A Simple Characterization for Truth-Revealing Single-Item Auctions

Kamal Jain[1], Aranyak Mehta[2], Kunal Talwar[3], and Vijay Vazirani[2]

[1] Microsoft Research, Redmond, WA
[2] College of Computing, Georgia Institute of Technology, Atlanta, USA
[3] Microsoft Research, Silicon Valley

Abstract. We give a simple characterization of all single-item truth-revealing auctions under some mild (and natural) assumptions about the auctions. Our work opens up the possibility of using variational calculus to design auctions having desired properties.

1 Introduction

The classic work of Vickrey [9] characterizes truth-revealing auctions when the allocation is required to be efficient, i.e., maximizes global welfare. The subsequent work of Clarke [3] and Groves [5] showed that a generalization of Vickrey's mechanism leads to truth-revealing mechanisms for a much wider class of applications. Although this so-called VCG mechanism stands as a pillar of auction theory, it suffers from the drawback that it designed for the implementation of efficient allocations. If the user utilities are unrestricted, then an efficient allocation (weighted VCG) is essentially the only possible implementable solution [8], but in more specific environments the goals of the auctioneer may be very different from efficiency.

In this paper, we wish to characterize single-item auctions in which efficient allocations are not necessarily required. Such situations are quite common in emerging applications of auctions, for instance, budget constrained ad auctions, as carried out by Internet search engine companies. These auctions consist of several micro-auctions, one for each search. Search engine companies want to maximize efficiency/revenue over the entire sequence of auctions. Simply maximizing efficiency/revenue for each micro-auction would entail awarding the ad to the highest bidder, and in the presence of budget constraint of the advertisers, this will not maximize efficiency/revenue for the entire sequence of auctions. On the other hand, the search engine companies would like to make each micro-auction truth-revealing [2, 1]. Such repeated auctions are called myopic truth-revealing.

In order to arrive at a simple characterization, we need to assume that the auction satisfies some natural constraints. Once we make these very reasonable assumptions, our characterization turns out to be simple enough to be practically useful, for example, in repeated ad auctions [6].

Given bids for the single item, an *auction* determines the winner and the price charged from her. Hence, we may assume that the auction is a function from

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bids to a profit vector whose components are zero for losers and zero/positive for the winner. We assume the following conditions on the auction:

1. **Truth-revealing:** Truth-revealing should be a dominant strategy of bidders.
2. **Continuity:** The auction function, as defined above, should be continuous.
3. **Autonomy:** For any bidder i, if all other bidders bid zero and i bids high enough, then i wins.
4. **Independence:** If bidder i is the winner and some other bidder j decreases her bid, then i still wins.

We say that two auction mechanisms are *equivalent* if for identical bid vectors their profit vectors are also identical. We show that any auction mechanism satisfying the above-stated conditions is equivalent to the following simple mechanism: For each bidder i there is a strictly monotonically increasing function $f_i : \mathbf{R}_+ \rightarrow \mathbf{R}$ which we call the *rank function of bidder i*. For a bid vector b, compute the rank of each bid, i.e. $f_i(b_i)$. The winner is a bidder with maximum rank, and she pays the least amount p such that if she had bid p she would still have maximum rank.

One may ask what is the use of assigning different rank functions to different users. For instance, in the repeated ad auctions, different bidders have different budgets and they may have different ad campaigns, i.e., they want to spend their money on different sets of key words. Clearly, in this case, we need to assign different rank functions to different bidders.

Truthful auctions of digital goods have a very simple characterization: for each bidder there is a threshold function, which depends only on the remaining bids, such that this bidder wins iff she bids at least the threshold function (see [4]). Under this characterization it is essential to award multiple items, if several bidders bid strictly more than their threshold amounts. Hence this characterization does not apply to single item auctions. Our characterization leads to a set of threshold functions, one for each bidder, such that the bid of at most one bidder can be strictly bigger than her threshold. Furthermore, in the characterization of digital good auctions, the threshold functions could be arbitrarily complex functions of the bids. However, in our characterization, the threshold functions are simply the maximum of single argument functions.

Characterizations such as ours usually lead to mathematical programs for finding one particular mechanism satisfying desired properties. For instance, Moulin and Shenker's [7] characterization of group strategyproof cost sharing methods, under natural assumptions, for submodular cost functions leads to a linear program for finding one such method. Our characterization leads to a variational calculus formulation of truth-revealing auction mechanisms satisfying our natrual assumptions. This may be useful for finding rank functions that lead to an auction mechanism with desired properties.

2 Model and Definitions

We are given a single item which we want to auction to a set of n bidders. Each bidder has a private evaluation of her worth of this item. An auction takes the bids of these bidders and assign the item to one of them and decides how much to charge her. The bidder who gets the item makes a profit of her evaluation minus the price charged. All other bidders make zero profit. We will only work with deterministic auctions.

Formally, we define an auction as a function $\boldsymbol{\alpha} : \mathbf{R}_+^n \to \mathbf{R}_+^n$ which maps a bid-vector to a profit-vector. The profit vector can have at most one non-zero entry. The assumptions in the Introduction lead to the following restrictions on $\boldsymbol{\alpha}$: $\boldsymbol{\alpha}$ is continuous, whenever $\boldsymbol{\alpha}$ outputs the all-zero vector then there are at least two bidders who can increase their bids to make their own profit positive, and if the input to $\boldsymbol{\alpha}$ has only one positive bid, then the profit vector is the same as the bid vector.

We will characterize this mechanism $\boldsymbol{\alpha}$ by another mechanism $\boldsymbol{\beta}$. In $\boldsymbol{\beta}$, each bidder has a strictly monotonically increasing function $f_i : \mathbf{R}_+ \to \mathbf{R}$ which we call the *rank function of bidder* i. For a bid vector b, $\boldsymbol{\beta}$ computes the rank of each bid, i.e. $f_i(b_i)$. The winner is a bidder with maximum rank, and she pays the least amount p such that if she had bid p she would still have maximum rank. Using assumptions on $\boldsymbol{\alpha}$ we show the existence of the rank functions.

3 Useful Properties

We first modify $\boldsymbol{\alpha}$ to massage it to a more useful form and show that it satisfies some useful properties.

Fix a bidder i. When all other bidders bid zero, there is a threshold t_i such that i wins if she bids higher than t_i (by autonomy and truthfulness). From the independence property, it follows that i cannot ever win if she bids lower than t_i. Thus we normalize her bid so that a bid of t_i corresponds to a bid of zero in a new mechanism $\boldsymbol{\alpha}'$. Formally, the mechanism $\boldsymbol{\alpha}'$ adds t_i to bidder i's bid and runs $\boldsymbol{\alpha}$ on the resulting bid vector. Clearly $\boldsymbol{\alpha}(\boldsymbol{b})$ is equivalent to $\boldsymbol{\alpha}'(\boldsymbol{b} - \boldsymbol{t})$ and $\boldsymbol{\alpha}'$ inherits truthfulness, continuity and independence. Further we argue that, $\boldsymbol{\alpha}'$ satisfies what we call non-favoritism:

Non-favoritism: The winner cannot have a zero bid unless all bidders have bid zero.

Indeed, suppose that bidder 1 wins with bid zero, when bidder 2 has a non zero bid in $\boldsymbol{\alpha}'$. Thus in $\boldsymbol{\alpha}$, 1 wins with bid t_1 when 2 has bid strictly more that t_2. By independence, we can assume that bidders $3, \ldots, n$ bid zero (in $\boldsymbol{\alpha}$).

First assume that $t_1 = 0$. Then bidder one wins when the bid vector is $(0, b_2, 0, \ldots, 0)$. But this contradicts the definition of t_2 (since $b_2 > t_2$). Thus t_1 must be strictly positive.

Consider the bid vector $(t_1 - \epsilon, t_2' + \epsilon, 0, \ldots, 0)$, for $\epsilon > 0$ being arbitrarily small. We first argue that bidder one cannot be the winner for this bid vector. Indeed, if bidder one was the winner, then by independence, she would be the winner for the bid vector $(t_1 - \epsilon, 0, \ldots, 0)$ as well contradicting the definition of t_1. Thus some other player i is the winner. By independence, i is the winner under the bid vector $(0, t_2 + \epsilon, 0, \ldots, 0)$ as well. But by definition of t_2 and truthfulness, i must be two and thus the threshold for player two, under the bid vector $(t_1 - \epsilon, \cdot, 0, \ldots, 0)$ is no larger than $t_2 + \epsilon$. Clearly bidder two still wins at bid vector $(t_1 - \epsilon, b_2, 0, \ldots, 0)$ and her profit must be at least $(b_2 - (t_2 + \epsilon))$ which is bounded away from zero when ϵ approaches zero. Hence by continuity, bidder two's profit under the bid vector $(t_1, b_2, 0, \ldots, 0)$ cannot be zero. This however contradicts that assumption that bidder one was the winner. Hence we have shown that

Lemma 1. *The auction $\boldsymbol{\alpha}'$ defined above satisfies non-favoritism.*

We also note that one of the t_i's must be zero. Indeed, consider the bidder that wins $\boldsymbol{\alpha}$ when all bids are zero. Clearly, this bidder has t_i equal to zero.

We are now ready to define $\boldsymbol{\beta}$. We shall define $\boldsymbol{\beta}$ so as to be equivalent to $\boldsymbol{\alpha}'$, clearly it can be easily massaged so as to be equivalent to $\boldsymbol{\alpha}$. To define $\boldsymbol{\beta}$, we need to specify functions $f_i(b_i)$ for each i. We define $f_1(\cdot)$ as the identity function. For any $i > 1$, we define $f_i(x)$ to be the infimum of all y such that bidder one wins when the bid vector is given by $b_1 = y, b_i = x$ and $b_j = 0$ for $j \neq 1, i$. It is easy to verify that the profit vector for bids $b_1 = f_i(x), b_i = x$ and $b_j = 0$ for $j \neq 1, i$ is the zero vector. In the next section, we show that $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}'$ are indeed equivalent mechanisms.

4 Proof of Equivalence

In this section we provide the details of the proof of the main result that $\boldsymbol{\beta}$ as defined above is equivalent to the given auction $\boldsymbol{\alpha}$'.

Lemma 2. *For all $i = 1, .., n$, $f_i(0) = 0$.*

Proof. $f_1(0)$ is clearly 0. We also have $\boldsymbol{\alpha}'(\delta, 0, 0, \ldots, 0) = (+, 0, \ldots, 0)$, $\forall\ \delta > 0$. By definition of f_i, we get $f_i(0) = 0$.

Lemma 3. *For all $i = 1, .., n$, f_i is strict monotonically increasing.*

Proof. f_1 is the identity function, so it is strictly increasing. We prove that f_2 is strictly increasing, the proof is the same for f_i, $i > 2$.

Suppose that there are two bid values for bidder 2, b_2 and b_2', $b_2 < b_2'$, such that $f_2(b_2) = q_2$, $f_2(b_2') = q_2'$, and $q_2 > q_2'$.

Then for the bid vector $(q_2 - \epsilon, b_2', 0, \ldots, 0)$ when ϵ is small enough so that $q_2 - \epsilon > q_2'$, the definition of $f_2(b_2')$ implies that bidder one must win. By independence then, bidder one wins for the bid vector $(q_2 - \epsilon, b_2, 0, \ldots, 0)$. This however contradicts the definition of $f_2(b_2)$.

This shows that f_2 is non-decreasing. To prove that it is strictly increasing, we use the other axioms. Assume that there are two bid values for bidder 2, $b_2 < b_2'$, such that $f_2(b_2) = f_2(b_2') = q_2$. We have:

$$\boldsymbol{\alpha}'(q_2, b_2, 0, \ldots, 0) = (0, 0, \ldots, 0)$$

and

$$\boldsymbol{\alpha}'(q_2, b_2', 0, \ldots, 0) = (0, 0, \ldots, 0)$$

Pick an arbitrarily small $\epsilon > 0$ and consider the bid vector $(q_2 - \epsilon, b_2, 0 \ldots, 0)$. By definition of f_2, bidder one loses and hence some bidder i must be the winner. By independence, i wins for the bid vector $(0, b_2, 0, \ldots, 0)$ as well, and by non-favoritism, i must be bidder two. Thus the threshold for bidder two, given the bids $(q_2 - \epsilon, -, 0, \ldots, 0)$ must be no larger than b_2. As a result, $\boldsymbol{\alpha}'(q_2 - \epsilon, b_2', 0, \ldots, 0) = (0, \delta, 0, \ldots, 0)$ for some $\delta \geq b_2' - b_2$ bounded away from zero. By continuity then, it follows that $\boldsymbol{\alpha}'(q_2, b_2', 0, \ldots, 0)$ must be at least $\delta > 0$. This however contradicts the definition of $f_2(b_2')$.

Theorem 1. *$\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}$ are equivalent auctions.*

Proof. It is easy to check that $\boldsymbol{\beta}$ as defined above satisfies all the axioms. We now argue that in fact $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}$ are equivalent.

Suppose $\boldsymbol{\alpha}$' is not equivalent to $\boldsymbol{\beta}$. Then there are four different ways in which they could differ. We consider each of these four cases, and obtain a contradiction. The only direct contradiction is obtained in Case 1, while in the rest of the cases we reduce to other cases, and eventually to case 1, taking care of avoiding logical cycles.

1. Case 1: There is a bid vector $(b_1, b_2, \ldots, b_n)$ s.t. two different bidders make positive profit in $\boldsymbol{\alpha}$' and $\boldsymbol{\beta}$. Assume first that

$$\boldsymbol{\alpha}'(b_1, b_2, \ldots, b_n) = (+, 0, \ldots, 0) \quad \textit{and} \quad \boldsymbol{\beta}(b_1, b_2, \ldots, b_n) = (0, +, 0, \ldots, 0)$$

   Since $\boldsymbol{\beta}(b_1, b_2, \ldots, b_n) = (0, +, 0, \ldots, 0)$, we have by definition of $\boldsymbol{\beta}$ that $\boldsymbol{\beta}(b_1, b_2, 0, \ldots, 0) = (0, +, 0, \ldots, 0)$. By definition of $f_2(b_2)$, it must be the case that $\boldsymbol{\alpha}'(b_1, b_2, 0, \ldots, 0) = (0, +, 0, \ldots, 0)$. On the other hand, by independence and the assumption that $\boldsymbol{\alpha}'(b_1, b_2, b_3, \ldots, b_n) = (+, 0, \ldots, 0)$, it follows that $\boldsymbol{\alpha}'(b_1, b_2, 0, \ldots, 0) = (+, 0, \ldots, 0)$. This however is a contradiction.

   Now assume w.l.o.g that

$$\boldsymbol{\alpha}'(b_1, b_2, \ldots, b_n) = (0, +, 0, 0 \ldots, 0)$$

$$\boldsymbol{\beta}(b_1, b_2, \ldots, b_n) = (0, 0, +, 0, \ldots, 0)$$

   From the definition of $\boldsymbol{\beta}$, it follows that $f_3(b_3) > f_2(b_2)$. Also, by independence, there is no loss of generality in assuming that $b_4, \ldots, b_n$ are all zero. Let b_1' be such that $f_2(b_2) < b_1' < f_3(b_3)$. Consider the behaviour of $\boldsymbol{\alpha}'$ under the bid vector $(b_1', b_2, b_3, 0, \ldots, 0)$. Let j be the winner. We split cases:
   (a) $j \geq 4$: This contradicts non-favoritism
   (b) $j = 1$: By independence, bidder one still wins $\boldsymbol{\alpha}'$ when the bid vector is $(b_1', 0, b_3, 0, \ldots, 0)$. This however contradicts the definition of $f_3(b_3)$.
   (c) $j = 2$: By independence, bidder two still wins $\boldsymbol{\alpha}'$ when the bid vector is $(b_1', b_2, 0, 0, \ldots, 0)$. This contradicts the definition of $f_2(b_2)$.
   (d) $j = 3$: Using independence, bidder three must win $\boldsymbol{\alpha}'$ for bid vector $(0, b_2, b_3, 0, \ldots, 0)$. On the other hand,
   $\boldsymbol{\alpha}'(b_1, b_2, b_3, 0, \ldots, 0) = (0, +, 0, \ldots, 0)$ along with independence implies that bidder two wins when the bid vector is $(0, b_2, b_3, 0, \ldots, 0)$. Hence we get a contradiction

Thus we have shown that when both $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}$ have a winner with positive profit, they must agree.

2. Case 2: There is a bid vector $(b_1, b_2, \ldots, b_n)$ s.t. there is a bidder who makes positive profit in $\boldsymbol{\beta}$, but no bidder makes positive profit in $\boldsymbol{\alpha}$', i.e.,

$$\boldsymbol{\alpha}'(b_1, b_2, \ldots, b_n) = (0, 0, 0, \ldots, 0) \quad \textit{and} \quad \boldsymbol{\beta}(b_1, b_2, \ldots, b_n) = (0, +, 0, \ldots, 0)$$

where we have taken the winner in $\boldsymbol{\beta}$ to be bidder 2 w.l.o.g.

For an arbitrarily small $\epsilon > 0$, consider the mechanism $\boldsymbol{\alpha}'$ when given the bid vector $(b_1, b_2 - \epsilon, b_3, \ldots, b_n)$. Let bidder i be the winner in this case; by truthfulness, i is different from two. Since ϵ can be made arbitrarily small, for any $\delta > 0$, bidder i is the winner in $\boldsymbol{\alpha}'$ when the bid vector is given by $b'_i = b_i + \delta$, $b'_j = b_i$ for $j \neq i$. Since δ can be made smaller, $\boldsymbol{\alpha}'(\boldsymbol{b'})$ has a positive ith component. Moreover, for small enough δ, $\boldsymbol{\beta}(\boldsymbol{b'})$ has a positive second component. Thus we have reduced this to the first case.

3. Case 3: There is a bid vector $(b_1, b_2, \ldots, b_n)$ s.t. there is a bidder who makes positive profit in $\boldsymbol{\alpha}$', but no bidder makes positive profit in $\boldsymbol{\beta}$, i.e.,

$$\boldsymbol{\alpha}'(b_1, b_2, \ldots, b_n) = (0, +, 0, \ldots, 0) \quad \textit{and} \quad \boldsymbol{\beta}(b_1, b_2, \ldots, b_n) = (0, 0, 0, \ldots, 0)$$

where we have taken the winner in $\boldsymbol{\alpha}$' to be bidder 2 w.l.o.g.

Since $\boldsymbol{\beta}(b_1, b_2, \ldots, b_n) = (0, 0, 0, \ldots, 0)$, we know that there are at least two bidders with the maximum value of their rank function. If there are exactly two such bidders, then consider two subcases: bidder 2 is one of the two, or bidder 2 is not one of the two.

In the first subcase, consider the bid vector obtained by reducing the bid of bidder 2 infinitesimally to $b_2 - \delta$, for some small $\delta > 0$. By axiom 3 (Continuity), we get that $\boldsymbol{\alpha}'(b_1, b_2 - \delta, b_3, \ldots, b_n)$ remains $(0, +, 0, \ldots, 0)$, but by definition of $\boldsymbol{\beta}$ (and Lemma 3) we see that$\boldsymbol{\alpha}'(b_1, b_2 - \delta, b_3, \ldots, b_n)$ gives a positive profit to the other bidder who had highest rank. This reduces to Case 1.

In the second subcase, when bidder 2 is not one of the two highest ranked bidders in $\boldsymbol{\beta}$, consider the bid vector obtained by reducing the bid of any one of the two bidders. $\boldsymbol{\beta}$ gives a positive profit to the other of the two bidders. But by independence, we see that $\boldsymbol{\alpha}$' still gives positive profit to bidder 2. This again reduces to Case 1.

In the case that there are more than two bidders with the highest value of rank function, we can choose a bidder who is not bidder 2, and reduce his bid to 0. By independence, we see that $\boldsymbol{\alpha}$' still gives positive profit to bidder 2. Since there were at least 3 bidders with highest rank in $\boldsymbol{\beta}$ before reducing the bid, there are now at least 2 bidders with highest rank. Hence the output of $\boldsymbol{\beta}$ remains $(0, 0, \ldots, 0)$. Thus we reduce to the same case again, but this time with a smaller number of positive bids, and we can use induction on the number of positive bids. The base case is that of two highest ranked bidders, which is taken care of above.

4. Case 4: There is a bid vector $(b_1, b_2, \ldots, b_n)$ s.t. the same bidder makes a positive profit in both $\boldsymbol{\alpha}$' and $\boldsymbol{\beta}$, but makes different amounts of profit.

We assume w.l.o.g. that the winning bidder is bidder 1. In the first subcase, we have $\boldsymbol{\alpha}'(b_1, b_2, \ldots, b_n) = (p + \Delta, 0, \ldots, 0)$ and $\boldsymbol{\beta}(b_1, b_2, \ldots, b_n) = (p, 0, \ldots, 0)$, for some $p, \Delta > 0$. Suppose we reduce the bid of bidder 1 from b_1 to $b_1 - p$. We get $\boldsymbol{\alpha}'(b_1 - p, b_2, \ldots, b_n) = (\Delta, 0, \ldots, 0)$ by Axiom 1 (Truthfulness), and we get $\boldsymbol{\beta}(b_1 - p, b_2, \ldots, b_n) = (0, 0, \ldots, 0)$ by the definition of $\boldsymbol{\beta}$. Thus we have reduced to Case 3.

In the second subcase, we have $\boldsymbol{\alpha}'(b_1, b_2, \ldots, b_n) = (p, 0, \ldots, 0)$ and $\boldsymbol{\beta}(b_1, b_2, \ldots, b_n) = (p + \Delta, 0, \ldots, 0)$, for some $p, \Delta > 0$. Again, we reduce bidder 1's bid to $b_1 - p$. By Axiom 1, we get that $\boldsymbol{\alpha}'(b_1 - p, b_2, \ldots, b_n)$ is either $(0, 0, \ldots, 0)$ or $(0, \ldots 0, +, 0, \ldots 0)$, where the positive entry is in some index other than 1. By the definition of $\boldsymbol{\beta}$ we get that $\boldsymbol{\beta}(b_1 - p, b_2, \ldots, b_n) = (\Delta, 0, \ldots, 0)$. Thus we have reduced to either Case 2 or to Case 1.

Finally, note that this mechanism $\boldsymbol{\beta}$, defined by the functions $f_i : \mathbf{R}_+ \to \mathbf{R}_+$ is equivalent to the mechanism $\boldsymbol{\alpha}'$. To get a similar mechanism equivalent to $\boldsymbol{\alpha}$, we define $g_i : \mathbf{R}_+ \to \mathbf{R}$ so that $g_i(x) = f_i(x - t_i)$ for $x \geq t_i$. When i bids less than t_i, she can never win irrespective of the other bids; we can thus define $g_i(x)$ arbitrarily in this range, e.g. $g_i(x) = (x - t_i)$ for $x \leq t_i$ works and makes g_i both continuous and strictly increasing. Also note that since for a bidder i such that $t_i = 0$ (such a bidder exists, as we argued earlier), $g_i(0) = 0$. Thus the winning bidder under g_i's always has a positive value of $g_i(b_i)$ and the negative values of g_i indeed are irrelevant.

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