM [Dempster et al., 1977] to maximize the likelihood of $\hat{\theta} = 1$
- Hidden Variables: $x_i, l$
- Observed Variables: $\hat{\theta} = 1$
- Unknown Parameters: $p_i(x_i)$
- Parameters constraints includes the resource, normalisation constraints

Mixture of Bayes Nets
Overview of EM for RC-DCOP

+ EM is an iterative approach consisting of E & M-step
  + E-step computes expectation over hidden variables
  + Implemented using message passing

+ M step maximises expected log-likelihood
  + A convex optimization problem
  + No analytical solution
  + Solved by iteratively optimising the dual of this convex M-step problem (**block coordinate descent strategy**)
  + Implementable using message-passing
Our Contribution

- Formulate QP for continuous relaxation of RC-DCOP
- QP formulation $\equiv$ Likelihood Maximization in a Bayes Net Mixture model
- Maximise the likelihood in the mixture of BN using Expectation Maximisation
- Develop Message-Passing based on EM
E-Step

Message:1 (Agent→Agent): \( \gamma_{i\rightarrow j}(x_j) \leftarrow \sum_{x_i} p_i(x_i) \hat{\theta}_{x_i x_j} \)

Message:2 (Agent→Resource): \( \delta_{i\rightarrow r}(x_i) \leftarrow p_i(x_i) \sum_{k \in Nb(i)} \gamma_{k\rightarrow i}(x_i) \)
M-Step

Message: 1 (Resource \rightarrow Agent): \( \mu_{r \rightarrow i} \leftarrow \max(0, \mu_r) \)

Message: 2 (Agent \rightarrow Resource): \( \nu_{i \rightarrow r}(x_i) \leftarrow \lambda_i + \sum_{r' \in Nr(i) \setminus \hat{r}} \mu_{r' \rightarrow i} u_i(r', x_i) \)

Parameter Update:

\[ p^*_i(x_i) \leftarrow \frac{p_i(x_i) \sum_{k \in Nb(i)} \gamma_k \rightarrow (x_i)}{\lambda_i + \sum_{r \in Nr(i)} \mu_{r \rightarrow i} \cdot u_i(r, x_i)} \]
Experimental Results

+ Experiment on Two benchmarks
+ Random Graphs (30 and 40 node):
  + Edge density is varied from 0.5 to 0.9
  + Random utility $\theta_{ij}$ between 1 to 10
  + Resource capacity is varied from 20%-60% of consumption
+ Graph Colouring:
  + # of Nodes is varied from 20 to 50
  + Use same settings provided in [Farinelli et al., AAMAS-08]
+ Comparison Algorithm:
  + Toulbar2 [Allouche et al., INRA-10]
  + Max-Sum in Frodo implementation [L'eaut'e et al., IJCAI-09]
Experimental Results(2)

- Failure - No resource feasible solution found
- EM has deterministic outcome, single run
- Max-Sum run multiple times due to variable outcome
+ EM outperforms toulbar2 as problem complexity increases.
+ EM solution quality is noticeably better than Max-Sum.
Experimental Results (4)

+ EM provides near-optimal solution for graph colouring problems
+ Toulbar2 finds optimal solution
+ Solution quality of EM is always better than Max-Sum for 30 node problems
EM almost always achieves solution within 3 minutes.

Although Max-Sum takes much lower time, its solution quality is worse.
Conclusion

+ Present a promising class of approximate algorithm for RC-DCOP using probabilistic inference
+ Solving RC-DCOP is equivalent to Likelihood Maximization
+ Use machine learning technique for likelihood maximization
+ Develop EM as message-passing algorithm for RC-DCOP
+ EM has much lower failure rate than Max-Sum, provides good quality solution
Questions???
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Thank you!
Extract integral solution from $p^*$ using rounding technique from [Ravikumar & Laffety, 2006]

If $p^*_i(x_i) \geq \delta$, set $p^\text{int}_i(x_i) = 1$ and $p^\text{int}_i(\hat{x}_i) = 0, \forall \hat{x}_i \setminus x_i$

For each unlabelled node $i \in V$ find $\arg\max_{x_i} \sum_{j \in Nb(i)} \sum_{x_j} \theta_{ij}(x_i, x_j)$

Label node with $x_i$ that satisfy the resource constraints

Iterate the process until convergence

Our Contributions

+ Provide a new class of approximate algorithm for RC-DCOP
+ Mapping of RC-DCOP to that of probabilistic inference in mixture of Bayes Nets
+ Maximises the likelihood (LM) in Bayes Nets that is equivalent to solve the RC-DCOP
+ Interpret LM as message-passing
Likelihood Maximisation in Mixture of BN

\[ P(\hat{\theta}, x_{l_1}, x_{l_2} | l; p) = P(\hat{\theta} | x_{l_1}, x_{l_2}, l) p_{l_1}(x_{l_1}; p)p_{l_2}(x_{l_2}; p) \]

Joint for Bayes net \( l \)

\[ \hat{\theta}_{x_l} \]

Likelihood for Bayes net \( l \): \( L^p_l = P(\hat{\theta} = 1 | l; p) = \sum_{x_l} P(\hat{\theta} = 1, x_{l_1}, x_{l_2} | l; p) = \sum_{x_l} \hat{\theta}_{x_l} p_{l_1}(x_{l_1}; p)p_{l_2}(x_{l_2}; p) \)

Likelihood for complete mixture: \( L^p = \sum_l P(l)L^p_l = \frac{1}{|E|} \sum_l \sum_{x_l} \hat{\theta}_{x_l} p_{l_1}(x_{l_1}; p)p_{l_2}(x_{l_2}; p) \)

\[ \sum_l \sum_{x_l} \theta_l(x_l)p_{l_1}(x_{l_1}; p)p_{l_2}(x_{l_2}; p) = |E|(\theta_{min} + (\theta_{max} - \theta_{min})L^p) \Rightarrow \text{QP formulation} \equiv \text{LM in BN mixture Model} \]
Optimisation problem in M-Step

\[
\max_{\{p_i^*\}} \sum_{i \in V} \sum_{x_i} p_i(x_i) \log p_i^*(x_i) \quad \sum_{j \in \text{Nb}(i)} \sum_{x_j} \hat{\theta}_{x_ix_j} p_j(x_j)
\]

Expected contribution for value \(x_i\)

subject to

\[
\sum_{x_i} p_i^*(x_i) = 1 \quad \forall i \in V
\]

\[
\sum_{i \in \text{Nb}(r)} \sum_{x_i} p_i^*(x_i) u_i(r, x_i) \leq C(r) \quad \forall r \in R
\]
Expectation Maximization

**Theorem 1:** Maximising the following log-likelihood $Q(p, p^*)$ w.r.t. $p^*$ iteratively finds the optimal solution for DCOP. [kumar et al. 2011]

$$Q(p, p^*) \propto \sum_{i \in V} \sum_{x_i} p_i(x_i) \log p_i^*(x_i) \sum_{j \in Nb(i)} \sum_{x_j} \hat{\theta}_{x_ix_j} p_j(x_j)$$

+ Need to satisfy resource & normalisation constraints for RC-DCOP

**Optimising the dual:**
+ Find the Lagrangian $L(p^*, \lambda, \mu)$ by using price variable $\lambda, \mu$
+ Find the dual function:

$$q(\lambda, \mu) = \min_{p^*} L(p^*, \lambda, \mu)$$

+ Parameter update:

$$p_i^*(x_i) = \frac{p_i(x_i) f_i(x_i)}{\lambda_i + \sum_r \mu_r \cdot u_i(r, x_i)}$$

M-Step

**Block coordinate descent (BCD):**
- Choose arbitrary variable $\lambda_i$, fix all other variables and optimise $q(.)$ w.r.t. $\lambda_i$

\[
\sum_{x_i} \frac{p_i(x_i)f_i(x_i)}{\lambda_i + \sum_r \mu_r \cdot u_i(r, x_i)} - 1 = 0
\]

- The largest root is the only feasible solution

- Minimise $q(.)$ w.r.t. $\mu_r$ to find the value of price variable $\mu_r$

- As Solution is uniquely determined, BCD is guaranteed to converge.

\[
g(x) = \sum_{t=1}^{T} \frac{a_t}{x + b_t}
\]
Maximizes the likelihood in mixture of bayes net

\[
\sum_l P(l) \sum_{x_1} P(\theta = 1, x_{l_1}, x_{l_2} | l; p) = \frac{1}{|E|} \sum_l \sum_{x_1} \hat{\theta}_{x_1} p_{l_1}(x_{l_1}; p)p_{l_2}(x_{l_2}; p)
\]

Utility function for graph colouring problem

\[
\theta_{ij}(x_i, x_j) = \begin{cases} 
0 & \text{if } x_i = x_j \\
1 + \gamma_i(x_i) + \gamma_j(x_j) & \text{Otherwise}
\end{cases}
\]