Critical slowdown for the Ising model on the 2D lattice

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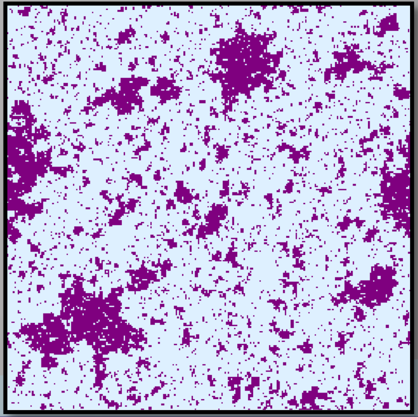
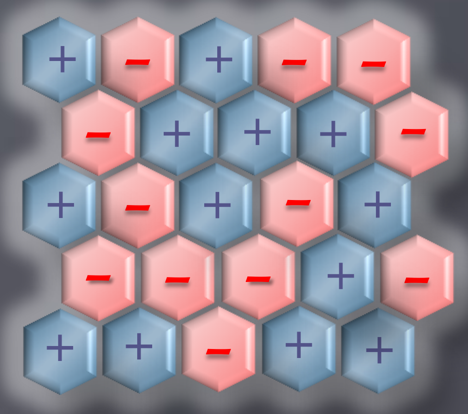


Joint work w. Eyal Lubetzky

Ising model

- Underlying geometry: finite graph $G=(V,E)$.
- Set of possible configurations:

$$\Omega=\{\pm 1\}^V$$



- Probability of a configuration $\sigma\in\Omega$ given by the *Gibbs distribution*

$$\mu(\sigma)=\frac{1}{Z(\beta)}\exp\ \beta\sum_{xy\in E}\sigma(x)\sigma(y)$$

Phase Transition

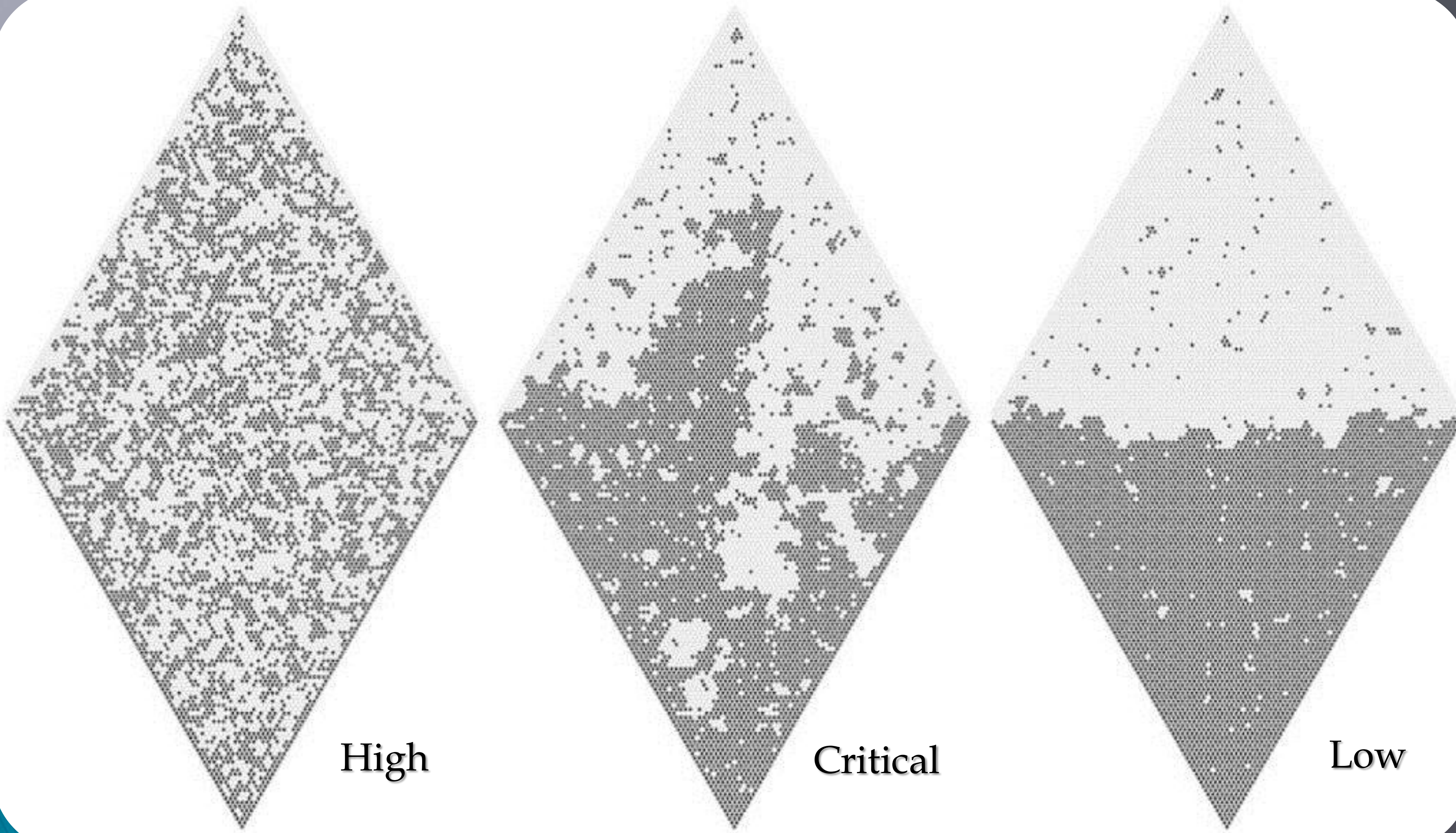
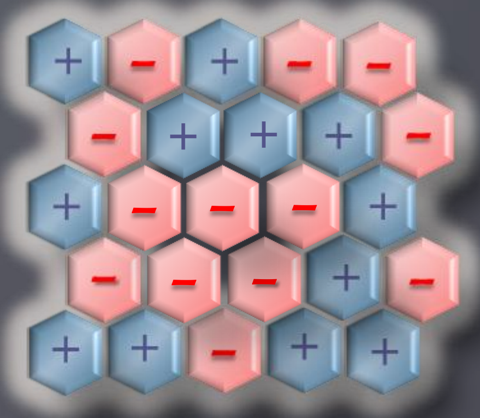


Image courtesy of David Wilson

Glauber dynamics for Ising

- One of the most commonly used MC samplers for the Gibbs distribution:
  - Update sites via *iid* Poisson(1) clocks
  - Each update replaces a spin at $u \in V$ by a new one $\sim \mu$ conditioned on $V \setminus \{u\}$ (heat-bath version).



- Ergodic reversible MC with stationary measure μ.
- *How fast does it converge to equilibrium?*

Rate of convergence to equilibrium

- Spectral gap in the spectrum of the generator:

  $$\text{gap} = \text{smallest positive eigenvalue the of heat-kernel } H.$$

  - Governs convergence in $L^2(\mu)$.
- Mixing time : standard measure of convergence:
  - The L^1 (total-variation) mixing time within ε is

    $$t_{\text{mix}}(\varepsilon) = \inf \ t : \max_{\sigma} \left\| H_t \ \sigma, \cdot \ - \mu \right\|_{\text{TV}} \le \varepsilon$$

    where H is the heat-kernel.

General (believed) picture for Glauber dynamics

- Setting: Ising model on the lattice $(\mathbb{Z}/n\mathbb{Z})^d$.
  Belief: For some critical inverse-temperature β_c :
- Low temperature: $(\beta > \beta_c)$
  gap^{-1} and t_{mix} are ***exponential*** in the surface area.
- Critical temperature: $(\beta = \beta_c)$
  gap^{-1} and t_{mix} are ***polynomial*** in the surface area.
- High temperature: $(\beta < \beta_c)$
  1. ***Rapid*** mixing: $\text{gap}^{-1} = O(1)$ and $t_{\text{mix}} \asymp \log n$
  2. Mixing occurs abruptly (***cutoff*** phenomenon).

Mixing time for Ising on 2D lattices

- Fast mixing for high temperatures:
  - [Aizenman, Holley ’84]
  - [Dobrushin, Shlosman ’87]
  - [Holley, Stroock ’87, ’89]
  - [Holley ’91]
  - [Stroock, Zegarlinski ’92a, ’92b, ’92c]
  - [Zegarlinski ’90, ’92]
  - [Lu, Yau ’93]
  - [Martinelli, Olivieri ’94a, ‘94b]
  - [Martinelli, Olivieri, Schonmann ’94]
- Slow Mixing for low temperatures:
  - [Schonmann ’87],
  - [Chayes, Chayes, Schonmann’87],
  - [Martinelli ’94],
  - [Cesi, Guadagni, Martinelli, Schonmann’96].

Mixing on the square lattice

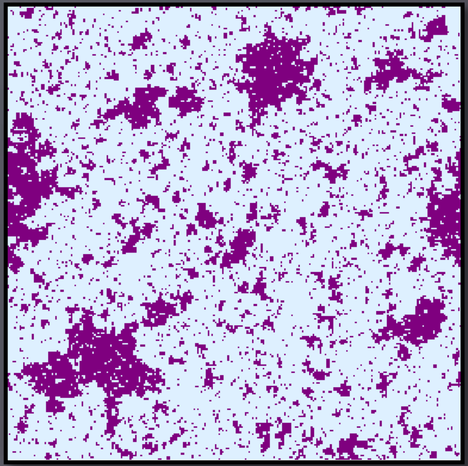
- High temperature: gap^{-1} is uniformly bounded, $O(\log n)$ mixing for all $\beta < \beta_c = \frac{1}{2}\log(1+\sqrt{2})$.
  - Dynamics conjectured to exhibit *cutoff* [Peres'04].
  - Recently confirmed [Lubetzky, S.]: $t_{\text{mix}} = \frac{1+o(1)}{\lambda_\infty}\log n$
- Low temperature: for $\beta > \beta_c$ both gap^{-1} and the mixing time are $\exp[(c(\beta)+o(1))n]$.
- Remains to verify power-law at critical $\beta = \beta_c$ …

Glauber dynamics at criticality

- Polynomial lower bound on gap^{-1} via the polynomial decay of spin-spin correlation whose asymptotics were established by [Onsager '44] ([cf. Holley '91]).
- Numerical experiments: $\exists$ universal exponent of $\sim$2.17
  - [Ito '93], [Wang, Hatano, Suzuki '95], [Grassberger '95], [Nightingale, Blöte '96], [Wang, Hu '97],…
- Compared to conjectured power-law behavior of gap^{-1} :
- ? No known *sub-exponential* upper bounds …
- *Only geometries* with proved power-law for critical Ising:
  - Mean-field [Ding, Lubetzky, Peres '09] (Curie-Weiss model)
  - Regular tree [Ding, Lubetzky, Peres '10] (Bethe lattice).

Scaling limit of critical Ising

- Understanding of the limit developed emerged with the advent of Schramm–Loewner evolution.



- Recent breakthrough results due to Smirnov describe full scaling limit of cluster interfaces as CLE_3.

- We use Russo-Seymour-Welsh type estimates for FK-Ising with arbitrary b.c.
  - [Chelkak, Smirnov '09]
  - [Camia, Newman '09]
  - [Duminil-Copin, Hongler, Nolin '09]

Main result: power-law at criticality

- THEOREM [Lubetzky, S.]: Critical slowdown verified in $\mathbb{Z}^2$:

Consider the critical Ising model on a finite box $\Lambda \subset \mathbb{Z}^2$ of side-length n. There exists an absolute constant C such that the spectral-gap of the Glauber dynamics under an arbitrary fixed boundary condition τ is bounded by $(\text{gap}_\Lambda^\tau)^{-1} \leq n^C$.

- COROLLARY:

Polynomial L^1 (total-variation) mixing time under any fixed boundary condition.

Further bounds on critical gap

- First polynomial upper bound for perfect simulation.
- A new lower bound (previously known lower bound was nearly linear due to [Holley '91]).

<u>THEOREM</u>

The spectral-gap of the Glauber dynamics for critical Ising on a finite box $\Lambda \subset \mathbb{Z}^2$ of side-length n with arbitrary boundary condition τ satisfies $(\mathrm{gap}_\Lambda^\tau)^{-1} \geq cn^{7/4}$

Main techniques

- Multi-scale estimates of the spectral gap.
- Approach for analyzing high temperature dynamics:
  - Control rate of mixing using exponential decay of correlation with distance.
- At criticality there are long range correlations foiling this approach.
- Alternative approach:
  - Use RSW estimates to get a spatial-mixing result, combine it with classical ingredients from MC analysis.
  - Analyze effect of an *entire face* of the boundary on spins (just enough spatial mixing to push this program through…)

Key spatial-mixing result

- THEOREM

Let $\Lambda = \ 1, r \ \times \ 1, r'$ for some integers r,r' satisfying $r'/r \geq \alpha > 0$ with α fixed and let $\Lambda_{\mathrm{T}} = \ 1, r \ \times \ \rho r, r'$ for some ρ satisfying $\alpha \leq \rho < r'/r$. Let ξ, η be two BC's on Λ that differ only on the bottom boundary $\ 1, r \ \times \{0\}$. Then

$$\left\| \mu_{\Lambda}^{\xi}(\sigma(\Lambda_{\mathrm{T}}) \in \cdot) - \mu_{\Lambda}^{\eta}(\sigma(\Lambda_{\mathrm{T}}) \in \cdot) \right\|_{\mathrm{TV}} \leq \exp(-\delta\rho),$$

Where $\delta > 0$ is a constant that depends only on α .

- Proof uses the RSW-estimate for critical crossing probabilities in a wired FK-Ising rectangle.

Single site vs. Block dynamics

- Classical tool in the analysis of Glauber dynamics:
  - Cover the sites using blocks $\mathcal{B} = \{B_i\}$.
  - Each block updates via a rate-1 Poisson clock.
  - Updates are $\sim$ stationary given the rest of the system.

- <u>PROPOSITION</u> (see, e.g. [Martinelli '97]):

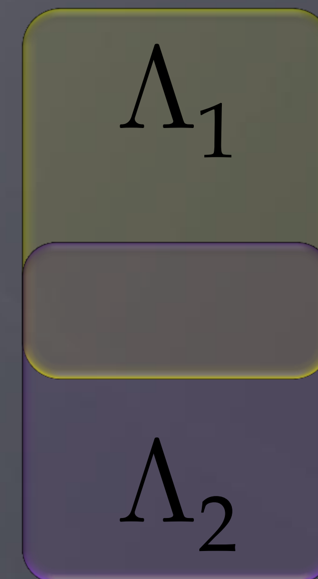
$$(\mathrm{gap}^{\tau}_{\Lambda})^{-1} \leq \max N_x (\mathrm{gap}^{\tau}_{\mathcal{B}})^{-1} \max_{i,\varphi} (\mathrm{gap}^{\varphi}_{B_i})^{-1}$$

where $(\mathrm{gap}^{\tau}_{\mathcal{B}})^{-1}$ *is the gap of the block-dynamics and* $N_x = \#\{i : B_i \ni x\}$

Upper bound via spatial-mixing

- Consider the following choice of blocks:

$$\Lambda_1 = 1, r \times \tfrac{1}{3} r', r' \;,$$
$$\Lambda_2 = 1, r \times 1, \tfrac{2}{3} r'$$



- The two blocks have a vertical overlap of height $r'/3$.

- As a result of the spatial-mixing theorem: For any boundary condition ξ on Λ we have

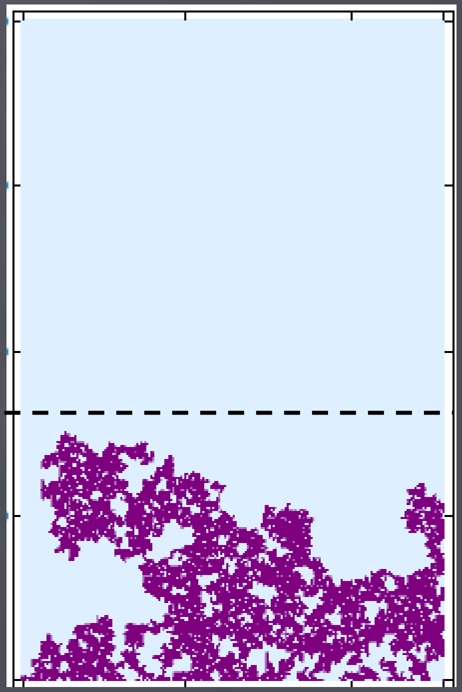
$$(\mathrm{gap}_{\mathcal{B}}^{\xi})^{-1} = O(1)$$

Upper bound via spatial-mixing (ctd.)

- The result is completed by induction on the block sizes.
- Each application decreases the volume of the blocks by a factor of $\frac{2}{3}$ at the cost of an absolute multiplicative constant in the gap.
- Iterating $2\log_{3/2} n$ steps completes the proof. ∎

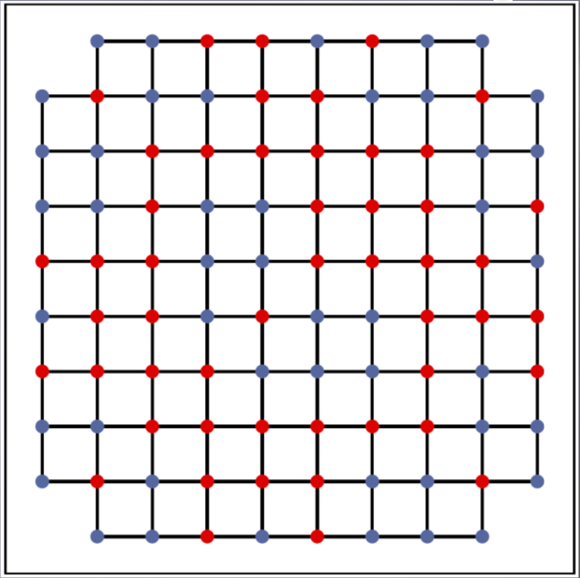
Intuition: spatial mixing proof

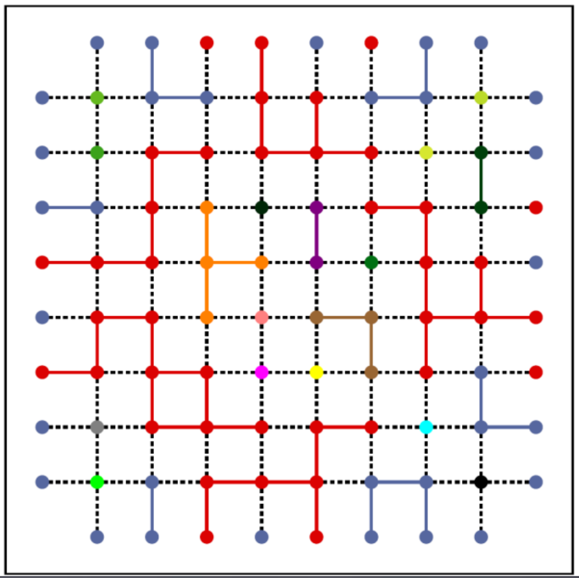
- Compare the all plus b.c with b.c with plus on 3 sides, minus on the bottom
  - Ising cluster adjacent to the bottom minus boundary converges to SLE_3 which does not climb past height ρr with positive probability.
  - In that case, the measures can be coupled.
- Actual setting:
  - For induction we need the result for arbitrary boundary conditions.



Solution: reduce to FK Ising

- Ising and its FK counterpart are coupled by the Edwards-Sokal coupling:

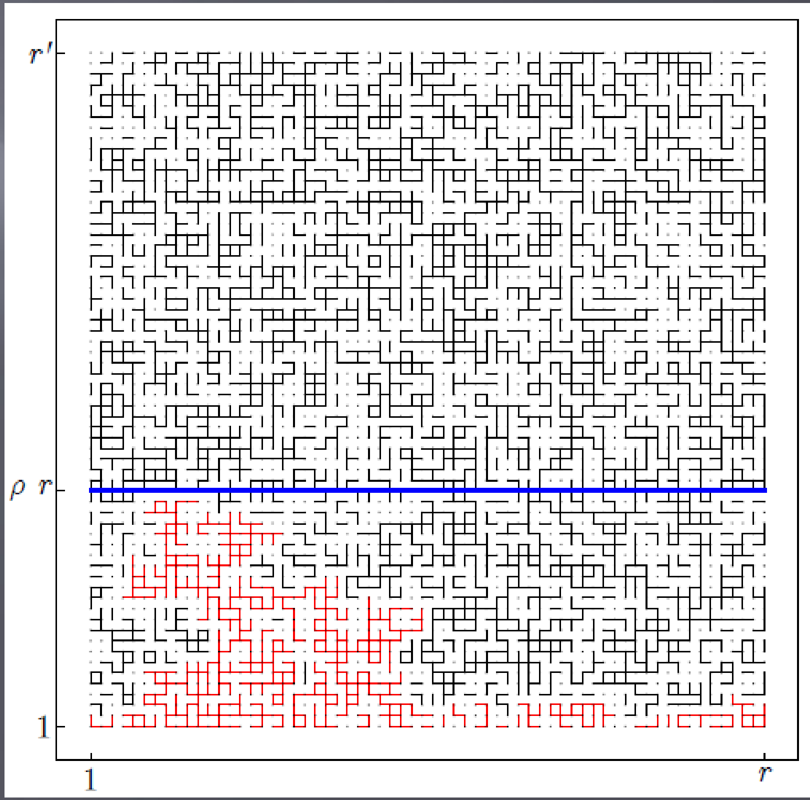




  - Under an arbitrary boundary condition ξ one can go from Ising to FK and back with conditioned on some event A_ξ which may have exponentially small probability…

Completing the proof

- Control crossing probabilities in the FK-Ising model conditioned on the event A_ξ .
- Utilize the recent RSW-type estimates with the FKG for the FK-model to derive the required coupling.
- Return to Ising via the Edwards-Sokal method to complete the proof.



Open problems

- Calculate the precise (universal) critical dynamical exponent.
- Establish power-law behavior on the lattice in 3 dimensions.

THANK YOU.