Strong and Pareto Price of Anarchy in Congestion Games

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Abstract

Strong Nash equilibria and Pareto-optimal Nash equilibria are natural and important strengthenings of the Nash equilibrium concept. We study these stronger notions of equilibrium in congestion games, focusing on the relationships between the price of anarchy for these equilibria and that for standard Nash equilibria (which is well understood). For *symmetric* congestion games with polynomial or exponential latency functions, we show that the price of anarchy for strong and Pareto-optimal equilibria is much smaller than the standard price of anarchy. On the other hand, for asymmetric congestion games with polynomial latencies the strong and Pareto prices of anarchy are essentially as large as the standard price of anarchy; while for asymmetric games with exponential latencies the Pareto and standard prices of anarchy are the same but the strong price of anarchy is substantially smaller. Finally, in the special case of linear latencies, we show that the strong and Pareto prices of anarchy coincide exactly with the known value $\frac{5}{2}$ for standard Nash, but are strictly smaller for symmetric games.

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1 Introduction

1.1 Background

In algorithmic game theory, the *price of anarchy* [14] is defined as the ratio of the social cost of a worst Nash equilibrium to that of a social optimum (i.e., an assignment of strategies to players achieving optimal social cost). This highly successful and influential concept is frequently thought of as the standard measure of the potential efficiency loss due to individual selfishness, when players are concerned only with their own utility and not with the overall social welfare. However, because a Nash equilibrium guarantees only that no single player (as opposed to no coalition) can improve his utility by moving to a new strategy, the price of anarchy arguably conflates the effects of selfishness and lack of coordination. Indeed, for several natural classes of games, the worst-case price of anarchy is achieved at a Nash equilibrium in which a *group* of selfish players can all improve their individual utilities by moving simultaneously to new strategies; in some cases, the worst Nash equilibrium may not even be *Pareto-optimal*—i.e., it may be possible that a group of players can move to new strategies so that *every* player is better off (or no worse off) than before.

In this context, two stronger equilibrium concepts perhaps better isolate the efficiency loss due only to selfishness. A *strong Nash equilibrium* [5] is defined as a state in which no subset of the players may simultaneously change their strategies so as to improve all of their costs. The *strong price of anarchy* (e.g., [3]) is the ratio between the cost of the worst strong equilibrium and the optimum cost. A weaker concept that is very widely studied in the economics literature (see, e.g., [15]) is that of a *Pareto-optimal Nash equilibrium*, which is defined as a Nash equilibrium for which there is no other state in which every player is better off. (Equivalently, one may think of a Pareto-optimal equilibrium as being stable under moves by single players or the coalition of all players, but not necessarily arbitrary coalitions.) One can argue that Pareto-optimality should be a minimum requirement for any equilibrium concept intended to capture the notion of selfishness, in that it should not be in every player's self-interest to move to another state. The *Pareto price of anarchy* is then defined in the obvious way.[†]

A natural question to ask is whether the strong and/or Pareto prices of anarchy are significantly less than the standard price of anarchy. In other words, does the requirement that the equilibrium be stable against coalitions lead to greater efficiency? We note that this question has been addressed recently for several specific families of games in the case of the strong (though not Pareto) price of anarchy [2, 3, 10]; see the related work section below. In this paper, we investigate the question for the large and well-studied class of *congestion games* with linear, polynomial or exponential latency functions.

A *congestion game* is an n-player game in which each player's strategy consists of a set of resources, and the cost of the strategy depends only on the number of players using each resource, i.e., the cost takes the form $\sum_r \ell_r(f(r))$, where $f(r)$ is the number of players using resource r, and ℓ_r is a non-negative increasing function. A standard example is a *network congestion game* on a directed graph, in which each player must select a path from some source to some destination, and each edge has an associated cost function, or "latency", ℓ_r that increases with the number of players using it. (Throughout, we shall use the term "latency" even though we will always be discussing general (non-network) congestion games.) Frequently the latencies are assumed to have a simple form, such as linear ($\ell_r(t) = \alpha_r + \beta_r t$ for $\alpha_r, \beta_r \geq 0$), polynomial ($\ell_r(t)$ is a degree-k polynomial with non-negative coefficients), or exponential ($\ell_r(t) = \alpha_r^t$ for $\alpha_r > 1$).

Congestion games have featured prominently in algorithmic game theory, partly because they capture a large class of routing and resource allocation scenarios, and partly because they are known to possess *pure* Nash equilibria [19]. The price of anarchy for congestion games is by now quite well understood, starting with Koutsoupias and Papadimitriou [14] who considered a (weighted) congestion game on a set of parallel edges. The celebrated work of Roughgarden and Tardos [21] established the value $\frac{4}{3}$ as the price of anarchy of network congestion games with linear latencies in the *nonatomic* case (where there are infinitely many

[†]Note that throughout we are assuming that cost (or utility) is *non-transferable*, i.e., players in a coalition cannot share their costs with each other. If costs can be shared, the situation is very different; see, e.g., [12] for a discussion of this alternative scenario.

players, each of whom controls an infinitesimal amount of traffic); this was extended to polynomial latencies in [22]. The more delicate n-player case was solved independently by Awerbuch, Azar and Epstein [6] and by Christodoulou and Koutsoupias [8], who obtained the tight value $\frac{5}{2}$ for the price of anarchy in the linear case, and a value $k^{k(1-o(1))}$ for the case of polynomial latencies. Subsequently Aland et al. [1] gave an exact value for the polynomial case. These works also handle the generalization to the case of weighted players.

Much less is known about strong or Pareto-optimal Nash equilibria in congestion games. Note that such equilibria need not exist. Holzman and Law-Yone [13] give a sufficient condition for the existence of a strong equilibrium based on the absence of a certain structural feature in the game, and also discuss the uniqueness and Pareto-optimality of Nash equilibria under the same condition. For the strong or Pareto price of anarchy, however, there appear to be no results for general congestion games.

1.2 Results

We investigate the strong and Pareto price of anarchy for congestion games with linear, polynomial and exponential latencies. Roughly speaking, we find that in symmetric[‡] games the resulting price of anarchy can be much less than the standard (Nash) price of anarchy, while in asymmetric games the behavior is more complicated: for linear and polynomial latencies, all three prices of anarchy are essentially the same, but for exponential latencies the standard and Pareto prices of anarchy are equal, while the strong price of anarchy is substantially smaller. (We note that this gap between symmetric and asymmetric games does not appear for standard Nash equilibria. Understanding the reason for this difference may be worthy of further study.)

More specifically, we show that the strong and Pareto prices of anarchy for symmetric congestion games with polynomial latencies of degree k are at most 2^{k+1} (and that this is tight up to a constant factor); this is in sharp contrast to the Nash price of anarchy of $k^{k(1-o(1))}$ obtained in [1, 6, 8]. In the special case of linear latency, we show that the strong and Pareto price of anarchy are strictly less than the exact value $\frac{5}{2}$ for standard Nash obtained in [6, 8]. For symmetric games with exponential latency α^k, we show that the strong and Pareto price of anarchy are at most n, while the standard Nash price of anarchy is at least β^n, where $\beta > 1$ is a constant that depends on α.

On the other hand, for asymmetric games with polynomial latency of degree k, we show that the strong (and therefore also the Pareto) price of anarchy is $k^{k(1-o(1))}$, matching the asymptotic value of the standard Nash price of anarchy in [6, 8]. Moreover, in the linear case all three prices of anarchy coincide exactly. For exponential latencies, we show that the Pareto price of anarchy is the same as for standard Nash (which we show to be exponentially large), and also that the strong price of anarchy is significantly smaller; thus we exhibit a separation between strong and Pareto prices of anarchy for a natural class of games.

Since strong and Pareto-optimal equilibria do not always exist, we should clarify the meaning of the above statements. An upper bound on the strong (respectively, Pareto) price of anarchy for a certain class of games bounds the price of anarchy *whenever a strong (respectively, Pareto-optimal) equilibrium exists*. A lower bound means that there is a specific game in the class that has a strong (respectively, Pareto-optimal) equilibrium achieving the stated price of anarchy.

We now briefly highlight a few of our proof techniques. To obtain upper bounds on the Pareto (and hence also strong) price of anarchy in symmetric games, we show that this price of anarchy can always be bounded above by the maximum ratio of the costs of individual players at equilibrium and the same ratio at the social optimum. This allows us to study the equilibrium and the optimum separately, greatly simplifying the analysis. We note that this fact holds for arbitrary symmetric games, not only congestion games, and thus may be of wider interest. Our upper bound on the Pareto price of anarchy for linear latencies requires a much more intricate analysis, and makes use of a matrix $M = (m_{ij})$, where m_{ij} is the relative cost increase to player i's cost at optimum when a new player moves to player j's strategy. This turns out to be a stochastic matrix with several useful properties. Finally, our lower bound arguments make use of constructions used

[‡]A game is *symmetric* if all players have the same sets of allowable strategies.

in [1, 6, 8], suitably modified so as to handle the stronger requirements of strong and Pareto equilibria. (These constructions typically have the property that the social optimum is a strong Nash equilibrium, so they are not applicable in our setting.)

1.3 Related work

Congestion games were introduced in Economics by Rosenthal [19], and further studied in an influential paper by Monderer and Shapley [16]. The concept of "price of anarchy"[§] was introduced by Koutsoupias and Papadimitriou [14], who analyzed a very simple weighted network congestion game on parallel links (with a different definition of social cost based on the *maximum*, rather than total or average, player cost). Roughgarden and Tardos [21, 22] gave tight bounds for the price of anarchy in congestion games with linear and polynomial latencies in the nonatomic or Wardrop model [7], in which there are infinitely many players each of whom controls a negligible amount of traffic. The papers [9, 20] consider the same scenario under maximum social cost.

Awerbuch et al. [6] and Christodoulou and Koutsoupias [8] consider the price of anarchy for congestion games with linear and polynomial latencies, obtaining the tight value $\frac{5}{2}$ for linear latencies and the approximate value $k^{k(1-o(1))}$ for polynomial latencies of degree k. Aland et al. [1] give an exact value for the polynomial case. Both [6] and [1] extend their results to congestion games with weighted players, while [8] also considers maximum social cost and mixed equilibria.

The strong equilibrium concept dates back to Aumann [5]. Holzman and Law-Yone [13] explore the question of the existence of strong equilibria in congestion games, and give a structural characterization of this property for the symmetric case. Rozenfeld and Tennenholtz [18] consider the analogous question in the case where the "latencies" are monotonically decreasing.

Several authors have considered the strong price of anarchy and the existence of strong Nash equilibria in various specific classes of games, often deriving significant gaps between the strong and standard price of anarchy. For example, Andelman et al. [3] study job scheduling and network creation games, Epstein et al. [10] cost-sharing connection games, and Albers [2] network design games.

Other measures stronger than the standard Nash price of anarchy have been studied recently by various authors. Anshelevich et al. [4] consider the *price of stability*, which is the ratio of the cost of a *best* Nash equilibrium to the social optimum, for network design games. And Hayrepeyan et al. [12] define and study the "price of collusion" in analogous fashion to the strong price of anarchy, with the crucial difference that coalitions aim to minimize not the cost of each of their members (as with the strong price of anarchy) but the combined cost of all members.

2 Preliminaries

2.1 Equilibrium concepts and congestion games

A *game* consists of a finite set of players $P = \{1, \ldots, n\}$, each of which is assigned a finite set of *strategies* S_i and a cost function $c_i : S_1 \times \cdots \times S_n \to \mathbb{N}$ that he wishes to minimize. A game is called *symmetric* if all of the S_i are identical. A *state* $s = (s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$ is any combination of strategies for the players. A state s is a *pure Nash equilibrium* if for all players i, $c_i(s_1, \ldots, s_i, \ldots, s_n) \leq c_i(s_1, \ldots, s_i', \ldots, s_n)$ for all $s_i' \in S_i$; thus at a Nash equilibrium, no player can improve his cost by unilaterally changing his strategy. It is well known that, while every (finite) game has a *mixed* Nash equilibrium[¶], not every game has a pure Nash equilibrium. A state $s = (s_1, \ldots, s_n)$ is a *Pareto-optimal Nash equilibrium* if it is a pure Nash equilibrium

[§]The term was in fact coined later by Papadimitriou [17]

[¶]In a mixed Nash equilbrium, a player's strategy can be any probability distribution over available strategies, and no individual player can improve his expected cost by choosing another probability distribution.

and there is no other state in which every player has lower cost than at s.[‖] Thus s is a Pareto-optimal Nash equilibrium if and only if, for all $s' = (s'_1, \ldots, s'_n) \in S_1 \times \cdots \times S_n$, there exists some player $j \in P$ such that $c_j(s') \geq c_j(s)$. A state $s = (s_1, \ldots, s_n)$ is a *strong Nash equilibrium* if there does not exist any *coalition* of players $C = \{i_1, ..., i_k\} \subseteq P$ that can move in such a way that every member of the coalition pays lower cost than at equilibrium. More formally, let $s'_{i_1}, \ldots, s'_{i_k} \in S_{i_1} \times \cdots \times S_{i_k}$ be any combination of strategies for the players in C, and let s' be the state reached from s when the players in C move to their corresponding strategies s'_{i_ℓ}. Then s is a strong Nash equilibrium if, for all coalitions C and all corresponding s', there exists some $j \in C$ such that $c_j(s') \geq c_j(s)$.

Finally, for any given state s, we will define the *social cost* $c(s)$ to be the sum of the players' costs in s, i.e., $c(s) = \sum_{i \in P} c_i(s)$. A state that minimizes the social cost in a game is called a *social optimum*.

We will focus on the class of games known as *congestion games*. These games are known to always possess a pure Nash equilibrium [19], though not necessarily a strong or Pareto-optimal equilibrium. In a congestion game, players' costs are based on the shared usage of a common set of *resources* $R = \{r_1, \ldots, r_m\}$. A player's strategy set $S_i \subseteq 2^R$ is an arbitrary collection of subsets of R; his strategy $s_i \in S_i$ will therefore be a subset of R. Each resource $r \in R$ has an associated non-decreasing cost or "latency" function $\ell_r : \{1, \ldots, n\} \to \mathbb{N}$; if t players are using resource r, they each incur a cost of $\ell_r(t)$. Thus in a state $s = (s_1, \ldots, s_n)$, the cost of player p_i is $c_i(s) = \sum_{r \in s_i} \ell_r(f_s(r))$, where $f_s(r)$ is the number of players using resource r under s (i.e., $f_s(r) = |\{j : r \in s_j\}|$).

Of particular interest are congestion games where the latency functions are linear ($\ell_r(t) = \alpha_r t + \beta_r$), polynomial ($\ell_r(t)$ is a degree-k polynomial in t with non-negative coefficients), or exponential ($\ell_r(t) = \alpha_r^t$ for $1 \leq \alpha_r \leq \alpha$.) For simplicity of notation, we shall assume that $\ell_r(t) = t$ for all r in the linear case, $\ell_r(t) = t^k$ for all r in the polynomial case, and $\ell_r(t) = \alpha^t$ for all r in the exponential case. This will not affect our lower bounds, which are based on explicit constructions of this restricted form, and it is not hard to check that the upper bounds go through as well; for example, it is straightforward to incorporate general non-negative coefficients by replicating resources. We omit the details, which are technical but standard.

2.2 Efficiency of equilibria

As is standard, we measure the relative efficiency loss for a specific type of equilibrium for a given family of games $\mathcal{G}$ as the maximum possible ratio, over all games in the family, of the social cost of an equilibrium state e in that game to the cost of a social optimum o of the same game, or

$$\sup_{\mathcal{G}, e} \frac{c(e)}{c(o)}.$$

This measure is known as the *price of anarchy* (or *coordination ratio*) in the case of Nash equilibria, and the *strong price of anarchy* when discussing strong Nash equilibria. In addition to these, we will also consider the case of Pareto-optimal Nash equilibria, in which case we call the above ratio the *Pareto price of anarchy*. Clearly the strong price of anarchy is no larger than the Pareto price of anarchy, which in turn is no larger than the standard (Nash) price of anarchy.

We note that, for the classes of congestion games we consider, strong and Pareto-optimal equilibria may not exist, and games may also have Pareto-optimal Nash equilibria but no strong Nash equilibria (see Section A.1 of the appendix for an example of the latter). Thus when we state an upper bound on the strong (respectively, Pareto) price of anarchy for a certain class of games, it should be understood that this bound holds for any game in which a strong (respectively, Pareto-optimal) equilibrium exists. When we state a lower bound, we mean that there exists a specific game in the class that has a strong (respectively, Pareto-optimal) equilibrium achieving the stated price of anarchy.

[‖]Some definitions of Pareto-optimality require there to be no other state in which no player has higher cost than at s and at least one player has lower cost. It is easy to check that our results carry over to this alternative definition with minor modifications to the proofs.

3 Symmetric games

In this section we prove upper bounds on the strong and Pareto price of anarchy for symmetric congestion games with polynomial and exponential latency functions. We shall see that these are much smaller than the known values for the standard Nash price of anarchy. Thus in the case of symmetric games, increased stability leads to greater efficiency.

3.1 The basic framework

The main vehicle for these proofs is a very simple framework that allows us to bound the price of anarchy in terms of the maximum ratio of the player costs at equilibrium and the maximum ratio of the player costs at a social optimum. This is the content of the following theorem, which we note applies to *all* symmetric games, not only congestion games.

Theorem 3.1 *Given a particular symmetric game with n players, let the state e be a Pareto-optimal Nash equilibrium and s be any other state. Let ρ_e be defined as $\max_{i,j} c_i(e)/c_j(e)$ over all players i, j, and ρ_s be similarly defined as $\max_{i,j} c_i(s)/c_j(s)$. Then*

$$\frac{c(e)}{c(s)} \leq \max\{\rho_e, \rho_s\}.$$

Proof: By symmetry, we can assume without loss of generality that the players are ordered by cost in both e and s: that is, $c_1(e) \leq \cdots \leq c_n(e)$ and $c_1(s) \leq \cdots \leq c_n(s)$. Thus $c_n(e)/c_1(e) = \rho_e$, and $c_n(s)/c_1(s) = \rho_s$. We now start from e, and consider the hypothetical move in which every player i moves from e_i to his corresponding strategy s_i in s. Since e is Pareto-optimal, there must exist some player j for whom $c_j(s) \geq c_j(e)$.

We now upper bound the social cost of equilibrium, $c(e) = \sum_i c_i(e)$, and lower bound the social cost $c(s) = \sum_i c_i(s)$ of state s. Consider first the $c_i(e)$ values. We have $c_1(e) \leq \cdots \leq c_j(e) \leq \cdots \leq c_n(e) = \rho_e c_1(e)$. The sum $\sum_i c_i(e)$ is therefore maximized when $c_1(e) = c_2(e) = \cdots = c_j(e)$ and $c_{j+1}(e) = \cdots = \rho_e c_1(e)$, giving an upper bound of $jc_j(e) + (n-j)\rho_e c_j(e)$. Similarly, for the $c_i(s)$ values, we have $c_1(s) \leq \cdots \leq c_j(s) \leq \cdots \leq \rho_s c_n(s)$. The sum $\sum_i c_i(s)$ is minimized when $c_1(s) = \cdots = c_{j-1}(s) = c_j(s)/\rho_s$ and $c_j(s) = \cdots = c_n(s)$, and is therefore at least $\frac{(j-1)c_j(s)}{\rho_s} + (n-j+1)c_j(s)$. Recalling that $c_j(s) \geq c_j(e)$ and combining the two bounds, we obtain

$$\frac{\sum_i c_i(e)}{\sum_i c_i(s)} \leq \frac{jc_j(s) + (n-j)\rho_e c_j(s)}{\frac{(j-1)c_j(s)}{\rho_s} + (n-j+1)c_j(s)} \leq \frac{j + (n-j)\rho_e}{\frac{(j-1)}{\rho_s} + (n-j+1)}. \tag{1}$$

Differentiating with respect to j, we find that this expression is maximized at $j = 1$ or $j = n$. In the former case the quotient is at most $\frac{1+(n-1)\rho_e}{n} \leq \rho_e$, while in the latter case it is at most $\frac{n}{(n-1)/\rho_s+1} \leq \rho_s$. □

By restricting s to be a social optimum in the above theorem, we obtain a natural approach to bounding the Pareto price of anarchy (and therefore also the strong price of anarchy) in a family of symmetric games. If we can find values ρ_e and ρ_o such that for any Pareto-optimal Nash equilibrium e we have $c_i(e)/c_j(e) \leq \rho_e$, and for any social optimum o we have $c_i(o)/c_j(o) \leq \rho_o$, then the Pareto price of anarchy for the family of games will be at most $\max\{\rho_e, \rho_o\}$. We now proceed to do this for the families of symmetric congestion games with polynomial and exponential latencies.

3.2 Polynomial latencies

For the case of polynomial latencies, where each resource r has latency function $\ell_r(t) = t^k$, we show the following:

Theorem 3.2 *For symmetric congestion games with polynomial latencies of degree k, the Pareto price of anarchy (and hence also the strong price of anarchy) is at most 2^{k+1}.*

Remarks: **(i)** Note that the Pareto price of anarchy is much smaller than the known value of $k^{k(1-o(1))}$ for the standard Nash price of anarchy [6, 8, 1]. **(ii)** It is not hard to verify that the upper bound in Theorem 3.2 is tight up to a constant factor. To see this, consider a 2-player symmetric game with $m = 2^k$ resources $\{r_1, \ldots, r_m\}$, and the following four strategies: $\{r_1, r_2\}$, $\{r_1, r_3, r_5\}$, $\{r_2, r_4, r_6\}$, and $\{r_3, r_4, \ldots, r_m\}$. The social optimum occurs when the two players choose $\{r_1, r_3, r_5\}$ and $\{r_2, r_4, r_6\}$, for a total cost of 6. On the other hand, the state in which the players choose $\{r_1, r_2\}$ and $\{r_3, r_4, \ldots, r_m\}$ is a strong equilibrium, and its cost is m. The price of anarchy is thus $\frac{m}{6} = \frac{2^k}{6}$.

Proof of Theorem 3.2: Following the framework of Theorem 3.1, it suffices to derive upper bounds on the ratios of player costs both at equilibrium and at a social optimum. This we do in the following two claims.

Claim 3.3 *In the situation of Theorem 3.2, we have* $\max_{i,j \in P} \frac{c_i(e)}{c_j(e)} \leq 2^k$.

Proof: Consider any two players i and j at an equilibrium e, and the hypothetical move in which player i switches from his current strategy e_i to j's strategy e_j, resulting in the new state e'.

We now bound $c_i(e')$ in terms of $c_j(e)$. Note that

$$c_i(e') = \sum_{r \in e_j} f_{e'}(r)^k = \sum_{r \in e_j \setminus e_i} (f_e(r)+1)^k + \sum_{r \in e_j \cap e_i} f_e(r)^k \leq \sum_{r \in e_j} (f_e(r)+1)^k.$$

(This captures the intuition that in switching to e_j, player i pays at most what player j would pay if there were one more player using each resource.) From this, it follows that

$$\frac{c_i(e')}{c_j(e)} \leq \frac{\sum_{r \in e_j} (f_e(r)+1)^k}{\sum_{r \in e_j} f_e(r)^k} \leq \max_{r \in e_j} \frac{(f_e(r)+1)^k}{f_e(r)^k} \leq 2^k.$$

Since e is a Nash equilibrium, we must have $c_i(e) \leq c_i(e')$, and thus $c_i(e) \leq 2^k c_j(e)$. □

Claim 3.4 *In the situation of Theorem 3.2, we have* $\max_{i,j \in P} \frac{c_i(o)}{c_j(o)} \leq 2^{k+1}$.

Proof: As in the proof of Claim 3.3, consider any two players i and j at a social optimum o. Assume that $c_i(o) \geq c_j(o)$ as the claim is immediately true otherwise. Again, consider the move in which i moves from his current strategy o_i to j's strategy o_j, resulting in the new state o'.

Since o is a social optimum, the social cost of o' must be at least that of o; i.e., $\sum_l c_l(o') - \sum_l c_l(o) \geq 0$. Using the fact that $\sum_l c_l(s) = \sum_r f_s(r)^{k+1}$ for any state s, we have

$$\begin{aligned}
0 &\leq \sum_r f_{o'}(r)^{k+1} - \sum_r f_o(r)^{k+1} \\
&= \sum_{r \in o_i \oplus o_j} f_{o'}(r)^{k+1} - \sum_{r \in o_i \oplus o_j} f_o(r)^{k+1} \\
&= \sum_{r \in o_j \setminus o_i} \left((f_o(r)+1)^{k+1} - f_o(r)^{k+1}\right) - \sum_{r \in o_i \setminus o_j} \left(f_o(r)^{k+1} - (f_o(r)-1)^{k+1}\right),
\end{aligned}$$

where the second line follows since $f_{o'}(r) = f_o(r)$ for $r \notin o_i \oplus o_j$.

We combine this with the observation that $c_i(o) = \sum_{r\in o_i} f_o(r)^k = \sum_{r\in o_i\setminus o_j} f_o(r)^k + \sum_{r\in o_i\cap o_j} f_o(r)^k$, which we add to both sides of the above:

$$\begin{aligned}
c_i(o) &\le \sum_{r\in o_j\setminus o_i}\left((f_o(r)+1)^{k+1} - f_o(r)^{k+1}\right) - \sum_{r\in o_i\setminus o_j}\left(f_o(r)^{k+1} - (f_o(r)-1)^{k+1} - f_o(r)^k\right) + \sum_{r\in o_i\cap o_j} f_o(r)^k \\
&\le \sum_{r\in o_j\setminus o_i}\left((f_o(r)+1)^{k+1} - f_o(r)^{k+1}\right) + \sum_{r\in o_i\cap o_j} f_o(r)^k,
\end{aligned}$$

as it can be verified that the second term in the first line is always at most zero.

Since $c_j(o) = \sum_{r\in o_j\setminus o_i} f_o(r)^k + \sum_{r\in o_i\cap o_j} f_o(r)^k$, we have

$$\begin{aligned}
\frac{c_i(o)}{c_j(o)} &\le \frac{\sum_{r\in o_j\setminus o_i}((f_o(r)+1)^{k+1} - f_o(r)^{k+1}) + \sum_{r\in o_i\cap o_j} f_o(r)^k}{\sum_{r\in o_j\setminus o_i} f_o(r)^k + \sum_{r\in o_i\cap o_j} f_o(r)^k} \\
&\le \frac{\sum_{r\in o_j\setminus o_i}((f_o(r)+1)^{k+1} - f_o(r)^{k+1})}{\sum_{r\in o_j\setminus o_i} f_o(r)^k} \\
&\le \max_{r\in o_j\setminus o_i} \frac{(f_o(r)+1)^{k+1} - f_o(r)^{k+1}}{f_o(r)^k},
\end{aligned}$$

where the second line follows because the ratio of the first sums in the numerator and denominator is greater than 1. This last quantity can be seen to be at most 2^{k+1}, proving the claim. □

Finally, combining Claims 3.3 and 3.4 with Theorem 3.1 completes the proof of Theorem 3.2. □

3.3 Exponential latencies

For the case of exponential latencies, where each resource has latency function $\ell_r(t) = \alpha^t$, we show the following upper bound on the Pareto and strong prices of anarchy. The proof of this follows the same structure as that of Theorem 3.2 and is left to the appendix.

Theorem 3.5 *For symmetric congestion games with n players and exponential latencies, the Pareto price of anarchy (and hence also the strong price of anarchy) is at most $\max\{\alpha, n\}$.*

For comparison purposes with the above upper bound, we now show that the price of anarchy for standard Nash equilibria in exponential congestion games is much larger—indeed, exponential in n.

Proposition 3.6 *For symmetric congestion games with exponential latencies α^t, the (standard Nash) price of anarchy is at least $(\frac{\alpha}{2})\alpha^{(\frac{\alpha/2-1}{\alpha-1})n}$, where n is the number of players.*

Proof: Our construction is based on that of Christodoulou and Koutsoupias [8] for the case of linear latencies. Our game contains m groups of resources, and $n = mt$ players. The players are divided evenly into m equivalence classes, labeled $\{1,\ldots,m\}$, with t players per class. Each of the m groups of resources consists of $\binom{m}{k}$ resources, each labeled with a different k-tuple of equivalence classes. The available strategies for all players are to take either (1) all resources in a single group of resources, or (2) for any i in $\{1,\ldots,m\}$, all resources that are labeled with i.

Given the value of α, we choose m and k such that $m \le 1 + (\frac{m}{k} - 1)\alpha$; for example, we can choose $k = \frac{\alpha}{2}$ and $m = \alpha - 1$ for integer $\alpha \ge 4$. It can be verified that, with these settings, the state in which each player takes all resources labeled with his equivalence class number is a Nash equilibrium, while the state in which each player takes the group of resources corresponding to his equivalence class (i.e., a player in class i takes all resources in group i) is a social optimum. A straightforward calculation then shows that the ratio of a player's Nash cost to his cost at social optimum is $k\alpha^{t(k-1)}$, which is $(\frac{\alpha}{2})\alpha^{(\frac{\alpha/2-1}{\alpha-1})n}$ for the above values of k and m. □

For completeness, we show that this same price of anarchy for standard Nash is upper bounded by α^n:

Proposition 3.7 *For asymmetric (and hence also symmetric) congestion games with exponential latencies, the (standard Nash) price of anarchy is at most α^n.*

Proof (sketch): As in the proofs of related results in [6, 8, 1], in a game with latency functions $\ell(t)$, we can prove an upper bound on the price of anarchy by finding $c_1, c_2 \geq 0$ such that the inequality $y\ell(x+1) \leq c_1 x\ell(x) + c_2 y\ell(y)$ holds for all $0 \leq x \leq n, 1 \leq y \leq n$; this implies a price of anarchy of at most $\frac{c_2}{1-c_1}$. When $\ell(t) = \alpha^t$, this clearly holds with $c_1 = 0$ and $c_2 = \alpha^x \leq \alpha^n$. □

Remark: With some extra work, the bound in Proposition 3.7 can be improved to $O(\alpha^{(1-\frac{1}{\alpha})n})$.

4 Asymmetric games

In this section we extend the investigation of the previous section to asymmetric games, and find that the situation is quite different. First we will see that, for asymmetric congestion games with polynomial latencies, the strong (and therefore also the Pareto) price of anarchy is essentially the same as the standard Nash price of anarchy. We will then go on to consider exponential latencies, where we find that the Pareto price of anarchy is the same as standard Nash, but the strong price of anarchy is significantly smaller.

We begin by considering polynomial latencies.

Theorem 4.1 *For asymmetric congestion games with polynomial latencies t^k, the strong price of anarchy is at least $\lfloor \Phi_k \rfloor^k$, where Φ_k is the positive solution of $(x+1)^k = x^{k+1}$.*

Remark: Φ_k is a generalization of the golden ratio (which is just Φ_1); its value is $\frac{k}{\log k}(1+o(1))$. Hence the lower bound of Theorem 4.1 is of the form $k^{k(1-o(1))}$, which is asymptotically the same value for the Nash price of anarchy obtained in [6, 8], and very close to the exact value obtained by Aland et al. [1].

Proof: Our lower bound construction is based on that of Aland et al., extended so as to handle the stricter requirement of a strong equilibrium. Consider an (asymmetric) game with n players (n assumed sufficiently large). Each player i has exactly two possible strategies, e_i and o_i. There are $n+m$ resources labeled $\{r_1, \ldots, r_{n+m}\}$, where m is a constant to be chosen later. For each player i, strategy o_i consists of the single resource r_i. (We shall modify this slightly for some of the players shortly.) Strategy e_i consists of the resources $\{r_{i+1}, \ldots, r_{i+m}\}$. (Thus for most players e_i consists of exactly m resources.)

We claim that the state $e = (e_1, \ldots, e_n)$ is a strong Nash equilibrium. To see this, note that under e the cost for player i is $c_i(e) = \sum_{j=i+1}^{i+m} \min\{j-1, m\}^k$. If now player i moves to his alternative strategy o_i, resulting in a new state $e^{(i)}$, his cost becomes $c_i(e^{(i)}) = \min\{i, m+1\}^k$. To show that e is a Nash equilibrium, we need to show that $c_i(e^{(i)}) \geq c_i(e)$ for all i.

Now note that, for all players $i \geq m+1$, we have $c_i(e^{(i)}) - c_i(e) = (m+1)^k - m^{k+1}$. Thus if we choose $m = \lfloor \Phi_k \rfloor$ to be the smallest integer such that $(m+1)^k \geq m^{k+1}$, we ensure that $c_i(e^{(i)}) \geq c_i(e)$ for all $i \geq m+1$. To obtain the same condition for players $1 \leq i \leq m$, we append to the strategy o_i the minimum number a_i of additional resources (unique to i) so that $c_i(e^{(i)}) = i^k + a_i \geq c_i(e)$. (Note that all the a_i are less than m^{k+1}.) This ensures that e is a Nash equilibrium.

To see that it is a strong equilibrium, consider a move by an arbitrary coalition of players to their alternative strategies o_i. We claim that the lowest numbered player in the coalition does not see an improvement in cost. This follows because the resource r_i, which i occupies under o_i, is still occupied by the same players as under e, so by the Nash property i's cost does not decrease.

Thus the strong price of anarchy is bounded below by $\frac{c(e)}{c(o)}$. But $c(e) \geq (n-m)m^k$, and $c(o) \leq mm^{k+1} + (n-m)$. Thus

$$\frac{c(e)}{c(o)} \geq \frac{(n-m)m^k}{m^{k+2}+n-m} \to \lfloor \Phi_k \rfloor^k \quad \text{as } n \to \infty.$$

This completes the proof. □

We now turn to exponential latencies. Our next result shows that the Pareto price of anarchy is equal to the standard Nash price of anarchy (which we showed to be exponential in n in Proposition 3.6).

Theorem 4.2 *For asymmetric congestion games with exponential latencies α^t, the Pareto price of anarchy is bounded below by, and hence is equal to, the standard Nash price of anarchy.*

Proof: Consider any n-player congestion game with exponential latencies α^t. Let state e be a Nash equilibrium for this game. We create a modified game in which the Pareto price of anarchy is only a $(1-O(\frac{1}{n}))$-factor smaller than the Nash price of anarchy of the original game.

To do this, we first replace each resource in the original game with a set of n resources in the modified game; strategies in the modified game correspond to those in the original game, except that the former include all n copies of the resources of the latter. This has the effect of multiplying player costs by a factor of n, but does not change the set of Nash and Pareto-optimal Nash equilibria.

We then add one more player, $n+1$, to the modified game; this player has a single strategy s_{n+1} consisting of new resources $\{\hat{r}_i : i = 1, \ldots, n\}$. Also, for players $1, \ldots, n$, we append resource $\hat{r}_i$ to every strategy of player i except for the equilibrium strategy e_i. Note that this makes the modified game asymmetric even if the original one is not. There is an obvious bijection between states of the original game and those of the modified game, and we shall abuse notation by identifying them. Also, we shall write $c(s)$ and $c'(s)$ for the social costs of state s in the original game and in the modified game respectively.

Now it is easy to see that the original Nash equilibrium e (together with strategy s_{n+1} for player $n+1$) is a Pareto-optimal equilibrium for the modified game: plainly it remains a Nash equilibrium, and any coalition move results in a cost increase for player $n+1$. Moreover, we have $c'(e) = nc(e) + n > nc(e)$, and for any state s, $c'(s) \leq nc(s) + 2n\alpha^2$, since the occupancy of each new resource is at most two. Thus the Pareto price of anarchy for the modified game is at least $\max_s \frac{c'(e)}{c'(s)} \geq \frac{c(e)}{c(o)+2\alpha^2}$, where o is a social optimum of the original game. But clearly $c(o) \geq n\alpha$, so the Pareto price of anarchy is at least $\frac{c(e)}{c(o)(1+\frac{2\alpha}{n})} = \frac{1}{1+\frac{2\alpha}{n}} \frac{c(e)}{c(o)}$. Thus the Pareto price of anarchy grows arbitrarily close to the Nash price of anarchy as n increases. This completes the proof. $\square$

Finally, we exhibit a separation between the Pareto and strong price of anarchy by showing that the latter (while still exponential) is significantly smaller than the value we obtained for the standard Nash price of anarchy in Proposition 3.6. We make the reasonable assumption that the number of resources is polynomially bounded in the number of players, as is the case in our lower bound construction in Proposition 3.6.

Theorem 4.3 *For asymmetric congestion games with n players and exponential latencies α^t, in which every strategy contains at most $p(n)$ resources for some fixed polynomial p, the strong price of anarchy is at most $\alpha^{(\frac{1}{3}+o(1))n}$.*

Proof: Let e be a strong equilibrium state and o be a social optimum state. As before, we will consider moves in which subsets of players move from their equilibrium strategies e_i to their strategies at optimum o_i. Let $c_*(o)$ denote the maximum cost of any player in state o.

We will make use of the following technical lemma:

Lemma 4.4 *Let S be a subset of players each of whose strategies at e contains at least one resource that is shared by at least u_e players at e (i.e., for all $i \in S$, $\exists r \in e_i$ such that $f_e(r) \geq u_e$). Then at least one of the following must be true: (1) $\alpha^{u_e} \leq c_*(o)$; or (2) there exist at least $u'_e = u_e - \log_\alpha p(n) - \log_\alpha c_*(o)$ players outside of S, each of whose strategies at e contains a resource that is shared by at least u'_e players not in S.*

Proof: Consider the move from e in which the players $i \in S$ each adopt their strategies o_i in o, resulting in a new state s in which their new costs are $c_i(s)$. For all $i \in S$, we will denote by $\hat{c}_i(s)$ the cost to player i at state s due only to the players in S; formally, $\hat{c}_i(s) = \sum_{r \in o_i} \alpha^{\hat{f}_s(r)}$, where $\hat{f}_s(r) = |\{j \in S : r \in s_j\}|$.

Since e is a strong equilibrium, there must be some player $i \in S$ for whom $c_i(s) \geq c_i(e)$. This might happen if $\hat{c}_i(s) \geq c_i(e)$, in which case since $c_i(o) \geq \hat{c}_i(s)$ and $c_i(e) \geq \alpha^{u_e}$, condition (1) of the lemma holds. Otherwise, we must have $\hat{c}_i(s) < c_i(e) \leq c_i(s)$. Since $c_i(e) \geq \alpha^{u_e}$, there must be some resource $r \in o_i$ for which $\alpha^{f_s(r)} \geq \frac{c_i(e)}{p(n)}$, or equivalently, for which $f_s(r) \geq \log_\alpha c_i(e) - \log_\alpha p(n)$. However, note that because $\hat{c}_i(s) \leq c_i(o) \leq c_*(o)$, we have $\hat{f}_s(r) \leq \log_\alpha c_*(o)$. Hence in state s there must be $f_s(r) - \hat{f}(r) \geq \log_\alpha c_i(e) - \log_\alpha p(n) - \log_\alpha c_*(o)$ players not in S that also use resource r. Since $c_i(e) \geq \alpha^{u_e}$, condition (2) of the lemma holds. $\square$

Suppose now that there exists a strong equilibrium e in a game fitting the description of the theorem such that $\frac{c_e}{c_*(o)} \geq \alpha^{\delta n}$ for a social optimum o. Then there must exist a player j for whom $c_j(e) \geq \frac{\alpha^{\delta n}}{n} c_*(o)$. Thus e_j must contain a resource r for which $f_r(e) \geq \log_\alpha \left(\frac{\alpha^{\delta n}}{np(n)} c_*(o) \right) = \delta n + \log_\alpha c_*(o) - \log_\alpha(np(n))$.

Let S_1 denote the players holding this resource. Consider the move in which we try to move all players in S_1 to their strategies at o. Applying Lemma 4.4 to these players, we find that either (1) $\alpha^{\delta n} \leq np(n)$, in which case the theorem is proven; or (2) there exist $\delta n - \log_\alpha(np^2(n))$ additional players, each of whose equilibrium strategies contains a resource shared by at least that many players not in S_1. Let these additional players form the set S_2. Since the game has n players, this implies that

$$n \geq |S_1 \cup S_2| \geq 2\delta n + \log_\alpha c_*(o) - \log_\alpha(n^2 p(n)^3). \tag{2}$$

We can then apply Lemma 4.4 again to $S_1 \cup S_2$, which again yields two possible outcomes. In case (1), we have that $\alpha^{\delta n} \leq c_*(o) np(n)^2$, or $\delta n \leq \log_\alpha c_*(o) + \log_\alpha(np(n)^2)$. Combining this with inequality (2) gives $3\delta n \leq n + \log_\alpha(n^3 p(n)^5)$, or $\delta \leq \frac{1}{3} + o(1)$, as claimed. In case (2), we are guaranteed the existence of $\delta n - \log_\alpha c_*(o) - \log_\alpha(np(n)^3)$ players not in $S_1 \cup S_2$. Combining this with the lower bound on $|S_1 \cup S_2|$ from (2), we must have at least $3\delta n - \log_\alpha(n^3 p(n)^6)$ players. Since this cannot exceed n, we find that $\delta \leq \frac{1}{3} + o(1)$, again as claimed. $\square$

5 Linear latencies

This section presents more detailed results for the special case of linear latencies.

Exact price of anarchy for asymmetric games. We first show that the strong (and thus also the Pareto) price of anarchy for asymmetric congestion games with linear latencies coincides exactly with the standard Nash price of anarchy, which is known to be $\frac{5}{2}$ [6, 8]. To do this, it is sufficient to exhibit a lower bound of $\frac{5}{2}$ on the strong price of anarchy. This is the content of the following theorem.

Theorem 5.1 *For asymmetric linear congestion games, the strong price of anarchy is at least* $\frac{5}{2}$.

The proof of this refines the construction in the proof of Theorem 4.1, and is left to the appendix.

Upper bound for symmetric games. We now show that, for *symmetric* linear congestion games, the Pareto (and hence also strong) price of anarchy is less than the known value $\frac{5}{2}$ for standard Nash equilibria in both symmetric and asymmetric games. For linear latencies, the framework of Theorem 3.1 only gives an upper bound of 3, so we must resort to a more involved analysis. We prove the following, stressing that our goal is not to find the best possible upper bound, but to show that the upper bound is strictly less than $\frac{5}{2}$.

Theorem 5.2 *For symmetric congestion games with linear latencies, the Pareto price of anarchy (and hence also the strong price of anarchy) is strictly less than* $\frac{5}{2}$.

The proof of this theorem is quite involved and is left to the appendix; we briefly outline the ideas here. We begin as in the proof of Theorem 3.1 by sorting the strategies at equilibrium e and optimum o by cost. A key ingredient in the earlier proof is the hypothetical move from e in which every player moves from his current strategy e_i to his strategy o_i at o, and the realization that at least one player must pay higher cost at o than at e. Here we extend that idea to a more complicated sequence of player moves, and again use the fact that at least one player must pay higher cost at the end of this sequence than he did at equilibrium.

One of the key concepts in this proof is the matrix $M = (m_{ij})$, where m_{ij} is the relative cost increase to player i at optimum when a new player moves to player j's strategy o_j. Using properties of M, we are able to identify three disjoint subsets of players L, H', and R satisfying $i < j < k$ for all $i \in L, j \in H'$ and $k \in R$ in such a way that, for all players in $i \in H'$, the ratio $q_i \stackrel{\text{def}}{=} \frac{c_i(e)}{c_i(o)} \geq \frac{5}{2}$, while for players $i \in L \cup R$, $q_i \leq 2$. Thus to bound the Pareto price of anarchy, we must upper bound the number of high-ratio players $|H'|$ relative to the number of low-ratio players $|L| + |R|$.

The moves from e we consider consist of two steps: first, each player i goes from his original strategy e_i to his strategy at social optimum o_i; then, we will try to take each of the players in R (which necessarily includes all players for whom $c_i(o) \geq c_i(e)$—these players are "unhappy" after the first step) and reassign them one-to-one to the strategies of players in H', giving a new state o'. It is possible to prove that, under this scheme, all players in R and H' are better off at o' than at o. Therefore, since e is Pareto-optimal, either this cannot be done (because $|R| > |H'|$), or some player in L ends up with higher cost than before.

We then split the proof into two cases. In the first case, players in H' can have ratios $q_i > \tau$ for some threshold τ. Here we can use the properties of the matrix M to show that every player in L will have lower cost at o' than at e, and thus it must be that $|R| > |H'|$, which gives us an overall price of anarchy below $\frac{5}{2}$. In the second case, all players in H' have ratios of at most τ. Here, in the most involved part of the analysis, we use the probabilistic method to lower bound $|R|$ in terms of $|L|$, $|H'|$ and τ, which we again show gives us a price of anarchy below $\frac{5}{2}$.

6 Open problems

We have left open a number of questions, including the following:

1. What is the exact strong (and Pareto) price of anarchy for symmetric congestion games with linear latencies? From Theorem 5.2 we know that this is less than the value $\frac{5}{2}$ of the standard Nash price of anarchy for these games. It is not too hard to obtain a lower bound of $\frac{3}{2}$ on this quantity, but we do not see how to obtain its exact value using the machinery of Section 5.
2. What is the computational complexity of deciding whether a congestion game possesses a strong or Pareto-optimal equilibrium, and if so of finding one? For standard Nash equilibria, the decision problem is trivial but finding an equilibrium is known to be in P for symmetric network congestion games and PLS-complete for general symmetric congestion games [11].

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Appendix

A.1 Example from Section 2.2

As advertised in Section 2.2, we give an example of asymmetric congestion games with linear, polynomial, and exponential latencies with integer base (i.e., $\ell_r(t) = \alpha^t$ for some positive integer α) that have Pareto-optimal Nash equilibria but no strong Nash equilibria.

Consider an n-player congestion game with latency functions $\ell(\cdot)$ which has a Nash equilibrium e but no strong Nash equilibrium; it is easy to construct a Prisoner's Dilemma-style game with this property. We use a similar construction to that in the proof of Theorem 4.2 to obtain a modified game that has a Pareto-optimal Nash equilibrium but still no strong Nash equilibrium, with the only difference being that here we replicate each resource m times (instead of n), for a constant $m > \ell(2)$; thus m is larger than $2, 2^k$ and α^2 for linear, polynomial, and exponential latencies respectively. We then add the extra player $n + 1$ and modify all strategies exactly as before.

Again, there is a natural bijection between states of the original game and those of the modified game, and we again abuse notation by identifying them. We write $c_i(s)$ and $c'_i(s)$ for the cost to player i of state s in the original game and in the modified game respectively.

Now it is easy to see that the original Nash equilibrium e (together with strategy s_{n+1} for player $n+1$) is a Pareto-optimal equilibrium for the modified game: plainly it remains a Nash equilibrium, and any coalition move results in a cost increase for player $n + 1$.

It remains to show that the modified game does not have any strong Nash equilibria. We do this by showing that if a state s is a strong Nash equilibrium in the modified game, it must correspond to a strong Nash equilibrium in the original game. Consider an arbitrary coalition of players in state s in the original game, and an arbitrary group move to some other strategies, resulting in a new state s'; we aim to show that one of the coalition members has cost in s' that is at least his cost in s. We do this by examining the corresponding move in the modified game. Since s is a strong Nash equilibrium there, we must have at least one player, say player i, for whom $c'_i(s') \geq c'_i(s)$. Since $c'_i(s') \leq mc_i(s') + \ell(2)$, and $c'_i(s) \geq mc_i(s)$, we obtain $mc_i(s') + \ell(2) \geq mc_i(s)$, implying that $c_i(s') + \frac{\ell(2)}{m} \geq c_i(s)$.

Since all player costs must be integral with the latency functions we consider, and $\frac{\ell(2)}{m} < 1$, it must be that $c_i(s') \geq c_i(s)$, which is what we needed.

A.2 Proof of Theorem 3.5

To prove Theorem 3.5, we again rely on the framework of Theorem 3.1 and establish the following two claims, which bound the ratios of player costs at equilibrium and at social optimum.

Claim A.2.1 *In the situation of Theorem 3.5, we have* $\max_{i,j \in P} \frac{c_i(e)}{c_j(e)} \leq \alpha$.

Proof: Consider an equilibrium state e and a move by player i from his current strategy e_i to player j's strategy e_j, resulting in the new state e'. Clearly for each resource $r \in e_j$, player i pays at most a factor of α more than j pays at e. Hence $c_i(e') \leq \alpha c_j(e)$. Since e is a Nash equilibrium, we must have $c_i(e) \leq c_i(e')$ and the claim follows. □

Claim A.2.2 *In the situation of Theorem 3.5, we have* $\max_{i,j \in P} \frac{c_i(o)}{c_j(o)} \leq \alpha(n + 1)$.

Proof: This proof follows along the same lines as that of Claim 3.4. Consider a social optimum o, and the move in which player i moves from o_i to player j's strategy o_j. Since o is a social optimum, this cannot decrease the overall social cost. Hence

$$
\begin{aligned}
0 &\le \sum_r f_{o'}(r)\alpha^{f_{o'}(r)} - \sum_r f_o(r)\alpha^{f_o(r)} \\
&= \sum_{r\in o_i \oplus o_j} f_{o'}(r)\alpha^{f_{o'}(r)} - \sum_{r\in o_i \oplus o_j} f_o(r)\alpha^{f_o(r)} \\
&= \sum_{r\in o_j\setminus o_i} \big((f_o(r)+1)\alpha^{f_o(r)+1} - f_o(r)\alpha^{f_o(r)}\big) - \sum_{r\in o_i\setminus o_j} \big(f_o(r)\alpha^{f_o(r)} - (f_o(r)-1)\alpha^{f_o(r)-1}\big).
\end{aligned}
$$

Writing $c_i(o) = \sum_{r\in o_i\setminus o_j} \alpha^{f_o(r)} + \sum_{r\in o_i\cap o_j} \alpha^{f_o(r)}$ and adding this to both sides, we obtain

$$
\begin{aligned}
c_i(o) &\le \sum_{r\in o_j\setminus o_i} \big((f_o(r)+1)\alpha^{f_o(r)+1} - f_o(r)\alpha^{f_o(r)}\big) - \sum_{r\in o_i\setminus o_j} (f_o(r)-1)(\alpha^{f_o(r)} - \alpha^{f_o(r)-1}) + \sum_{r\in o_i\cap o_j} \alpha^{f_o(r)} \\
&\le \sum_{r\in o_j\setminus o_i} \big((f_o(r)+1)\alpha^{f_o(r)+1} - f_o(r)\alpha^{f_o(r)}\big) + \sum_{r\in o_i\cap o_j} \alpha^{f_o(r)}.
\end{aligned}
$$

Combining this with the fact that $c_j(o) = \sum_{o_j\setminus o_i} \alpha^{f_o(r)} + \sum_{o_i\cap o_j} \alpha^{f_o(r)}$, we get

$$
\begin{aligned}
\frac{c_i(o)}{c_j(o)} &\le \frac{\sum_{r\in o_j\setminus o_i}((f_o(r)+1)\alpha^{f_o(r)+1} - f_o(r)\alpha^{f_o(r)}) + \sum_{r\in o_i\cap o_j} \alpha^{f_o(r)}}{\sum_{o_j\setminus o_i} \alpha^{f_o(r)} + \sum_{o_i\cap o_j} \alpha^{f_o(r)}} \\
&\le \frac{\sum_{r\in o_j\setminus o_i}((f_o(r)+1)\alpha^{f_o(r)+1} - f_o(r)\alpha^{f_o(r)})}{\sum_{o_j\setminus o_i} \alpha^{f_o(r)}} \\
&\le (f_o(r)+1)\alpha.
\end{aligned}
$$

Since $f_o(r) \le n$, this completes the proof of the claim. $\square$

A direct application of Theorem 3.1 to the results of these claims gives an overall bound of $\alpha(n+1)$, which we now improve to $\max\{\alpha, n\}$. We first follow the proof of Theorem 3.1 through inequality (1), where one can see that the ratio on the right-hand side is maximized when the player j whose cost increases is either player 1 or player n. In the former case, the resulting ratio is at most ρ_e, which is α for exponential latencies. In the latter case, the resulting ratio becomes $\frac{n\rho_o}{n-1+\rho_o}$, or $\frac{\alpha n(n+1)}{n-1+\alpha(n+1)}$ for exponential latencies. This is clearly at most n. $\square$

A.3 Proof of Theorem 5.1

We construct a family of n-player games with latencies $\ell_r(t) = t$ for all r whose strong price of anarchy approaches $\frac{5}{2}$ as $n \to \infty$. Our construction will be a refined version of that in the proof of Theorem 4.1. For simplicity, assume n is even. There will be $n+3$ resources $\{r_1, \ldots, r_{n+3}\}$. Each player i has two possible strategies, e_i and o_i. Strategy o_i consists of the single resource r_i, except that we add resources r_{n+2} and r_{n+3} to player 1's strategy o_1. The strategies e_i are defined as follows:

$$
e_i = \begin{cases} \{r_{i+1}, r_{i+2}\} & \text{if } i \text{ is odd;} \\ \{r_{i+1}\} & \text{if } i \text{ is even.} \end{cases}
$$

We claim that the state $e = (e_i)$ forms a strong Nash equilibrium. To see it is a Nash equilibrium, suppose player i switches from e_i to o_i. If $i > 1$ is odd then his strategy switches from $\{r_{i+1}, r_{i+2}\}$, at a cost of

$1 + 2 = 3$, to $\{r_i\}$ at a cost of 3. (This calculation follows from the fact that the players occupying r_i, r_{i+1} and r_{i+2} under e are respectively $\{i-2, i-1\}$, $\{i\}$ and $\{i+1, i+2\}$.) Hence there is no cost improvement. Similarly, if i is even then his strategy switches from $\{r_{i+1}\}$, at a cost of 2, to $\{r_i\}$, at a cost of 2, again giving no improvement. Finally, if $i = 1$ then his strategy switches from $\{r_{i+1}, r_{i+2}\}$, at a cost of $1+2 = 3$, to $\{r_i, r_{n+2}, r_{n+3}\}$, at a cost of $1+1+1 = 3$, again giving no improvement. Hence e is a Nash equilibrium.

To see that it is a strong equilibrium, suppose that an arbitrary subset of the players switch from their strategies e_i to o_i. Then the lowest-numbered player i in the subset experiences no cost improvement. This follows as in the proof of Theorem 4.1 because the resources which i occupies under o_i are still occupied by the same players as under e, so by the Nash property i's cost does not decrease.

The price of anarchy of this game is therefore at least $\frac{c(e)}{c(o)}$. But in o each player occupies one resource alone (except for player 1, who occupies three resources), so $c(o) = n + 2$. And in e each odd-numbered player occupies one resource alone and one resource shared with another player, while each even-numbered player occupies one resource shared with another player. Thus $c_i(e) = 3$ if i is odd, and $c_i(e) = 2$ if i is even, so $c(e) = \frac{5n}{2}$. Hence the price of anarchy is at least $\frac{5n/2}{n+2} \to \frac{5}{2}$ as $n \to \infty$. $\square$

A.4 Proof of Theorem 5.2

Setting. From this point on, as in the proof of Theorem 3.1, we will consider an equilibrium state e and a social optimum state o, both of which have the players sorted in increasing order of cost; that is, $c_1(e) \leq \cdots \leq c_n(e)$ and $c_1(o) \leq \cdots \leq c_n(o)$.

A key consideration in the earlier proof is the hypothetical move from e in which every player moves from his current state e_i to his corresponding strategy o_i at o, and the realization that at least one player must pay higher cost at o. Here we extend that idea to a more complicated set of moves to be described later, and again use the fact that at least one player must still pay higher cost at the end of this sequence than he did at equilibrium.

We now describe a number of preliminary concepts and claims that will set up the proof of Theorem 5.2. First, for the state o, we will define an associated matrix $M = (m_{ij})$, where $m_{ij} = \frac{\sum_{e \in o_i \cap o_j} ((f_o(r)+1) - f_o(r))}{\sum_{r \in o_i} f_o(r)}$. This can be seen to be $\frac{|o_i \cap o_j|}{c_i(o)}$, where $|o_i \cap o_j|$ is the number of resources used in both o_i and o_j. Intuitively, m_{ij} is the relative cost increase to player i's cost $c_i(o)$ when a new player moves to strategy o_j. Further, we will denote by q_i the ratio $\frac{c_i(e)}{c_i(o)}$, and let $q_{\max}$ and $q_{\min}$ be the maximum and minimum such values of q_i. Note that $q_{\min} \leq 1$ since e is Pareto-optimal, and we can assume that $q_{\max} \geq \frac{5}{2} - \varepsilon$ for some small $\varepsilon > 0$ or else the theorem is trivially true; let $i_{\max}$ and $i_{\min}$ refer to a player that has ratio $q_{\max}$ and a player that has ratio $q_{\min}$, respectively. As a matter of notation, for an integer i and a set of integers S, we will say that $i < S$ (or $i > S$) if i is smaller (or larger) than every element of S. Similarly we will write $S_1 < S_2$ if every element of S_1 is less than every element of S_2.

We begin with the following lemma, which spells out some key properties of this setup.

Lemma A.4.1 *Consider an equilibrium state e and a social optimum state o in the setting described above. Then the following properties hold.*

- *(a) For all i, j, $2c_i(e) \geq c_j(e)$.*
- *(b) For all i, $\sum_j m_{ij} = 1$, and for all i, j, $m_{ii} \geq m_{ij}$.*
- *(c) For all i, j, $(2 + m_{ii})c_i(o) \geq c_j(o)$.*
- *(d) $i_{\max} < i_{\min}$, and hence $q_{\max} \leq 3$.*
- *(e) There exists $\varepsilon > 0$ such that, if players i, j with $i < j$ have ratios $q_i, q_j \geq \frac{5}{2} - \varepsilon$, then for all players k with $i < k < j$, $q_k > 2$.*

Proof:

(a) At the equilibrium state e, consider the move in which player j moves to player i's strategy. The new cost for player j will be at most $\sum_{r \in e_i}(f_e(r)+1) \leq 2\sum_{r \in e_i} f_e(r) \leq 2c_i(e)$. Since e is a Nash equilibrium, this must be at least $c_j(e)$.

(b) Since $\sum_j |o_i \cap o_j| = \sum_{r \in o_i} f_o(r) = c_i(o)$, we have $\sum_j m_{ij} = 1$. It is also evident that $|o_i \cap o_j|$ is maximized when $j = i$, and hence $m_{ii} \geq m_{ij}$ for all j.

(c) Starting from the social optimum state o, consider the move in which player j moves from o_j to o_i; the total social cost cannot decrease as a result. We think of this move as occurring in two phases: when player j moves away from strategy o_j, the total social cost decreases by at least $c_j(o)$, when he moves to strategy o_i, the social cost increases by $\sum_{r \in o_i}((f_o(r)+1)^2 - f_o(r)^2) = \sum_{r \in o_i}(2f_o(r) + 1) = 2c_i(o) + |o_i|$, where $|o_i|$ is the number of resources in o_i. Using the fact from (b) above that $|o_i| = m_{ii}c_i(o)$, we put this together to find that $c_j(o) \leq (2 + m_{ii})c_i(o)$.

(d) Note that for any player j with $j \leq i_{\max}$, we have that $2c_j(e) = 2q_jc_j(o) \geq q_{\max}c_{i_{\max}}(o)$ by (a) above. Thus $q_j \geq \frac{q_{\max}}{2}\frac{c_{i_{\max}}(o)}{c_j(o)} \geq \frac{q_{\max}}{2} > 1$. Since $q_{\min} \leq 1$, $i_{\min}$ must be larger than $i_{\max}$.

To see that $q_{\max} \leq 3$, observe that $c_{i_{\max}}(e) \leq c_{i_{\min}}(e) \leq c_{i_{\min}}(o) \leq (2 + m_{i_{\max}i_{\max}})c_{i_{\max}}(o)$. Thus $\frac{c_{i_{\max}}(e)}{c_{i_{\max}}(o)} \leq 3$, as $m_{ii} \leq 1$ for all i by (b).

(e) Note that $j < i_{\min}$, by part (a). Hence $c_{i_{\min}}(o) \geq c_{i_{\min}}(e) \geq c_j(e) \geq (\frac{5}{2} - \varepsilon)c_j(o) \geq (\frac{5}{2} - \varepsilon)c_k(o) = \frac{1}{q_k}(\frac{5}{2} - \varepsilon)c_k(e) \geq \frac{1}{q_k}(\frac{5}{2} - \varepsilon)c_i(e) \geq \frac{1}{q_k}(\frac{5}{2} - \varepsilon)^2 c_i(o)$. By (d), we have that $c_{i_{\min}}(o) \leq 3c_i(o)$ and thus $q_k \geq \frac{1}{3}(\frac{5}{2} - \varepsilon)^2 > 2$ for sufficiently small $\varepsilon > 0$.

□

These results allow us to define the sets $L = \{i < i_{\max} : q_i \leq 2\}$, $R = \{i > i_{\max} : q_i \leq 2\}$, $H = \{i : q_i > 2\}$, and $H' = \{i \in H : L < i < R\}$; thus H' is the subset of H satisfying $L < H' < R$. Note that the only players not in $L \cup H' \cup R$ are players in $H \setminus H'$; further, by part (e) above, all of these players i must satisfy $q_i < \frac{5}{2} - \varepsilon$.

The idea behind the above definitions is the following: We will start by considering moves from e in which all players i move from e_i to their corresponding strategies at social optimum o_i. At least one player (necessarily in R) will pay a higher cost at o than at e. We try to rectify this by reassigning the players in R to new strategies, namely some of the o_h for players $h \in H'$. As we will see now, this results in a state o' in which all players in H and all players who have been reassigned are better off than at e.

Lemma A.4.2 *Consider the two-step sequence of moves in which (1) each player moves from e_i to o_i, and (2) a subset of players $R' \subseteq R = \{i_1, \ldots, i_\ell\}$ are reassigned to strategies held by (distinct) players $\{h_1, \ldots, h_\ell\} \subseteq H'$; call the resulting state o'. Then for all players i in $R' \cup H$, $c_i(o') < c_i(e)$.*

Proof: By the definition of the matrix M and Lemma A.4.1 (b), we have that, for any player i, $c_i(o') \leq (1 + \sum_{h_j} m_{ih_j})c_i(o) \leq 2c_i(o)$. But if $i \in H$ then $c_i(e) > 2c_i(o) \geq c_i(o')$, as needed.

For the case of $i \in R'$, we need only observe that at o', player i is playing the same strategy as a player in H', and that $c_i(e) \geq c_h(e)$ for all $h \in H'$, and apply the above reasoning. □

Generally, we will be considering moves in which we try to reassign all of the players in R (i.e., $R' = R$ in the above Lemma). If we are successful in doing so, then since e is Pareto-optimal, there must be a player $i \in L$ who was not reassigned who must now be paying at least as much at o' as at e. This happens if the strategies in H' occupied by the reassigned players have combined m_{ih} values of at least $q_i - 1$. Thus, for any player i, we define its *capacity* γ_i to be $\frac{q_i - 1}{\sum_{h \in H} m_{ih}}$. For $i \notin H$, this is at least $\frac{q_i - 1}{1 - m_{ii}}$. The following claim will prove useful later.

Claim A.4.3 *For any $i \in L$ and $h \in H'$, $\gamma_i \geq \frac{q_h - 2}{6 - 2q_h}$.*

Proof: As just stated, $\gamma_i \geq \frac{q_i-1}{1-m_{ii}}$. We first lower bound the numerator by observing that $q_i \geq \frac{q_h}{2}$. For the denominator, note that $i_{\min} > h$. Thus $(2+m_{ii})c_i(o) \geq c_{i_{\min}}(o) \geq c_{i_{\min}}(e) \geq c_h(e) = q_h c_h(o)$, and so $2 + m_{ii} \geq \frac{q_h c_h(o)}{c_i(o)} \geq q_h$ and $1 - m_{ii} \leq 3 - q_h$.

Combining these bounds yields $\gamma_i \geq \frac{\frac{q_h}{2}-1}{3-q_h}$, as needed. □

At this point it is convenient to split the proof of Theorem 5.2 into two cases: one for when $q_{\max} > \frac{8}{3}$, and the other for when $q_{\max} \leq \frac{8}{3}$. We handle the former case first.

Proof of Theorem 5.2 when $q_{\max} > \frac{8}{3}$: First, we claim that for all $i \in L$, $\gamma_i > 1$ by applying Claim A.4.3 with $q_h = q_{\max}$. But $\gamma_i > 1$ immediately implies that $c_i(o') < c_e(o)$.

Now, starting from e, suppose that we first move all players from their equilibrium strategies to their social optimum strategies, and then try to reassign all players in R to the strategies of distinct players in the set H'. If we can do this, then by Lemma A.4.2 and the above fact that $c_i(o') < c_e(o)$ for all players $i \in L$ we will have achieved a situation in which all players are better off than in e, thus contradicting the Pareto-optimality of e. But the only way this reassignment can fail is if $|H'| < |R|$, so that there are not enough slots to which to assign players in R. Thus we may assume $|H'| < |R|$.

We now compute the price of anarchy for just the players in $R \cup H'$, or $\frac{\sum_{i\in R\cup H'} c_i(e)}{\sum_{i\in R\cup H'} c_i(o)}$. Noting that for all $i \in R$, $c_i(o) \geq \frac{q_{\max}}{2} c_{i_{\max}}(o)$, it can be verified that $\frac{\sum_{i\in R\cup H'} c_i(e)}{\sum_{i\in R\cup H'} c_i(o)} \leq \frac{|H'|\times q_{\max} + |R|\frac{q_{\max}}{2}\times 2}{|H'| + \frac{q_{\max}}{2}|R|}$. This is maximized when $|H'| = |R|$ and $q_{\max} = 3$, giving us a ratio of $\frac{12}{5}$. (We have used the fact that $q_{\max} \leq 3$ from Lemma A.4.1 (d).)

We complete the proof by observing that all of the players i not in $H' \cup R$ have q_i strictly less than $\frac{5}{2} - \varepsilon$, and hence including them cannot increase the Pareto price of anarchy to $\frac{5}{2}$. □

We now continue with the proof for the case when $q_{\max} \leq \frac{8}{3}$. For this we require a technical lemma.

Lemma A.4.4 *Let i be a specific player in L. Suppose we "mark" each player in H' independently with probability p. Let X be the minimum number of players we must unmark so that the set of remaining marked players H'' satisfies $\sum_{h\in H''} m_{ih} \leq q_i - 1$. Then $\mathrm{E}[X] < \frac{p}{\gamma_i} + p\frac{\beta_i}{1-\beta_i}$, where $\beta_i = \exp(-8\gamma_i(\gamma_i - p))$. In particular, if $p \leq \frac{\gamma_i}{2}$ then $\mathrm{E}[X] \leq \frac{2p}{\gamma_i}$.*

Proof: We recast the setting as follows: For $j \in \{1, \ldots, |H'|\}$ define $\delta_j = \frac{m_{ih_j}}{\sum_h m_{ih_j}}$, where h_j is the jth largest value of m_{ih} among all $h \in H'$. Thus the δ_j are sorted in decreasing order. For each j, let X_j be a random variable that is δ_j with probability p, and 0 otherwise; thus $\mathrm{E}[\sum_j X_j] = p$. Our goal is then to bound the expected minimum number of X_j which we have to discard (i.e., change the value to 0) so that $\sum_j X_j \leq \gamma_i$. Note that we may assume $\gamma_i \leq 1$ since otherwise trivially $X = 0$.

The number of such X_j is determined by the following greedy procedure. Once the X_j are fixed, we go through those that were selected (i.e., for which $X_j = \delta_j$) in increasing order of δ_j (descending values of j), keeping a running sum. Once this sum overflows γ_i, we must discard the current selected X_j and all remaining selected X_j. Therefore,

$$\begin{aligned}
\mathrm{E}[X] &= \sum_{j\geq 1} \Pr[X_j \text{ causes overflow}](1 + p(j-1)) \\
&= \Pr[\text{overflow occurs at all}] + p\sum_{j\geq 2} \Pr[X_j \text{ causes overflow}](j-1) \\
&= \Pr[\text{overflow occurs at all}] + p\sum_{j\geq 2} \Pr[\text{overflow occurs for some } j' \geq j] \\
&= \Pr[\text{overflow occurs at all}] + p\sum_{j\geq 2} \Pr[\sum_{k\geq j} X_k \geq \gamma_i]
\end{aligned}$$

where in the third line we have used the fact that for positive integer-valued random variables Z, $\mathrm{E}[Z] = \sum_i \Pr[Z \geq i]$. Continuing,

$$\begin{aligned}\Pr[\sum_{k\geq j} X_k \geq \gamma_i] &= \Pr[\sum_{k\geq j} X_k - pT_j \geq \gamma_i - pT_j], \text{ where } T_j = \textstyle\sum_{k\geq j}\delta_k \\ &\leq \exp\left(-\frac{2(\gamma_i - pT_j)^2}{\sum_{k\geq j}\delta_k^2}\right),\end{aligned}$$

using a Chernoff-Hoeffding bound. Since δ_j decreases with j, we have that $\sum_{k\geq j}\delta_k^2 \leq T_j\delta_j$, and that $\delta_j \leq \frac{1-T_j}{j-1}$. Thus we get $\Pr[\sum_{k\geq j} X_k \geq \gamma_i] \leq \exp\left(-\frac{2(\gamma_i-pT_j)^2(j-1)}{T_j(1-T_j)}\right)$. This yields

$$\mathrm{E}[X] \leq \Pr[\text{overflow occurs at all}] + p\sum_{j\geq 2}\exp\left(-\frac{2(\gamma_i - pT_j)^2(j-1)}{T_j(1-T_j)}\right).$$

The first term is bounded by $\frac{p}{\gamma_i}$ using Markov's inequality. The second term can be bounded as follows:

$$\begin{aligned}p\sum_{j\geq 2}\exp\left(-\frac{2(\gamma_i - pT_j)^2(j-1)}{T_j(1-T_j)}\right) &\leq p\sum_{j\geq 1}[\exp(-8\gamma_i(\gamma_i - p))]^j \\ &\leq p\frac{\beta_i}{1-\beta_i}, \text{ where } \beta_i = \exp(-8\gamma_i(\gamma_i - p)),\end{aligned}$$

where in the first line we used the fact that the quotient $\frac{(\gamma_i - pz)^2}{z(1-z)}$ is maximized at $z = \frac{\gamma_i}{2\gamma_i - p}$. This finishes the general case of the lemma.

To see the special case, if $p \leq \frac{\gamma_i}{2}$ then $\beta_i \leq \exp(-4\gamma_i^2)$, and $\frac{\beta_i}{1-\beta_i} \leq \frac{1}{\gamma_i}$ for $\gamma_i \in [1/4, 1]$. But we know that γ_i lies in this range, since $\gamma_i \leq 1$ by our observation at the beginning of the proof, and $\gamma_i \geq \frac{1}{4}$ by Claim A.4.3 with $q_h = q_{\max} \geq \frac{5}{2} - \varepsilon$. □

We are now ready to prove Theorem 5.2 for the case of $q_{\max} \leq \frac{8}{3}$.

Proof of Theorem 5.2 when $q_{\max} \leq \frac{8}{3}$: As in the proof for the previous case, it suffices to bound the Pareto price of anarchy for the players in $L \cup R \cup H'$ since the remaining players cannnot cause the value to reach $5/2$. Accordingly, we begin with the following:

Claim A.4.5 *The Pareto price of anarchy over only the players in $L \cup R \cup H'$ satisfies*

$$\frac{\sum_{i\in L\cup R\cup H'} c_i(e)}{\sum_{i\in L\cup R\cup H'} c_i(o)} \leq \max\left\{\frac{|L|\frac{2q_h}{3} + |H'|q_h + |R|q_h}{|L|\frac{q_h}{3} + |H'| + |R|\frac{q_h}{2}}, 2\right\},$$

where the maximum is also taken over all players $h \in H'$.

Proof: Note that there must exist some $h \in H'$ for which $\frac{\sum_{i\in L\cup R\cup H'} c_i(e)}{\sum_{i\in L\cup R\cup H'} c_i(o)} \leq \frac{\sum_{i\in L\cup R} c_i(e) + |H'|c_h(e)}{\sum_{i\in L\cup R} c_i(o) + |H'|c_h(o)}$. Therefore, we have

$$\frac{\sum_{i\in L\cup R\cup H'} c_i(e)}{\sum_{i\in L\cup R\cup H'} c_i(o)} \leq \frac{\sum_{i\in L\cup R} c_i(e) + |H'|q_h c_h(o)}{\sum_{i\in L\cup R} c_i(o) + |H'|c_h(o)} \leq \frac{\sum_{i\in L\cup R} 2c_i(o) + |H'|q_h c_h(o)}{\sum_{i\in L\cup R} c_i(o) + |H'|c_h(o)},$$

using the fact that $q_i \leq 2$ for $i \in L \cup R$. If the left-hand side is at least 2, then we can maximize this quotient by keeping $c_i(o)$ as small as possible for $i \in L \cup R$. For $i \in R$, we must have $c_i(e) = q_i c_i(o) \geq c_h(e) = q_h c_h(o)$, and thus $c_i(o) \geq \frac{q_h c_h(o)}{2}$. For $i \in L$, we must have $3c_i(o) \geq c_{i_{\min}} \geq q_h c_h(o)$, so $c_i(o) \geq \frac{q_h c_h(o)}{3}$. Substituting these lower bounds for $c_i(o)$ into the quotient, and dividing through by $c_h(o)$, we get the bound in the claim. □

We will consider moves that begin with each player moving from his equilibrium strategy e_i to his social optimum strategy o_i, followed by an attempt to reassign the players in R to the strategies of distinct players in the set H' to reach a new state o'. Because of the Pareto-optimality of e, it is impossible to do this in such a way that $c_i(o') < c_i(e)$ for all players in the game.

The specific moves we consider will take advantage of Lemma A.4.4. Suppose we mark each player in H' independently with probability p. We will choose $p = \alpha\gamma$ where $\gamma = \frac{q_h - 2}{6 - 2q_h}$, which from Claim A.4.3 is a lower bound on the capacity of any player $i \in L$, and $\frac{1}{4} \le \alpha < \frac{1}{3}$. The expected number of players marked is then $p|H'|$, and by Lemma A.4.4, for each $i \in L$, the expected number of marked players that need to be unmarked so that $\sum_{h \text{ marked}} m_{ih} \le q_i - 1$, is at most $\frac{2p}{\gamma} = 2\alpha$. Thus, the expected number of marked players that need to be unmarked so that this is true for all players $i \in L$ is at most $2\alpha|L|$. The probabilistic method then implies there exists some set of players in H' of cardinality $p|H'| - 2\alpha|L|$ satisfying this property. From this we can conclude that $|R| > p|H'| - 2\alpha|L|$, or else e is not Pareto-optimal.

We now have two final cases, according to the sign of $p|H'| - 2\alpha|L|$.

If $p|H'| - 2\alpha|L| \ge 0$, we can bound the Pareto price of anarchy from $L \cup R \cup H'$ by

$$\frac{\sum_{i \in L \cup R \cup H'} c_i(e)}{\sum_{i \in L \cup R \cup H'} c_i(o)} \le \frac{|L|\frac{2q_h}{3} + |H'|q_h + (p|H'| - 2\alpha|L|)q_h}{|L|\frac{q_h}{3} + |H'| + (p|H'| - 2\alpha|L|)\frac{q_h}{2}}.$$

Setting this to be less than $\frac{5}{2}$ is equivalent to

$$\alpha < \frac{2q_h|L| + 3pq_h|H'| - 12q_h|H'| + 30|H'|}{6q_h|L|} = \frac{1}{3} + \frac{|H'|}{|L|}\frac{pq_h - 4q_h + 10}{2q_h}. \tag{3}$$

The last quotient in inequality (3) can be seen to be non-negative so long as $\alpha \ge \frac{1}{4}$ and $q_h \le q_{\max} \le \frac{8}{3}$, so the inequality is certainly satisfied for $\frac{1}{4} \le \alpha < \frac{1}{3}$. Thus for α in this range we get a Pareto price of anarchy less than $\frac{5}{2}$.

Finally, if $p|H'| - 2\alpha|L| < 0$, then we can bound the price of anarchy by $\frac{|L|\frac{2q_h}{3} + |H'|q_h}{|L|\frac{q_h}{3} + |H'|}$. This is less than $\frac{5}{2}$ when $\frac{|L|}{|H'|} > 6 - \frac{15}{q_h}$. Using the fact that $\frac{p}{\alpha} = \gamma = \frac{q_h - 2}{6 - 2q_h}$, we can verify that this is indeed the case whenever $p|H'| - 2\alpha|L| < 0$. $\square$