A renormalisation group analysis of the 4-dimensional continuous-time weakly self-avoiding walk

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Abstract
We prove $|x|^{-2}$ decay of the critical two-point function for the continuous-time weakly self-avoiding walk on $\mathbb{Z}^4$. The walk two-point function is identified as the two-point function of a supersymmetric field theory with quartic self-interaction, and the field theory is then analysed using renormalisation group methods.

This is joint work with David Brydges.


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Self-avoiding walk

Discrete-time model: Let \( S_n(x) \) be the set of \( \omega : \{0, 1, \ldots, n\} \rightarrow \mathbb{Z}^d \) with:
\( \omega(0) = 0, \omega(n) = x, |\omega(i + 1) - \omega(i)| = 1, \) and \( \omega(i) \neq \omega(j) \) for all \( i \neq j \).
Let \( S_n = \bigcup_{x \in \mathbb{Z}^d} S_n(x) \).

Let \( c_n(x) = |S_n(x)| \). Let \( c_n = \sum_x c_n(x) = |S_n| \). Easy: \( c_n^{1/n} \rightarrow \mu \).
Declare all walks in \( S_n \) to be equally likely: each has probability \( c_n^{-1} \).

Two-point function: \( G_z(x) = \sum_{n=0}^{\infty} c_n(x) z^n \), radius of convergence \( z_c = \mu^{-1} \).

Predicted asymptotic behaviour:
\[ c_n \sim A \mu^n n^{\gamma - 1}, \quad \mathbb{E}_n |\omega(n)|^2 \sim D n^{2\nu}, \quad G_{z_c}(x) \sim c |x|^{-(d-2+\eta)}, \]
with universal critical exponents \( \gamma, \nu, \eta \) obeying \( \gamma = (2 - \eta)\nu \).
A random SAW on $\mathbb{Z}^2$ with $10^6$ steps

(Figure by T. Kennedy)
**Dimensions** $d \geq 4$

**Theorem.** (Brydges, Spencer (1985); Hara, Slade (1992); Hara (2008)...)  
For $d \geq 5$,  

\[
c_n \sim A \mu^n, \quad \mathbb{E}_n |\omega(n)|^2 \sim Dn, \quad G_{zc}(x) \sim c|x|^{-(d-2)}, \quad \frac{1}{\sqrt{Dn}} \omega(\lfloor nt \rfloor) \Rightarrow B_t.
\]

**Prediction** is that upper critical dimension is 4, and asymptotic behaviour for $\mathbb{Z}^4$ has log corrections (Brézin, Le Guillou, Zinn-Justin 1973):

\[
c_n \sim A \mu^n (\log n)^{1/4}, \quad \mathbb{E}_n |\omega(n)|^2 \sim Dn (\log n)^{1/4}, \quad G_{zc}(x) \sim c|x|^{-2}.
\]

Also, for susceptibility and correlation length, as $z \uparrow z_c$,

\[
\chi(z) \sim \frac{A' |\log(1 - z/z_c)|^{1/4}}{1 - z/z_c}, \quad \xi(z) \sim \frac{D' |\log(1 - z/z_c)|^{1/8}}{(1 - z/z_c)^{1/2}},
\]

where

\[
\chi(z) = \sum_{n=0}^{\infty} c_n z^n, \quad \frac{1}{\xi(z)} = - \lim_{n \to \infty} \frac{1}{n} \log G_z(ne_1).
\]
Continuous-time weakly self-avoiding walk

This is a modification of the SAW model. We are interested in dimensions $d \geq 4$. Let $E_0$ denote the expectation for continuous-time nearest-neighbour simple random walk $X(t)$ on $\mathbb{Z}^d$ started from 0 (with $\text{Exp}(1)$ holding times), and let

$$L_{u,T} = \int_0^T \delta_{u,X(s)} ds, \quad I(0,T) = \sum_{u \in \mathbb{Z}^d} L_{u,T}^2.$$

Then

$$I(0,T) = \int_0^T \int_0^T \delta_{X(s),X(t)} ds dt.$$

Let $g \in (0, \infty)$. The two-point function is defined to be

$$G_{g,\nu}(x) = \int_0^\infty E_0 \left( e^{-gI(0,T)} \delta_{X(T),x} \right) e^{-\nu T} dT.$$

(role of $z$ now played by $e^{-\nu}$). A subadditivity argument shows that the susceptibility $\chi_g(\nu) = \sum_{x \in \mathbb{Z}^d} G_{g,\nu}(x)$ is finite if $\nu > \nu_c(g)$ and is infinite if $\nu < \nu_c(g)$. 
Main result

**Theorem** (Brydges–Slade 2010). Let $d \geq 4$. There exists $g_0$ such that for $0 < g \leq g_0$,

$$G_{g,\nu_c}(x) = \frac{c}{|x|^{d-2}} + o\left(\frac{1}{|x|^{d-2}}\right).$$

**Outlook:** The method of proof (RG) has the potential to (but has not yet fully achieved):

- prove logarithmic corrections for susceptibility and correlation length for $d = 4$
- prove same result also with small nearest-neighbour attraction (Bauerschmidt)
- prove same result for a particular spread-out model of discrete-time strictly self-avoiding walk with exponentially decaying step weights
  (explicitly the weight of a step is $(1 - a^{-1}\Delta)^{-1}(x, y)$ with $0 < a \ll 1$)

**Related results:**

- weakly SAW on 4-dimensional hierarchical lattice: Brydges, Evans, Imbrie (1992); Brydges, Imbrie (2003); and with different RG approach Ohno, Hara (2010+). The hierarchical lattice is a replacement of $\mathbb{Z}^4$ by a recursive structure which is well-suited to the RG.
- weakly self-avoiding Lévy walk on $\mathbb{Z}^3$ ($\alpha = \frac{3+\epsilon}{2}$, $d_c = 3 + \epsilon$): Mitter, Scoppola (2008).
Finite-volume approximation

Now we fix $g > 0$ and usually drop it from the notation.

Standard methods (Simon–Lieb inequality) show that

$$G_{\nu_c}(x) = \lim_{\nu \downarrow \nu_c} \lim_{\Lambda \uparrow \mathbb{Z}^d} G_{\Lambda, \nu}(x),$$

where $\Lambda = \mathbb{Z}^d / R \mathbb{Z}$ is a torus approximating $\mathbb{Z}^d$ and

$$G_{\Lambda, \nu}(x) = \int_0^\infty E_0^\Lambda \left( e^{-g I_{\Lambda}[0, T]} \delta_{X(T), x} \right) e^{-\nu T} dT,$$

with $E_0^\Lambda$ the expectation for the continuous-time simple random walk on $\Lambda$, and $I_{\Lambda}[0, T] = \sum_{v \in \Lambda} L_{v,T}^2$.

Thus we can work in finite volume, and slightly subcritical, as long as we maintain sufficient uniformity to take the limits.
Let $\varphi : \Lambda \to \mathbb{C}$. Let $\bar{\varphi}_x = u_x - iv_x$ denote the complex conjugate of $\varphi_x = u_x + iv_x$. Let $\Delta$ denote the discrete Laplacian on $\Lambda$, i.e., $\Delta \varphi_x = \sum_{y: |y-x|=1} (\varphi_y - \varphi_x)$. Let

$$\psi_x = \frac{1}{\sqrt{2\pi i}} d\varphi_x, \quad \bar{\psi}_x = \frac{1}{\sqrt{2\pi i}} d\bar{\varphi}_x,$$

$$\tau_x = \varphi_x \bar{\varphi}_x + \psi_x \wedge \bar{\psi}_x = u_x^2 + v_x^2 + \frac{1}{\pi} du_x \wedge dv_x,$$

$$\tau_{\Delta, x} = \frac{1}{2} \left( \varphi_x (\Delta \bar{\varphi})_x + (-\Delta \varphi)_x \bar{\varphi}_x + \psi_x \wedge (-\Delta \bar{\psi})_x + (-\Delta \psi)_x \wedge \bar{\psi}_x \right),$$

where $\wedge$ is the standard anti-commutative wedge product. Then

$$G_{\Lambda, \nu}(x) = \int_{\mathbb{C}^\Lambda} e^{-\sum_{u \in \Lambda} (\tau_{\Delta, u} + g\tau^2 + \nu \tau_u)} \bar{\varphi}_0 \varphi_x.$$

RHS is the two-point function of a supersymmetric field theory with boson field $(\varphi, \bar{\varphi})$ and fermion field $(\psi, \bar{\psi})$.

Meaning of the integral

The definition of an integral such as

\[ G_{\Lambda, \nu}(x) = \int_{\mathbb{C}^{\Lambda}} e^{-\sum_{u \in \Lambda}(\tau_{\Delta} u + g\tau u^2 + \nu \tau u)} \Phi_0 \varphi_x \]

is as follows:

• expand entire integrand in power series about degree-zero part (finite sum), e.g.,

\[ e^{\tau v} = e^{\varphi_x \bar{\varphi}_x + \psi_x \bar{\psi}_x} = e^{\varphi_x \bar{\varphi}_x} \left( 1 + \psi_x \bar{\psi}_x \right), \]

• keep only terms with one factor \( d\varphi_x \) and one \( d\bar{\varphi}_x \) for each \( x \in \Lambda \),

• write \( \varphi_x = u_x + i\nu_x, \bar{\varphi}_x = u_x - i\nu_x \) and similarly for differentials,

• then use anti-commutativity to rearrange the differentials to \( \prod_{x \in \Lambda} du_x dv_x \),

• and finally perform Lebesgue integral over \( \mathbb{R}^{2|\Lambda|} \).

Such integrals have nice properties. Let \( S(\Lambda) = \sum_{x \in \Lambda}(\tau_{\Delta,x} + m^2 \tau_x) \). Then:

\[ \int e^{-S(\Lambda)} F(\tau) = F(0), \quad \int e^{-S(\Lambda)} \Phi_0 \varphi_x = (-\Delta + m^2)^{-1}(0, x). \]

Now we study the integral and forget about the walks.
The change of variable $\varphi_x \mapsto \sqrt{1 + z_0 \varphi_x}$, with $z_0 > -1$, gives

$$G_{\Lambda, \nu}(x) = (1 + z_0) \int_{C^\Lambda} e^{-\sum u \left( (1 + z_0) \tau \Delta, u + g(1 + z_0)^2 \tau^2 + \nu(1 + z_0) \tau u \right)} \bar{\varphi}_0 \varphi_x.$$  

Introducing an external field $\sigma \in \mathbb{C}$, let

$$S(\Lambda) = \sum_{u \in \Lambda} \left( \tau \Delta, u + m^2 \tau u \right),$$

$$V_0(\Lambda) = \sum_{u \in \Lambda} \left( g_0 \tau^2 + \nu_0 \tau u + z_0 \tau \Delta, u \right) + \sigma \bar{\varphi}_0 + \bar{\sigma} \varphi_x,$$

$$g_0 = (1 + z_0)^2 g, \quad \nu_0 = (1 + z_0) \nu_c, \quad m^2 = (1 + z_0)(\nu - \nu_c).$$

Then

$$G_{\Lambda, \nu}(x, y) = (1 + z_0) \left. \frac{\partial}{\partial \sigma} \frac{\partial}{\partial \bar{\sigma}} \right|_0 \int e^{-S(\Lambda) - V_0(\Lambda)}.$$

Want to show that $\exists z_0$ such that first part of $V_0$ is a small perturbation and use

$$\lim_{m^2 \downarrow 0} \lim_{\Lambda \uparrow \mathbb{Z}^d} \left. \frac{\partial}{\partial \sigma} \frac{\partial}{\partial \bar{\sigma}} \right|_0 \int e^{-S(\Lambda)} e^{-\sigma \bar{\varphi}_0 - \bar{\sigma} \varphi_x} = (-\Delta)^{-1}(0, x) \sim \text{const} |x|^{-(d-2)}.$$
Gaussian “expectation”

For a positive definite $\Lambda \times \Lambda$ matrix $C$, and $A = C^{-1}$, let

$$S_A(\Lambda) = \sum_{x,y \in \Lambda} \left( \varphi_x A_{xy} \bar{\varphi}_x + \psi_x A_{xy} \bar{\psi}_y \right)$$

and, for a form $F$,

$$\mathbb{E}_C F = \int_{C^\Lambda} e^{-S_A(\Lambda)} F.$$  

Then $\mathbb{E}_C 1 = 1$. With $C = (-\Delta + m^2)^{-1}$, our goal is to compute

$$\lim_{m^2 \downarrow 0} \lim_{\Lambda \uparrow \mathbb{Z}^4} G_{\Lambda,\nu}(x, y) = \lim_{m^2 \downarrow 0} \lim_{\Lambda \uparrow \mathbb{Z}^4} (1 + z_0) \frac{\partial}{\partial \sigma} \frac{\partial}{\partial \bar{\sigma}} \bigg|_0 \mathbb{E}_C e^{-V_0(\Lambda)}.$$

These integrals have much in common with standard Gaussian integrals. However, this is not ordinary probability theory and and in general $\mathbb{E}_C$ will be a Grassmann integral that take values in a space of differential forms.
Convolution integrals

Write $\phi = (\varphi, \bar{\varphi})$, $d\phi = (d\varphi, d\bar{\varphi})$.

Recall that $X \sim N(0, \sigma_1^2 + \sigma_2^2)$ has the same distribution as $X_1 + X_2$ where $X_1 \sim N(0, \sigma_1^2)$ and $X_2 \sim N(0, \sigma_2^2)$ are independent.

This finds expression for $\mathbb{E}_C$ via the following fact:

$$\mathbb{E}_{C_2+C_1} F = \mathbb{E}_{C_2} \circ \mathbb{E}_{C_1} \theta F,$$

where

$$(\theta F)(\phi, \xi, \psi, \eta) = F(\phi + \xi, \psi + \eta)$$

and $\mathbb{E}_{C_1}$ integrates out $\xi$ and $\eta = \frac{1}{\sqrt{2\pi i}}d\xi$, leaving $\phi$ and $\psi = \frac{1}{\sqrt{2\pi i}}d\phi$ fixed. Then $\mathbb{E}_{C_2}$ integrates out $\phi$ and $\psi$. 
Finite-range decomposition of covariance

Theorem (Brydges, Guadagni, Mitter 2004). Let $d > 2$. Fix a large $L$ and suppose $|\Lambda| = L^{Nd}$. Let $C = (-\Delta + m^2)^{-1}$. It is possible to write:

$$C = \sum_{j=1}^{N} C_j$$

with $C_j$ positive definite,

$$C_j(x, y) = 0 \quad \text{if} \quad |x - y| \geq \frac{1}{2}L^j$$

and, for $j = 1, \ldots, N - 1$ and with $[\phi] = \frac{1}{2}(d - 2)$ (so $[\phi] = 1$ for $d = 4$),

$$|C_j(x, x)| \leq O(L^{-2[\phi](j-1)}),$$

$$|\nabla_x^\alpha \nabla_y^\beta C_j(x, x)| \leq O(L^{-(2[\phi]+|\alpha|_1+|\beta|_1)(j-1)}).$$
The RG map

The covariance decomposition induces a field decomposition and allows the expectation to be done iteratively:

\[ \phi = \sum_{j=1}^{N} \xi_j, \quad d\phi = \sum_{j=1}^{N} d\xi_j, \quad \mathbb{E}_C = \mathbb{E}_{C_N} \circ \cdots \circ \mathbb{E}_{C_2} \circ \mathbb{E}_{C_1}. \]

Write \( \phi_j = \sum_{i=j+1}^{N} \xi_i \), with \( \phi_0 = \phi, \phi_N = 0 \). Then \( \phi_j = \phi_{j+1} + \xi_{j+1} \). Let

\[ Z_0 = Z_0(\phi, d\phi) = e^{-V_0(\Lambda)}, \]

and

\[ Z_j(\phi_j, d\phi_j) = \mathbb{E}_{C_j} \cdots \mathbb{E}_{C_1} Z_0. \]

In particular, our goal is to compute

\[ Z_N = \mathbb{E}_C Z_0 = \mathbb{E}_C e^{-V_0(\Lambda)} \]

and we are led to study the RG map:

\[ Z_{j+1} = \mathbb{E}_{C_{j+1}} Z_j. \]
Relevant, marginal, irrelevant directions

Let \( d = 4 \). The covariance estimates suggest that \( \xi_{j+1,x} \approx L^{-j}[\phi] = L^{-j} \) and that this field is approximately constant over distance \( L^j \). Thus, for a block \( B \) of side \( L^j \),

\[
\sum_{x \in B} |\xi_{j+1,x}|^p \approx |B|L^{-jp} = L^{j(4-p)},
\]

which is relevant for \( p < 4 \), marginal for \( p = 4 \), irrelevant for \( p > 4 \).

Taking symmetries and derivatives into account, the relevant and marginal monomials are:

\[
\tau, \quad \tau_\Delta, \quad \tau^2.
\]

The role of \( d = 4 \): \( \tau^2 \) is relevant for \( d < 4 \) and irrelevant for \( d > 4 \):

\[
\sum_{x \in B} |\xi_{j+1,x}|^4 \approx |B|L^{-4j[\phi]} = L^{j(4-d)}.
\]
The map $\mathbb{E}_{C_1} : Z_0 \mapsto Z_1$

This map takes a function of $\phi = \phi_1 + \xi_1$ to a function of $\phi_1$ by integrating out $\xi_1$.

Write $Z_0(x) = I_0(x) = e^{-V_0(x)}$, and, for $X \subset \Lambda$, write

$$I_0(X) = \prod_{x \in X} I_0(x) = e^{-V_0(X)}.$$

This is a function of $\phi$.

Let $V_1$ be a version of $V_0$ with modified coupling constants $(g_1, \nu_1, z_1)$ and regarded as a function of $\phi_1$ (and $d\phi_1$). Let $I_1(x) = e^{-V_1(x)}$, this will be an approximation to $Z_1$. Let

$$\delta I_{1,x}(\phi_1, \xi_1) = I_{0,x}(\phi_1 + \xi_1) - I_{1,x}(\phi_1).$$

Then

$$Z_1(\Lambda) = \mathbb{E}_{C_1} I_0(\Lambda) = \mathbb{E}_{C_1} \prod_{x \in \Lambda} (I_{1,x} + \delta I_{1,x})$$

$$= \mathbb{E}_{C_1} \sum_{X \subset \Lambda} I_1^{\Lambda \setminus X} \delta I_1^X = \sum_{X \subset \Lambda} I_1^{\Lambda \setminus X} \mathbb{E}_{C_1} \delta I_1^X.$$
**The $I \circ K$ representation**

We write this as

$$Z_1(\Lambda) = \sum_{X \subset \Lambda} I_1^{\Lambda \setminus X} E_{C_1} \delta I_1^X = \sum_{U \in \mathcal{P}_1} I_1^{\Lambda \setminus U} K_1(U),$$

where

$$K_1(U) = \sum_{X \in \mathcal{P}_0(U)} I_1^{U \setminus X} E_{C_1} \delta I_1^X$$

with factorisation property.
The $I \circ K$ representation

The formula

$$Z_1(\Lambda) = \sum_{X \subset \Lambda} I_1^{A \setminus X} \mathbb{E}_{C_1} \delta I_1^X = \sum_{U \in \mathcal{P}_1} I_1^{A \setminus U} K_1(U),$$

is an instance of the following “circle product.”

Let $B_j$ represent the blocks in a paving of $\Lambda$ by blocks of side $L^j$, and let $\mathcal{P}_j$ denote the set of finite unions of such blocks. Given even forms $F, G$ defined on $\mathcal{P}_j$, let

$$(F \circ G)(\Lambda) = \sum_{U \in \mathcal{P}_j} F(\Lambda \setminus U)G(U).$$

This defines an associative and commutative product. For $X \in \mathcal{P}_0$, let $K_0(X) = \delta_{X, \emptyset}$. Let $K_1$ be defined as above and let $I_1(U) = \prod_{x \in U} I_{1,x}$ for $U \in \mathcal{P}_1$. Then

$$Z_0(\Lambda) = I_0(\Lambda) = (I_0 \circ K_0)(\Lambda), \quad Z_1(\Lambda) = (I_1 \circ K_1)(\Lambda).$$
Flow of coupling constants

**Theorem.** Let \( d = 4 \) (\( d > 4 \) is simpler). There is a choice of

\[
V_{j,u} = g_j \tau_u^2 + \nu_j \tau_u + z_j \tau_{\Delta,u} + \lambda_j (\delta_{u,0} \sigma \tilde{\varphi}_0 + \delta_{u,x} \tilde{\varphi}_x) + q_j \frac{1}{2} (\delta_{u,0} + \delta_{u,x}) \sigma \tilde{\sigma}
\]

which determines \( I_j \), and of \( K_j \), such that

\[
Z_j(\Lambda) = (I_j \circ K_j)(\Lambda), \quad Z_{j+1}(\Lambda) = \mathbb{E}_{C_{j+1}} Z_j(\Lambda) = (I_{j+1} \circ K_{j+1})(\Lambda),
\]

and moreover

\[
\begin{align*}
g_{j+1} &= g_j - c_j g_j^2 + r_{g,j} \\
\nu_{j+1} &= \nu_j + 2g_j C_{j+1}(0, 0) + r_{\mu,j} \\
z_{j+1} &= z_j + r_{z,j} \\
K_{j+1} &= r_{K,j},
\end{align*}
\]

with additional equations for \( \lambda_j \) and \( q_j \), such that the \( r \)'s are error terms within an appropriately defined Banach space, and Lipschitz in \((g_j, \nu_j, z_j, K_j)\). \( K_j \) enters only in the error terms and these are independent of \( \lambda_j, q_j \).
Fixed point theorem

We prove that there is a choice of initial conditions $z_0$ (which occurs in $c$ of $c|x|^{-2}$) and $\nu_0$ (which puts us at the critical point) such that the solution $(g_j, \nu_j, z_j, K_j)_{0 \leq j \leq N}$, in the limits $N \to \infty$ and $m^2 \to 0$, has limit

$$(g_j, \nu_j, z_j, K_j) \to (0, 0, 0, 0) \quad \text{“infrared asymptotic freedom.”}$$

From this, estimates on $K_N$, and the specific form $q_j \approx \sum_{i=1}^{j} C_i(0, x) \to C_{\mathbb{Z}^4}(0, x)$ we obtain

$$G_{\nu_c}(x) = \lim_{\nu \downarrow \nu_c} (1 + z_0) \lim_{N \to \infty} \frac{\partial}{\partial \sigma} \frac{\partial}{\partial \bar{\sigma}} \bigg|_{0} Z_N(\Lambda)$$

$$= \lim_{m^2 \downarrow 0} (1 + z_0) \lim_{N \to \infty} \frac{\partial}{\partial \sigma} \frac{\partial}{\partial \bar{\sigma}} \bigg|_{0} (I_N(\Lambda) + K_N(\Lambda))$$

$$= \lim_{m^2 \downarrow 0} (1 + z_0) \lim_{N \to \infty} \frac{\partial}{\partial \sigma} \frac{\partial}{\partial \bar{\sigma}} \bigg|_{0} (e^{-q_N \sigma \bar{\sigma}} + 0)$$

$$= \lim_{m^2 \downarrow 0} (1 + z_0) \lim_{N \to \infty} q_N$$

$$= c'(\Delta_{\mathbb{Z}^4})^{-1}(0, x) \sim c|x|^{-2}.$$