Supplement: Radial Mean and Anisotropy of White Noise Samples

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In this supplemental document for [Bowers et al. 2010], we derive the analytic radial means and anisotropy of 1D white noise samples. We show that when using the real form of Fourier basis, using the radial mean is the same as using the complex form of Fourier basis; however, the anisotropy when using the former is twice as much as when using the latter, as is manifested in our experimental results in the paper. The definition of radial means and anisotropy can be found in literature work such as [Lagae and Dutré 2008]. The proof can be straightforwardly extended to 2D white noise samples.

We assume there are N independent and uniformly random samples x_j distributed between [0, 1] (i.e. these are white noise samples). In the following we derive the analytic radial mean and anisotropy of the samples' power spectrum, and compare the results using 1D complex Fourier basis vs. 1D real Fourier basis.

1 Using 1D Complex Fourier Basis

It's well-known that each 1D complex Fourier basis is in the form:

$$f_\omega(x) = e^{-i\,2\pi\,\omega\,x} = \cos 2\pi\omega x + i\,\sin 2\pi\omega x \tag{1}$$

where ω is the frequency. The Fourier transform of a set of impulse samples is the sum of the Fourier basis evaluated at each sample:

$$F(\omega) = \sum_{j=1}^{N} f_\omega(x_j) = \sum_{j=1}^{N} \cos 2\pi\omega x_j + i \sum_{j=1}^{N} \sin 2\pi\omega x_j \tag{2}$$

Power Spectrum. The power spectrum is defined as the squared magnitude of $F(\omega)$. We further divide it by N in order to normalize it. Therefore the power spectrum $P(\omega)$ is:

$$P(\omega) = \frac{1}{N}|F(\omega)|^2 = \frac{1}{N}\left[\left(\sum_{j=1}^{N}\cos 2\pi\omega x_j\right)^2 + \left(\sum_{j=1}^{N}\sin 2\pi\omega x_j\right)^2\right] \tag{3}$$

Radial Mean. The radial mean R_m is defined as the average of $P(\omega)$ within a given frequency range $[\omega_1, \omega_2]$. When the range being considered is sufficiently small, it suffices to examine the expected value of $P(\omega)$ at a specific frequency ω. In order words, $R_m = E[P(\omega)]$, where E denotes the expected value with respect to the joint distribution of the samples. It is calculated as:

$$E[P(\omega)] = \int_0^1 ... \int_0^1 P(\omega)\,\mathrm{d}x_1\,\mathrm{d}x_2 ...\,\mathrm{d}x_N \tag{4}$$

While we could expand the quadratic terms in Eq. 3 to evaluate the integral, doing so makes it hard to compute the anisotropy later. So instead, we use an inductive approach. Let R_m^N denote the expectation of Eq. 3 with the sum running from $j = 1$ to N, and R_m^{N-1} denote that with the sum running from $j = 2$ to N, and so on. By integrating Eq.4 over $\mathrm{d}x_1$, we find that:

$$R_m^N = \frac{1}{N} + R_m^{N-1} + \Delta_{rem} \tag{5}$$

where Δ_{rem} is a residual term:

$$\Delta_{rem} = \frac{2\,(N-1)\,(\sin\pi\omega)^2}{N\,(\pi\,\omega)^2} \tag{6}$$

For any given finite N, Δ_{rem} will rapidly fall off to zero for as the frequency ω becomes large. Thus for a sufficiently large ω:

$$R_m^N = \frac{1}{N} + R_m^{N-1} \tag{7}$$

Using induction, and the fact that $R_m^1 = \frac{1}{N}$, we get:

$$E[P(\omega)] = R_m^N = \frac{1}{N} + \frac{1}{N} + ... + \frac{1}{N} = 1 \tag{8}$$

In other words, the power spectrum has an expected value of 1.0, and this is independent of the frequency ω. This conforms with the observation that the radial mean plot is centered around the 1.0 line.

Anisotropy. The anisotropy A_r is defined as the variance of $P(\omega)$ relative to the squared radial mean R_m^2. Since we have already calculated R_m, we now only need to compute the expected value of $P^2(\omega)$, which is:

$$P^2(\omega) = \frac{1}{N^2}\left[\left(\sum_{j=1}^{N}\cos 2\pi\omega x_j\right)^2 + \left(\sum_{j=1}^{N}\sin 2\pi\omega x_j\right)^2\right]^2 \tag{9}$$

Similarly to before, we use an inductive approach. Let S_m^N denote the expected value of the above equation with the sum running from $j = 1$ to N, and S_m^{N-1} denote that with the sum running from $j = 2$ to N, and so on. By integrating Eq. 9 over $\mathrm{d}x_1$, we find that:

$$S_m^N = \frac{1}{N^2} + S_m^{N-1} + \frac{4}{N}R_m^{N-1} + \Delta_{rem} \tag{10}$$

where Δ_{rem} is again a residual term that can be ignored for a sufficiently large ω. Using induction:

$$S_m^N = \frac{N}{N^2} + \frac{4}{N}\left(\frac{N-1}{N} + \frac{N-2}{N} + ... + \frac{1}{N}\right) = 2 - \frac{1}{N} \tag{11}$$

For a large enough N, we can see $S_m^N \approx 2$.

Now, the variance of $P(\omega)$ is simply $E[P^2(\omega)] - E^2[P(\omega)]$, or in other words $S_m - R_m^2$, thus

$$\mathrm{Var}(P(\omega)) = S_m - R_m^2 = 2 - 1^2 = 1 \tag{12}$$

Again, this conforms with our observation that the anisotropy plot is centered around the 1.0 line.

2 Using 1D Real Fourier Basis

The 1D real-form Fourier basis consists of only the real portion of the 1D complex Fourier basis:

$$f_\omega(x) = \sqrt{2}\cos 2\pi\omega x \tag{13}$$

where ω is integer frequency: $\omega = 0, 1, 2, ...$, and $\sqrt{2}$ is a normalization factor for the basis set to remain orthonormal. Such a basis can be derived from the eigenfunctions of the Laplacian operator, as described in our paper. Given the form of the basis, the Fourier transform of a set of samples now becomes:

$$F(\omega) = \sum_{j=1}^{N} \sqrt{2}\cos 2\pi\omega x_j \tag{14}$$

Therefore the **power spectrum** is:

$$P(\omega) = \frac{1}{N}|F(\omega)|^2 = \frac{2}{N}\left(\sum_{j=1}^{N}\cos 2\pi\omega x_j\right)^2 \tag{15}$$

Radial Mean and Anisotropy. We use the same inductive method as before. For radial mean, the induction formula is:

$$R_m^N = \frac{1}{N} + R_m^{N-1} + \Delta_{rem} \tag{16}$$

This is the same with Eq. 5, thus the **radial mean** remains the same:

$$E[P(\omega)] = R_m = 1$$

To compute anisotropy, we first compute expected value of $P^2(\omega)$ which we denote as S_m. In this case, the induction formula is:

$$S_m^N = \frac{3}{2N^2} + S_m^{N-1} + \frac{6}{N}R_m^{N-1} + \Delta_{rem} \tag{17}$$

Therefore:

$$S_m = \frac{3N}{2N^2} + \frac{6}{N}(\frac{N-1}{N} + \frac{N-2}{N} + ... + \frac{1}{N}) = 3 - \frac{3}{2N} \tag{18}$$

For a sufficiently large N, we have $S_m^N \approx 3$.

Given R_m and S_m, the variance of $P(\omega)$ is:

$$\mathrm{Var}(P(\omega)) = S_m - R_m^2 = 3 - 1^2 = 2 \tag{19}$$

This conforms with our observation that the anisotropy plot in this case is centered around the 2.0 line. Note that this is twice as much as the anisotropy when using the complex 1D Fourier basis.

2.1 Multiple Experimental Runs

In practice, we typically perform multiple experimental runs, each run generating N random samples. We compute the power spectrum of each run, average the results over all runs, and then use the averaged power spectrum to compute radial mean and anisotropy. Typically we use $K = 10$ runs.

Under multiple runs, the radial mean still has an expected value of 1.0, since the expected value of each run is 1.0. On the other hand, the anisotropy is inversely proportional to K. This is because the variance of the average of K independent identically-distributed (i.i.d.) variables is $\frac{1}{K}$ of the variance of each random variable by itself. Therefore, with $K = 10$ runs, the anisotropy using real-form Fourier basis is centered around the 0.2 (or -7dB) line. For comparison: the anisotropy using complex 1D Fourier basis is centered around the 0.1 (or -10dB) line.

References

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