Settling the Complexity of Arrow-Debreu Equilibria in Markets with Additively Separable Utilities

Xi Chen[*] Decheng Dai[†] Ye Du[‡] Shang-Hua Teng[§]

Abstract

We prove that the problem of computing an Arrow-Debreu market equilibrium is PPAD-complete even when all traders use additively separable, piecewise-linear and concave utility functions. In fact, our proof shows that this market-equilibrium problem does not have a fully polynomial-time approximation scheme unless every problem in PPAD is solvable in polynomial time.

[*]Department of Computer Science, Princeton University.

[†]Department of Computer Science, Tsinghua University. Work done while visiting Princeton University. This work was supported in part by the National Natural Science Foundation of China Grant 60553001, and the National Basic Research Program of China Grant 2007CB807900, 2007CB807901.

[‡]Department of Electrical Engineering and Computer Science, University of Michigan.

[§]Microsoft Research New England. Affiliation starting from Fall 2009: Department of Computer Science, University of Southern California.

1 Introduction

One of the central developments in mathematical economics is the general equilibrium theory, which provides the foundation for competitive pricing [1, 34]. When specialized to exchange economies, it considers an *exchange market* in which there are m traders and n divisible goods, where trader i has an *initial endowment* of $w_{i,j} \geq 0$ of good j and a *utility function* $u_i : \mathbb{R}^n_+ \rightarrow \mathbb{R}$. The *individual goal* of trader i is to obtain a new bundle of goods that maximizes her utility. This new bundle can be specified by a column vector $\mathbf{x}_i \in \mathbb{R}^n_+$, where the j^{th} entry $x_{i,j}$ is the amount of good j that trader i is able to obtain after the exchange. Naturally, the exchange should satisfy $\sum_i x_{i,j} \leq \sum_i w_{i,j}$, for all good j.

The pioneering equilibrium theorem of Arrow and Debreu [1] states that if all the utilities $u_1, ..., u_m$ are quasi-concave, then under some mild conditions, the market has an *equilibrium price* $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}^n_+$: At this price, independently, each trader can sell her endowment virtually to the market to obtain a budget and then buys a bundle of goods with this budget from the market — which contains the union of all goods — that maximizes her utility. The equilibrium condition guarantees that the supply equals the demand and hence the market clears: Every good is sold and every trader's budget is completely spent.

The existence proof of Arrow and Debreu, based on Kakutani's fixed point theorem [27], is non-constructive in the view of polynomial-time computability. Despite the progress both on algorithms for and on the complexity-theoretic understanding of market equilibria, several fundamental questions concerning market equilibria, including some seemingly simple ones, remain unsettled.

Vijay Vazirani [30] wrote:

> *"Concave utility functions, even if they are additively separable over the goods, are not easy to deal with algorithmically. In fact, obtaining a polynomial time algorithm for such functions is a premier open question today."*

A function $u(x_1, ..., x_n)$ is an *additively separable* and *concave* function if there exist n real-valued concave functions $f_1, ..., f_n$ such that $u(x_1, ..., x_n) = \sum_{j=1}^{n} f_j(x_j)$. Noting that every concave function can be approximated by a piecewise linear and concave (PLC) function, Vazirani [30] further asked whether one can find an equilibrium in a market with additively separable PLC utility functions in polynomial time; or whether the problem is PPAD-hard. This open question has been echoed in several work since then [12, 23, 19, 36].

1.1 Our Contribution

In this paper, we settle the complexity of computing an Arrow-Debreu equilibrium in an exchange market with additively separable PLC utilities. We show that this equilibrium problem is PPAD-complete.

For an integer $t > 0$, a real-valued function $f(\cdot)$ is t-segment piecewise linear over $\mathbb{R}_+ = [0, +\infty)$ if f is continuous and $\mathbb{R}_+$ can be divided into t sub-intervals such that f is linear over every sub-interval. If each trader's utility is an additively separable t-segment PLC function, then we refer to the market as a *t-linear market*. Clearly, a market with linear utilities is a 1-linear market. In contrast to the fact that an Arrow-Debreu market equilibrium of a 1-linear market can be found in polynomial time [17, 29, 11, 13, 24], we show that even computing an Arrow-Debreu equilibrium in a 2-linear market is PPAD-complete, via a reduction from Sparse Bimatrix [5]: the problem of finding an approximate Nash equilibrium in a sparse two-player game (see Section 2.1 for the definition).

Our construction of the PPAD-complete markets has several nice technical elements. First we introduce a sequence of simple *linear* markets $\{\mathcal{M}_n\}$ with n goods, which we refer to as the *price-regulating* markets. $\mathcal{M}_n$ has the following nice *price-regulation property*: If $\mathbf{p}$ is a *normalized*[1] approximate equilibrium price vector of $\mathcal{M}_n$, then $p_k \in [1, 2]$ for all $k \in [n]$. This price-regulation property allows us to encode n free variables $x_1, ..., x_n$ between 0 and 1 using the n entries of $\mathbf{p}$ by setting $x_k = p_k - 1$.

As a key step in our analysis, we show that the price-regulation property is *stable* with respect to "*small perturbations*" to $\mathcal{M}_n$: When new traders are added to $\mathcal{M}_n$ (without introducing new goods), this property

[1] We say a price vector $\mathbf{p}$ is normalized if the smallest nonzero entry of $\mathbf{p}$ is equal to 1.

remains hold as long as the total amount of goods these traders carry with them is *small* compared to those of the traders in $\mathcal{M}_n$. We then show how to set the initial endowments and utility functions of new traders so that we can control the flows of goods in the market and set new requirements that every approximate equilibrium price vector $\mathbf{p}$ has to satisfy.

Using the stability of the price-regulating markets $\{\mathcal{M}_n\}$, we give a reduction from a two-player game to a 2-linear market $\mathcal{M}$: Given an $n \times n$ two-player game $(\mathbf{A}, \mathbf{B})$, we construct an additively separable PLC market by adding new traders — whose initial endowments are relatively small — to $\mathcal{M}_{2n+2}$, the price-regulating market with $2n+2$ goods. We use the first $2n$ entries of $\mathbf{p}$ to encode a pair of probability vectors $(\mathbf{x}, \mathbf{y})$: $x_k = p_k - 1$ and $y_k = p_{n+k} - 1$, $k \in [n]$. We then develop a novel way to enforce the Nash equilibrium constraints over $\mathbf{A}$, $\mathbf{B}$, $\mathbf{x}$ and $\mathbf{y}$ by carefully specifying the behaviors of the new traders. In doing so, we get a market $\mathcal{M}$ with the property that from every approximate market equilibrium $\mathbf{p}$ of $\mathcal{M}$, the pair $(\mathbf{x}, \mathbf{y})$ obtained above (after normalization) is an approximate Nash equilibrium of $(\mathbf{A}, \mathbf{B})$. Moreover, if $(\mathbf{A}, \mathbf{B})$ is a sparse game, then the relation of which traders are interested in which goods in $\mathcal{M}$ is also sparse.

In the construction of $\mathcal{M}$, the price-regulation property plays a critical role. It enables us to design the utility functions of the new traders so that we know exactly their preferences over the goods with respect to any approximate equilibrium price $\mathbf{p}$, even though we have no idea in advance about the entries of $\mathbf{p}$ when constructing $\mathcal{M}$.

We anticipate that our reduction techniques will help to resolve more complexity questions concerning other families of exchange markets such as the general CES and the hybrid linear-Leontief markets [6].

1.2 Related Work

The computation of an equilibrium price in an exchange market has been a challenging problem in mathematical economics [30]. The matter is more complex because some markets only have irrational equilibria, making the computation of exact equilibria with a finite-precision algorithm impossible. One alternative approach to handle irrationality is to express equilibria in some simple algebraic form. However, it turns out that finding an exact market equilibrium in general is not computable [32].

To circumvent the irrationality, one usually uses some notion of approximate market equilibria. There are various notions of approximate equilibria: some require that the approximation solution is within a small geometric distance from an exact equilibrium, while others only require that the supply-demand condition and/or the individual optimality condition are approximately satisfied. In this paper, following Scarf [33], we consider the latter notion of approximate market equilibria.

1.2.1 Algorithms for Market Equilibria

Scarf pioneered the algorithmic development of computing general competitive equilibria [33]. His approach combined numerical approximation with combinatorial insights used in Sperner's lemma [35] for fixed points and in Lemke and Howson's algorithm for two-player games. Although his algorithm may not always run in polynomial time, Scarf's work has profound impact to computational economics.

Building on the success of convex programming [17], polynomial-time algorithms have been developed for special markets whose sets of equilibria enjoy some degree of convexity. For Arrow-Debreu markets with linear utility functions, Nenakov and Primak gave a polynomial-time algorithm [29], and there are now several polynomial-time algorithms for computing or approximating market equilibria with linear utility functions [11, 13, 24, 18, 25, 14, 38]. Other polynomial-time algorithms for special markets include Eaves's algorithm for Cobb-Douglas markets [16] and Devanur and Vazirani's algorithm for markets with spending constraint utilities [15] (also see [36]). The convex programming based approach has been extended to all markets whose utilities satisfy weak gross substitutability (WGS) by Codenotti, Pemmaraju, and Varadarajan [9]. In [8], Codenotti, McCune, and Varadarajan showed that for markets that satisfy WGS, there is a price-adjustment mechanism called *tâonnement* that converges to an approximate equilibrium efficiently.

A closely related market model is Fisher's model [2]. In this model, there are two types of traders in the market: *producers* and *consumers*. Each consumer comes to the market with a budget and a utility function. Each producer comes to the market with an endowment of goods, and will sell them to the consumers for money. A market equilibrium is then a price vector for goods so that if each consumer spends all her budget to maximize her utility, then the market clears. An (approximate) market equilibrium in a Fisher's market with CES (constant elasticity of substitution) utility functions [17, 38, 37, 13, 26] or with piecewise linear utility functions [37] can be found in polynomial time.

However, progress on Arrow-Debreu exchange markets whose sets of equilibria do not enjoy convexity has been relatively slow. There are only a few algorithms in this category. Devanur and Kannan [12] gave a polynomial-time algorithm for PLC markets with a constant number of goods. Codenotti et. al. [7] gave a polynomial-time algorithm for CES markets when the elasticity of substitution $s \geq 1/2$.

1.2.2 The Complexity of Equilibrium Problems

Papadimitriou initiated the complexity-theoretic study of fixed-point computations [31]. He introduced the complexity class PPAD, and proved that the problem of finding a Nash equilibrium in a two-player game, the computational version of Sperner's Lemma, and the problem of computing an approximate fixed point are members of PPAD.

Recently, there was a series of developments that characterized the computational complexity of several equilibrium problems in game theory. Daskalakis, Goldberg and Papadimitriou [21] proved that the problem of finding an exponentially-precise Nash equilibrium of a four-player game is PPAD-complete. Chen and Deng [3] then proved that finding a two-player Nash equilibrium is also PPAD-complete. Chen and Deng's result, together with an earlier reduction of [10], implies that computing a market equilibrium in an Arrow-Debreu market with Leontief utilities is PPAD-hard. On the approximation front, Chen, Deng, and Teng [4] proved that it is PPAD-complete to find a polynomially-precise approximate equilibrium in two-player or multi-player games. Huang and Teng [23] then extended this approximation result to Leontief market equilibria. Their approximation result also implies that the market equilibrium problem with CES utility functions is PPAD-hard, if the elasticity of substitution s is sufficiently small.

2 Preliminaries

2.1 Complexity of Nash Equilibria in Sparse Two-Player Games

A two-player game is defined by the payoff matrices $(\mathbf{A}, \mathbf{B})$ of its two players. In this paper, we assume that both players have n choices of actions and hence $\mathbf{A}$ and $\mathbf{B}$ are square matrices with n rows and columns. We use $\Delta^n \subset \mathbb{R}^n$ to denote the set of probability distributions of n dimensions.

A pair of probability vectors $(\mathbf{x}, \mathbf{y})$ (i.e., $\mathbf{x} \in \Delta^n$ and $\mathbf{y} \in \Delta^n$) is a Nash equilibrium of $(\mathbf{A}, \mathbf{B})$, if for all i and j in $[n] = \{1, 2, ..., n\}$, $\mathbf{A}_i \mathbf{y}^T < \mathbf{A}_j \mathbf{y}^T \Rightarrow x_i = 0$ and $\mathbf{x}\mathbf{B}_i < \mathbf{x}\mathbf{B}_j \Rightarrow y_i = 0$, where we use $\mathbf{A}_i$ and $\mathbf{B}_i$ to denote the ith row vector of $\mathbf{A}$ and the ith column vector of $\mathbf{B}$, respectively.

Definition 1 (Well-Supported Nash Equilibria)**.** *For $\epsilon > 0$, $(\mathbf{x}, \mathbf{y})$ is an ϵ-well-supported Nash equilibrium of $(\mathbf{A}, \mathbf{B})$, if $\mathbf{x}, \mathbf{y} \in \Delta^n$ and for all $i, j \in [n]$,*

$$\mathbf{A}_i \mathbf{y}^T + \epsilon < \mathbf{A}_j \mathbf{y}^T \implies x_i = 0, \quad \textit{and} \quad \mathbf{x}\mathbf{B}_i + \epsilon < \mathbf{x}\mathbf{B}_j \implies y_i = 0. \tag{1}$$

Definition 2 (Sparse Normalized Two-Player Games)**.** *A two-player game $(\mathbf{A}, \mathbf{B})$ is* normalized *if every entry of $\mathbf{A}$ and $\mathbf{B}$ is between -1 and 1. We say a two-player game $(\mathbf{A}, \mathbf{B})$ is* sparse *if every row and every column of $\mathbf{A}$ and $\mathbf{B}$ have at most 10 nonzero entries.*

Let Sparse Bimatrix denote the search problem of finding an n^{-6}-well-supported Nash equilibrium in an $n \times n$ sparse normalized two-player game, then by [5], Sparse Bimatrix is PPAD-complete.

2.2 Markets with Additively Separable PLC Utilities

Let $\mathcal{G} = \{G_1, ..., G_n\}$ denote a set of n divisible goods and $\mathcal{T} = \{T_1, ..., T_m\}$ denote a set of traders. For each trader T_i, we use $\mathbf{w}_i \in \mathbb{R}^n_+$ to denote her initial endowment and $u_i : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+$ to denote her utility function. In this paper, we will focus on markets with additively separable piecewise linear and concave utilities.

A continuous function $r(\cdot)$ over $\mathbb{R}_+$ is said to be *t-segment* piecewise linear and concave (PLC), if $r(0) = 0$ and there exists a tuple $[\theta_0 > \theta_1 > ... > \theta_t \geq 0; 0 < a_1 < a_2 < ... < a_t]$ of length $2t + 1$, such that

1. For any $i \in [0 : t - 1]$, the restriction of f over $[a_i, a_{i+1}]$ ($a_0 = 0$) is a segment of slope θ_i;
2. The restriction of f over $[a_t, +\infty)$ is a ray of slope θ_t.

The $(2t + 1)$-tuple $[\theta_0, \theta_1, ..., \theta_t; a_1, a_2, ..., a_t]$ is also called the *representation* of $r(\cdot)$. Moreover, we say $r(\cdot)$ is *strictly monotone* if $\theta_t > 0$, and is *α-bounded* for some $\alpha \geq 1$ if $\alpha \geq \theta_0 > \theta_1 > ... > \theta_t \geq 1$.

Definition 3. *A function $u(\cdot) : \mathbb{R}^n_+ \to \mathbb{R}_+$ is said to be an* additively separable PLC *function if there exist a set of n* PLC *functions $r_1(\cdot), ..., r_n(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ such that $u(\mathbf{x}) = \sum_{j \in [n]} r_j(x_j)$, for all $\mathbf{x} \in \mathbb{R}^n_+$.*

In such a market, we use, for each trader T_i, $r_{i,j}(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ to denote her PLC function with respect to good $G_j \in \mathcal{G}$. In another word, we have $u_i(\mathbf{x}) = \sum_{j \in [n]} r_{i,j}(x_j)$, for all $\mathbf{x} \in \mathbb{R}^n_+$. We use $\mathbf{p} \in \mathbb{R}^n_+$ to denote a price vector, where $\mathbf{p} \neq \mathbf{0}$ and p_j is the price of G_j (we always assume that $\mathbf{p}$ is *normalized*: the smallest nonzero entry of $\mathbf{p}$ equals 1). Given $\mathbf{p}$, we let $\mathrm{OPT}(i, \mathbf{p})$ denote the set of allocations that maximize $u_i(\cdot)$:

$$\mathrm{OPT}(i, \mathbf{p}) = \mathrm{argmax}_{\,\mathbf{x} \in \mathbb{R}^n_+,\ \mathbf{x} \cdot \mathbf{p} \leq \mathbf{w}_i \cdot \mathbf{p}}\ u_i(\mathbf{x}).$$

We use $\mathcal{X} = \{\mathbf{x}_i \in \mathbb{R}^n_+ : i \in [m]\}$ to denote an allocation of the market: For each trader $T_i \in \mathcal{T}$, $\mathbf{x}_i \in \mathbb{R}^n_+$ is the amount of goods that T_i receives. In particular, the amount of G_j that T_i receives in $\mathcal{X}$ is $x_{i,j}$.

Definition 4 (Arrow-Debreu [1])**.** *A market equilibrium is a non-zero price vector $\mathbf{p} \in \mathbb{R}^n_+$ such that there exists an allocation $\mathcal{X}$ which has the following properties:* 1)*. Every trader gets an optimal bundle: For every $T_i \in \mathcal{T}$, we have $\mathbf{x}_i \in \mathrm{OPT}(i, \mathbf{p})$; and* 2)*. The market clears:*

For every good $G_j \in \mathcal{G}$, $\sum_{i \in [m]} x_{i,j} \leq \sum_{i \in [m]} w_{i,j}$; If $p_j > 0$, then $\sum_{i \in [m]} x_{i,j} = \sum_{i \in [m]} w_{i,j}$.

In general, not every market has an equilibrium price vector. For the additively separable PLC markets considered here, the following condition guarantees the existence of an equilibrium. Theorem 1 is a corollary of Maxfield [28], and the proof can be found in Appendix A.

Definition 5 (Economy Graphs [28])**.** *Given an additively separable* PLC *market, we define a directed graph $G = (\mathcal{T}, E)$ as follows. The vertex set of the graph is exactly $\mathcal{T}$, the set of traders in the market. For every two traders $T_i \neq T_j \in \mathcal{T}$, we have an edge from T_i to T_j if there exists an integer $k \in [n]$ such that $w_{i,k} > 0$ and $r_{j,k}(\cdot)$ is strictly monotone. In another word, trader T_i possesses a good that T_j wants. G is called the* economy graph *of the market* [28, 7]*. We say the market is* strongly connected *if G is* strongly connected*.*

Theorem 1. *Let $\mathcal{M}$ be a market with additively separable* PLC *utilities. If it is strongly connected, then a market equilibrium $\mathbf{p}$ exists. Moreover, if all the parameters of $\mathcal{M}$ are rational numbers, then it must have a rational market equilibrium $\mathbf{p}$. The number of bits we need to describe $\mathbf{p}$ is polynomial in the input size of $\mathcal{M}$ (that is, the number of bits we need to describe the market $\mathcal{M}$).*

2.3 Definition of the Sparse Market Equilibrium Problem

By Theorem 1, the following problem Market is well defined: The input of the problem is an additively separable PLC market $\mathcal{M}$ that is both rational and strongly connected; and the output is a rational market equilibrium of $\mathcal{M}$. In the rest of the section, we define a much more restricted version of Market: Sparse Market. The main result of the paper is that Sparse Market is PPAD-complete.

First, the input of Sparse Market is an additively separable PLC market which not only is strongly connected, but also satisfies the following three conditions:

Definition 6 (α-Bounded Markets)**.** *We say an additively separable* PLC *market is* α-bounded*, for some $\alpha \geq 1$, if for all T_i and G_j, $r_{i,j}(\cdot)$ is either the zero function ($r_{i,j}(x) = 0$ for all x) or α-bounded.*

Definition 7 (2-Linear Markets). *We call an additively separable* PLC *market* $\mathcal{M}$ *a* 2-linear *market, if for all* $T_i \in \mathcal{T}$ *and* $G_j \in \mathcal{G}$, $r_{i,j}(\cdot)$ *has at most two segments.*

Definition 8 (t-Sparse Markets). *We say an additively separable* PLC *market is* t-sparse*, for some integer* $t > 0$*, if* 1) *For every* $T_i \in \mathcal{T}$, $|\text{supp}(\mathbf{w}_i)| \le t$; *and* 2) *For every* $T_i \in \mathcal{T}$*, the number of* $j \in [n]$ *such that* $r_{i,j}(\cdot)$ *is not the zero function is at most* t*. In another word, every trader owns at most* t *goods at the beginning and is interested in at most* t *goods.*

We use the following definition of approximate market equilibria:

Definition 9 (ϵ-Approximate Market Equilibrium). *Given an additively separable* PLC *market* $\mathcal{M}$*, we say* $\mathbf{p}$ *is an* ϵ*-approximate market equilibrium of* $\mathcal{M}$*, for some* $\epsilon \ge 0$*, if there is an allocation* $\mathcal{X} = \{\mathbf{x}_i \in \mathbb{R}^n_+ : i \in [m]\}$ *such that every trader gets an optimal bundle with respect to* $\mathbf{p}$*:* $\mathbf{x}_i \in \text{OPT}(i, \mathbf{p})$ *for all* $i \in [m]$*; and*

The market clears approximately: For every $G_j \in \mathcal{G}$, $\left| \sum_{i\in[m]} x_{i,j} - \sum_{i\in[m]} w_{i,j} \right| \le \epsilon \cdot \sum_{i\in[m]} w_{i,j}$.

We remark that there are various notions of approximate market equilibria. The reason we adopted the one above is to simplify the analysis. The construction in Section 4 actually works for some other notions of approximate equilibria, e.g., the one that allows the allocation to be approximately optimal for each trader.

Finally, we let Sparse Market denote the following search problem:

> The input of the problem is a 2-linear market $\mathcal{M}$ that is strongly connected, 27-bounded, and 23-sparse; and the output is an n^{-13}-approximate market equilibrium of $\mathcal{M}$, where n is the number of goods in the market.

It is tedious but not hard to show that Sparse Market is a problem in PPAD[2]. One can actually replace the constant 27 here by any constant larger than 1 and our main result, Theorem 2, below still holds. The constant 23, however, is related to the constant 10 in Definition 2. The main result of the paper is

Theorem 2 (Main). Sparse Market *is* PPAD*-complete.*

3 A Price-Regulating Market

We now construct the family of price-regulating market $\{\mathcal{M}_n\}$. For each positive integer $n \ge 2$, $\mathcal{M}_n$ has n goods and satisfies the following strong price regulation property.

Property 1 (Price Regulation). *A price vector* $\mathbf{p}$ *is a normalized* n^{-1}*-approximate equilibrium of* $\mathcal{M}_n$ *if and only if* $1 \le p_k \le 2$*, for all* $k \in [n]$*.*

We start with some notation. The goods in $\mathcal{M}_n$ are $\mathcal{G} = \{G_1, ..., G_n\}$, and the traders in $\mathcal{M}_n$ are

$$\mathcal{T} = \{T_{\mathbf{s}} : \mathbf{s} \in S\}, \quad \text{where} \quad S = \{\mathbf{s} = (i,j) : 1 \le i \ne j \le n\}.$$

For every trader $T_{\mathbf{s}} \in \mathcal{T}$, we use $\mathbf{w}_{\mathbf{s}} \in \mathbb{R}^n_+$ to denote her initial endowment, $u_{\mathbf{s}} : \mathbb{R}^n_+ \to \mathbb{R}_+$ to denote her utility function, $r_{\mathbf{s},k}(\cdot)$ to denote her PLC function with respect to G_k, and $\text{OPT}(\mathbf{s}, \mathbf{p})$ to denote the set of bundles that maximize her utility with respect to $\mathbf{p}$.

$\mathcal{M}_n$ is *linear*, in which for all $\mathbf{s} \in S$ and $k \in [n]$, $r_{\mathbf{s},k}(\cdot)$ is a ray starting at $(0,0)$. In the construction below, we use $r_{\mathbf{s},k}(\cdot) \Leftarrow [\theta]$ to denote the action of setting $r_{\mathbf{s},k}(\cdot)$ to be the linear function of slope $\theta \ge 0$.

[2] In [20], the author showed how to construct a continuous map from any market with quasi-concave utilities such that the set of fixed points of the map is precisely the set of equilibria of the market. When the market is additively separable PLC, one can show that the continuous map is indeed Lipschitz continuous. As a result, one can reduce the problem of finding an approximate market equilibrium to the problem of finding an approximate fixed point in a Lipschitz continuous map. This implies a reduction from Sparse Market to the discrete fixed point problem studied in [22] (also see [4] for the high-dimensional version) which is in PPAD, and thus, the former is also in PPAD.

Construction of $\mathcal{M}_n$: First, we set the initial endowment vectors $\mathbf{w_s}$: For every $\mathbf{s} = (i, j) \in S$, we have $w_{\mathbf{s},k} = 1/n$ if $k = i$; and $w_{\mathbf{s},k} = 0$ otherwise. Second, we set the PLC functions $r_{\mathbf{s},k}(\cdot)$: For all $\mathbf{s} = (i, j) \in S$ and $k \in [n]$, we set $r_{\mathbf{s},k}(\cdot) \Leftarrow [\theta]$ and $\theta = 0$ if $k \neq i, j$; $\theta = 1$ if $k = j$; and $\theta = 2$ if $k = i$.

It is easy to check that $\mathcal{M}_n$ constructed above is strongly connected, 2-bounded, and 2-sparse.

Proof of Property 1. The first direction is trivial. If $1 \leq p_k \leq 2$ for all $k \in [n]$, then $\mathcal{X} = \{\mathbf{x_s} = \mathbf{w_s} : \mathbf{s} \in S\}$ is a market clearing allocation that provides an optimal bundle of goods for each trader at price $\mathbf{p}$.

The second direction is less trivial. Let $\mathbf{p}$ be a normalized n^{-1}-approximate market equilibrium of $\mathcal{M}_n$, and $\mathcal{X}$ be an optimal allocation that clears the market. First, it is easy to check that p_k must be positive for all $k \in [n]$ since otherwise, we have $x_{\mathbf{s},k} = +\infty$ for all $\mathbf{s} = (i, j)$ such that $k = i$ or j, which contradicts the assumption that $\mathbf{p}$ is an approximate equilibrium.

Since $\mathbf{p}$ is normalized, we have $p_k \geq 1$ for all $k \in [n]$. Now assume for contradiction that Property 1 is not true, then without loss of generality, we may assume that $p_1 = \max_k p_k > 2$ and $p_2 = \min_k p_k = 1$. To reach a contradiction, we focus on the amount of G_1 each trader gets in the allocation $\mathcal{X}$. First, if $1 \notin \{i, j\}$ where $\mathbf{s} = (i, j)$, then we have $x_{\mathbf{s},1} = 0$; Second, if $i = 1$ and $j = 2$, then $x_{\mathbf{s},1} = 0$ since $2/p_1 < 1/p_2$ and $T_\mathbf{s}$ likes G_2 better than G_1 with respect to the price vector $\mathbf{p}$; Third, if $j = 1$, then $x_{\mathbf{s},1} = 0$ since $1/p_1 < 2/p_i$ and $T_\mathbf{s}$ likes G_i better than G_1; Finally, for all $\mathbf{s} = (i, j)$ such that $i = 1$ and $j \neq 2$, we have $x_{\mathbf{s},1} \leq 1/n$ since the budget of $T_\mathbf{s}$ is exactly $(1/n) \cdot p_1$. As a result, we have

$$\sum_{\mathbf{s} \in S} x_{\mathbf{s},1} \leq (n-2)/n \quad \text{while} \quad \sum_{\mathbf{s} \in S} w_{\mathbf{s},1} = (n-1)/n, \quad \text{but} \quad \left| \frac{n-2}{n} - \frac{n-1}{n} \right| > \frac{1}{n} \cdot \frac{n-1}{n},$$

which contradicts the assumption that $\mathbf{p}$ is an n^{-1}-approximate market equilibrium. □

4 Reduction from Sparse Bimatrix to Sparse Market

In this section, we give a polynomial-time reduction from Sparse Bimatrix to Sparse Market. Given an $n \times n$ sparse two-player game $(\mathbf{A}, \mathbf{B})$, where $\mathbf{A}, \mathbf{B} \in [-1, 1]^{n \times n}$, we build an additively separable PLC market $\mathcal{M}$ by adding more traders to the price-regulating market $\mathcal{M}_{2n+2}$. There are $2n+2$ goods $\mathcal{G} = \{G_1, ..., G_{2n}, G_{2n+1}, G_{2n+2}\}$ in $\mathcal{M}$, and the traders $\mathcal{T}$ in $\mathcal{M}$ are

$$\mathcal{T} = \{T_\mathbf{s}, T_\mathbf{u}, T_\mathbf{v}, T_i : \mathbf{s} \in S, \mathbf{u} \in U, \mathbf{v} \in V, i \in [2n]\},$$

where $S = \{(i, j) : 1 \leq i \neq j \leq 2n+2\}$, $U = \{(i, j, 1) : 1 \leq i \neq j \leq n\}$ and $V = \{(i, j, 2) : 1 \leq i \neq j \leq n\}$. The traders $T_\mathbf{s}$, where $\mathbf{s} \in S$, have almost the same initial endowments $\mathbf{w_s}$ and PLC functions $r_{\mathbf{s},k}(\cdot)$ as in $\mathcal{M}_{2n+2}$. We only slightly modify the parameters to ease the analysis.

For each agent $T \in \mathcal{T}$, we will set her PLC function $r(\cdot)$ with respect to G_k, $k \in [2n+2]$, to one of the following functions: *1).* $r(\cdot)$ is the zero function: $r(x) = 0$ for all $x \geq 0$ (denoted by $r(\cdot) \Leftarrow [0]$); or *2).* $r(\cdot)$ is a ray: $r(x) = \theta \cdot x$ for all $x \geq 0$ (denoted by $r(\cdot) \Leftarrow [\theta]$); or *3).* $r(\cdot)$ is a 2-segment PLC function with representation $[\theta_0, \theta_1; a_1]$ (denoted by $r(\cdot) \Leftarrow [\theta_0, \theta_1; a_1]$).

4.1 Setting up the Market

4.1.1 Traders $T_\mathbf{s}$, where $\mathbf{s} \in S$

For each trader $T_\mathbf{s} \in \mathcal{T}$, where $\mathbf{s} = (i, j) \in S$, we set her initial endowment $\mathbf{w}_s$ and her PLC functions $r_{\mathbf{s},k}(\cdot)$ almost the same as hers in $\mathcal{M}_{2n+2}$. The initial endowment $\mathbf{w_s}$ is set as: $w_{\mathbf{s},k} = 1/n$ if $k = i$; and $w_{\mathbf{s},k} = 0$ otherwise, where $k \in [2n+2]$. The PLC function $r_{\mathbf{s},k}(\cdot)$ is set as: $r_{\mathbf{s},k}(\cdot) \Leftarrow [\theta]$ and $\theta = 0$ if $k \notin \{i, j\}$; $\theta = 1$ if $k = j$; and $\theta = 2$ if $k = i$, where $k \in [2n+2]$.

4.1.2 Traders $T_\mathbf{u}$, where $\mathbf{u} \in U$

Now let $\mathbf{u} = (i, j, 1)$ be a triple in U with $1 \leq i \neq j \leq n$. We use $\mathbf{A}_i$ and $\mathbf{A}_j$ to denote the ith and jth row vectors of $\mathbf{A}$, respectively. We define $\mathbf{C}$ and $\mathbf{D}$ to be the following two n-dimensional vectors: For $k \in [n]$,

$$(C_k, D_k) = (A_{i,k} - A_{j,k}, 0) \text{ if } A_{i,k} - A_{j,k} \geq 0; \text{ and } (C_k, D_k) = (0, A_{j,k} - A_{i,k}) \text{ otherwise.}$$

By definition, we have $\mathbf{C} - \mathbf{D} = \mathbf{A}_i - \mathbf{A}_j$ while both vectors $\mathbf{C}$ and $\mathbf{D}$ are non-negative. Moreover, because $\mathbf{A}$ is sparse, the number of nonzero entries in either $\mathbf{C}$ or $\mathbf{D}$ is at most 20, and each entry is between 0 and 2. We also let E, F be the following non-negative numbers: Let $C = \sum_{k\in[n]} C_k$ and $D = \sum_{k\in[n]} D_k$, then

$$(E, F) = (D - C, 0) \text{ if } D \geq C; \text{ and } (E, F) = (0, C - D) \text{ otherwise.} \tag{2}$$

Accordingly we have $E, F \geq 0$ and $E + C = F + D$. Moreover, since $\mathbf{C}, \mathbf{D}$ are sparse, we have $E, F \leq 20 \cdot 2$.

Using $\mathbf{C}$ and E, we set the initial endowment $\mathbf{w_u} = (w_{\mathbf{u},1}, ..., w_{\mathbf{u},2n+1}, w_{\mathbf{u},2n+2})$ of $T_\mathbf{u}$ as follows:

1. $w_{\mathbf{u},i} = 1/n^4$; $w_{\mathbf{u},k} = w_{\mathbf{u},2n+2} = 0$ for all other $k \in [n]$;
2. $w_{\mathbf{u},n+k} = C_k/n^5$ for all $k \in [n]$; and $w_{\mathbf{u},2n+1} = E/n^5$.

It is easy to verify that the number of nonzero entries in $\mathbf{w_u}$ is at most 22.

Using $\mathbf{D}$ and F, we set the PLC utility functions $r_{\mathbf{u},k}(\cdot)$, where $k \in [2n+2]$, of $T_\mathbf{u}$ as follows:

1. $r_{\mathbf{u},i}(\cdot) \Leftarrow [9, 1; 1/n^4]$; and $r_{\mathbf{u},k}(\cdot) \Leftarrow [0]$ for all other $k \in [n]$;
2. $r_{\mathbf{u},2n+2}(\cdot) \Leftarrow [3]$;
3. $r_{\mathbf{u},n+k}(\cdot) \Leftarrow [0]$ for all $k \in [n]$ such that $D_k = 0$;
4. $r_{\mathbf{u},n+k}(\cdot) \Leftarrow [27, 1; D_k/n^5]$ for all $k \in [n]$ such that $D_k > 0$; and
5. $r_{\mathbf{u},2n+1}(\cdot) \Leftarrow [0]$ if $F = 0$; and $r_{\mathbf{u},2n+1}(\cdot) \Leftarrow [27, 1; F/n^5]$ if $F > 0$.

Note that the number of $k \in [2n+2]$ such that $r_{\mathbf{u},k}(\cdot)$ is not the zero function is at most 23.

The constants 1, 3, 9 and 27 in the construction may look strange at first sight. The motivation is that, if the price regulation property still holds for the new market $\mathcal{M}$ (which turns out to be true), then we know exactly the preference of $T_\mathbf{u}$ over the goods since $3 > 2$. See the proof of Lemma 4 for more details.

4.1.3 Traders $T_\mathbf{v}$, where $\mathbf{v} \in V$

The behavior of $T_\mathbf{v}$, $\mathbf{v} \in V$, is very similar to that of $T_\mathbf{u}$ except that it works on the second matrix $\mathbf{B}$.

Let $\mathbf{v} = (i, j, 2)$ be a triple in V with $1 \leq i \neq j \leq n$. We let $\mathbf{B}_i$ and $\mathbf{B}_j$ denote the ith and jth column vectors of $\mathbf{B}$, respectively. Similarly, we define the following n-dimensional vectors $\mathbf{C}$ and $\mathbf{D}$: For $k \in [n]$,

$$(C_k, D_k) = (B_{k,i} - B_{k,j}, 0) \text{ if } B_{k,i} - B_{k,j} \geq 0; \text{ and } (C_k, D_k) = (0, B_{k,j} - B_{k,i}) \text{ otherwise.}$$

As a result, we have $\mathbf{C} - \mathbf{D} = \mathbf{B}_i - \mathbf{B}_j$, while $\mathbf{C}, \mathbf{D}$ are non-negative and sparse. We also define $E, F \geq 0$ in a similar way so that $E + \sum_{k\in[n]} C_k = F + \sum_{k\in[n]} D_k$ and $0 \leq E, F \leq 40$.

Using $\mathbf{C}$ and E, we set the initial endowment vector $\mathbf{w_v} = (w_{\mathbf{v},1}, ..., w_{\mathbf{v},2n+1}, w_{\mathbf{v},2n+2})$ of $T_\mathbf{v}$ to be

1. $w_{\mathbf{v},n+i} = 1/n^4$; $w_{\mathbf{v},n+k} = w_{\mathbf{v},2n+2} = 0$ for all other $k \in [n]$;
2. $w_{\mathbf{v},k} = C_k/n^5$ for all $k \in [n]$; and $w_{\mathbf{v},2n+1} = E/n^5$.

Using $\mathbf{D}$ and F, we set the PLC utility functions $r_{\mathbf{v},k}(\cdot)$, where $k \in [2n+2]$, of $T_\mathbf{v}$ as follows:

1. $r_{\mathbf{v},n+i}(\cdot) \Leftarrow [9, 1; 1/n^4]$; and $r_{\mathbf{v},n+k}(\cdot) \Leftarrow [0]$ for all other $k \in [n]$;
2. $r_{\mathbf{v},2n+2}(\cdot) \Leftarrow [3]$;
3. $r_{\mathbf{v},k}(\cdot) \Leftarrow [0]$ for all $k \in [n]$ such that $D_k = 0$;
4. $r_{\mathbf{v},k}(\cdot) \Leftarrow [27, 1; D_k/n^5]$ for all $k \in [n]$ such that $D_k > 0$; and
5. $r_{\mathbf{v},2n+1}(\cdot) \Leftarrow [0]$ if $F = 0$; and $r_{\mathbf{v},2n+1}(\cdot) \Leftarrow [27, 1; F/n^5]$ if $F > 0$.

Again, $|\text{supp}(\mathbf{w_v})| \leq 22$ and the number of indices k such that $r_{\mathbf{v},k}(\cdot)$ is not the zero function is at most 23.

4.1.4 Traders T_i, where $i \in [2n]$

Finally for each $i \in [2n]$ we set the initial endowment vector $\mathbf{w}_i = (w_{i,1}, ..., w_{i,2n+1}, w_{i,2n+2})$ of T_i as follows: $w_{i,2n+1} = 1/n^{12}$; and $w_{i,k} = 0$ for all other $k \in [2n+2]$. We set the PLC functions $r_{i,k}(\cdot)$, $k \in [2n+2]$, of T_i as follows: $r_{i,i}(\cdot) \Leftarrow [1]$; and $r_{i,k}(\cdot) \Leftarrow [0]$ for all other $k \in [2n+2]$.

4.2 From Approximate Market Equilibria to Approximate Nash Equilibria

By definition, it is easy to verify that $\mathcal{M}$ constructed above is a 2-linear additively separable PLC market which is strongly connected, 27-bounded and 23-sparse. Let $N = 2n+2$, the number of goods in $\mathcal{M}$. Then to prove Theorem 2, we only need to show that from every N^{-13}-approximate market equilibrium $\mathbf{p}$ of $\mathcal{M}$, one can construct an n^{-6}-well-supported Nash equilibrium $(\mathbf{x}, \mathbf{y})$ of $(\mathbf{A}, \mathbf{B})$ in polynomial time.

To this end, let $(\mathbf{x}', \mathbf{y}')$ denote the following two n-dimensional vectors: $x'_k = p_k - 1$ and $y'_k = p_{n+k} - 1$, for all $k \in [n]$. Then, we normalize $(\mathbf{x}', \mathbf{y}')$ to get a pair $(\mathbf{x}, \mathbf{y})$ (we will show later that $\mathbf{x}', \mathbf{y}' \neq \mathbf{0}$):

$$x_k = x'_k \Big/ \sum_{i \in [n]} x'_i \quad \text{and} \quad y_k = y'_k \Big/ \sum_{i \in [n]} y'_i, \quad \text{for all } k \in [n]. \tag{3}$$

Theorem then 2 follows directly from Theorem 3 below, which we will prove in the next section (Note that if $\mathbf{p}$ is a N^{-13}-approximate equilibrium, then it is also an n^{-13}-approximate equilibrium by definition).

Theorem 3. *If $\mathbf{p}$ is an n^{-13}-approximate market equilibrium of $\mathcal{M}$, then $(\mathbf{x}, \mathbf{y})$ constructed above must be an n^{-6}-well-supported Nash equilibrium of $(\mathbf{A}, \mathbf{B})$.*

5 Correctness of the Reduction

In this section, we prove Theorem 3. Let $\mathbf{p} = (p_1, ..., p_{2n+2})$ be a normalized n^{-13}-approximate equilibrium of $\mathcal{M}$. By the same argument used earlier, we can show that $p_k > 0$ for all $k \in [2n+2]$. Therefore, we have $p_k \geq 1$ for all k and $\min_k p_k = 1$. Let $\mathcal{X}$ be an optimal allocation with respect to $\mathbf{p}$ that clears the market approximately: $\mathcal{X} = \{\mathbf{a_s}, \mathbf{a_u}, \mathbf{a_v}, \mathbf{a}_i : \mathbf{s} \in S, \mathbf{u} \in U, \mathbf{v} \in V, i \in [2n]\}$.

We start with the following notation. Let $\mathcal{T}' \subseteq \mathcal{T}$ be a subset of traders and $k \in [2n+2]$. Then we use $w_k[\mathcal{T}']$ to denote the amount of good G_k that traders in $\mathcal{T}'$ possess at the beginning; and $a_k[\mathcal{T}']$ to denote the amount of good G_k that $\mathcal{T}'$ receives in the final allocation $\mathcal{X}$.

According to our construction, $w_k[\mathcal{T}] \in [2, 3]$ for any $k \in [2n+2]$. We further divide the traders $\mathcal{T}$ into two groups: $\mathcal{T}_1 = \{T_\mathbf{s} : \mathbf{s} \in S\}$ and $\mathcal{T}_2 = \mathcal{T} - \mathcal{T}_1$. Then by the definition of approximate market equilibria,

$$\big| w_k[\mathcal{T}_1] - a_k[\mathcal{T}_1] + w_k[\mathcal{T}_2] - a_k[\mathcal{T}_2] \big| \leq 3/n^{13}, \quad \text{for all } k \in [2n+2]. \tag{4}$$

First, we prove that, the price vector $\mathbf{p}$ must still satisfy the price-regulation property as in the price-regulating market $\mathcal{M}_{2n+2}$. The proof can be found in Appendix B which is similar to the proof of Property 1 and mainly uses the fact that traders in $\mathcal{T}_1$ possess almost all the goods in $\mathcal{M}$.

Lemma 1 (Price Regulation)**.** *For all $k \in [2n+2]$, $1 \leq p_k \leq 2$.*

Next, we prove two useful relations between p_k and $w_k[\mathcal{T}_2] - a_k[\mathcal{T}_2]$, $k \in [2n+2]$.

Lemma 2. *Let $\mathbf{p}$ be a normalized n^{-13}-approximate market equilibrium and $\mathcal{X}$ be an optimal allocation that clears the market approximately. If $w_k[\mathcal{T}_2] - a_k[\mathcal{T}_2] > 3/n^{13}$ for some $k \in [2n+2]$, then $p_k = 1$.*

Proof Sketch. Without loss of generality, we prove the lemma for the case when $k = 1$.

By (4), $w_1[\mathcal{T}_1] - a_1[\mathcal{T}_1] < 0$. This means that, in the market participated by traders $T_\mathbf{s}$, the amount of G_1 that they would like to buy is strictly more than the amount of G_1 they possess at the beginning. Intuitively this implies that the price p_1 of G_1 is lower than what it should be. See Appendix C for details. □

Lemma 3. *Let* $\mathbf{p}$ *be a normalized* n^{-13}*-approximate market equilibrium and* $\mathcal{X}$ *be an optimal allocation that clears the market approximately. If* $w_k[\mathcal{T}_2] - a_k[\mathcal{T}_2] < -3/n^{13}$ *for some* $k \in [2n+2]$*, then* $p_k = 2$.

The proof of Lemma 3 is similar and can be found in Appendix D. We also need the following lemma.

Lemma 4. *Let* $\mathbf{u} = (i, j, 1) \in U$ *and* $\mathbf{u}' = (j, i, 1) \in U$*, then* $w_{\mathbf{u},k} + w_{\mathbf{u}',k} \geq a_{\mathbf{u},k} + a_{\mathbf{u}',k}$ *for all* $k \in [2n+1]$*.*
Let $\mathbf{v} = (i, j, 2) \in V$ *and* $\mathbf{v}' = (j, i, 2) \in V$*, then* $w_{\mathbf{v},k} + w_{\mathbf{v}',k} \geq a_{\mathbf{v},k} + a_{\mathbf{v}',k}$ *for all* $k \in [2n+1]$*.*

Proof. Without loss of generality, we only prove the first part of Lemma 4 for the case when $\mathbf{u} = (1, 2, 1)$ and $\mathbf{u}' = (2, 1, 1)$. Let $\mathbf{C}$ and $\mathbf{D}$ denote the following two n-dimensional vectors: For $k \in [n]$,

$$(C_k, D_k) = (A_{1,k} - A_{2,k}, 0) \text{ if } A_{1,k} - A_{2,k} \geq 0; \text{ and } (C_k, D_k) = (0, A_{2,k} - A_{1,k}) \text{ otherwise.} \tag{5}$$

We also define E and F as in (2). By the construction, $w_{\mathbf{u},n+k} = C_k/n^5$, $w_{\mathbf{u}',n+k} = D_k/n^5$ for all $k \in [n]$,

$$w_{\mathbf{u},1} = w_{\mathbf{u}',2} = 1/n^4, \quad w_{\mathbf{u},2n+1} = E/n^5, \quad w_{\mathbf{u}',2n+1} = F/n^5,$$

and all other entries of $\mathbf{w_u}$ and $\mathbf{w_{u'}}$ are 0.

We now focus on the preference of $T_\mathbf{u}$. After selling its initial endowment, the budget of $T_\mathbf{u}$ is $\Theta(1/n^4)$ by Lemma 1, since the total amount of goods she possesses is $\Theta(1/n^4)$. The PLC utility functions $r_{\mathbf{u},k}(\cdot)$ of $T_\mathbf{u}$ are designed carefully, so that even though we do not know what exactly $\mathbf{p}$ is, we know the behavior of $T_\mathbf{u}$ due to the price-regulation property: $T_\mathbf{u}$ first buys the following bundle of goods from the market

$$\big\{D_k/n^5 \text{ amount of } G_{n+k} \text{ and } F/n^5 \text{ amount of } G_{2n+1} : k \in [n]\big\}. \tag{6}$$

As $\mathbf{D}$ has at most 20 nonzero entries and each entry is between 0 and 2, the cost of this bundle is $O(1/n^5)$. $T_\mathbf{u}$ then buys as much G_1 as it can up to $1/n^4$, and spends all the money left, if any, on G_{2n+2}.

The behavior of $T_{\mathbf{u}'}$ is similar. It first buys the following bundle of goods from the market:

$$\big\{C_k/n^5 \text{ amount of } G_{n+k} \text{ and } E/n^5 \text{ amount of } G_{2n+1} : k \in [n]\big\}. \tag{7}$$

It then buys as much G_2 as it can up to $1/n^4$, and spends all the money left, if any, on G_{2n+2}.

Now we are ready to prove the lemma. The case when $k \in [n]$ but $k \neq 1, 2$ is trivial because $w_{\mathbf{u},k} = w_{\mathbf{u}',k}$ $= a_{\mathbf{u},k} = a_{\mathbf{u}',k} = 0$. When $k = 1$, we have $w_{\mathbf{u},1} + w_{\mathbf{u}',1} = 1/n^4$, $a_{\mathbf{u}',1} = 0$, $a_{\mathbf{u},1} \leq 1/n^4$ and thus, Lemma 4 follows. $k = 2$ can be proved similarly. For the case of $n + k$, $k \in [n]$, and for the case of $2n + 1$, we have

$$w_{\mathbf{u},n+k} = C_k/n^5, \quad w_{\mathbf{u}',n+k} = D_k/n^5, \quad a_{\mathbf{u},n+k} = D_k/n^5, \quad \text{and} \quad a_{\mathbf{u}',n+k} = C_k/n^5,$$
$$w_{\mathbf{u},2n+1} = E/n^5, \quad w_{\mathbf{u}',2n+1} = F/n^5, \quad a_{\mathbf{u},2n+1} = F/n^5, \quad \text{and} \quad a_{\mathbf{u}',2n+1} = E/n^5,$$

and Lemma 4 follows. This finishes the proof of the lemma. □

By Lemma 4 and Lemma 2, we immediately get the following important corollary concerning p_{2n+1}.

Corollary 1. $p_{2n+1} = 1$.

Proof. By Lemma 4, we have $w_{2n+1}[T_\mathbf{u}, T_\mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V] - a_{2n+1}[T_\mathbf{u}, T_\mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V] \geq 0$. However,

$$w_{2n+1}\big[T_i : i \in [2n]\big] = 2n \cdot (1/n^{12}) = 2/n^{11} \quad \text{and} \quad a_{2n+1}\big[T_i : i \in [2n]\big] = 0.$$

Thus, $w_{2n+1}[\mathcal{T}_2] - a_{2n+1}[\mathcal{T}_2] > 3/n^{13}$. It then follows from Lemma 2 that $p_{2n+1} = 1$. □

Now let $\mathbf{x}'$ and $\mathbf{y}'$ denote the vectors where $x'_k = p_k - 1$ and $y'_k = p_{n+k} - 1$. By Lemma 1, $x'_k, y'_k \in [0, 1]$ for all $k \in [n]$. We prove the following two properties of $(\mathbf{x}', \mathbf{y}')$ and use them to prove Theorem 3.

Property 2. *For all* $1 \le i \neq j \le n$, *we have*

$$(\mathbf{A}_i - \mathbf{A}_j)\mathbf{y}'^T < -\epsilon \implies x'_i = 0 \quad \text{and} \quad \mathbf{x}'(\mathbf{B}_i - \mathbf{B}_j) < -\epsilon \implies y'_i = 0, \tag{8}$$

where $\epsilon = n^{-6}$, $\mathbf{A}_i$ *denotes the* ith *row vector of* $\mathbf{A}$, *and* $\mathbf{B}_i$ *denotes the* ith *column vector of* $\mathbf{B}$.

Property 3. *There exist* i *and* $j \in [n]$ *such that* $x'_i = 1$ *and* $y'_j = 1$.

Now assume that $\mathbf{x}'$ and $\mathbf{y}'$ satisfy both properties. In particular, Property 3 implies that $\mathbf{x}', \mathbf{y}' \neq \mathbf{0}$. As a result, we can normalize them to get two probability distribution $\mathbf{x}$ and $\mathbf{y}$ using (3). Theorem 3 then follows, and the proof can be found in Appendix E. Finally, we prove Property 2 and Property 3.

Proof of Property 2. We only prove the first part of (8) for the case when $i = 1$ and $j = 2$. The other part can be proved similarly. Let $\mathbf{u} = (1, 2, 1)$ and $\mathbf{u}' = (2, 1, 1)$. Let $\mathbf{C}$ and $\mathbf{D}$ be the two nonnegative vectors defined in (5), and E and F be the two nonnegative numbers defined in (2). Assume $(\mathbf{A}_1 - \mathbf{A}_2)\mathbf{y}'^T < -\epsilon$. Then the money of $T_\mathbf{u}$ left after purchasing the bundle in (6) is

$$p_1 \cdot \frac{1}{n^4} + \sum_{k \in [n]} p_{n+k} \cdot \frac{C_k}{n^5} + p_{2n+1} \cdot \frac{E}{n^5} - \sum_{k \in [n]} p_{n+k} \cdot \frac{D_k}{n^5} - p_{2n+1} \cdot \frac{F}{n^5}. \tag{9}$$

By Corollary 1, $p_{2n+1} = 1$. Using $\mathbf{C} - \mathbf{D} = \mathbf{A}_1 - \mathbf{A}_2$ and $E + \sum_k C_k = F + \sum_k D_k$, we simplify (9) to be

$$p_1 \cdot \frac{1}{n^4} + \frac{1}{n^5} \sum_{k \in [n]} y'_k \cdot (C_k - D_k) = p_1 \cdot \frac{1}{n^4} + \frac{1}{n^5}(\mathbf{A}_1 - \mathbf{A}_2)\mathbf{y}'^T < p_1 \cdot \frac{1}{n^4} - \frac{\epsilon}{n^5}. \tag{10}$$

This implies that the amount $a_{\mathbf{u},1}$ of G_1 that $T_\mathbf{u}$ buys is smaller than $1/n^4 - \epsilon/(p_1 n^5) \le 1/n^4 - 1/(2n^{11})$, since $\epsilon = n^{-6}$. However, we have $w_{\mathbf{u},1} = 1/n^4$ and thus, $w_{\mathbf{u},1} - a_{\mathbf{u},1} > 1/(2n^{11})$.

On the other hand, it is easy to check that $w_{\mathbf{u}',1} = 0$ and $a_{\mathbf{u}',1} = 0$. By Lemma 4, we have

$$w_1[T_\mathbf{u}, T_\mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V] - a_1[T_\mathbf{u}, T_\mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V] > 1/(2n^{11}). \tag{11}$$

Next we bound $w_1\big[T_i : i \in [2n]\big] - a_1\big[T_i : i \in [2n]\big]$. By the construction, we have $a_1\big[T_i : i \in [2n], i \neq 1\big] = 0$,

$$a_{1,1} = 1 \cdot w_{1,2n+1}/p_1 \le 1/n^{12} \quad \text{and thus,} \quad w_1\big[T_i : i \in [2n]\big] - a_1\big[T_i : i \in [2n]\big] \ge -1/n^{12}.$$

Combining (11), we have $w_1[\mathcal{T}_2] - a_1[\mathcal{T}_2] \gg 3/n^{13}$. It then follows from Lemma 2 that $x'_1 = 0$. □

Proof of Property 3. Let $\ell \in [n]$ be one of the indices that maximizes $\mathbf{A}_\ell \mathbf{y}'^T$, then we prove Property 3 by showing that $x'_\ell = 1$. Without loss of generality, we may assume that $\ell = 1$.

First, we consider a pair $\mathbf{v} = (i, j, 2)$, $\mathbf{v}' = (j, i, 2)$ in V. In the proof of Lemma 4, we actually showed that $w_{\mathbf{u},n+k} + w_{\mathbf{u}',n+k} = a_{\mathbf{u},n+k} + a_{\mathbf{u}',n+k}$, for all pairs $\mathbf{u} = (i, j, 1)$ and $\mathbf{u}' = (j, i, 1)$ in U, and all $k \in [n]$. Similarly, we can prove that $w_{\mathbf{v},1} + w_{\mathbf{v}',1} = a_{\mathbf{v},1} + a_{\mathbf{v}',1}$. Second, for every $\mathbf{u} = (i, j, 1) \in U$, we always have $w_{\mathbf{u},1} = a_{\mathbf{u},1}$. This is because: If $i \neq 1$, then $w_{\mathbf{u},1} = a_{\mathbf{u},1} = 0$; and if $i = 1$, then by (10), the money of $T_\mathbf{u}$ left after purchasing the bundle of goods in (6) is at least p_1/n^4, so $w_{\mathbf{u},1} = a_{\mathbf{u},1} = 1/n^4$. As a result, we have

$$w_1[T_\mathbf{u}, T_\mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V] = a_1[T_\mathbf{u}, T_\mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V].$$

However, the amount of G_1 that T_1 buys is $p_{2n+1} \cdot w_{1,2n+1}/p_1 \ge 1/(2n^{12})$. As a result,

$$w_1\big[T_i, i \in [2n]\big] - a_1\big[T_i, i \in [2n]\big] \le -1/(2n^{12}).$$

Finally, we have $w_1[\mathcal{T}_2] - a_1[\mathcal{T}_2] \ll -3/n^{13}$. By Lemma 3, we conclude that $p_1 = 2$ and thus, $x'_1 = 1$. □

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A Proof of Theorem 1

In this section, we prove Theorem 1. To this end, we first show that under the conditions of Theorem 1, $\mathcal{M}$ has at least one *quasi-equilibrium* (see the definition below). Then we show that any quasi-equilibrium of $\mathcal{M}$ is indeed a market equilibrium.

Definition 10. *A quasi-equilibrium of $\mathcal{M}$ is a* (normalized) *price vector $\mathbf{p} \in \mathbb{R}_+^n$ such that there exists an allocation $\mathcal{X} = \{\mathbf{x}_i \in \mathbb{R}_+^n : i \in [m]\}$ which has the following properties:*

1. *The market clears: For every good $G_j \in \mathcal{G}$,*
$$\sum_{i\in[m]} x_{i,j} \le \sum_{i\in[m]} w_{i,j};$$
In particular, if $p_j > 0$, then
$$\sum_{i\in[m]} x_{i,j} = \sum_{i\in[m]} w_{i,j};$$
2. *For every trader $T_i \in \mathcal{T}$, at least one of the following is true:*
   (a) $\mathbf{x}_i \in \mathrm{OPT}(i, \mathbf{p})$;
   (b) $\mathbf{p} \cdot \mathbf{x}_i = \mathbf{p} \cdot \mathbf{w}_i = 0$ *(zero income).*

The difference between market equilibria and quasi-equilibria is that in the latter, we *do not* require the optimality of allocations for traders with a zero income: If a trader has a zero income, then we can assign her any bundle of zero cost. However, if $\mathbf{p}$ is a quasi-equilibrium and the income of every trader is positive with respect to $\mathbf{p}$, then by definition $\mathbf{p}$ must be a market equilibrium.

In [28] Maxfield gave a set of conditions that are sufficient for the existence of a quasi-equilibrium in an exchange market. We use the following simplified version:

Theorem 4 ([28])**.** *An exchange market $\mathcal{M}$ has a quasi-equilibrium* $\mathbf{p}$ *if*

1. *For each trader $T_i \in \mathcal{T}$, its utility function $u_i : \mathbb{R}_+^n \to \mathbb{R}$ is both continuous and quasi-concave; and*
2. *For each trader $T_i \in \mathcal{T}$, u_i is non-satiable, i.e., for any $\mathbf{x} \in \mathbb{R}_+^n$, there exists a vector $\mathbf{y} \in \mathbb{R}_+^n$ such that $u_i(\mathbf{y}) > u_i(\mathbf{x})$.*

Now we use Theorem 4 to prove Theorem 1.

Proof of Theorem 1. First, it is not hard to check that if $\mathcal{M}$ is an additively separable PLC market that is strongly connected, then it satisfies both conditions in Theorem 4. In particular, u_i is non-satiable since the economy graph of $\mathcal{M}$ is strongly connected and thus, there exists a $j \in [n]$ such that $r_{i,j}(\cdot)$ is *strictly monotone*. As a result, $\mathcal{M}$ must have a quasi-equilibrium $\mathbf{p}$. We use $\mathcal{X} = \{\mathbf{x}_i \in \mathbb{R}_+^n : i \in [m]\}$ to denote an allocation that clears the market. Since $\mathbf{p} \neq \mathbf{0}$, there is at least one trader in $\mathcal{T}$, say $T_1 \in \mathcal{T}$, has a positive income.

Second, we show that for every trader, its income is positive and thus, $\mathbf{p}$ is indeed an equilibrium of $\mathcal{M}$. Suppose this is not true, then there is at least one trader T_2 whose income is zero. Since the economy graph is strongly connected, there is a directed path from T_2 to T_1. As a result, there must be a directed edge T_3T_4 on the path such that the income of T_3 is zero and the income of T_4 is positive. By definition, there exists a $j \in [n]$ such that the amount of G_j that T_3 owns at the beginning is positive and the PLC utility function of T_4 with respect to G_j is strictly monotone. However, since the income of T_3 is zero, we have $p_j = 0$ and thus, the amount of G_j that T_4 wants to buy is $+\infty$, contradicting the assumption that $\mathbf{p}$ is a quasi-equilibrium of $\mathcal{M}$ (since the income of T_4 is positive but the bundle she receives is not optimal).

Now we have proved the existence of a market equilibrium $\mathbf{p}$. The second part of Theorem 1 follows from the work of Devanur and Kannan [12]. In [12], the authors proposed an algorithm for computing a market equilibrium in an additively separable PLC market[3]. They divide the whole search space $\mathbb{R}_+^n$ of $\mathbf{p}$ into "cells" $C \subset \mathbb{R}_+^n$ using hyperplanes. Then for each cell C, there is a rational linear program LP_C that characterizes the set of market equilibria in C: $\mathbf{p} \in C$ is an equilibrium of $\mathcal{M}$ if and only if it is a feasible solution to LP_C (In particular, if LP_C has no feasible solution then there is no equilibrium in C). Moreover, the size of LP_C, for any cell C, is polynomial in the input size of $\mathcal{M}$.

Now let $\mathbf{p}$ be a market equilibrium of $\mathcal{M}$, which is not necessarily rational. We let C^* denote the cell that $\mathbf{p}$ lies in, then $\mathbf{p}$ must be a feasible solution to LP_{C^*}. Since LP_{C^*} is rational, it must have a rational solution $\mathbf{p}^*$ and the number of bits one needs to describe $\mathbf{p}^*$ is polynomial in the size of LP_{C^*} and thus, is polynomial in the input size of $\mathcal{M}$. Theorem 1 then follows since $\mathbf{p}^*$ is also an equilibrium of $\mathcal{M}$. □

B Proof of Lemma 1: The Price-Regulation Property

Proof. Assume for contradiction that $\mathbf{p}$ does not satisfies the price-regulation property. Then without loss of generality, we assume that $p_1 = \max_k p_k > 2$ and $p_2 = 1$.

By the same argument used in the proof of Property 1, we have

$$w_1[\mathcal{T}_1] = (2n+1) \cdot \frac{1}{n}, \quad a_1[\mathcal{T}_1] \le 2n \cdot \frac{1}{n}, \quad \text{and thus,} \quad w_1[\mathcal{T}_1] - a_1[\mathcal{T}_1] \ge \frac{1}{n}.$$

By (4), we have

$$w_1[\mathcal{T}_2] - a_1[\mathcal{T}_2] \le -\frac{1}{n} + \frac{3}{n^{13}} \implies a_1[\mathcal{T}_2] \ge w_1[\mathcal{T}_2] + \frac{1}{n} - \frac{3}{n^{13}} \ge \frac{1}{n} - \frac{3}{n^{13}} \tag{12}$$

because $w_1[\mathcal{T}_2] \ge 0$. However, this cannot be true since the amount of goods the traders in $\mathcal{T}_2$ possess at the beginning is much smaller compared to $1/n$. Even if they spend all the money on G_1, we still have

$$a_1[\mathcal{T}_2] \le \frac{\sum_{k\in[2n+2]} p_k \cdot w_k[\mathcal{T}_2]}{p_1} \le \sum_{k\in[2n+2]} w_k[\mathcal{T}_2] = O(n^{-2}) \ll \frac{1}{n},$$

since we assumed that $p_1 = \max_k p_k$. This contradicts with (12). □

C Proof of Lemma 2

Proof. Without loss of generality, we prove the lemma for the case when $k = 1$. By (4), we have $w_1[\mathcal{T}_1] - a_1[\mathcal{T}_1] < 0$. This means that, in the market participated by traders $T_{\mathbf{s}}$, the amount of G_1 which they would like to buy is strictly more than the amount of G_1 they possess at the beginning. Intuitively, this implies that the price p_1 of G_1 is lower than what it should be, and indeed we show below that $p_1 = \min_k p_k = 1$.

[3] When the number of goods is constant, the algorithm is polynomial-time.

On one hand, by the construction, only the following traders $T_{\mathbf{s}}$ are interested in G_1:

$$S_1 = \{\mathbf{s} = (1, j) : j \neq 1\} \quad \text{and} \quad S_2 = \{\mathbf{s} = (i, 1) : i \neq 1\}.$$

On the other hand, we have

$$a_1[T_{\mathbf{s}}, \mathbf{s} \in S_1] \leq w_1[T_{\mathbf{s}}, \mathbf{s} \in S_1] = w_1[\mathcal{T}_1]$$

due to the budget limitation. As a result, there must exist an $\mathbf{s} = (i, 1) \in S_2$ such that $a_{\mathbf{s},1} > 0$. Since $\mathbf{a_s}$ is an optimal bundle for $T_{\mathbf{s}}$ with respect to $\mathbf{p}$, we have

$$\frac{1}{p_1} \geq \frac{2}{p_i} \implies p_1 \leq \frac{p_i}{2}.$$

By Lemma 1, the price-regulation property, we conclude that $p_1 = 1$ and the lemma is proved. □

D Proof of Lemma 3

Proof. Without loss of generality, we prove the lemma for the case when $k = 1$. By (4), we have $w_1[\mathcal{T}_1] - a_1[\mathcal{T}_1] > 0$. This means that, in the market participated by traders $T_{\mathbf{s}}$, the amount of G_1 which they would like to buy is strictly less than the amount of G_1 they possess at the beginning. Intuitively this implies that the price p_1 of G_1 is higher than what it should be, and indeed we show below that $p_1 = 2 = \max_k p_k$.

Since $a_1[\mathcal{T}_1] < w_1[\mathcal{T}_1]$, there must exist a $j \in [2n + 2]$ with $j \neq 1$ such that $\mathbf{s} = (1, j)$ and

$$a_{\mathbf{s},1} < w_{\mathbf{s},1}.$$

(Otherwise $a_1[\mathcal{T}_1] \geq w_1[\mathcal{T}_1]$). This means that $T_{\mathbf{s}}$ spends some of its money to buy G_j and thus,

$$\frac{1}{p_j} \geq \frac{2}{p_1} \implies p_1 \geq 2p_j.$$

By Lemma 1, the price-regulation property, we conclude that $p_1 = 2$ and the lemma is proved. □

E Proof of Theorem 3

Proof of Theorem 3. Since both $\mathbf{x}$ and $\mathbf{y}$ are probability distributions, we only need to prove that $(\mathbf{x}, \mathbf{y})$ satisfies (1) for all $1 \leq i \neq j \leq n$. We only prove the first part of (1) here. Assume $\mathbf{A}_i\mathbf{y}^T + \epsilon < \mathbf{A}_j\mathbf{y}^T$, then

$$(\mathbf{A}_i - \mathbf{A}_j)\mathbf{y}'^T = (\mathbf{A}_i - \mathbf{A}_j)\mathbf{y}^T \cdot \left(\sum_{k \in [n]} y'_k\right) < -\epsilon$$

since $\sum_{k\in[n]} y'_k \geq 1$ by Property 3. As a result, by Property 2 we have $x'_i = 0$ and thus, $x_i = 0$. □