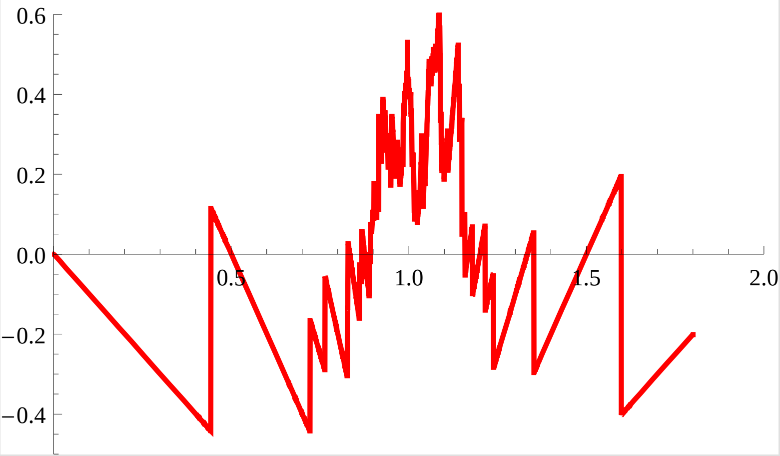
Martingales from pairs of randomized Poisson, Gamma, negative binomial and hyperbolic secant processes

Włodek Bryc[1]

Cincinnati

October 16, 2010

[1]Based on joint work in progress with Jacek Wesołowski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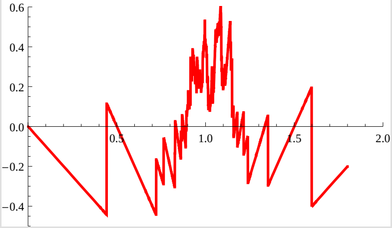
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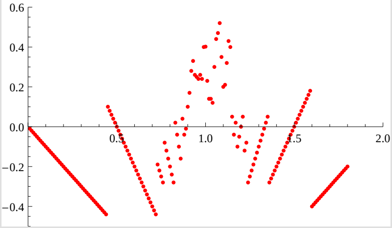
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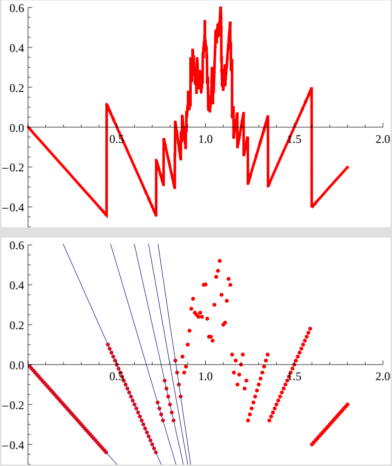
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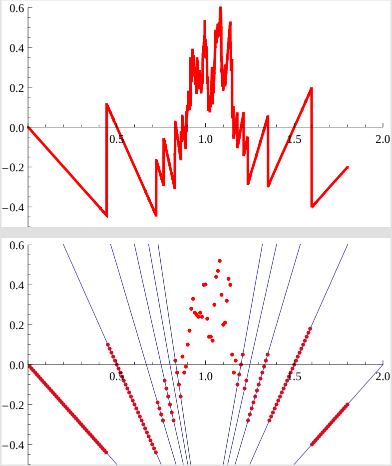
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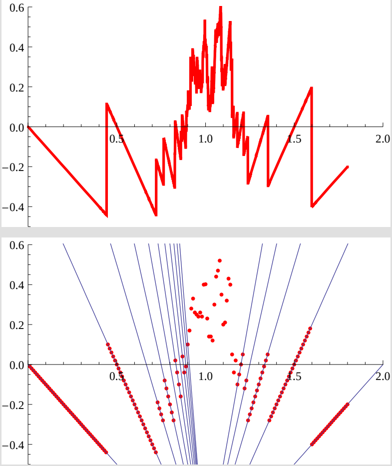
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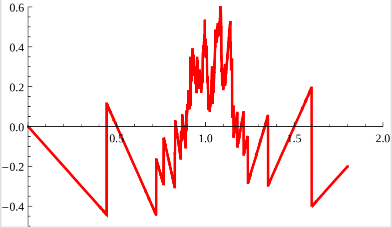
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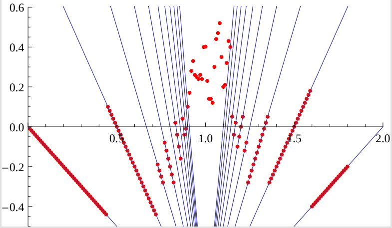
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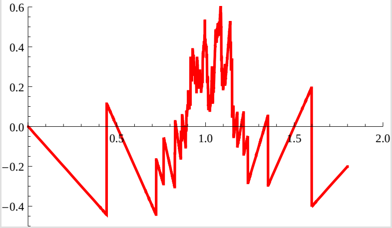
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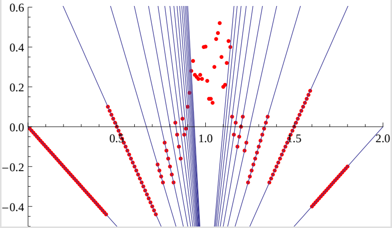
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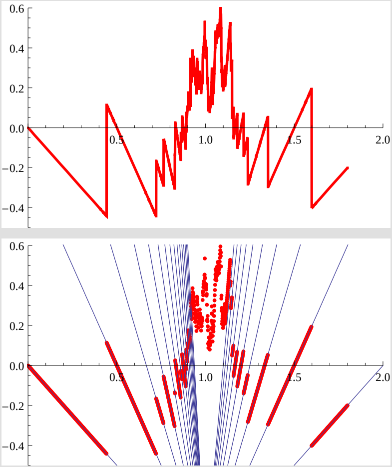
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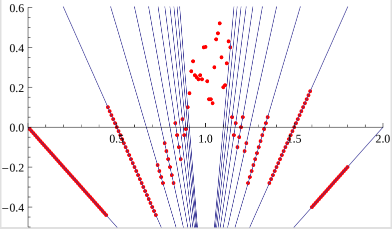

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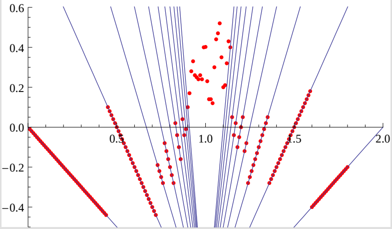

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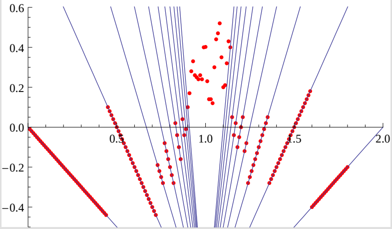

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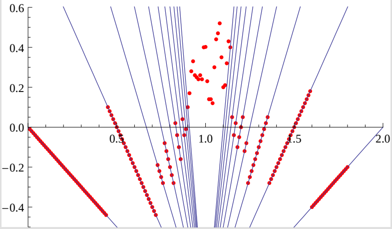

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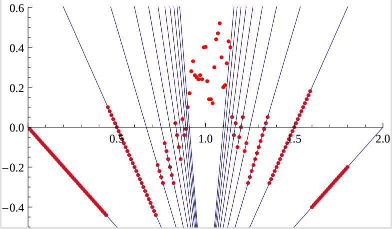

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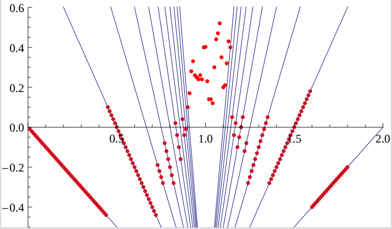

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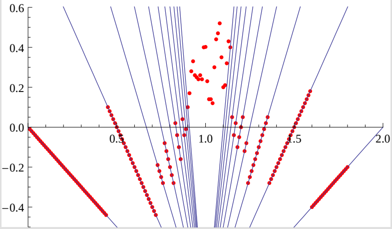

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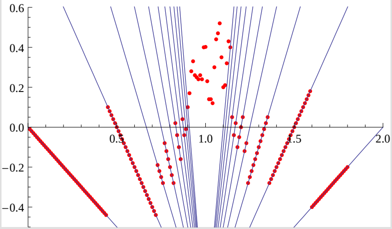

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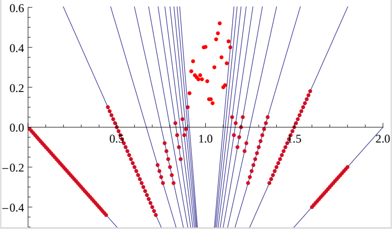

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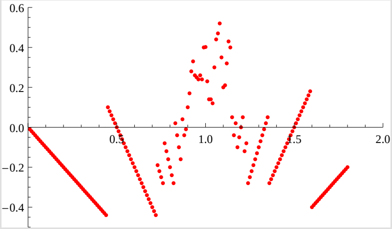
- $t = 1$ should be hidden in these formulas!



Conditionally on X_a and X_b

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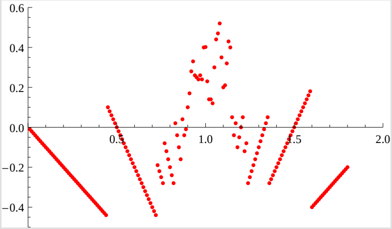

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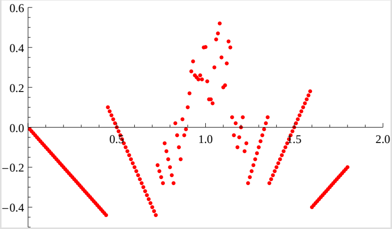

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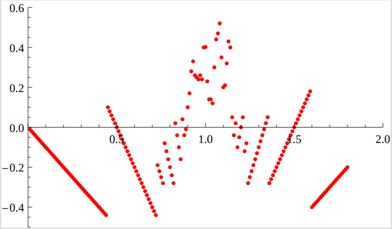

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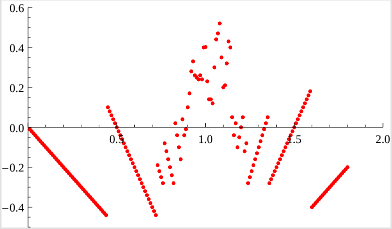

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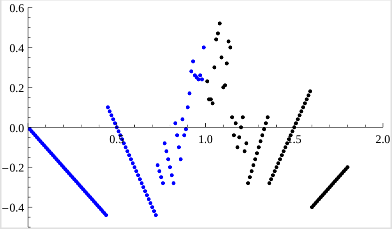

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- $\mathrm{Var}[X_t|\mathcal{F}_{s,u}]$ is the same quadratic polynomial in X_s, X_u
- But after conversion to "standard form", two-sided conditional variances of bridges based on X_a, X_1 or on X_1, X_b are simpler than for bridges based on X_a, X_b.

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3. there exist numerical constants $\eta, \theta \in \mathbb{R}$, $\sigma, \tau \geq 0$ and $\rho \in [-2, 2\sqrt{\sigma\tau}]$ such that for all $s < t < u$,

$$\mathrm{Var}[Z_t|\mathcal{F}_{s,u}] = F_{t,s,u} \Bigg(1 + \eta \frac{uZ_s - sZ_u}{u - s} + \theta \frac{Z_u - Z_s}{u - s} + \sigma \frac{(uZ_s - sZ_u)^2}{(u - s)^2} + \tau \frac{(Z_u - Z_s)^2}{(u - s)^2} + \rho \frac{(Z_u - Z_s)(uZ_s - sZ_u)}{(u - s)^2}\Bigg),$$

where $F_{t,s,u}$ is an explicit non-random constant.

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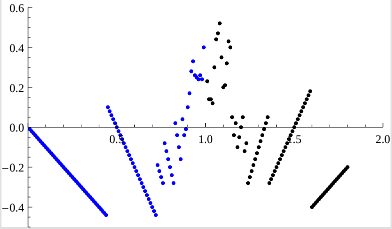
- The inverse transformation

$$X_t = (1-t) W_{t/(1-t)}$$

  represents the Brownian bridge in terms of the Wiener process.



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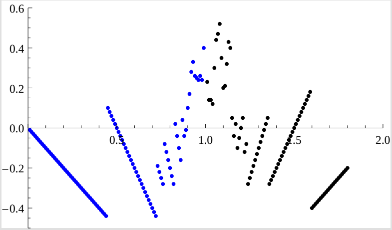

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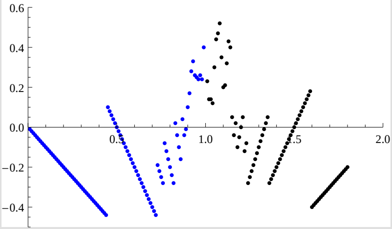

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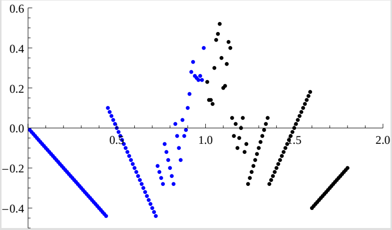

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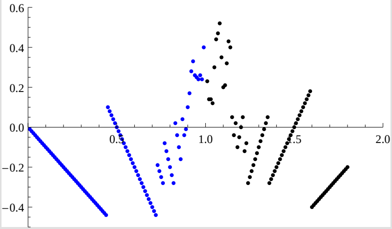

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Question

Which other Lévy processes could be "put together" into a quadratic harness? Into a martingale?

$$\mathrm{Var}[Z_t|\mathcal{F}_{s,u}] \sim 1 + \eta\frac{uZ_s - sZ_u}{u-s} + \theta\frac{Z_u - Z_s}{u-s} + \sigma\frac{(uZ_s - sZ_u)^2}{(u-s)^2} + \tau\frac{(Z_u - Z_s)^2}{(u-s)^2} + \rho\frac{(Z_u - Z_s)(uZ_s - sZ_u)}{(u-s)^2}$$

Proposition (B..-Wesolowski arxiv 2009)

Suppose $\exists T > 0$ *such that* $\{Z_t : t < T\}$ *conditioned on* Z_T*, leads to QH* Y *with*

$$\mathrm{Var}(Y_t|\mathcal{F}_{s,u}) \sim 1 + \theta_Y\frac{Y_u - Y_s}{u-s} + \tau_Y\frac{(Y_u - Y_s)^2}{(u-s)^2} + \rho_Y \dots.$$

Then one of the following cases must happen:

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$$f(x; t, \alpha) = \frac{(2\cos\frac{\alpha}{2})^{2t}}{2\pi\Gamma(2t)}|\Gamma(t+ix)|^2 e^{\alpha x},\ t > 0.$$

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For random $\Pi \in (0,1)$, *define* Y_t *as* Π*-conditionally negative binomial process* $\Pr(Y_t = k | \Pi = p) = \frac{\Gamma(t+k)}{\Gamma(t)k!} p^t (1-p)^k, \; k = 0, 1, \dots$.

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1. *Then* $Y = (Y_t)_{t\geq 0}$ *is Markov.*
2. *Assume* $\beta = \mathbf{E}(1/\Pi) - 1$ *and* $v^2 = \mathrm{Var}(1/\Pi) > 0$. *Then*

$$Z_t = c(1-t)Y_{\frac{t}{cv(1-t)}} - t\frac{\beta}{v}\,.$$

is a quadratic harness on $(0,1)$ *with parameters*

$$\eta = \theta = \frac{(2\beta+1)\,v}{(\beta+1)\,\beta},\ \sigma = \tau = \frac{v^2}{\beta(\beta+1)},\ \rho = 2\sqrt{\sigma\tau}.$$

Here $c = \frac{v}{v^2+\beta^2+\beta}$.

Proposition (folklore? Poisson case: Nekrutkin(2007))

Let $\Pi \in (0,1)$ *be a random variable such that* $\mathbf{E}(1/\Pi) < \infty$. *Suppose* Y *is a* Π*-conditionally negative binomial process and* $Z_t = c(1-t)Y_{\frac{t}{cv(1-t)}} - t\frac{\beta}{v}$ *with some coefficients* $\beta, v > 0$. *Then the following are equivalent:*

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2. Π *has the beta* $B_I(a,b)$ *density*

$$h(p) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p^{a-1}(1-p)^{b-1} 1_{(0,1)}(p), \tag{1}$$

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Proof: Special case of Diaconis-Ylvisaker, *Conjugate priors for exponential families*, Ann. Statist., 1979.

Let Y and Y' be Π-conditionally independent negative binomial processes. With $\beta = \mathbf{E}(1/\Pi) - 1$, $v^2 = \mathrm{Var}(1/\Pi)$, define

$$Z_t = \begin{cases} c(1-t)Y_{\frac{t}{cv(1-t)}} - t\frac{\beta}{v} & \text{if } 0 \le t < 1, \\ \left(\frac{1-\Pi}{\Pi} - \beta\right)/v & \text{if } t = 1, \\ c(t-1)Y'_{\frac{1}{cv(t-1)}} - \frac{\beta}{v} & \text{if } t > 1. \end{cases} \tag{2}$$

$c = \frac{v}{v^2+\beta^2+\beta}$.
Time-inversion: $(Z_t) \sim (tZ_{1/t})$.

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- the "correct law" gives us a quadratic harness.
- laws for randomization (except Poisson) can be deduced from results in [Diaconis-Ylvisaker (1979)]

- A pair of Λ-conditionally independent Poisson processes (N_t), $(\tilde{N}_t)$ makes a martingale on $(0, \infty)$ iff Λ has gamma density $h(d\lambda) = C\lambda^{p-1}e^{-r\lambda}1_{(0,\infty)}(\lambda)d\lambda$

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Conclusions

1. For a process $(X_t)_{t\in(0,\infty)}$ with linear two-sided regressions, non-constant quadratic conditional variances, and product covariance $\mathrm{cov}(X_s, X_t) = (as + b)(ct + d)$ for $s < t$, the existence of a "special time" T can be recognized by a calculation.

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2. Then (X_t) is put together from one of the "randomized pairs" of Poisson, negative binomial, gamma, or hyperbolic secant processes.

Thank you

Abstract

Consider a pair of independent Poisson processes, or a pair of Negative Binomial processes, or Gamma, or hyperbolic secant processes with a shared randomly selected parameter. Under appropriate randomization, one can deterministically re-parametrize the time and scale for both processes so that the first process runs on time interval $(0, 1)$, the second process runs on time interval $(1, \infty)$, and the two processes seamlessly join into one Markov martingale on $(0, \infty)$. In fact, a property stronger than martingale holds: we stitch together two processes into a single quadratic harness on $(0, \infty)$

◂ Overview ▸▸ End now