Interfacial Phenomena & Skew Diffusion

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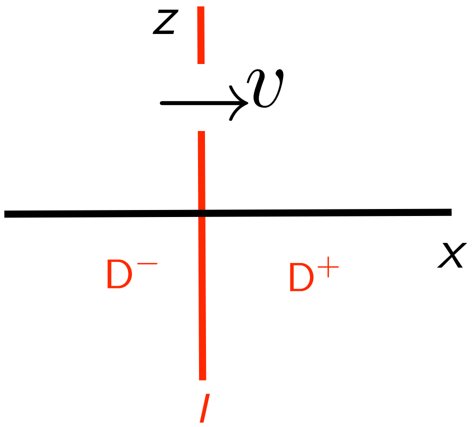
Example 1: Dispersion in Heterogeneous Media

B. Wood, CBEE LAB, OSU



$$D(x) = \begin{cases} D^{+} & x \geq 0 \\ D^{-} & x < 0. \end{cases}$$

c = concentration of injected dye

Fickean Dispersion Model:

$$\frac{\partial c}{\partial t} = \frac{1}{2}\nabla \cdot (D(x)\nabla c) - v\frac{\partial c}{\partial x}$$

Experimentally Observed Asymmetry

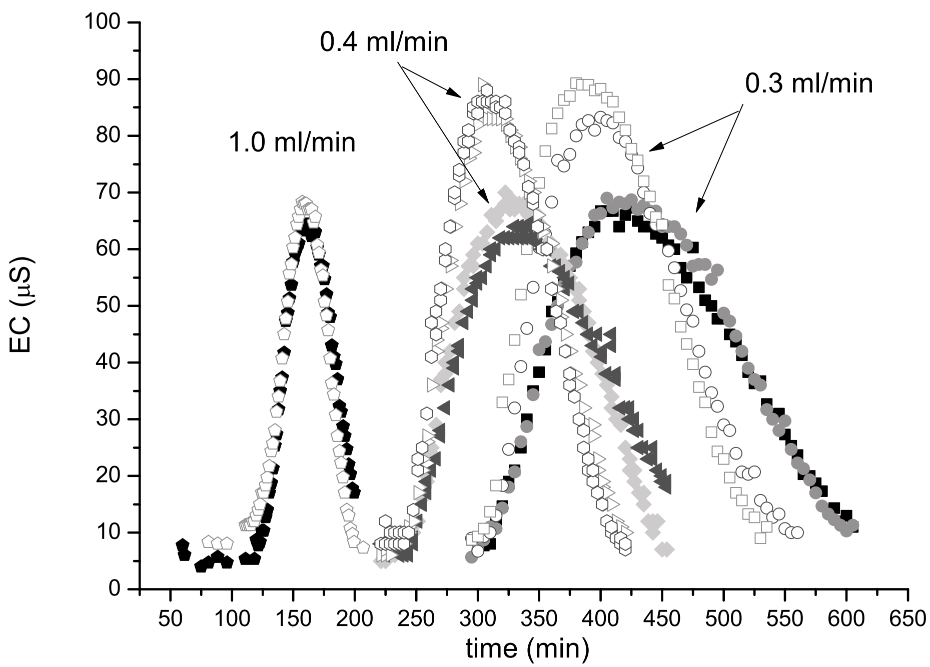
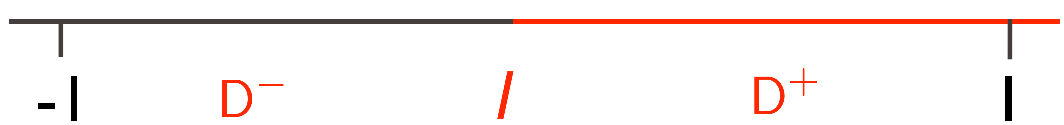


Figure 2. Experiments showing differences in breakthrough behavior for coarse to fine (C-F) and fine-to-coarse (F-C) directions of flow at flow rates of 0.3, 0.4, and 1 mL/min. The tracer input pulse times are 5, 3.75, and 1.5 min, and the sample collection intervals are 5, 3.75, and 1.5 min, respectively. Solid symbols represent C-F direction, and open symbols represent F-C direction. Two experiments in each direction were measured for the 0.3 and 0.4 mL/min flow rates. Each point represents an average of three measurements. The vertical axis shows electrical conductivity (EC), which is directly proportional to concentration.

B. Berkowitz, A. Cortis, I Dror and H Scher - 2009., B.Wood -2010.

PROBLEM: Explain the asymmetry from the Fickian Dispersion Model ?

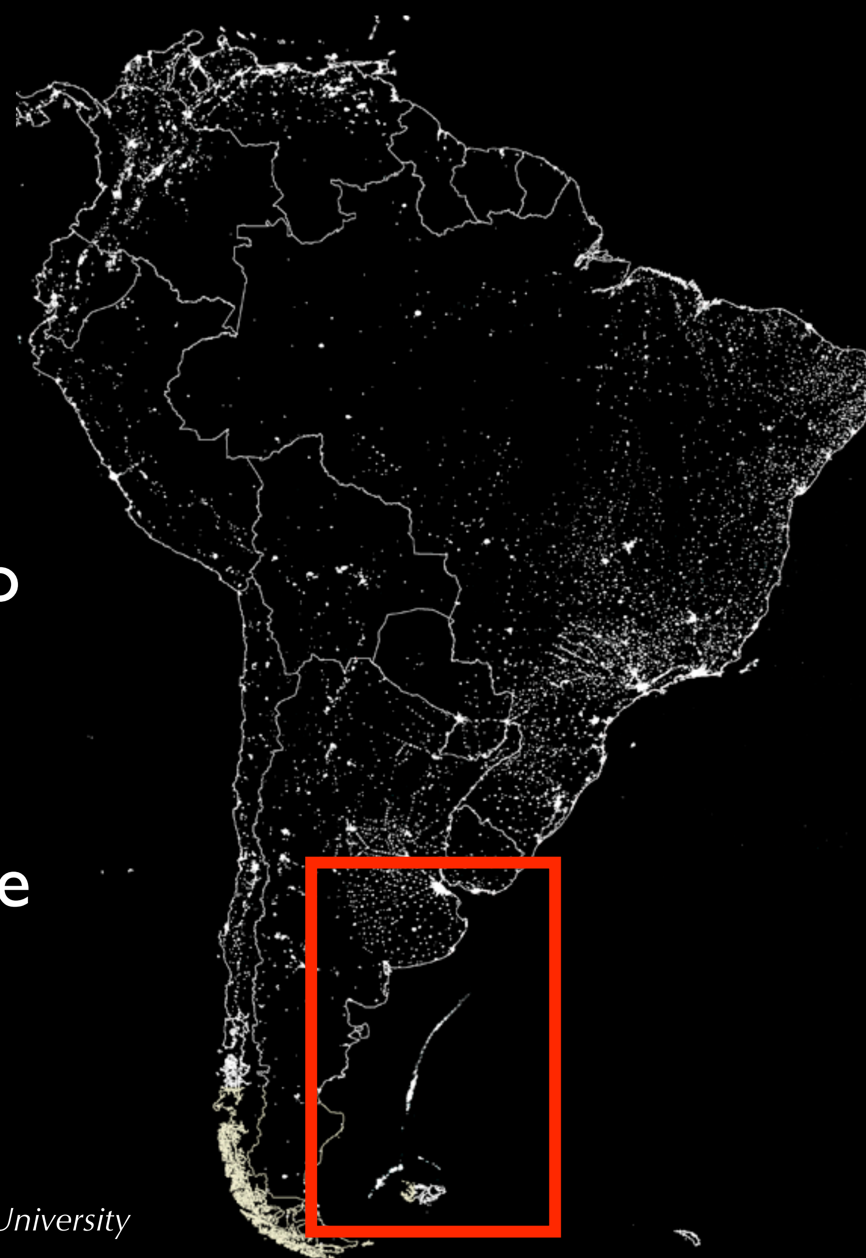


Q: Assume that D- < D+. Which is more likely removed first, a particle injected at -l and removed at l, or a particle injected at l and removed at -l ?

Example 2: Physical Oceanography - Upwelling of the Malvinas Current off the coast of Argentina

Highlighted region points to flotilla of ‘fishing factories’

Concentration of ships on the continental break where upwelling occurs.

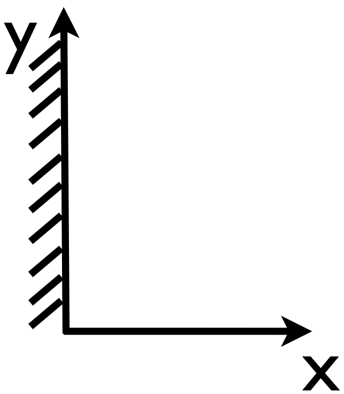


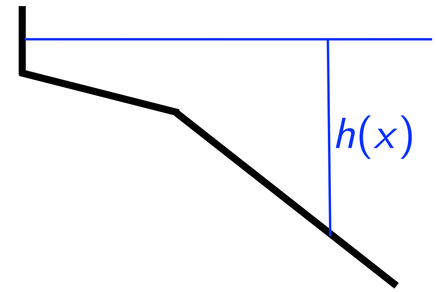
Acknowledgment: Ricardo Matano - COAS - Oregon State University

Arrested Topographic Wave Model

Quasi geostrophic balance - Hydrostatic Approximation.

$$
\begin{aligned}
-fv &= -g\frac{\partial \eta}{\partial x} \\
fu &= -g\frac{\partial \eta}{\partial y} - \frac{rv}{h} \\
0 &= \frac{\partial(uh)}{\partial x} + \frac{\partial(vh)}{\partial y}
\end{aligned}
$$





$\eta(x, y)$ = free surface, $h(x)$ = ocean depth

Eliminate velocity in terms of free surface

$$
\frac{\partial \eta}{\partial y} + \frac{r}{f}\left(\frac{\partial h}{\partial x}\right)^{-1}\frac{\partial^2 \eta}{\partial x^2} = 0
$$

$f < 0$ in Southern hemisphere

Arrested Topographic Wave Model

Transport = velocity times height of water column

Height of water column = $h(x) + \eta(x, y)$

$s^{\pm}$ = slope of continental shelf (-)and slope (+)

$$D^{+} = \frac{2r}{|f|s^{+}}, \quad D^{-} = \frac{2r}{|f|s^{-}}$$ (Shelf Break Interface)

Arrested Topographic Wave Model:

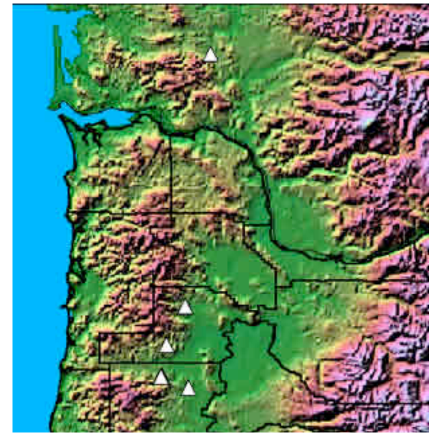
$$\frac{\partial \eta}{\partial t} = \frac{1}{2} D^{\pm} \frac{\partial^2 \eta}{\partial x^2}$$

Example 3: FENDERS BLUE BUTTERFLY

Fender's Blue

Kincaid's Lupin

Patch Distribution



Ecology, 82(7), 2001, pp. 1879–1892

EDGE-MEDIATED DISPERSAL BEHAVIOR IN A PRAIRIE BUTTERFLY

CHERYL B. SCHULTZ[1] AND ELIZABETH E. CRONE[2]

``Given past research on the Fender's blue and the potential to investigate response to patch boundaries in this system, we ask two central questions. First, how do organisms respond to habitat edges? Second, what are the implications of this behavior for residence time?"

INTERFACES AND PROBABILITY

- Stochastic Versions of Interface Conditions ?

Example 1:

Interface conditions: Continuity of c and of flux

$$c\Big|_{I+} = c\Big|_{I-} \qquad D^+ \frac{\partial c}{\partial x}\Big|_{I+} = D^- \frac{\partial c}{\partial x}\Big|_{I-}$$

Example 2:

Interface conditions: Continuity of c and of derivative

$$\eta\Big|_{I+} = \eta\Big|_{I-} \qquad \frac{\partial \eta}{\partial x}\Big|_{I+} = \frac{\partial \eta}{\partial x}\Big|_{I-}$$

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Example 3: ?

General Interface Conditions:

$$\lambda f'(0^+) - (1-\lambda) f'(0^-) = 0 \quad 0 \le \lambda \le 1$$

FELLER'S CLASSIFICATION : Given msble coefficients

$$\sigma^2(x) > 0, \qquad \mu(x)$$

there is a unique diffusion process X with generator

$$A = \frac{d^2}{dsdm} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$$

where

$$\mathcal{D}_A = \{f \in C(J) : \frac{d^2 f}{dmds} \in C(J)\}$$

$$ds(x) = e^{-\int_{\cdot}^{x} \frac{2\mu(y)}{\sigma^2(y)}dy} dx$$

$$dm(x) = \frac{2}{\sigma^2(x)} e^{\int_{\cdot}^{x} \frac{2\mu(y)}{\sigma^2(y)}dy} dx$$

STROOCK-VARADHAN MARTINGALE: Given measurable coefficients

$$\sigma^2(x) > 0, \qquad \mu(x)$$

there is a unique continuous process X such that

$$M_t(f) = f(X_t) - \int_0^t Af(X_s)ds, \quad t \geq 0$$

is a martingale for all $f \in C^\infty$.

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- Let $c(x,t) = E_x c_0(X_t)$

$$c_0 \in \mathcal{D}_A \Rightarrow \frac{\partial c}{\partial x}(0^+, t) = \frac{\partial c}{\partial x}(0^-, t), \quad t \geq 0.$$

i.e. $\lambda = \frac{1}{2}$

WHERE'S THE WIGGLE ROOM FOR INTERFACES ?

Analytic Remedy

Define
$$\Lambda(x) = \begin{cases} \lambda x & \text{if} \quad x \geq 0 \\ (1-\lambda)x & \text{if} \quad x < 0. \end{cases}$$

Write $c_0 \in \mathcal{I}_\lambda$ if

$$c_0 \in C(R), \quad \lambda c_0'(0^+) - (1-\lambda)c_0'(0^-) = 0$$

Then $c_0 \circ \Lambda \in \mathcal{I}_{\frac{1}{2}}$ (Space Change)

Adjust Coefficients (Time Change)

A_λ :
$$D_\lambda(x) = \begin{cases} (1-\lambda)^2 D^- & \text{if} \quad x < 0. \\ \lambda^2 D^+ & \text{if} \quad x \geq 0 \end{cases}$$

Get Adjusted S-V Process: $X_\lambda \sim A_\lambda$

Then $x \to E_{\Lambda^{-1}(x)} c_0(\Lambda(X_\lambda(t))) \in \mathcal{I}_\lambda$ (Ouknine 1987)

On Brownian Motion Observations

``The trajectories are confused and complicated so often and so rapidly that it is impossible to follow them; the trajectory actually measured is very much simpler and shorter than the real one. Similarly, the apparent mean speed of a grain during a given time varies in a wildest way in magnitude and direction, and does not tend to a limit as the time taken for an observation decreases, as may be easily shown by noting, in the camera lucida, the positions occupied by a grain from minute to minute, and then every five seconds, or, better still, by photographing them every twentieth of a second, as has been done by Victor Henri Comandon, and de Broglie when kinematographing the movement. ***It is impossible to fix a tangent, even approximately, at any point on a trajectory, and we are thus reminded of the continuous underived functions of the mathematicians.***"

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Q: So what would Perrin see when there is an interface ?

SKEW BROWNIAN MOTION $B^\alpha(t)$ $0 \le \alpha \le 1$

Ito-McKean (1963): Construct diffusion defined by:

Infinitesimal Generator $A = \frac{1}{2}\frac{d^2}{dx^2}$

$\mathcal{D}_A$: $f \in \mathcal{C}(R) \cap \mathcal{C}^2(R\backslash\{0\})$ $\alpha f'(0^+) = (1-\alpha) f'(0^-)$

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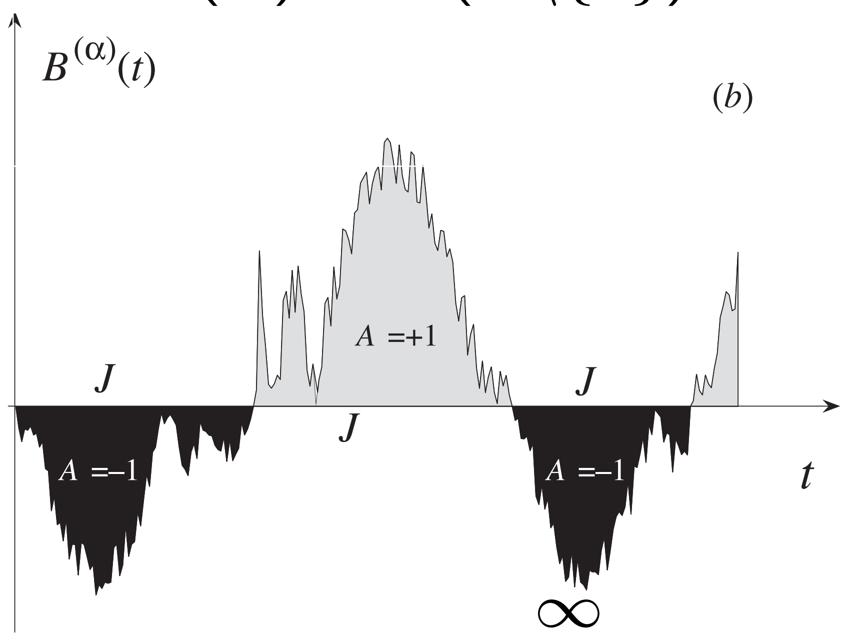

IID ± 1 Valued A_n

$P(A_n = 1) = \alpha$

J_n Excursion Intervals

$$B^{(\alpha)}(t) = \sum_{n=1}^{\infty} \mathbf{1}_{J_n}(t) A_n |B(t)|$$

GENERAL MARTINGALE FORMULATION

Consider $$\frac{\partial c}{\partial t} = \frac{D^{\pm}}{2}\frac{\partial^2 c}{\partial x^2}$$

with the interface condition

$$c(0^+) = c(0^-), \quad \lambda c'(0^+) = (1-\lambda)c'(0^-)$$

Re-scaled Skew Brownian Motion:

$$s(B^\alpha(t)) = \sqrt{D^+}B^\alpha(t)\mathbf{1}(B^\alpha(t)) + \sqrt{D^-}B^\alpha(t)\mathbf{1}(B^\alpha(t))$$

Martingale Problem Given $D^{\pm}$ and $0 < \lambda < 1$, determine α so that

$$f(s(B^\alpha(t))) - \frac{1}{2}\int_0^t D(s(B^\alpha)(u))f''(s(B^\alpha(u))du$$

is a martingale for all $f \in \mathcal{D}_\lambda$ where

$$\mathcal{D}_\lambda = \{g \in \mathcal{C}^2(R\backslash\{0\}) \cap \mathcal{C}(R) : \lambda g'(0^+) = (1-\lambda)g'(0^-)\}$$

SOLUTION

$$dB^\alpha = dB + (2\alpha - 1)dl = dB + \frac{2\alpha - 1}{2\alpha} dA^+_\alpha$$

(Le Gall 1982)

$$A^+_\alpha(t) = \lim_{\epsilon \downarrow 0} \int_0^t \mathbf{1}[0 \leq B^{(\alpha)}(s) < \epsilon] ds$$

local time from the right

Itô-Tanaka applied to $Y(t) = s(B^\alpha(t))$ gives

$$dY = \sqrt{D(Y)} dB + \frac{1}{2}\left(1 + \frac{\sqrt{D^-}}{\sqrt{D^+}} - \frac{\sqrt{D^-}}{\sqrt{D^+}\alpha}\right) dA^+_Y$$

For $f \in \mathcal{D}_\lambda$, and $Z(t) = f(Y(t))$ one has

$$dZ = f'(Y)\sqrt{D(Y)} dB + \frac{1}{2} D(Y) f''(Y) dt + \frac{1}{2f'(0^-)}\left(\frac{\sqrt{D^-}}{\sqrt{D^+}} - \frac{\sqrt{D^-}}{\sqrt{D^+}\alpha} + \frac{1-\lambda}{\lambda}\right) dA^+_Y$$

Proposition:

$$f(Y(t)) - \frac{1}{2}\int_0^t D(Y(s)) f''(Y(s)) ds \text{ martingale} \iff \alpha = \frac{\lambda\sqrt{D^-}}{\lambda\sqrt{D^-} + (1-\lambda)\sqrt{D^+}}$$

Back To Examples

$$\alpha = \frac{\lambda\sqrt{D^-}}{\lambda\sqrt{D^-} + (1-\lambda)\sqrt{D^+}}$$

Example 1 - Continuity of Flux

$$\lambda = \frac{D^+}{D^+ + D^-} \qquad \alpha \equiv \alpha^* = \frac{\sqrt{D^+}}{\sqrt{D^+} + \sqrt{D^-}}$$

Example 2 - Continuity of Derivative

$$\lambda = 1/2, \qquad \alpha = \frac{\sqrt{D^-}}{\sqrt{D^+} + \sqrt{D^-}}$$

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``MODIFICATION'' -- Integrate w.r.to Lebesgue in place of QV

APPLICATION TO EXAMPLE I

$$D^- < D^+$$

Q: Assume that D- < D+. Which is more likely removed first, a particle injected at -1 and removed at 1 or a particle injected at 1 and removed at -1 ?

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Q: Assume that D- < D+. Which is more likely removed first, a particle injected at -1 and removed at 1 or a particle injected at 1 and removed at -1 ?

A: The experiments show that in this configuration, a particle injected at -1 arrives faster at 1 than when the particle is injected at 1 and removed at -1 .

i.e. FINE TO COARSE IS FASTER THAN COARSE TO FINE

APPLICATION TO EXAMPLE I

$$D^- < D^+$$

Impact on asymmetry of break-through curves. Fine to Coarse corresponds to $D^- < D^+$ and thus α > 1/2. Coarse to Fine corresponds to α < 1/2.

Proposition: Assume $\sqrt{D^-} < \sqrt{D^+}$, $\alpha^* = \sqrt{D^+}/(\sqrt{D^+} + \sqrt{D^-})$ Let $Y = s(B^{\alpha^*})$, $T_y^* = \inf\{t \geq 0 : Y_t = y\}$. Then for each $t > 0, y > 0$,

$$P_{-y}(T_y^* > t) < P_y(T_{-y}^* > t)$$

SKEW BROWNIAN MOTION

Proposition: Let $T_a^\alpha = \inf\{t \geq 0 : B^\alpha(t) = a\}$. If $\alpha > 1/2$, then for any $t > 0$, $a > 0$, $P_{-a}(T_a^\alpha > t) < P_a(T_{-a}^\alpha > t)$.

Pf: Note $T_0 = T_0^{1/2}$ and $T_0^\alpha \stackrel{d}{=} T_0$ under P_a, $a \neq 0$. so
$P_a(T_0^\alpha > t) = P_a(T_0 > t) = P_{-a}(T_0 > t) = P_{-a}(T_0^\alpha > t)$.

$$P_{-a}(T_a^\alpha > t)) = \int_0^t P_0(T_a^\alpha > t - s) P_{-a}(T_0 \in ds)$$

$$P_a(T_{-a}^\alpha > t)) = \int_0^t P_0(T_a^{1-\alpha} > t - s) P_{-a}(T_0 \in ds)$$

Strong Markov Property of Skew BM

Natural Coupling:

$$B^\alpha(t) = \sum_{m=1}^\infty \mathbf{1}_{J_m}(t)[2\mathbf{1}_{[0,\alpha)}(U_m) - 1]|B(t)|$$

U_m are iid unif dist on $[0, 1]$.

Note if $1 > \alpha > 1/2$, $\quad [T_a^{1-\alpha} \leq t] \subset [T_a^\alpha \leq t]$

$$P_0(T_a^\alpha > t - s) \leq \frac{1-\alpha}{\alpha} P_0(T_a^{1-\alpha} > t - s) < P_0(T_a^{1-\alpha} > t - s)$$

Proof of Ordering

Scaling properties $B^{\alpha}_{ct} \overset{d}{=} \sqrt{c} B^{\alpha}_{t}$

$$T_0^* =^{P_1-dist} \frac{1}{D^+} T_0 \qquad T_0^* =^{P_{-1}-dist} \frac{1}{D^-} T_0$$

$$T_1^* =^{P_0-dist} \frac{1}{D^+} T_1^{\alpha^*} \qquad T_{-1}^* =^{P_0-dist} \frac{1}{D^-} T_1^{(1-\alpha^*)}$$

$$P_{-1}(T_1^* > t) = \int_0^t P_0(\frac{1}{D^+} T_1^{\alpha^*} > t-s) P_0(\frac{1}{D^-} T_1 \in ds)$$

$$\overset{S}{<} \qquad \overset{S}{<}$$

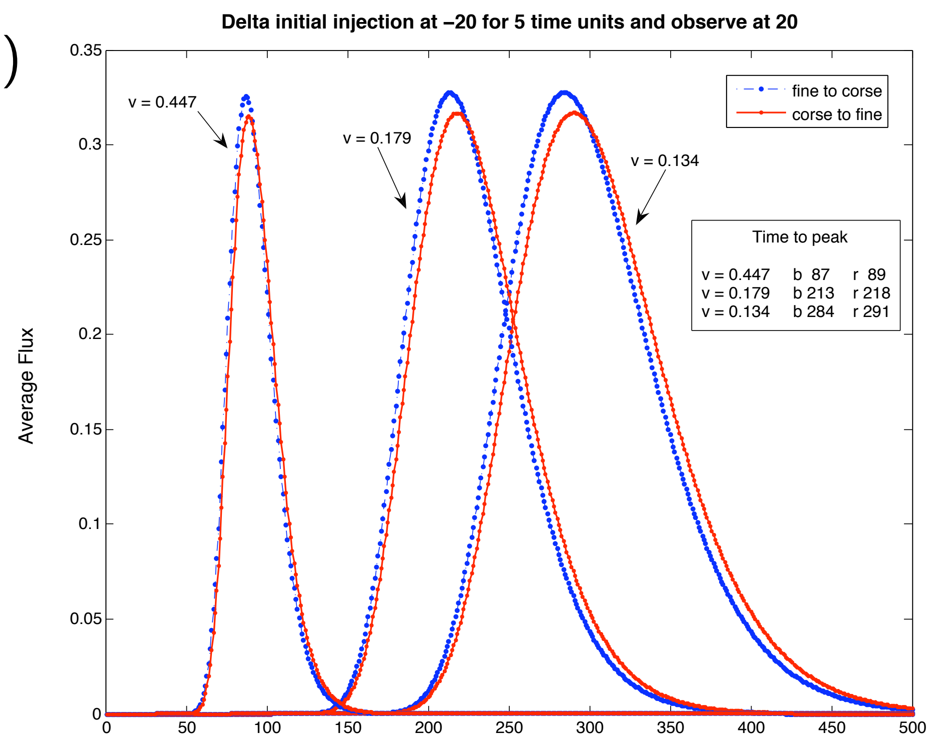
$$P_1(T_{-1}^* > t) = \int_0^t P_0(\frac{1}{D^+} T_1 > t-s) P_0(\frac{1}{D^-} T_1^{(1-\alpha^*)} \in ds)$$

Alternative: **Resident Concentration Curve**

KEY: $$P_0(B_t^{(\alpha)} > b, L_t^0(B^{(\alpha)}) \in dl, \Gamma_+^\alpha(t) \in d\tau) =$$

$$\frac{(1-\alpha)l}{2\pi(t-\tau)^{3/2}\tau^{1/2}} \exp\left(-\frac{((1-\alpha)l)^2}{2(t-\tau)} - \frac{(b+\alpha l)^2}{2\tau}\right) dl d\tau$$

$$\frac{A}{U}\left(Uc(b,t) + D(b)\frac{\partial c}{\partial x}\right) = Q(t)$$



PROBLEM: First passage time distribution for skew Brownian motion

SOLUTION:

- Appuhamillage, T., D. Sheldon (2010): **ArXiv1008.2989**
- Skew Brownian Motion with Drift - **OPEN**

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