Posted Prices vs. Negotiations: An Asymptotic Analysis

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Abstract

Full revelation of private values is impractical in many large-scale markets, where posted price mechanisms are a simpler alternative. In this work, we compare the asymptotic behavior of full revelation auctions to posted price auctions in a Bayesian model. We show that posted-price auctions that use discriminatory (i.e., personalized) prices can be asymptotically equivalent to optimal full revelation auctions with the right choice of prices. On the other hand, posted price auctions with one symmetric price are asymptotically inferior to optimal full revelation auctions. Our results are given for general independent distribution functions under standard weak conditions (called the von Mises conditions) that correspond to inherent properties of the distribution functions (e.g., if the support is bounded or unbounded, the shape of the distribution tail, etc). Our results apply to other settings like online algorithms and secretary problems.

1 Introduction

A substantial part of the literature on auctions explored the design of *optimal* auctions, and focused on ways to negotiate with the bidders for eliciting the relevant information that they hold. Sometimes, however, decisions should be taken very quickly, and the auctioneer cannot allow an iterative procedure of bidding or waiting for bidders to determine their exact valuation. One prominent example are *electricity markets* (see, e.g., [34]); when a sudden drop in the electricity supply takes place, the allocation of the remaining electricity should be determined immediately. One solution that have been used in practice is to post prices for the bidders, and ask for their immediate take-it-or-leave-it response. Another example where full negotiation is not necessarily the right thing to do, is in *large-scale e-commerce markets*. Selling items by a full negotiation process (like eBay auctions) clearly provides the seller with sufficient information for determining the bidders with the highest values (and it can thus use optimal mechanisms like Vickrey's [32] and Myerson's [23]). However, such mechanisms incur other costs, like the need to maintain more complex software. With the presence of numerous bidders, it may be reasonable not to negotiate with the bidders at all, but simply post high-enough prices and with high probability at least some of the bidders will be willing to pay these high prices. Moreover, it is shown in several works that bidders prefer simple mechanisms with simple rules (see, e.g., the behavioral research in [15]); in simple mechanisms they tend to act more rationally and they are more likely to participate.

The obvious questions now are understanding how posted price auctions compare with full-revelation auctions, and what are the optimal prices to use. These are indeed the main questions this paper aims to address.

Our paper compares the expected revenue in full-revelation auctions to that in optimal posted-price auctions. We focus on single-item auctions in a Bayesian model where the values that the bidders are willing to pay for the item are independently distributed according to a distribution function that is known to the seller; this distribution is identical for all bidders. We study two natural families of posted-price auctions. Auctions where the seller posts a single price for all the bidders, referred to as *symmetric* posted-price auctions, and auctions with personalized prices which we call *discriminatory* posted-price auctions. Note that although the distributions are ex-ante identical, the seller still may gain more power by publishing discriminatory prices. We stress that in both families of posted-price auctions the seller does not collect any information from the bidders, and just uses the ex-ante commonly known distributions on their values to determine take-it-or-leave-it offers.

Therefore, we compare the following three auctions:

1. **Full-revelation auctions.** Auctions where each bidder reveals his exact private value to the seller. An *optimal* full-revelation auction maximizes the expected revenue in equilibrium; Myerson's auction that allocates the item to the bidder with the highest virtual valuation $v_i - \frac{1-F(v_i)}{f(v_i)}$ is an optimal full-revelation auction.

2. **Posted-price auction.** The seller publishes a price p, and the item is sold to any one of the bidders that is willing to buy the item at the price p. In an *optimal* posted-price auction, the seller determines the price p such that his expected revenue is maximized.

3. **Discriminatory posted-price auctions.** Auctions where the seller publishes an individual price p_i for each bidder, and the item is sold to the bidder with the highest price among those who accepted their offer. In an *optimal* discriminatory posted-

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price auction, the seller publishes prices that optimize the expected revenue.

Note that the 3 auctions admit dominant-strategy equilibrium. In the full-revelation case, the dominant-strategy equilibrium is achieved under Myerson's regularity condition (i.e., that the virtual-valuation function is monotone non-decreasing). In the posted-price auctions it is clearly a dominant strategy do accept only offers smaller than one's value. We note that our results directly apply for the goal of social-welfare maximization in addition to revenue maximization, via the classic reduction of Myerson [23].[1]

1.1 Our Results

This paper provides an exact asymptotic characterization of the optimal expected revenue achieved by each one of the above auctions. Since single-item auction is the most fundamental problem in mechanism design, and also two exactly quantify the differences between the auctions, we present the exact constants when expressing the optimal revenue. For the posted price auctions, we also present the exact prices that achieve the optimal results. Our results are given for general distribution functions, under weak standard conditions on the shape of the distribution at the right extreme of its support. Our results are given up to terms with lower asymptotic order, that is, up to factor of $1 - o(1)$. We provide two sets of results; one set of results for distribution on a support that is bounded from above, and a second set for distributions on unbounded supports. In the first case we require a mild assumption on the way the distribution function approaches the end of the support, called the *first von Mises* condition; the latter case requires a similar mild condition on the shape of the tail of the distribution which is called the *second von Mises* condition. These conditions are taken from works in statistics on extreme-value theory and highest-order statistics and, intuitively, guarantee that the distribution of the highest order statistics behaves in a non-degenerate manner when the number of samples is large. The conditions are very weak, and it seems hard to come up with natural distributions for which they do not hold. Among others, they include all distributions with a differentiable density function and always positive support (on bounded intervals) and power-law distributions or log-normal distributions (unbounded support). We will formally present these conditions in Section 2. More details can be found in the tomes [7, 8].

Our result derive stark conclusions on how the above auctions compare, and we have different conclusion for the bounded support and the unbounded support cases.

[1] Myerson [23] showed that maximizing the revenue is actually equivalent to maximizing the social welfare after performing a transformation on the valuations (i.e., when considering the *virtual* valuations. This reduction actually requires an additional regularity condition on the distributions, that is, that the transformed valuation (the "virtual" valuations) will be a non-decreasing function of the original valuation.

Bounded supports.

Theorem: *(informal) Discriminatory posted-price auctions with the right choice of prices can be almost equivalent in terms of revenue to optimal full-revelation auctions. On the other hand, symmetric posted-price auctions incur an asymptotically greater loss by a logarithmic factor in the number of bidders.*

For example, in the case where the values of the bidders are distributed uniformly on $[0, 1]$, the optimal expected revenue in the full revelation auction (i.e., the Myerson auction) is around $1 - \frac{2}{n}$. The optimal revenue with a symmetric posted price is around $1 - \frac{\log n}{n}$. With discriminatory prices, however, although they do not collect any information from the bidders, the expected revenue becomes very close to the full-revelation result, i.e., $1 - \frac{4}{n}$. (All the above expressions of the expected revenue are given up to lower order asymptotic terms.)

Unbounded supports.

Theorem: *(informal) Optimal full revelation auctions, symmetric posted-price auctions and discriminatory posted-price auctions do not converge to the same revenue when the number of bidders grow. (We present an exact characterization of the ratio between the expected revenue in these auctions.)*

For example, consider power-law distributions on the open interval $[1, \infty)$, i.e., where the cumulative distribution function is $F(x) = 1 - \frac{1}{x^k}$ $(k > 1)$. For example, if $k = 2$, the expected revenue on the optimal full-revelation auction turns out to be $0.88\sqrt{n}$, while the optimal expected revenue in symmetric posted-price auctions is $0.64\sqrt{n}$ and in discriminatory posted-price auctions it is $0.7\sqrt{n}$ (always omitting lower order terms). Note that the ratio between those revenues is not converging to 1.

1.2 Related work

Asymptotic comparison of symmetric and discriminatory posted-price auctions for the uniform distribution was briefly given in the context of auctions with bounded communication in [3, 5]. Subsequent papers also studied simple nearly-optimal economic mechanisms that collect minimal information from the bidders, e.g., [4, 27, 20].

Our model also provides solutions to other well-studied models. One closely related problem is the *secretary problem*. In secretary problems, a sequence of agents that hold values is presented to the algorithm, and the algorithm can stop and choose a value any time. The goal of the algorithm is to maximize the expected value he stopped at (other goals have been also studied for this model over the years). The algorithm has no knowledge on the future values and he cannot return to values he passed in the past. We consider the "full-information" version of the secretary problem, where the samples are drawn independently from the same distribution function.

The optimal posted-price auctions is algorithmically similar to the secretary problem, and our solutions actually provide an optimal algorithm for maximizing the expected value gained in full-information secretary problems.[2] Some of the classic solutions for the secretary problems (e.g., [12]) presented results in the same spirit as ours, but did not handle the asymptotic analysis and the generality of the distribution functions in our paper. Variants of the secretary problem were studied, in auction settings or general settings, for example, in [1, 19, 14]. More relevant literature studied statistical stopping rules (or "prophet inequalities") in similar settings [31, 17], and more recently in [18].

Analogously, posted-price auctions are associated with the study of online auctions and online algorithms. It was observed (e.g., in [3, 5]) that optimal posted-price auctions can be implemented in an online manner. Along these lines, an easy observation is that in optimal Bayesian online auctions (or secretary problems), each agent is presented a single take-it-or-leave-it threshold, and thus these are actually posted-price implementations. See the survey [24] and the references within for more information on online auctions.

Asymptotic analysis of the revenue in single-item auctions appeared in [9, 6], with the emphasis on the asymptotic strategies of the bidders in first-price auctions. None of these works studied posted-price auctions, neither symmetric nor discriminatory. More information on extreme-value theory and the theory of highest-order statistics can be found in [8, 7]. A related line of research is by [30, 26, 28, 29] and others, who studied the loss in double auctions as the number of bidders grow. More related recent work is by [16] who studied discriminatory prices when the number of items is some non-trivial fraction of the number of bidders.

2 The von Mises-Conditions

Consider a seller that wishes to sell one item to a set of n bidders. Each bidder has a private value v_i that she is willing to pay for this item. Each v_i is drawn from a distribution function F. The density function (if exists) is denoted by f.

Our analysis requires some weak assumptions on the distribution functions. These assumptions describe how the distribution function behaves in the extreme part of the support; since our analysis closely relates to highest-order statistics analysis, the behavior on the lower parts of the support will not matter when the number of bidders is large. We will consider different assumptions for distributions with bounded and unbounded support.

[2]In many classic paper on secretary problems the goal is to maximize the probability that the secretary with the highest value will be chosen, rather than maximizing the expected value, see, e.g., [12, 1].

Historically, the conditions were introduced by von Mises in 1936 [33]. Together with a third, they are a slight relaxation of the necessary and sufficient conditions of Gnedenko [13] under which the maximal value of n distribution approaches a single distribution after renormalization. While the von Mises conditions are slightly stronger than Gnedenko's condition (they require, for example, that the density exists), every function which satisfies Gnedenko's condition is tail-equivalent to one which satisfies the von Mises condition [2].

2.1 The first von Mises-Condition (Unbounded Support)

We will first consider the unbounded support case, and we will denote the following condition as the first von Mises condition (or VM1).

Definition 1. We say that F satisfies the VM1 property with parameter α if F has unbounded support, is eventually differentiable[3], and satisfies, for some $\alpha > 0$,

$$\lim_{x\to\infty} \frac{xf(x)}{1-F(x)} = \alpha. \tag{1}$$

Remark: If $\alpha \le 1$, then the expectation $E[X]$ may not exist. We will therefore only consider VM1 with $\alpha > 1$.

One example that we will use throughout the paper for a VM1 distribution function is the family of power-law distributions. That is, when $F(x) = 1 - \frac{1}{x^k}$ for some $k > 1$. It is easy to see that power-law distributions admits the VM1 property with the parameter k (the parameter that defines the power-law distribution). (One can check this by observing that $f(x) = \frac{k}{x^{k+1}}$ and thus $\frac{xf(x)}{1-F(x)}$ is exactly the constant k.)

VM1 holds for distributions with a "heavy tail", for example, log-normal, Pareto, Burr, log-Gamma, and, as mentioned, power-law distributions (for any k).

2.2 The second von Mises-Condition (Bounded Support)

We now present the second von Mises condition for distribution functions on bounded supports. That is, $F(x)$ is defined for x's smaller than some upper bound b.

Definition 2 (VM2)**.** We say that F has the VM2 property with parameter α if the support of F has an upper bound $b < \infty$, if F is eventually differentiable, and satisfies, for some $\alpha > 0$,

$$\lim_{x\to b} \frac{(b-x)f(x)}{1-F(x)} = \alpha. \tag{2}$$

[3]That is, the density function exists for all the points in the support that are greater than some x_0.

We will use the uniform distribution on the support $[0,1]$ as the leading example for the VM2 property. It is easy to verify that this distribution indeed has the VM2 property with the parameter $\alpha = 1$ (clearly, $b = 1$, $f(x) = 1$ and $F(x) = x$ for every x).

VM2 is a weak property for distributions on bounded supports. It is a weaker assumption than, for example, distributions with differentiable always-positive density functions. It is also weaker than the demand that a distribution function has a Taylor expansion around the upper bound of the support b.[4]

We would like to note that there also exists a third von Mises condition, for "light-tailed" distributions like the exponential distribution. Results for this condition are out of the scope of this conference version. See [8] for more details on the von Mises conditions.

3 Exposition of our Results

In this section, we will present our formal results without the proofs, which will be given later, and demonstrate these results using simple distribution functions. Section 3.1 will discuss our results for distributions with unbounded supports (under VM1) and section 3.2 will present our results for bounded supports (under VM2). For each family of distributions we will actually present three theorems, characterizing optimal results for (1) full-revelation auctions, (2) symmetric posted-price auctions, and (3) discriminatory posted-price auctions. Note that each of these two theorems actually contains two results, a lower bound and an upper bound. For example, for the results about posted prices, we show that using certain prices generates a certain amount of expected revenue, but also that no other scheme of posted prices can achieve a better expected result.

We will first explain two standard notations that we use in the statements of the theorems:

- We denote by $F^{-1}(t)$ the value x for which $F(x) = t$.
- Let $\Gamma(\cdot)$ denote the Gamma function which is the extension of the factorial function to real numbers.[5]

The results we have for the full-information case actually describe the expected social-welfare in the auction, that is, the expectation of the highest value (the highest-order statistic). It is easy to conclude the expected revenue from this result using one of the following ways:

1. *Reduction via virtual valuations.* Namely, optimal revenue is equal to the optimal social-welfare when we perform the following transformation on the values (and consider an additional bidder in the new model, the seller, with a zero value): $\tilde{v}_i(v_i) = v_i - \frac{1-F(v_i)}{f(v_i)}$. $\tilde{v}_i$ is often called the *virtual valuation* of player i.
2. *Order statistics properties.* Since under standard conditions (non-decreasing virtual valuations) second-price auctions achieves optimal revenue, we should actually calculate the expectation of the second-highest order statistic. Denote the i'th order statistic from a sample of n bidders by $Y_i^{(n)}$. The following property enables us to compute the expectation of the second-order statistics from the first-order statistics (see, e.g., [21]):[6]

$$E\left[Y_2^{(n)}\right] = nE\left[Y_1^{(n-1)}\right] - (n-1)E\left[Y_1^{(n)}\right]$$

3.1 Distributions with Unbounded Support

We now present our results for models where the values of the players are drawn from a cumulative distribution function (cdf) F which satisfies the first von Mises property.

3.1.1 Full-revelation auctions

We will first present the expression for the optimal expected social welfare. Note that $\Gamma(\frac{\alpha-1}{\alpha})$ is a constant that only depends on the parameter of the VM1 function, and therefore the optimal result is proportional to $F^{-1}(1-\frac{1}{n})$. It is well known that a social welfare that exactly equals the expectation of the highest-order statistic can be achieved, even in dominant strategies, using second-price auctions.

Theorem 1. *Let F be a cdf satisfying VM1 for $\alpha > 1$. Then, the expectation of the maximum is of n random variables chosen according to F is*

$$\Gamma(\tfrac{\alpha-1}{\alpha})F^{-1}\left(1-\frac{1}{n}\right)(1+o(1))$$

For example, recall that the power-law distribution $F(x) = 1 - \frac{1}{x^k}$ is VM1 with parameter k. It is easy to see that $F^{-1}(1-\frac{1}{n})$ for this distribution equals $\sqrt[k]{n}$ (by solving $1-\frac{1}{x^k} = 1-\frac{1}{n}$). Therefore, the expected social welfare equals $\Gamma(\frac{k-1}{k})\sqrt[k]{n}(1-o(1))$. When $k=2$, it follows that the expected-social welfare is approximately $1.77\sqrt{n}$.

We will now show how this result can easily derive the optimal expected revenue. The virtual valuation of player i is $\tilde{v}_i(v_i) = v_i - \frac{1-F(v_i)}{f(v_i)} = v_i(1-\frac{1}{k})$. Therefore, the corresponding cdf in the virtual-valuation domain is $\tilde{F}(x) = F(\frac{x}{1-\frac{1}{k}}) = 1 - \frac{(1-\frac{1}{k})^k}{x^k}$, and it is easy to see that the von Mises parameter for $\tilde{F}$ remains k. Since $\tilde{F}^{-1}(1-\frac{1}{n}) = (1-\frac{1}{k})\sqrt[k]{n}$, it follows that the expectation of the virtual valuation is $\Gamma(\frac{k-1}{k})(1-\frac{1}{k})\sqrt[k]{n}$, which is different than the optimal expected social welfare by a factor of $1-\frac{1}{k}$. For $k=2$ the expected revenue is therefore about $0.88\sqrt{n}$.

[4] We would like to emphasize that VM2 is strictly weaker than the above examples. For example, the function $1-(1-x)^k \cdot \log x$ has no Taylor expansion around 1 but the VM2 still holds.

[5] $\Gamma(x) = \int_0^\infty t^{z-1}e^{-t}dt$, and for integer x it holds that $\Gamma(x) = (x-1)!$.

[6] Note that the second-price auction is not exactly the optimal auction, but a second-price auction with a reserve price. However, since the optimal reserve price is independent of the number of players (e.g., [10, 21]) with numerous players this becomes negligible.

3.1.2 Symmetric posted-price auctions

We now move to explore symmetric posted-price auctions. We will first define a new notation. For $k > 1$, the equation $\exp(x) = 1 + kx$ has a unique positive solution.[7] We will denote this solution by $\eta(k)$ (and thus, $\exp(\eta(k)) = 1 + k\eta(k)$). The theorem shows that the optimal expected revenue is proportional to $F^{-1}(1 - \frac{\eta(\alpha)}{n})$, and shows a family of prices that achieve this optimal revenue. Note that $\eta(\alpha)$ depends only on the von Mises parameter α and is independent of the number of bidders.

Theorem 2. *Let F be a cdf satisfying VM1 for $\alpha > 1$. Then, in a symmetric posted-price auction the optimal expected revenue satisfies,*

$$r_n = \frac{\alpha\eta(\alpha)}{1 + \alpha\eta(\alpha)} F^{-1}\left(1 - \frac{\eta(\alpha)}{n}\right)(1 + o(1)) \ . \tag{3}$$

Furthermore, the optimal prices satisfy

$$p_n = F^{-1}\left(1 - \frac{\eta(\alpha)}{n}\right)(1 + o(1)) \ . \tag{4}$$

Finally, all prices of the form (4) achieve a revenue of the form (3).

Note that the second statement on the theorem claims that *the* optimal price belongs to the family described in Equation (4), and the final statement shows that actually *any* price of this form achieve asymptotically optimal results. Therefore, we can use prices $p_n = F^{-1}(1 - \frac{\eta(\alpha)}{n})$ if we want to run such an auction in practice, and there is no need to find *the* optimal price for achieving the revenue in Equation 3.

Lets consider again the example of the power-law distributions. It is easy to see that $F^{-1}(1 - \frac{\eta(\alpha)}{n}) = \frac{\sqrt[k]{n}}{\sqrt[k]{\eta(k)}}$, and therefore the expected revenue is about $\frac{k\eta(k)}{1+k\eta(k)} \frac{\sqrt[k]{n}}{\sqrt[k]{\eta(k)}}$. When $k = 2$, $\eta(2) \cong 1.26$ and the optimal revenue is about $0.64\sqrt{n}$.

3.1.3 Discriminatory posted-price auctions

We finally present the expression for the optimal expected revenue in discriminatory posted-price auctions.

Theorem 3. *Let F be a cdf satisfying VM1 for $\alpha > 1$. Then, the optimal expected revenue in a discriminatory posted-price auction satisfies*

$$r_n = \frac{\alpha - 1}{\alpha} F^{-1}\left(1 - \frac{\alpha - 1}{\alpha n}\right)(1 + o(1)) .$$

Consider again power-law distributions of the form $F(x) = 1 - \frac{1}{x^k}$. We know that the VM1 parameter is k, and $F^{-1}(1 - \frac{\alpha-1}{\alpha n})$ in this case equals $\sqrt[k]{\frac{k}{k-1}} \sqrt[k]{n}$, and therefore the optimal expected revenue here is $\frac{k-1}{k} \sqrt[k]{\frac{k}{k-1}} \sqrt[k]{n}$. When $k = 2$, the optimal revenue is therefore about $0.7\sqrt{n}$.

To conclude the discussion on distributions with unbounded support, we consider the power-law distribution example. It is shown that as expected, the expected revenue from the discriminatory posted-price auctions ($0.7\sqrt{n}$) is somewhere between the full revelation optimum ($0.88\sqrt{n}$) and the symmetric posted-price auction ($0.64\sqrt{n}$)). One interesting conclusion is that while the asymptotic behavior of all these functions is similar (ignoring constants), they do not converge to the same value as n grows. In other words, the ratio between the revenue obtained by any two of these auctions converges to some constant other than 1.

[7] To see this, it is sufficient to note that the C^∞ function $g(x) = \exp(x) - 1 - kx$ is zero at $x = 0$ and has negative derivative, it's derivative has exactly one zero, and $\lim_{x\to\infty} g(x) = \infty$.

3.2 Distributions with Bounded Support

We now describe the results for distributions on the bounded support that satisfy the second von Mises conditions. In these results, both the constants and the $1 - o(1)$ term apply on the gap between the highest point in the support b and the actual revenue (see results below). We will use the uniform distribution on $[0, 1]$ as a leading example for the results below. For such distributions, the upper bound on the support is $b = 1$ and the von Mises parameter is $\alpha = 1$. Also, $F^{-1}(x) = x$ for every $x \in [0, 1]$.

3.2.1 Full-revelation auctions

Again we start with an upper bound on the expected revenue - the maximal expected social welfare.

Theorem 4. *Let F be a cdf satisfying VM2 for $\alpha > 0$ with $b := F^{-1}(1) < \infty$. Then, the expectation of the maximum is*

$$b - \Gamma(\frac{\alpha + 1}{\alpha})\Big(b - F^{-1}(1 - \tfrac{1}{n})\Big)(1 + o(1))$$

For example, for the uniform distribution we have $\Gamma(\frac{\alpha+1}{\alpha}) = \Gamma(2) = 1$, and $F^{-1}(1 - \frac{1}{n}) = 1 - \frac{1}{n}$ and it follows that the expected social welfare is about $1 - (1 - (1 - \frac{1}{n})) = 1 - \frac{1}{n}$ (this, of course, is a well known fact). The optimal expected revenue can be shown by the methods mentioned above (see also [9]) to be about $1 - \frac{2}{n}$.

3.2.2 Posted-price auctions

Using a single posted price, the expected revenue that an auction can obtain decreases. We show that the loss (the term that depends on n) increases by a factor of $\log n$.

Theorem 5. *Let F be a cdf satisfying VM2 for $\alpha > 0$ with $b := F^{-1}(1) < \infty$. Then, the maximal expected revenue achievable in a symmetric posted price auction with n bidders is*

$$r_n = b - \Big(b - F^{-1}\Big(1 - \frac{\log(\alpha n)}{\alpha n}\Big)\Big)(1 + o(1)).$$

Again, for the uniform distribution the optimal expected revenue from the optimal symmetric posted-price auction is about $1 - \frac{\log(n)}{n}$.

3.2.3 Discriminatory posted-price auctions

It turns out that with discriminatory prices, the expected revenue returns to be very close to the one in the full information case, up to a constant factor. This comes in contrast to the case of a single posted price. Note that in discriminatory auctions the seller determines the prices based only on the ex-ante distribution and do not elicit any information from the bidders.

Theorem 6. *Let F be a cdf satisfying VM2 for $\alpha > 0$ with $b := F^{-1}(1) < \infty$. Then, the expected revenue in a discriminatory auction satisfies*

$$r_n = b - \frac{\alpha+1}{\alpha}\left(b - F^{-1}\left(1 - \frac{\alpha+1}{\alpha n}\right)\right)(1 + o(1)).$$

For example, for the uniform distribution where $\alpha = 1$ we have that $r_n = 1 - \frac{1}{4n}$.

3.3 Techniques used

In order to prove our results, we heavily borrow tools from Karamata's theory of regularly varying functions. This theory is heavily used by statisticians in extremal value theory (i.e., the theory where one is concerned about the maximal or minimal outcome in a sequence of random variables, see, e.g. [8]). Karamata's theory parametrizes distribution functions roughly one parameter α, which roughly describes how the distribution "behaves around $F^{-1}(1)$".

4 Proofs

4.1 Regularly Varying Functions

In our proofs, we will use the theory of regularly varying functions. An excellent introduction to this subject can be found in Appendix B of [8] (which has been taken in large parts from [11]).

Definition 3. The set RV_α consists of the Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{R}$ which are eventually positive and satisfy

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^\alpha.$$

We say that f is *regularly varying* with index α.

We will use the folowing representation theorem for functions from RV_α. A proof can be found in [8] (Theorem B.1.6).

Proposition 4 (Representation theorem)**.** *If $g \in \mathrm{RV}_\alpha$, there exists t_0 and measurable functions $a : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $c : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $\lim_{t\to\infty} c(t)$ exists and is positive and finite, and $\lim_{t\to\infty} a(t) = \alpha$, and such that for $t > t_0$*

$$g(t) = c(t) \exp\left(\int_{t_0}^{t} \frac{a(s)}{s} ds\right).$$

We will further use the following theorem due to Potter [25]. A proof can be found in [8] (Proposition B.1.9, Part 5).

Proposition 5. *If $g \in \mathrm{RV}_\alpha$, $\varepsilon > 0$, then there exists t_ε such that, if $t \geq t_\varepsilon$, $tx \geq t_\varepsilon$:*

$$(1-\varepsilon)x^\alpha \min(x^\varepsilon, x^{-\varepsilon}) < \frac{g(tx)}{g(t)} < (1+\varepsilon)x^\alpha \max(x^\varepsilon, x^{-\varepsilon}).$$

4.2 Basics Properties

Let $F : \mathbb{R} \to [0,1]$ be a cdf. The function U maps n to $F^{-1}(1 - \frac{1}{n})$, formally $U(n) := \sup\{x \in \mathbb{R} : \frac{1}{1-F(x)} \leq n\}$. The function U plays a key role in extremal value theory (consider, e.g., [8]). It's relevance to our results is clear (just note that we could replace $F^{-1}(1 - \frac{1}{n})$ with $U(n)$ in all our theorems).

We further define the function, $\widetilde{U}$, which maps n to $F^{-1}(e^{-1/n})$, formally $\widetilde{U}(n) := \sup\{x \in \mathbb{R} : \frac{1}{-\ln(F(x))} \leq n\}$. Since for n large enough $e^{-1/n} \approx 1 - 1/n$, the functions U and $\widetilde{U}$ behave roughly the same for large n (as we show in Claim 6). The advantage of $\widetilde{U}$ is that the probability that the maximal element of n random variables chosen according to F is at most $\widetilde{U}(n/t)$ is e^{-t}, as one checks using the simple calculation: $\Pr[\max(X_1, \ldots, X_n) \leq \widetilde{U}(\frac{n}{t})] = (F(\widetilde{U}(\frac{n}{t})))^n = \exp(n \ln(F(\widetilde{U}(\frac{n}{t})))) = \exp(-t)$.

Claim 6. *If F is VM1 with index α then $U \in \mathrm{RV}_{1/\alpha}$, $\widetilde{U} \in \mathrm{RV}_{1/\alpha}$. If F is VM2 with upper bound $b = 0$ for index α, then $U \in \mathrm{RV}_{-1/\alpha}$ and $\widetilde{U} \in \mathrm{RV}_{-1/\alpha}$. Furthermore, in both cases $\lim_{x\to\infty} U(x)/\widetilde{U}(x) = 1$.*

Proof. The proof of the statement if F is VM1 can be found in [8], Theorem 1.1.11 and Corollary 1.2.10. If F is VM2, we have $\lim_{x\to 0} \frac{f(x)x}{1-F(x)} = \frac{g(y)y}{1-G(y)}$ where $G(y) = F(1/x)$, which implies the claim.

Furthermore, we get

$$\begin{aligned}
\lim_{n\to\infty} \frac{U(n)}{\widetilde{U}(n)} &= \lim_{x\to F^{-1}(1)} \frac{U(\frac{1}{1-F(x)})}{\widetilde{U}(\frac{1}{1-F(x)})} \\
&= \lim_{x\to F^{-1}(1)} \frac{x}{\widetilde{U}(\frac{1}{1-F(x)})} \\
&= \lim_{x\to F^{-1}(1)} \frac{1-F(x)}{-\ln(F(x))} = F(F^{-1}(1)) = 1.
\end{aligned}$$

$\square$

4.3 Theorems 1 and 4

Theorems 1 and 4 can be proven together. For this, we assume without loss of generality that $F^{-1}(1) = 0$ in case of Theorem 4 (otherwise apply the argument on $F(x + F^{-1}(1)))$. Then, we can write

$$\lim_{x\to F^{-1}(1)} \frac{xf(x)}{1-F(x)} = \alpha,$$

where α changed the sign compared to Theorem 4.

As described above, we let $\widetilde{U}$ be the inverse of the function $x \to \frac{1}{-\ln(F(x))}$. We recall from Claim 6 that $\widetilde{U} \in \mathrm{RV}_{1/\alpha}$ and $\lim_{n\to\infty} \frac{\widetilde{U}(n)}{F^{-1}(1-\frac{1}{n})} = 1$. Therefore, it is sufficient to show that for $n \to \infty$:

$$\int_{F^{-1}(0)}^{F^{-1}(1)} \big(F^n(x)\big)' x dx = \Gamma(\tfrac{\alpha-1}{\alpha})\widetilde{U}(n)(1+o(1)),$$

or, equivalently

$$\Delta^{(n)} := \Big(\frac{\int_{F^{-1}(0)}^{F^{-1}(1)} \big(F^n(x)\big)' x}{\widetilde{U}(n)} - \Gamma(\frac{\alpha-1}{\alpha})\Big) \in o(1) \quad (5)$$

We substitute $x = \widetilde{U}(n/y)$ in the above integral

$$\int_{F^{-1}(0)}^{F^{-1}(1)} (F^n(x))' x dx \quad (6)$$

$$= \int_{\widetilde{U}(0)}^{\widetilde{U}(\infty)} \big(\exp(n \ln(F(\widetilde{U}(n/y))))\big)' x dx \quad (7)$$

$$= \int_{\widetilde{U}(0)}^{\widetilde{U}(\infty)} \frac{d}{dx}\big(\exp(-y)\big)\widetilde{U}(\tfrac{n}{y}) dx \quad (8)$$

$$= \int_0^\infty \exp(-y)\widetilde{U}(\tfrac{n}{y}) dy. \quad (9)$$

Fix now $\varepsilon > 0$, and apply Proposition 5. In this case, we see that there exists $t(\varepsilon)$ such that for $nx \geq t(\varepsilon)$

$$(1-\varepsilon)x^{1/\alpha}\min(x^\varepsilon, x^{-\varepsilon}) < \frac{\widetilde{U}(nx)}{\widetilde{U}(n)} \quad (10)$$

$$< (1+\varepsilon)x^{1/\alpha}\max(x^\varepsilon, x^{-\varepsilon}). \quad (11)$$

We assume $n > t(\varepsilon)$, and with $\Gamma(\frac{\alpha-1}{\alpha}) = \int_0^\infty y^{-1/\alpha}e^{-y}$ in mind we express $\Delta^{(n)}$ as

$$\Delta^{(n)} = \int_0^\infty e^{-y}\frac{\widetilde{U}(\frac{n}{y})}{\widetilde{U}(n)} dy - \Gamma(\frac{\alpha-1}{\alpha}) \quad (12)$$

$$= \int_0^1 e^{-y}\Big(\frac{\widetilde{U}(\frac{n}{y})}{\widetilde{U}(n)} - y^{-1/\alpha}\Big) dy \quad (13)$$

$$+ \int_1^{\frac{n}{t(\varepsilon)}} e^{-y}\Big(\frac{\widetilde{U}(\frac{n}{y})}{\widetilde{U}(n)} - y^{-1/\alpha}\Big) dy \quad (14)$$

$$+ \int_{\frac{n}{t(\varepsilon)}}^\infty e^{-y}\Big(\frac{\widetilde{U}(\frac{n}{y})}{\widetilde{U}(n)} - y^{-1/\alpha}\Big) dy. \quad (15)$$

First, using (10)

$$\begin{aligned}
&\Big|\int_0^1 \exp(-y)\Big(\frac{\widetilde{U}(\frac{n}{y})}{\widetilde{U}(n)} - y^{-1/\alpha}\Big) dy\Big| \\
&\leq \Big|\int_0^1 \exp(-y)\Big((1+\varepsilon)(y^{-1/\alpha}y^{-\varepsilon}) - y^{-1/\alpha}\Big) dy\Big| \\
&\leq \underbrace{\Big|\int_0^1 \exp(-y)\Big(y^{-1/\alpha-\varepsilon} - y^{-1/\alpha}\Big) dy\Big|}_{\leq \int_0^1 y^{-1/\alpha-\varepsilon} - y^{-1/\alpha} dy = \frac{1}{1-\frac{1}{\alpha}-\varepsilon} - \frac{1}{1-\frac{1}{\alpha}}} \\
&\quad + \underbrace{\Big|\int_0^1 \exp(-y)\Big(\varepsilon(y^{1/\alpha-\varepsilon})\Big) dy\Big|}_{\leq \varepsilon\Gamma(\frac{\alpha-1}{\alpha}-\varepsilon)} \\
&\leq \Delta_1(\varepsilon)
\end{aligned}$$

with $\Delta_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Second, again with (10)

$$\begin{aligned}
&\Big|\int_1^{\frac{n}{t_0}} \exp(-y)\Big(\frac{\widetilde{U}(\frac{n}{y})}{\widetilde{U}(n)} - y^{-1/\alpha}\Big) dy\Big| \\
&\leq \Big|\int_1^{\frac{n}{t_0}} \exp(-y)\Big((1+\varepsilon)y^{-1/\alpha+\varepsilon} - y^{-1/\alpha}\Big) dy\Big| \\
&= \underbrace{\Big|\int_1^\infty \exp(-y)\Big(y^{-1/\alpha+\varepsilon} - y^{-1/\alpha}\Big) dy\Big|}_{=\Gamma(\frac{\alpha-1}{\alpha}+\varepsilon) - \Gamma(\frac{\alpha-1}{\alpha}) - \int_0^1 e^{-y}(y^{1/\alpha} - y^{1/\alpha-\varepsilon}) dy} \\
&\quad + \underbrace{\Big|\int_1^{\frac{n}{t_0}} \exp(-y)\Big(\varepsilon y^{-1/\alpha+\varepsilon}\Big) dy\Big|}_{\leq \varepsilon\Gamma(\frac{\alpha-1}{\alpha}+\varepsilon)} \\
&= \Delta_2(\varepsilon),
\end{aligned}$$

and again $\Delta_2(\varepsilon) \to 0$ as $\varepsilon \to 0$, since the $\int_0^1 e^{-y}(y^{1/\alpha} - y^{1/\alpha-\varepsilon})$ can be bounded as in the previous case.

Finally, consider the last term

$$\begin{aligned}
&\Big|\int_{\frac{n}{t(\varepsilon)}}^\infty \exp(-y)\Big(\frac{\widetilde{U}(\frac{n}{y})}{\widetilde{U}(n)} - y^{-1/\alpha}\Big) dy\Big| \\
&\leq \Big|\int_{\frac{n}{t(\varepsilon)}}^\infty \exp(-y)\frac{\widetilde{U}(\frac{n}{y})}{\widetilde{U}(n)} dy\Big| \\
&\quad + \Big|\int_{\frac{n}{t(\varepsilon)}}^\infty \exp(-y)y^{-1/\alpha} dy\Big|
\end{aligned}$$

Since $\widetilde{U}$ is monotone, we get

$$\begin{aligned}
&\Big|\int_{\frac{n}{t(\varepsilon)}}^\infty \exp(-y)\frac{\widetilde{U}(\frac{n}{y})}{\widetilde{U}(n)} dy\Big| \\
&\leq \frac{\max(|\widetilde{U}(0)|, |\widetilde{U}(t(\varepsilon))|)}{|\widetilde{U}(n)|}\int_{\frac{n}{t(\varepsilon)}}^\infty \exp(-y) dy \\
&= \frac{\max(|\widetilde{U}(0)|, |\widetilde{U}(t(\varepsilon))|)}{|\widetilde{U}(n)|}\exp(-\frac{n}{t(\varepsilon)}).
\end{aligned}$$

Furthermore, if n is big enough $\int_{\frac{n}{t(\varepsilon)}}^{\infty} \exp(-y)y^{-1/\alpha} < e^{-\frac{n}{2t(\varepsilon)}}$.

Collecting, these bounds we get that there are functions Δ_3, γ_1, γ_2

$$|\Delta^{(n)}| \leq \Delta_3(\varepsilon) + \frac{\gamma_1(\varepsilon)}{\widetilde{U}(n)} \exp(-n\gamma_3(\varepsilon))$$

where $\Delta_3 \in o(1)$.

In order to see that $\Delta^{(n)} \in o(1)$, let now $\delta > 0$ and fix ε small enough such that $\Delta_3(\varepsilon) < \frac{\delta}{2}$.

Using the representation theorem, the last term can then be bounded as

$$\frac{\Delta_3(\varepsilon)}{\widetilde{U}(n)} \exp(-n/t_0(\varepsilon)) = \frac{\Delta_3(\varepsilon)\exp(-n/t_0 - \int_{t_1}^{n} \frac{a(s)}{s} ds)}{c(t)}$$

which goes to zero as $n \to \infty$.

This proves Theorems 1 and 4.

4.4 Theorems 3 and 6

Again, Theorems 3 and 6 can be proven simultaneously. We use the following claim:

Claim 7. *Let $b \in \{0, \infty\}$, $g > 0$ differentiable and assume that*

$$\lim_{x \to b} \frac{xg'(x)}{g(x)} = \alpha.$$

Consider sequences $x_i \to b$, $\delta_i \in o(1)$. Then,

$$g(x_i(1 + \delta_i)) = g(x_i) \cdot (1 + \alpha\delta_i(1 + o(1))).$$

Proof. The claim follows straightforward from Proposition 5 in case $b = \infty$. In case $b = 0$ it follows by considering the function $g(1/x)$. □

Lemma 8. *Let F be a cdf with upper bound $b \in \{0, \infty\}$, and let*

$$\lim_{x \to b} \frac{xf(x)}{1 - F(x)} = \alpha. \tag{16}$$

(Here, $\alpha > 0$ if $b = \infty$ and $\alpha < 0$ if $b = 0$). Assume that the seller obtains a constant value $v < b$ if he does not sell the item. Then, the expected revenue in a discriminatory auction satisfies

$$r_i = \frac{\alpha - 1}{\alpha} F^{-1}(1 - \frac{\alpha - 1}{\alpha i})(1 + o(1)).$$

Proof. Let r_{i+1} be the revenue achieved with i bidders, and let p_i be the highest price offered to the first of those bidders. We have the recursion

$$r_{i+1} = F(p_i)r_i + (1 - F(p_i))p_i, \tag{17}$$

and we obtain

$$0 = \frac{dr_{i+1}}{dp_i} = f(p_i)r_i + 1 - F(p_i) - f(p_{i+1})p_i.$$

We let $\alpha_i := \frac{p_i f(p_i)}{1-F(p_i)}$, and observe for later use that the von-Mises condition (16) implies $\alpha_i \to \alpha$ as $i \to \infty$ (since the prizes must approach the upper limit $F^{-1}(1)$ if it exists, or diverge otherwise). We get

$$0 = f(p_i)r_i + \frac{p_i f(p_i)}{\alpha_i} - f(p_i)p_i,$$

and thus $p_i = r_i \frac{\alpha_i}{\alpha_i - 1}$. We insert this in (17) and formulate everything with the notation $p_i = U(m_i)$ (i.e., $F(p_i) = 1 - \frac{1}{m_i}$). We get

$$\begin{aligned} r_{i+1} &= \Big(1 - \frac{1}{m_i}\Big)r_i + \frac{1}{m_i} r_i \frac{\alpha_{i+1}}{\alpha_{i+1} - 1} \\ &= r_i(1 + \frac{1}{(\alpha_i - 1)m_i}), \end{aligned}$$

we can write this as

$$\frac{\alpha}{\alpha - 1} r_{i+1} = \frac{\alpha}{\alpha - 1} r_i(1 - \frac{1}{(\alpha_i - 1)m_i}), \tag{18}$$

Let ε_i be such that $\frac{\alpha}{\alpha-1} = (1 + \varepsilon_i)\frac{\alpha_i}{\alpha_i - 1}$. Then, (18) can be written as

$$p_{i+1}(1 + \varepsilon_{i+1}) = p_i(1 + \varepsilon_i)(1 - \frac{1}{(\alpha_i - 1)m_i})$$

Let $\widetilde{m}_i := U^{-1}(p_i(1 + \varepsilon_i))$. We apply U^{-1} on both sides and use Claim 7 to get

$$\begin{aligned} \widetilde{m}_{i+1} &= \widetilde{m}_i\big(1 - \frac{\alpha}{(\alpha_i - 1)m_i}(1 + o(1))\big) \\ &= \widetilde{m}_i - \frac{\widetilde{m}_i}{m_i} \frac{\alpha}{\alpha_i - 1}(1 + o(1)) \\ &= \widetilde{m}_i - \frac{\alpha}{\alpha - 1}(1 + o(1)). \end{aligned}$$

Since $\widetilde{m}_0$ is some constant, we get

$$\widetilde{m}_i = i\frac{\alpha}{\alpha - 1}(1 + o(1)).$$

Unrolling everything finishes the proof. □

4.5 Theorem 2

We first show that any price of the form (4) achieves an expected revenue of the form (3). Fix a constant $\delta \in [-1, 1]$ and let $p = \widetilde{U}(\frac{n}{\eta(\alpha)})(1 + \delta)$. We can see that this implies $p = \widetilde{U}(\frac{n}{\eta(\alpha)}(1 + \alpha\delta(1 + o(1))))$.

Thus, the revenue is at least, for $n > n_\delta$:

$$\begin{aligned} r &= \Big(1 - \exp(-\frac{\eta(\alpha)}{1 + \alpha\delta(1 + o(1))})\Big)p \\ &= \Big(1 - e^{-\eta(\alpha)}e^{-\eta(\alpha)\delta(1+o(1))}\Big)p \\ &\geq \Big(1 - e^{-\eta(\alpha)}(1 - 2\eta(\alpha)\delta)\Big)p = \Big(\frac{k\eta(k)}{1 + k\eta(k)}\Big)p(1 - O(\delta)). \end{aligned}$$

Since δ can be chosen arbitarily small, this proves this part of the theorem.

We now show that no higher revenue can be achieved. For this, define $p_x := \widetilde{U}(\frac{n}{x})$. We get for the revenue $r_x = (1-e^{-x})p_x$, and thus

$$\rho_x := \frac{r_x}{r_1} = \frac{1-e^{-x}}{1-e}\frac{p_x}{p_1} = \frac{1-e^{-x}}{1-e}\frac{\widetilde{U}(\frac{n}{x})}{\widetilde{U}(n)}.$$

We fix $\delta > 0$, apply Proposition 5. for $n > t_\delta$, $t_\delta n > x$ we get the upper bound

$$\rho_x \leq (1+\delta)\frac{1-e^{-x}}{1-e}x^{-1/\alpha}\max(x^\delta, x^{-\delta}).$$

We first consider the case $x \leq 1$. Then,

$$\rho_x \leq (1+\delta)\frac{1-e^{-x}}{1-e}x^\gamma.$$

where $\gamma = -\frac{1}{\alpha} - \delta$. The derivative of this upper bound is

$$(1+\delta)x^{\gamma-1}\frac{\gamma(e^{-x}-1) - e^{-x}x}{e-1}$$

This is zero if and only if $1 - \frac{x}{\gamma} = e^x$, which has unique positive solution $x = \eta(-\frac{1}{\gamma}) = \eta(\alpha - \delta)$. Thus, x in this case, the revenue can not exceed $r_1(1+\delta)\frac{1-e^{-x}}{1-e}x^\gamma$.

If $x > 1$ we get the bound

$$\rho_x \leq (1+\delta)\frac{1-e^{-x}}{1-e}x^\gamma$$

where now $\gamma = -\frac{1}{\alpha} + \delta$. Again, the zero is good.

Furthermore, we need to show that $x < t_\delta n$ is not a restriction. Thus, assume $x \geq t_\delta n$, i.e., the auctioneer selects a price smaller than $\widetilde{U}(\frac{1}{t_\delta})$. In this case, however,the revenue can be at most this constant.

4.6 Theorem 5

We now prove Theorem 5. For this, we first assume that the upper bound of the distribution function is 1.

A proof of the following claim can be found, for example, in [22].

Claim 9. *If* $\lim_{x\to 1}\frac{f(x)(1-x)}{1-F(x)} = \alpha$, *then for any* $\delta > 0$, *for any* $\varepsilon < \varepsilon_\delta$,

$$1 - \varepsilon^{\frac{1-\delta}{\alpha}} \leq F^{-1}(1-\varepsilon) \leq 1 - \varepsilon^{\frac{1+\delta}{\alpha}}$$

Lemma 10. *Let* F *be a cdf with* $F^{-1}(1) = 1$ *such that*

$$\lim_{x\to 1}\frac{f(x)(1-x)}{1-F(x)} = \alpha \in (0,\infty).$$

For any $\delta > 0$ *there exists an* n_δ *such that the expected revenue achieved by posting the symmetric price*

$$p_n = F^{-1}\big(1 - (1+\delta)\tfrac{\log(\alpha n)}{\alpha n}\big) \tag{19}$$

is at least

$$1 - (1-p_n)(1+\delta) = p_n - \delta + p_n\delta$$

for $n > n_\delta$.

Proof. The expected revenue is for a price p as in (19)

$$\begin{aligned} r &= (1-F^n(p))p \\ &= (1 - \Big(1 - (1+\delta)\frac{\log(\alpha n)}{\alpha n}\Big)^n)p \\ &\geq (1 - \exp(-(1+2\delta)\frac{\log(\alpha n)}{\alpha}))p \\ &\geq p - (\alpha n)^{-\frac{1+2\delta}{\alpha}} \end{aligned}$$

It remains to be shown that $(\alpha n)^{\frac{1+2\delta}{\alpha}} \leq \delta(1-p)$, for n large enough, which is equivalent to $1 - p \geq \frac{(\alpha n)^{\frac{1+2\delta}{\alpha}}}{\delta}$, which follows from Claim 9. □

Lemma 11. *Let* F *be a cdf with* $F^{-1}(1) = 1$ *such that*

$$\lim_{x\to 1}\frac{f(x)(1-x)}{1-F(x)} = \alpha \in (0,\infty).$$

For any $\delta > 0$ *there exists an* n_δ *such that the expected revenue in a posted price auction is at most*

$$1 - \Big(1 - F^{-1}\big(1 - \tfrac{\log(\alpha n)}{\alpha n}\big)\Big)(1-\delta)$$

for any $n > n_\delta$.

Proof. Consider a price p_δ of

$$p_\delta := p = F^{-1}\big(1 - (1-\delta)\tfrac{\log(\alpha n)}{\alpha n}\big).$$

If $p \leq p_\delta$, we have for the revenue r

$$r \leq p_\delta = F^{-1}\big(1 - (1-\delta)\tfrac{\log(\alpha n)}{\alpha n}\big) < \big(1 - \tfrac{\log(\alpha n)}{\alpha n}\big)(1 - 2\alpha\delta).$$

If $p > p_\delta$ the revenue is at most the probability that someone accepts. Thus,

$$\begin{aligned} r &< 1 - F^n(p_\delta) = 1 - \big(1 - (1-\delta)\tfrac{\log(\alpha n)}{\alpha n}\big)^n \\ &= 1 - \exp(-(1-\delta)\tfrac{\log(\alpha n)}{\alpha}) = 1 - (\alpha n)^{-\frac{1-\delta}{\alpha}} \\ &\leq F^{-n}(\frac{1}{\alpha n}). \end{aligned}$$

This is way smaller than what we want in the lemma. □

Together, we can now prove Theorem 5.

Proof (of Theorem 5). Let F be a function as in the Theorem, $b = F^{-1}(1)$. Define the function $\widetilde{F}(x) := F(bx)$. Assume the bidders valuation is chosen according to F. The expected revenue in a posted-price auction with price p is $(1-F^n(p))p = b\cdot(1-\widetilde{F}(p/b))\frac{p}{b}$, which is b times the expected revenue in a posted-price auction for distribution function $\widetilde{F}$. Note that $\widetilde{F}^{-1}(1) = 1$, and $b\widetilde{F}^{-1}(y) = F^{-1}(y)$. Furthermore

$$\begin{aligned} \lim_{x\to 1}\frac{(1-x)\tilde{f}(x)}{1-\widetilde{F}(x)} &= \lim_{x\to 1}\frac{(1-x)bf(bx)}{1-F(bx)} \\ &= \lim_{x\to b}\frac{(b-x)f(x)}{1-F(x)} \\ &= \alpha. \end{aligned}$$

Thus, the optimal revenue is $b\cdot(1-(1-\widetilde{F}^{-1}(1-(\frac{\log(\alpha n)}{n}))))(1+o(1)) = b - (b - b\widetilde{F}^{-1}(1-(\frac{\log(\alpha n)}{n})))(1+o(1)) = b - (b - F^{-1}(1-(\frac{\log(\alpha n)}{n})))(1+o(1))$. □

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