A SUPPLY AND DEMAND FRAMEWORK FOR TWO-SIDED MATCHING MARKETS

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Abstract. We propose a new model of two-sided matching markets, which allows for complex heterogeneous preferences, but is more tractable than the standard model, yielding rich comparative statics and new results on large matching markets.

We simplify the standard Gale and Shapley (1962) model in two ways. First, we assume that a finite number of agents on one side (colleges or firms) are matched to a continuum mass of agents on the other side (students or workers). Second, we show that stable matchings can be characterized in terms of supply and demand equations.

We show that, very generally, the continuum model has a unique stable matching, which varies continuously with the underlying fundamentals. Moreover, stable matchings in the continuum model are the limit of the set of stable matchings in large discrete economies, so that the continuum model is an approximation of the standard Gale and Shapley model in markets where agents on one side are matched to many agents on the other side.

We apply the model to a price-theoretic analysis of how competition induced by school choice gives schools incentives to invest different aspects of quality, and of the distortions in privately optimal investments. As another application, we characterize the asymptotics of school choice mechanisms used in practice, generalizing previous results of Che and Kojima (2010).

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1. Introduction

1.1. **Overview.** In two-sided matching markets buyers and sellers have preferences over *who* they interact with on the other side of the market. For example, consulting firms competing for college graduates care about which workers they hire. Such a market clears not only through wages, as a college graduate cannot simply demand the firm he prefers - he must also be chosen by the firm. These are key features of many important markets, and matching markets have been extensively studied in economics. Much of the literature is based on one of two classic frameworks, each with distinct advantages and limitations.[1]

One strand of the literature follows Becker's (1973) marriage model. These models often assume simple preferences, with men and women being ranked from best to worst. Moreover, utility is transferable, so that a couple may freely divide the gains from marriage. These stark assumptions lead to simple models with rich comparative statics that have been applied to diverse problems such as explaining sex differences in educational attainment, changes in CEO wages, and the relationship between the distribution of talent and international trade.[2]

Another line of research follows Gale and Shapley's (1962) college admissions model. These models allow for complex heterogeneous preferences, and (possibly) for limitations on how parties may split the surplus of a relationship. Due to its generality, this model is a cornerstone of market design, and has been applied to the study and design of market clearinghouses (e.g. the National Resident Matching Program, that matches 30,000 doctors and hospitals per year; the Boston and New York City public school matches, which match over 100,000 students per year),[3] the use of signaling in labor markets, the relationship between matching and auctions, and supply chain networks.[4] This framework has had less success in empirical applications, where the multiplicity of solutions is an issue,[5] and in obtaining comparative statics results. In contrast to the

[1]In this paper we focus on frictionless matching markets, where the appropriate equilibrium concept is the notion of stability defined below. Another important literature, pioneered by Mortensen and Pissarides (1994), considers matching markets with frictions. We do not pursue this line of inquiry.

[2]See respectively Chiappori et al. (2009); Gabaix and Landier (2008) and Tervio (2008); Grossman (2004), and the related paper by Grossman and Maggi (2000).

[3]The redesign of the National Resident Matching Program is described in Roth and Peranson (1999). School choice was introduced as a mechanism design problem in the seminal paper of Abdulkadiroglu and Sönmez (2003), and the redesign of the Boston and New York City matches is described in Abdulkadiroglu et al. (2005a,b).

[4]See respectively Coles et al. (2010), Coles et al. (Forthcoming) and Lee and Niederle (2011); Hatfield and Milgrom (2005); Ostrovsky (2008).

[5]See Sørensen (2007) for a discussion of the issue of multiplicity of stable matchings for the estimation of matching models, and a set of assumptions under which it can be circumvented. The Gale and Shapley model is often used for the simulation of outcomes once preferences have been estimated, as in Lee and Niederle (2011) and Hitsch et al. (2010).

former strand of the literature, comparative statics results are often difficult and of a limited nature in this framework.[6]

This paper proposes a new model of matching markets based on Aumann's (1964) insight that markets with a continuum of traders may be considerably simpler than those with a finite number of traders. Like the Gale and Shapley (1962) framework, the proposed model allows for rich heterogeneous preferences and (possible) restrictions on transfers. However, like the Becker (1973) model, it also permits straightforward derivation of comparative statics.

The basic features of our model follow the standard Gale and Shapley college admissions model. Agents on one side (colleges or firms) are to be matched to many agents on the other side (students or workers). Throughout most of the paper we consider the extreme case where there are no transfers between agents. As in the standard model, the solution concept is stability. A matching between students and colleges is stable if no pair of a student and a college would like to break away from a match partner and match to each other.

We make two key simplifications to the standard Gale and Shapley model. First, we apply the logic of *supply and demand* to matching markets. We give a new characterization of stable matchings that allows us to use standard techniques from competitive equilibrium models in a matching setting. We show that, even in the standard discrete model, stable matchings have a very simple structure, given by admission thresholds P_c at each college c. We term such a threshold a cutoff, and colleges are said to accept students who are ranked above the cutoff. Given a vector of cutoffs P, a student's demanded college is defined as her favorite college that would accept her. We show that, for every stable matching, there exists a vector of cutoffs such that each student demands the college she is matched to. Moreover, at any vector of cutoffs P that clears supply and demand for colleges, the demand function yields a stable matching. Therefore, finding stable matchings in the Gale and Shapley model is equivalent to finding a solution P to a set of market clearing equations[7]

$$D(P) = S.$$

[6]For example, Crawford's (1991) main results are comparative statics theorems. He shows, in a generalization of the Gale and Shapley model, that adding a firm to the economy makes the worker-optimal and the firm-optimal stable matchings better for all workers. Hatfield and Milgrom's (2005) Theorem 6 shows that, in their model, a similar result is true when a worker leaves the market. Both these results, although unsurprising, are nontrivial, and the proofs depend on the Gale Shapley algorithm and its generalizations. In contrast, in our model comparative statics are obtained simply by differentiating the market clearing equations. This reveals not only the direction but also the magnitude of changes, and gives a systematic technique to derive comparative statics, which works in cases that may be less straightforward than the addition of a worker or firm.

[7]As we discuss formally below, the formula $D(P) = S$ only holds when there is excess demand for all colleges. In general the system of equations to be solved is for every college c: $D_c(P) \leq S_c$, with equality if $P_c > 0$.

Mathematically, this is equivalent to solving a general equilibrium model, with cutoffs P in the stead of prices.

The second key simplification is an assumption. In the model each of a finite number of colleges is matched, not to a discrete number of students, but to a *continuum mass of students*. Therefore, our model approximates a situation where each agent on one side is matched to a large number of agents on the other side. This is the case in a number of important matching markets, such as college admissions, school choice clearinghouses, diverse labor markets (e.g. the market for associates at major US law firms),[8] and debt and equity underwriting.[9] The assumption of a continuum of students and finite number of colleges is similar to Aumann's (1964) use of a continuum of consumers trading finitely many types of goods.

The continuum assumption considerably simplifies the analysis. We show that, with great generality, the continuum model has a *unique stable matching*, corresponding to the unique solution of the market clearing equations, and that this stable matching varies *continuously* with the underlying fundamentals. This result, and its proof, are similar in spirit to Debreu's (1970) finding that typical Walrasian economies have a finite number of competitive equilibria.[10] The result implies that *comparative statics* may be derived from the *market clearing equations*, as in standard price-theoretic arguments. This is in contrast with the standard discrete model, that possibly has multiple stable matchings, which are typically found using the Gale and Shapley algorithm, rendering techniques such as the implicit function theorem not directly applicable.

To justify using the simpler continuum model as an approximation of real markets, we give a set of convergence results, which are our main theoretical contribution. A sequence of increasingly large discrete economies is said to converge to a continuum economy if the empirical distribution of student types converges to the distribution in the continuum economy, and the number of seats per student in each college converges. Whenever the continuum economy has a unique stable matching, all of the stable matchings of the discrete economies converge to this unique stable matching of the continuum economy. Moreover, we show that all stable matchings of the large discrete economies become very similar (in a sense we make precise). Therefore, even in a large discrete economy, the concept of stability clears the market in a way that is essentially unique.

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[8] See Oyer and Schaefer (2010) for a description of institutional features of this market.

[9] This market is discussed in Asker and Ljungqvist (2010).

[10] Our uniqueness proof follows Debreu's insight of applying differential topology techniques to establish properties valid in generic economies. Although Debreu's result only shows that equilibria are isolated and finite in number, as opposed to unique as is the case in matching markets, both results justify using market clearing equations, and the implicit function theorem when demand is continuously differentiable, to derive comparative statics.

We illustrate the usefulness of the model in two different applications: a set of new asymptotic results in market design, and with a novel analysis of the effects of competition between schools. In market design, our paper is the first to characterize the asymptotics of stable matchings in a class of large matching markets.[11] In particular, we characterize the asymptotics of school choice mechanisms used in practice to match students to schools, such as deferred acceptance with single tie-breaking. Previously, Che and Kojima (2010) had characterized the limit of the random serial dictatorship mechanism, and shown that it was identical to the probabilistic serial mechanism. This corresponds to the particular case of deferred acceptance with single tie-breaking where students do not have priorities to schools. We extend existing results to the case where schools do give priorities to some students, as in many existing school choice systems. As corollaries, it follows that, in large markets, the deferred acceptance mechanism displays small aggregate randomness, and may produce Pareto inefficient outcomes with high probability. In addition, the characterization gives a unified formula that describes the asymptotics of the important mechanisms of deferred acceptance with single tie-breaking, random serial dictatorship, probabilistic serial, and others that are covered by our model.

Another application is a price-theoretic analysis of the effects of school competition, a classic question in public economics. We consider how competition among schools induced by flexible school choice, as practiced in cities such as New York, gives incentives for schools to invest in quality. This problem has been previously studied in the discrete Gale and Shapley framework by Hatfield et al. (2011a), and in a simplified model by Hoxby (1999). We consider a setting where schools compete for students, and determine how much schools benefit from investing in quality, in terms of attracting a stronger entering class. The continuum model clarifies how incentives depend on the types of students catered to, the distribution of preferences, and on market structure. We show that schools have muted, or possibly even negative incentives to perform quality improvements that target lower-ranked students. Moreover, such concerns are exacerbated when schools have market power. Therefore, while school choice might give schools incentives to improve, our results raise the concern that such improvements will disproportionately benefit top students. We view this analysis as complementary to the analyses with discrete models, where it is not possible to decompose comparative statics in terms of the distribution of characteristics of agents in a market.

As this example indicates, our model can be applied to derive Nash equilibria of games in matching markets where firms behave strategically. Crucially for these applications, the comparative statics in the continuum model deliver *magnitudes* of how a firm's

[11] This sharp characterization given by Che and Kojima (2010) and our paper stands in contrast to other contributions on large matching markets, which focused on incentives (Lee, 2011; Kojima and Pathak, 2009; Immorlica and Mahdian, 2005), or existence of a stable matching (Ashlagi et al., 2011).

actions impact market equilibrium, which yield simple first order conditions for optimal play, under any particular assumptions on the nature of competition.[12]

We derive additional robustness results and extensions that are particularly relevant for applications. First, to apply the continuum model to real-life markets, it is important to determine how good an approximation it affords. We give bounds on how close the set of stable matchings of a large economy are to the stable matching in the continuum model. We consider the case where student types in a discrete economy are independently identically distributed according to a given measure. We show that the probability that cutoffs that clear the market deviate by more than a given constant from their limit value decreases exponentially with the size of the economy. Moreover, we bound the probability that the number of students who receive different matches than what is expected in the limit is greater than a constant, also by an exponentially decreasing quantity.

Likewise, it is important to guarantee that comparative statics results derived in the continuum model can be applied to discrete economies. We give a result showing that comparative statics in the continuum model extend to large discrete economies. Using cutoffs, we define orderings over stable matchings. We show that if the unique stable matchings of two continuum economies are ordered in a particular way, then the sets of stable matchings of two approximating large discrete economies are strongly set ordered in the same direction. Therefore, our model can be used to derive comparative statics in discrete economies. While directly proving such results often involves complex arguments,[6] our model allows an analyst to simply apply the implicit function theorem to the continuum model, and the results are guaranteed to follow for large discrete economies.

Finally, in several markets wages or prices are personalized. The supplementary appendix extends the model to a setting where parties (e.g., workers and firms) are free to negotiate contract terms, possibly including wages, in the spirit of Hatfield and Milgrom (2005) and Kelso and Crawford (1982). The set of possible contracts may have restrictions, so that the model encompasses the case of no transfers, as in the Gale and Shapley (1962) model, and the case of transferable utility, as in most of Becker's (1973) analysis. We define stability in terms of firm preferences over individual contracts, which is justified when preferences are responsive. We find that the simple characterization of stable matchings in terms of a small set of cutoffs still holds. Moreover, under fairly general conditions analogous to the case without transfers, the model yields a unique allocation of firms to workers in every stable matching. However, the division of surplus between firms and workers is no longer uniquely determined.

[12]See Azevedo (2011) for an illustration, in a model of imperfect competition in a matching market.

The paper is organized as follows. Section 1.2 clarifies the connections of our paper with previous work. Section 2 introduces our model (2.1), gives the new characterization of stable matchings in terms of supply and demand (2.2), and illustrates our main results through a simple example (2.3). Section 2 and the application in Section 5.1 are self-contained and sufficient for readers interested in applying our model.

Section 3 reviews the discrete Gale and Shapley model, and defines the notions of convergence we use. Section 4 proves our main theoretical results, giving conditions for the continuum economy to have a unique stable matching, and for convergence of the discrete model to our model. Section 5 discusses applications and extensions. Section 5.1 applies the model to understand how school competition gives schools incentives to improve quality, and what types of students are targeted by these improvements. Section 5.2 discusses bounds on our convergence results, 5.3 shows that comparative statics in the continuum model imply comparative statics in the discrete model, 5.4 derives asymptotic characterizations of assignment and school choice mechanisms, 5.5 considers economies with multiple stable matchings. Omitted proofs are in the Appendix.

1.2. **Related Literature.** Our paper is related to several active lines of research. First, although we are interested in matching markets more generally, our contribution is directly motivated by the literature on the design of large matching clearinghouses, such as the school matches in Boston and New York City (Abdulkadiroglu et al. 2005a,b; Sönmez and Unver 2009; Kesten 2010; Kesten and Ünver 2010). Such markets are both organized around clearinghouses, so that the resulting allocations are indeed stable, and have each school being matched to many students. One direct application of our model is as a framework to understand the properties of different mechanisms. In Section 5.4 we use our model to derive an asymptotic characterization of school choice mechanisms used in practice.

More broadly, in the mechanism design literature, there is a long tradition of studying the properties of large markets (Hurwicz 1972; Roberts and Postlewaite 1976; Jackson and Manelli 1997; Reny and Perry 2006; Pesendorfer and Swinkels 2000; Swinkels 2001; Bodoh-Creed 2010). In the market design literature, many recent papers have focused on large markets (Kojima and Manea 2009; Manea 2009; Budish and Cantillon 2012; Budish 2011; Che and Kojima 2010; Ünver 2010; Azevedo and Budish 2011; Liu and Pycia 2011).

More closely related to our model, are the contributions in market design that study large matching markets (Roth and Peranson 1999; Immorlica and Mahdian 2005; Kojima and Pathak 2009; Lee 2011). The focus of these papers is quite different from ours. Roth and Peranson (1999), using simulations and data from the National Resident Matching Program observed that, even though stable matching mechanisms are manipulable in theory, they seem to be very close to strategy-proof in large markets.

This spurred several subsequent papers that theoretically evaluate this proposition. Our work differs from this literature in two key aspects. The first is that previous work has focused on showing approximate incentive compatibility of stable matching mechanisms.[13] In contrast, we characterize the limit of the set of stable matchings in large matching markets, which is important for applications both in market design and in the analysis of matching markets, such as in Sections 5.1 and 5.4. Second, the type of limit we take is different. While papers in this literature consider the limit where both sides of the market grow, we consider the case where there is a fixed finite number of colleges, and the number of students grows. The two different limits are suited for analyzing different situations. The former models a situation where each college or firm is a small part of a thick market, and its choices have a vanishing effect on other firms. In contrast, our model is better suited for situations where firms are differentiated, and may have some market power. In such settings, firms do have incentives to misreport preferences to a mechanism, for example by reducing quantity as in a standard oligopoly model (Azevedo, 2011).

One paper that does give a sharp asymptotic characterization of the outcome of a mechanism is Che and Kojima (2010). Their main result is that the random serial dictatorship mechanism is asymptotically equivalent to Bogomolnaia and Moulin's (2001) probabilistic serial mechanism. As we discuss below, this and other more general characterizations can be obtained from our results. The derivation of such results is considerably simpler within our framework, as instead of employing combinatorial arguments like the extant literature, we apply differential topology techniques, an approach introduced by Debreu (1970) in general equilibrium theory.

Another central contribution of our paper is the characterization of stable matchings in terms of cutoffs clearing supply and demand. We highlight several related results in the literature. An early result by Roth and Sotomayor (1989) shows that different entering classes in a college at different stable matchings are ordered in the sense that, save for students which are in both entering classes, all students of an entering class are better than those in the other entering class. This suggests that parametrizing the set of

[13]These papers are typically concerned with showing that in large matching markets the set of stable matchings is small, what is referred to as core convergence results. Likewise, establishing that supply and demand uniquely clear the market in the continuum model is one of our main results, due to its importance for applications, as it adds tractability to the continuum model, and because it is an important feature of matching markets. As a consequence of our the uniqueness result, we prove a core convergence result showing that the set of stable matchings does become small in large economies. From a technical perspective this result is interesting because, while our setting is different, we prove this result without the assumptions of short preference lists, or a specific probability generating process for preferences, which is necessary for the combinatorial arguments commonly used in this literature. Another related strand of the literature imposes conditions on preferences to guarantee uniqueness of a stable matching in a given finite economy (Eeckhout 2000; Clark 2006; Niederle and Yariv 2009). Recent work has sought to bound the number of stable matchings given restrictions on preferences (Samet 2010).

stable matchings by the lowest ranked student is possible, though their result does not describe such a parametrization, and the proof is independent from ours. Balinski and Sönmez (1999) give a characterization of fair allocations[14] based on threshold scores. Sönmez and Ünver (2010) propose a mechanism for solving the course allocation problem, where students place bids for courses and report preferences. Their Proposition 1 shows that running a deferred acceptance algorithm while using bids as preferences for the courses leads to thresholds such that students are matched to their preferred affordable bundle of courses. Biró (2007) studies the algorithm used to compute the outcome of college admissions in Hungary. The algorithm, while very similar to the Gale and Shapley algorithm, uses scores at each stage. Biró (2007) states without proof that a definition of stability in terms of cutoffs is equivalent to the standard definition. Abdulkadiroglu et al. (2012) consider a particular continuum economy, motivated by a school choice problem. They define the continuum deferred acceptance algorithm, and show that it converges to an allocation with a cutoff structure. Moreover, in the course of the proof of their Lemma 4, they show that in their model there is a unique vector of cutoffs that equate demand and supply. Their ingenious proof uses different ideas than our paper, and could prove useful in other settings.

An influential paper by Adachi (2000) gives a characterization of stable matchings in terms of fixed points of an operator over pre-matchings.[15] This insight has been widely applied in the matching with contracts literature.[16] This characterization is very different from the one in terms of cutoffs, as pre-matchings are considerably more complex than cutoffs. In fact, a pre-matching specifies a college for each student, and a set of students for each college, so that the very dimensionality of the set of pre-matchings is much higher than the set of cutoffs. As such, this characterization is more useful for deriving theoretical results in generalized matching models (as in Echenique and Oviedo, 2006), than the simple comparative statics that follow from the cutoff approach. In the supplementary appendix we discuss in detail the connection between the two models. We show that it is possible to write intermediate steps of the fixed point operator proposed by Adachi (2000), and use this fact prove a limited set of results about pre-matchings using cutoffs, although the results from the two approaches are largely independent.

Our characterization Lemma 2 is analogous to the Fundamental Theorems of Welfare Economics. Segal (2007) shows that these theorems may be stated for a wide class of

[14] Formally, fairness is a concept that applied to the "student placement problem" as defined by Balinski and Sönmez (1999). A matching is individually rational, fair, and non-wasteful if and only if it is stable for the "college admissions problem" that is associated to the original problem, as defined by Balinski and Sönmez (1999).

[15] These ideas have been extended to many-to-one and many-to-many matching markets in important papers by Echenique and Oviedo (2004, 2006).

[16] See for example Ostrovsky (2008); Hatfield and Milgrom (2005); Echenique (2012).

social choice problems: namely, socially optimal outcomes can be decentralized with a notion of price equilibrium that is appropriate for the problem. Furthermore, he characterizes the prices that verify a problem's solutions with minimal communication (Segal's Theorems 2, 3). Applied to stable many-to-one matching, his characterization yields pre-matchings as the appropriate prices (Segal's Proposition 5). In our model, where colleges' preferences are defined by students' scores, the minimally informative prices in Segal's Theorem 3 coincide with our notion of market clearing cutoffs.

Chade et al. (2011), Avery and Levin (2010) and Hickman (2010) consider non-cooperative models of college admissions, where colleges use admission thresholds. The Chade et al. (2011) model has a continuum of students and two colleges. They require identical preferences for students and colleges, but allow for costly applications and incomplete information, which we do not. Interestingly, although they use a non-cooperative model, they find that colleges optimally admit students with quality signals above a threshold that equates demand and supply.

Finally, some recent papers explore connections between the matching theoretic concept of stability and Walrasian Equilibrium (Hatfield and Milgrom 2005; Hatfield et al. 2011b), and their equivalence in certain settings.[17] While we do make such a connection, the scope and spirit of our result is quite different. Our results imply that stable matchings may be found by solving for selectivity cutoffs P that equate demand and supply for each college. This means that stable matchings can be found by solving market clearing equations for a given excess demand curve $D(P) - S$. This is equivalent to solving for equilibrium prices in a general equilibrium economy. This is a useful connection, and cutoffs do share some properties with prices. However, we highlight that cutoffs are *not* prices, and there is no equivalence between stability and Walrasian equilibrium in our model.

2. The Continuum Model

2.1. **Definitions.** We begin the exposition with the simpler, and novel continuum model, and examine its connection with the standard discrete Gale and Shapley (1962) model in Sections 3 and 4. The model follows closely the Gale and Shapley (1962) college admissions model. The main departure is that a finite set of colleges $C = \{1, 2, \ldots, C\}$ are matched to a continuum mass of students. A student is described by her **type** $\theta = (\succ^\theta, e^\theta)$. $\succ^\theta$ is the student's strict **preference** ordering over colleges. The vector $e^\theta \in [0,1]^C$ describes the colleges' ordinal preferences for the student. We refer to e^θ_c as student θ's **score** at college c. Colleges prefer students with higher scores. That is,

[17] Specifically, Hatfield et al. (2011b) show that, in a trading network with quasilinear utilities, free transfers of a numeraire between agents, and substitutable preferences the set of stable outcomes is essentially equivalent to the set of Walrasian equilibria.

college c prefers student θ over θ' if $e_c^\theta > e_c^{\theta'}$.[18] To simplify notation we assume that all students and colleges are acceptable.[19] Let $\mathcal{R}$ be the set of all strict preference orderings over colleges. We denote the set of all student types by $\Theta = \mathcal{R} \times [0,1]^C$.

A **continuum economy** is given by $E = [\eta, S]$, where η is a probability measure[20] over Θ and $S = (S_1, S_2, \ldots, S_C)$ is a vector of strictly positive **capacities** for each college. We make the following assumption on η, which corresponds to colleges having strict preferences over students in the discrete model.

Assumption 1. ***(Strict Preferences)*** *Every college's indifference curves have η-measure 0. That is, for any college c and real number x we have $\eta(\{\theta : e_c^\theta = x\}) = 0$. The set of all economies satisfying this assumption is denoted by $\mathcal{E}$.*

A **matching** μ describes an allocation of students to colleges. Formally, a matching in a continuum economy $E = [\eta, S]$ is a function $\mu : C \cup \Theta \to 2^\Theta \cup (C \cup \Theta)$, such that

(1) For all $\theta \in \Theta$: $\mu(\theta) \in C \cup \{\theta\}$.
(2) For all $c \in C$: $\mu(c) \subseteq \Theta$ is measurable, and $\eta(\mu(c)) \le S_c$.
(3) $c = \mu(\theta)$ iff $\theta \in \mu(c)$.
(4) (Right continuity) For any sequence of student types $\theta^k = (\succ, e^k)$ and $\theta = (\succ, e)$, with e^k converging to e, and $e_c^k \ge e_c^{k+1} \ge e_c$ for all k, c, we can find some large K so that $\mu(\theta^k) = \mu(\theta)$ for $k > K$.

The definition of a matching is analogous to that in the discrete model. Conditions 1-3 mirror those in the discrete model. (1) states that each student is matched to a college or to herself, which represents being unmatched. (2) that colleges are matched to sets of students with measure not exceeding its capacity. (3) is a consistency condition, requiring that a college is matched to a student iff the student is matched to the college.

The technical Condition (4) is novel. It states that given a sequence of students $\theta^k = (e^k, \succ)$, which are decreasingly desirable, with scores $e^k \to e$, then for large enough k all students $(\succ, e^k)$ in the sequence receive the same allocation, and the limit

[18]We take college's preferences over students as primitives, rather than preferences over sets of students. It would have been equivalent to start with preferences over sets of students that were responsive to the preferences over students, as in Roth (1985).

[19]This assumption is without loss of generality. If some students find some colleges unacceptable we can generate an equivalent economy where all colleges are acceptable. Add a fictitious "unmatched" college with a large capacity and set student preferences to rank it as they would rank being unmatched. Set student preferences to rank all unacceptable colleges as acceptable, but ranked below the fictitious college. Since the fictitious college never reaches its capacity, any student that is matched to an unacceptable college can form a blocking pair with the fictitious college. Therefore stable matching of the resulting economy are equivalent to stable matching of the original one. Likewise, we can add a fictitious mass of students that would be ranked bellow all acceptable students and ranked above the unacceptable students.

[20]We must also specify a σ-algebra where η is defined. The set Θ is the product of $[0,1]^C$ and the finite set of all possible orderings over C. We take the Borel σ-algebra of the product topology (the standard topology for $\mathbb{R}^C$ times the set of all subsets topology for the finite set of preference orderings).

student $(e, \succ)$ receives this allocation too. The reason why we impose this condition is that in the continuum model it is always possible to add an extra measure 0 set of students to a college without having it exceed its capacity. This would generate multiplicities of stable matchings that differ only in sets of measure 0. Condition 4 rules out such multiplicities. The intuition is that right continuity implies that a stable matching always allows an extra measure 0 set of students into a college when this can be done without compromising stability. Other than eliminating such multiplicities up to a measure 0 set of students, the condition does not affect the set of stable matchings.

A student-college pair (θ, c) **blocks** a matching μ at economy E if the student θ prefers c to her match and either (i) college c does not fill its quota or (ii) college c is matched to another student that has a strictly lower score than θ. Formally, (θ, c) blocks μ if $c \succ^{\theta} \mu(\theta)$ and either (i) $\eta(\mu(c)) < S_c$ or (ii) there exists $\theta' \in \mu(c)$ with $e_c^{\theta'} < e_c^{\theta}$.

Definition 1. A matching μ for a continuum economy E is **stable** if it is not blocked by any student-college pair.

A stable matching always exists. The proof is similar to Gale and Shapley's (1962) classic existence proof in the discrete case, and works by showing that a deferred acceptance procedure converges to a stable matching. The result is formally stated and proved in Appendix A.

We refer to the stable matching correspondence as the correspondence associating each economy in $\mathcal{E}$ with its set of stable matchings. In some sections in the paper the economy is held fixed. Whenever there is no risk of confusion we will omit dependence of certain quantities on the economy.

2.2. **The Supply and Demand Characterization of Stable Matchings.** In the theory of competitive equilibrium, prices play a key role. In this section we show that selectivity thresholds at each college, which we term cutoffs, play a a similar role in matching markets, with respect to two key dimensions. One important property of prices is decentralizing the equilibrium allocation. If agents' optimal choices are unique, prices determine the allocation, with each agent simply choosing her favorite bundle that is affordable. Another property is that the task of finding an equilibrium is reduced to finding a vector of prices that clears demand and supply. As the dimensionality of this vector is often small compared to the number of agents in the economy, prices play an important computational role. It is important to highlight that cutoffs are very different than prices, and prices have many properties that cutoffs do not share. For example, in general equilibrium the price ratio of two goods gives the marginal rate of substitution of a consumer who buys a positive amount of each. No such relationship for cutoffs holds in our setting, where the allocation depends exclusively on ordinal preferences. Yet, cutoffs and prices do share the two properties outlined above.

Throughout this subsection, we fix an economy E, and abuse notation by omitting dependence on E when there is no risk of confusion. A **cutoff** is a minimal score $P_c \in [0,1]$ required for admission at a college c. We say that a student θ can **afford** college c if $P_c \leq e^\theta_c$, that is c would accept θ. A student's **demand** given a vector of cutoffs is her favorite college among those she can afford. That is,

$$D^\theta(P) = \arg\max_{\succ^\theta}\{c | P_c \leq e^\theta_c\} \cup \{\theta\}. \tag{2.1}$$

Aggregate demand for college c is the mass of students that demand it,

$$D_c(P) = \eta(\{\theta : D^\theta(P) = c\}).$$

The aggregate demand vector $\{D_c(P)\}_{c\in C}$ is denoted by $D(P)$.

A market clearing cutoff is a vector of cutoffs that clears supply and demand for colleges.

Definition 2. A vector of cutoffs P is a **market clearing cutoff** if it satisfies the following market clearing equations.

$$D_c(P) \leq S_c$$

for all c, and $D_c(P) = S_c$ if $P_c > 0$.

There is a natural one-to-one correspondence between stable matchings and market clearing cutoffs. To describe this correspondence, we define two operators. Given a market clearing cutoff P, we define the associated matching $\mu = \mathcal{M}P$ using the demand function:

$$\mu(\theta) = D^\theta(P).$$

Conversely, given a stable matching μ, we define the associated cutoff $P = \mathcal{P}\mu$ by the score of a marginal students matched to each college:

$$P_c = \inf_{\theta\in\mu(c)} e^\theta_c. \tag{2.2}$$

The operators $\mathcal{M}$ and $\mathcal{P}$ form a bijection between stable matchings and market clearing cutoffs.

Lemma 1. ***(Supply and Demand Lemma)*** *If μ is stable matching, then $\mathcal{P}\mu$ is a market clearing cutoff. If P is a market clearing cutoff, then $\mathcal{M}P$ is a stable matching. In addition, the operators $\mathcal{P}$ and $\mathcal{M}$ are inverses of each other.*

The lemma subsumes two useful facts. First, stable matchings all have a very special structure. Given any stable matching μ, there must exist a corresponding vector of cutoffs such that each student is matched to $\mu(\theta) = D^\theta(P)$. Therefore, any stable matching corresponds to each student choosing her favorite college conditional on being accepted at a vector of cutoffs P. Therefore, if one is interested in stable matchings, it is not necessary to consider all possible matchings, but only those that have this very special structure. This is similar to the decentralization property that prices have in competitive equilibrium, where each agent's allocation is determined by her preferences, endowment, and market prices.[21]

Second, the lemma implies that computing stable matchings is equivalent to finding market clearing cutoffs, as $\mathcal{M}$ and $\mathcal{P}$ are a one-to-one correspondence between these two sets. Therefore, stable matchings can be found by solving market clearing equations, balancing demand $D(P)$ and supply S. In particular, finding stable matchings is equivalent to finding competitive equilibria of an economy with quasilinear preferences and aggregate demand function $D(P)$.

2.3. **Example: The Supply and Demand Characterization and Convergence of Discrete Economies to Continuum Economies.** This simple example illustrates the supply and demand characterization of stable matchings and previews our convergence results. There are two colleges $c = 1, 2$, and the distribution of students η is uniform. That is, there is a mass $1/2$ of students with each preference list $1, 2$ or $2, 1$, and each mass has scores distributed uniformly over $[0, 1]^2$ (Figure 1). If both colleges had capacity $1/2$, the unique stable matching would have each student matched to her favorite college. To make the example interesting, assume $S_1 = 1/4$, $S_2 = 1/2$. That is, college 2 has enough seats for all students who prefer college 2, but college 1 only has capacity for half of the students who prefer it.

A familiar way of finding stable matchings is using the student-proposing deferred acceptance algorithm. In Appendix A we formally define the algorithm, and prove that it converges to a stable matching. Here, we informally follow the algorithm for this example, to build intuition on the special structure of stable matchings. At each step of the algorithm unassigned students propose to their favorite college out of the ones that have not rejected them yet. If a college has more students than its capacity assigned to it, it rejects the lower ranked students currently assigned, to stay within its capacity. Figure 1 displays the trace of the algorithm in our example. In the first step, all students apply to their favorite college. Because college 1 only has capacity $1/4$, and each square has mass $1/2$, it then rejects half of the students who applied. The rejected students then apply to their second choice, college 2. But this leaves college

[21] P decentralizes the allocation in the sense that, as in competitive equilibrium, the allocation received by $\theta \in \Theta$ is determined solely by $(\succ^\theta, e^\theta)$ and P. Thus, P summarizes the effect of aggregate market conditions on θ's allocation. Note, however, that e^θ represents colleges' preferences. Therefore, if colleges have private information over their preferences, it is not true that P decentralizes the allocation in the sense that a student's allocation only depends on her private information and P. See Segal (2007) for a discussion of related characterizations for general social choice functions.

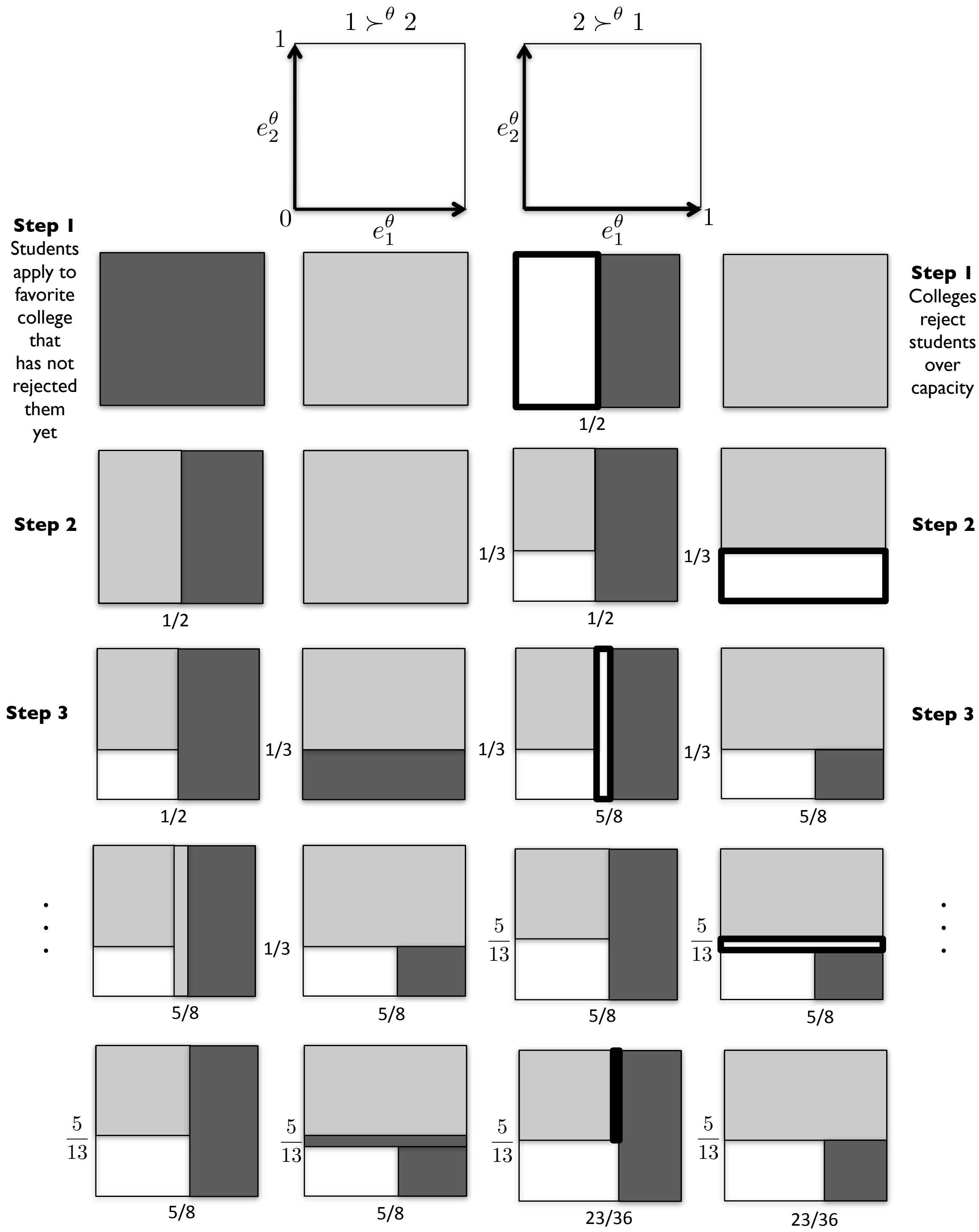

FIGURE 1. The set of student types Θ is represented by the two squares on the top panel. The left square represents students that prefer college 1, and the right square students who prefer college 2. Scores at each college are represented by the (x, y) coordinates. The lower panels show the first five steps of the Gale-Shapley student-proposing algorithm. In each line, students apply to their favorite colleges that have not yet rejected them in the left panel, and colleges reject students to remain within capacity in the right panel. Students in dark gray are tentatively assigned to college 1, and in light gray tentatively assigned to college 2.

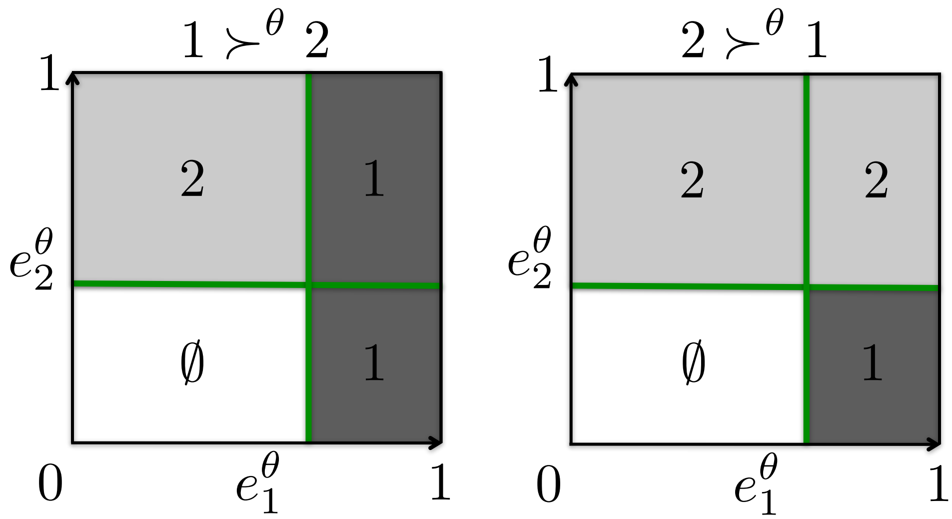

FIGURE 2. A stable matching in a continuum economy with two colleges. The two squares represent the set of student types Θ. The left square represents students that prefer college 1, and the right square students who prefer college 2. Scores at each college are represented by the (x, y) coordinates. The white area represents unmatched students, dark gray are matched to college 1, and light gray to college 2.

2 with $1/2 + 1/4 = 3/4$ students assigned to it, which is more than its quota. College 2 then rejects its lower ranked students. Those who had already been rejected stay unmatched. But those who hadn't been rejected by college 1 apply to it, leaving it with more students than capacity, and the process continues. Although the algorithm does not reach a stable matching in a finite number of steps, it always converges, and its pointwise limit (shown in Figure 2) is a stable matching (this is proven in Appendix A). Figure 1 hints at this, as the measure of students rejected in each round is becoming smaller and smaller.

However, Figures 1 and 2 give much more information than simply convergence of the deferred acceptance mechanism. We can see that cutoffs yield a simpler decentralized way to describe the matching. Note that all students accepted to college 1 have a score e_1^θ above a cutoff of $P_1 \approx .640$, and those accepted to college 2 have a score e_2^θ above some cutoff $P_2 \approx .390$. Hence, had we known these numbers in advance, it would have been unnecessary to run the deferred acceptance algorithm. All we would have to do is assign each student to her favorite college such that her score is above the cutoff, $e_c^\theta \geq P_c$ (supply and demand Lemma 1).

Additionally, the supply and demand lemma gives another way of finding stable matchings. Instead of following the deferred acceptance algorithm, one may simply look for cutoffs that equate supply and demand $D(P) = S$. Consider first demand for college 1. The fraction of students in the left square of Figure 2 demanding college 1 is

$1-P_1$. And in the right square it is $P_2(1-P_1)$. Therefore $D_1(P) = (1+P_2)(1-P_1)/2$. D_2 has an analogous formula, and the market clearing equations are

$$\begin{aligned} S_1 &= 1/4 = && (1+P_2)(1-P_1)/2 \\ S_2 &= 1/2 = && (1+P_1)(1-P_2)/2. \end{aligned}$$

Solving this system, we get $P_1 = (\sqrt{17}+1)/8$ and $P_2 = (\sqrt{17}-1)/8$. In particular, because the market clearing equations have a unique solution, the economy has a unique stable matching (Theorem 1 shows this is a much more general phenomenon).

We show below that the cutoff characterization is also valid in the discrete Gale and Shapley (1962) model, save for the fact that in discrete model each stable matching may correspond to more than one market clearing cutoff (Lemma 2). Figure 3 illustrates cutoffs for a stable matching in a discrete economy with $1,000$ students. The 1,000 students were assigned random types, drawn from the distribution η used in the continuum example. For that reason, the empirical distribution of types of the discrete economy approximates that of the continuum economy. In this sense, this discrete economy approximates the continuum economy. Note that the cutoffs in the discrete economy are approximately the same as the cutoffs in the continuum economy. Theorem 2 shows that, generically, the market clearing cutoffs of approximating discrete economies approach market clearing cutoffs of the continuum economy.

3. The Discrete Model and Convergence Notions

This section reviews the discrete Gale and Shapley (1962) college admissions model, states the discrete supply and demand lemma, and defines convergence notions used to state our main results.

3.1. **The Discrete Gale and Shapley Model.**

The set of colleges is again C. A finite economy $F = [\tilde{\Theta}, \tilde{S}]$ specifies a finite set of students $\tilde{\Theta} \subset \Theta$, and an integer vector of capacities $\tilde{S}_c > 0$ for each college. We assume that no college is indifferent between two students in $\tilde{\Theta}$. A **matching** for finite economy F is a function $\tilde{\mu} : C \cup \tilde{\Theta} \to 2^{\tilde{\Theta}} \cup (C \cup \tilde{\Theta})$ such that

(1) For all θ in $\tilde{\Theta}$: $\mu(\theta) \in C \cup \{\theta\}$.
(2) For all $c \in C$: $\mu(c) \in 2^{\tilde{\Theta}}$ and $\#\mu(c) \leq \tilde{S}_c$.
(3) For all $\theta \in \tilde{\Theta}, c \in C$: $\mu(\theta) = c$ iff $\theta \in \mu(c)$.

These conditions may be interpreted as follows. (1) Each student is matched to a college or to herself, (2) each college is matched to at most $\tilde{S}_c$ students, and (3) the consistency condition that a college is matched to a student iff the student is matched to the college.

The definition of a **blocking pair** is the same as in Section 2.1. A matching $\tilde{\mu}$ is said to be **stable** for finite economy F if it has no blocking pairs.

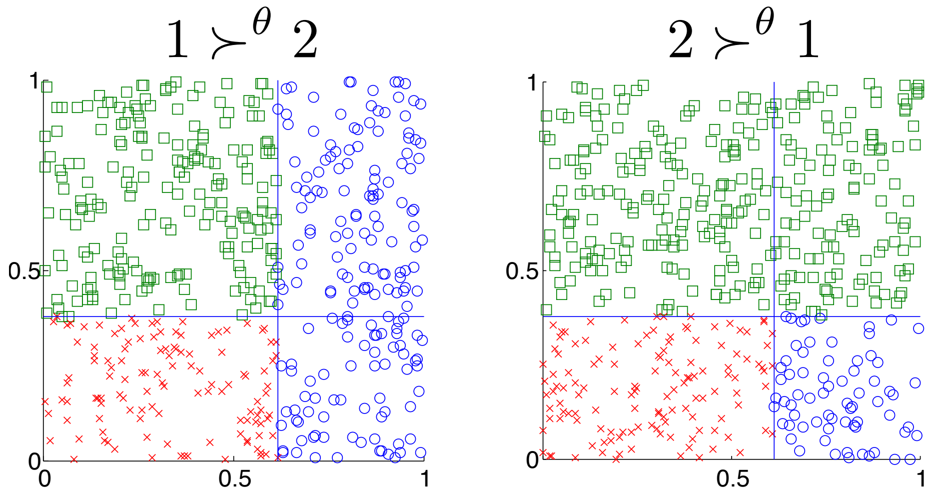

FIGURE 3. Cutoffs of a stable matching in a discrete economy approximating the continuum economy in the example. The two squares represent the set of student types Θ. The left square represents students prefer college 1, and the right square students who prefer college 2. Scores at each college are represented by the (x, y) coordinates. There are 2 colleges, with capacities $q_1 = 250$, $q_2 = 500$. 500 students have preferences $\succ^{\theta}= 1, 2, \emptyset$ and 500 students have preferences $2, 1, \emptyset$. Scores e^{θ} were drawn independently according to the uniform distribution in $[0, 1]^2$. The Figure depicts the student-optimal stable matching. Balls represent students matched to college 1, squares to college 2, and Xs represent unmatched students.

Given a finite economy $F = [\tilde{\Theta}, \tilde{S}]$, we may associate with it the empirical distribution of types

$$\eta = \sum_{\theta \in \tilde{\Theta}} \frac{1}{\#\tilde{\Theta}} \delta_{\theta},$$

where δ_{θ} denotes is the probability distribution placing probability 1 on the point θ. The supply of seats per student is given by $S = \tilde{S}/\#\tilde{\Theta}$. Note that $[\eta, S]$ uniquely determine a discrete economy $F = [\tilde{\Theta}, \tilde{S}]$, as $\tilde{\Theta} = \text{support}(\eta)$ and $\tilde{S} = S \cdot \#\tilde{\Theta}$. Therefore, either pair $[\tilde{\Theta}, \tilde{S}]$ or $[\eta, S]$ uniquely determine a finite economy F. Throughout the remainder of the text we will abuse notation and refer to finite economies simply as

$$F = [\eta, S].$$

This convention will be useful for stating our convergence results below, as it makes finite economies F comparable to continuum economies E.

3.1.1. *Cutoffs.* In this section, we fix a finite economy $F = [\eta, S]$, and will omit dependence on F in the notation. A cutoff is a number P_c in $[0, 1]$ specifying an admission threshold for college c. Given a vector of cutoffs P, a student's individual demand

$D^\theta(P)$, the aggregate demand function $D(P)$, and market clearing cutoffs are defined as in Section 2.2.

In the discrete model, we define the operators $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{P}}$, which have essentially the same definitions as $\mathcal{M}$ and $\mathcal{P}$. Given market clearing cutoffs P, $\tilde{\mu} = \tilde{\mathcal{M}}P$ is the matching such that for all $\theta \in \tilde{\Theta}$: $\tilde{\mu}(\theta) = D^\theta(P)$. Given a stable matching $\tilde{\mu}$, $P = \tilde{\mathcal{P}}\tilde{\mu}$ is given by $P_c = 0$ if $\eta(\tilde{\mu}(c)) < S_c$ and $P_c = \min_{\theta \in \tilde{\mu}(c)} e_c^\theta$ otherwise.

In the discrete case, we have the following analogue of the supply and demand lemma.

Lemma 2. ***(Discrete Supply and Demand Lemma)*** *In a discrete economy, the operators $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{P}}$ take stable matchings into market clearing cutoffs, and vice versa. Moreover, $\tilde{\mathcal{M}}\tilde{\mathcal{P}}$ is the identity.*

Proof. Consider a stable matching $\tilde{\mu}$, and let $P = \tilde{\mathcal{P}}\tilde{\mu}$. Any student θ who is matched to a college $c = \tilde{\mu}(\theta)$ can afford her match, as $P_c \leq e_c^\theta$ by the definition of $\tilde{\mathcal{P}}$. Likewise, students who are unmatched may always afford being unmatched. Note that no student can afford a college $c' \succ^\theta \tilde{\mu}(\theta)$: if she could, then $P_{c'} \leq e_{c'}^\theta$, and by the definition of $\tilde{\mathcal{P}}$, there would be another student θ' matched to c' with $e_{c'}^{\theta'} < e_{c'}^\theta$, or empty seats at c', which would contradict $\tilde{\mu}$ being stable. Consequently, no student can afford any option better than $\tilde{\mu}(\theta)$, and all students can afford their own match $\tilde{\mu}(\theta)$. This implies $D^\theta(P) = \tilde{\mu}(\theta)$. This proves both that $\tilde{\mathcal{M}}\tilde{\mathcal{P}}$ is the identity, and that P is a market clearing cutoff.

In the other direction, let P be a market clearing cutoff, and $\tilde{\mu} = \tilde{\mathcal{M}}P$. By the definition of the operator $\tilde{\mathcal{M}}$ and the market clearing conditions, $\tilde{\mu}$ is a matching, so we only have to show there are no blocking pairs. Assume by contradiction that (θ, c) is a blocking pair. If c has empty slots, then $P_c = 0 \leq e_c^\theta$. If c is matched to a student θ' that is less preferred than θ, then $P_c \leq e_c^{\theta'} \leq e_c^\theta$. Hence, we must have $P_c \leq e_c^\theta$. However, this implies that $c \preceq^\theta D^\theta(P) = \tilde{\mu}(\theta)$, so (θ, c) cannot be a blocking pair, reaching a contradiction. □

The lemma guarantees that stable matchings always have a cutoff structure $\tilde{\mu} = \tilde{\mathcal{M}}P$ even in a discrete market. Therefore, it is still true that in the discrete model cutoffs decentralize the allocation, and that for finding all stable matchings one only has to consider market clearing cutoffs. The only difference between the lemmas for the discrete and continuous case, is that in the continuum model the correspondence between market clearing cutoffs and stable matchings is one-to-one. In the discrete model, in contrast, each stable matching may be associated with several market clearing cutoffs. The reason is that changing a particular P_c in a range where there are no students with scores e_c^θ does not affect the demand function.

3.2. **Convergence Notions.** To describe our convergence results, we must define notions of convergence for economies and stable matchings. We will say that a sequence of continuum economies $\{E^k\}_{k\in\mathbb{N}}$, $E^k = [\eta^k, S^k]$ converges to a limit economy $E = [\eta, S]$ if the measures η^k converge in the weak sense to η,[22] and if the vectors S^k converge to S.

Throughout the paper, we will use the sup norm whenever considering vectors in Euclidean space, and denote the sup norm by $\|\cdot\|$. The sup norm will be used to evaluate the distance between vectors of cutoffs, supply vectors, and demand vectors. We take the distance between stable matchings to be the distance between their associated cutoffs in the supremum norm in $\mathbb{R}^C$. That is, the distance between two stable matchings μ and μ' is

$$d(\mu, \mu') = \|\mathcal{P}\mu - \mathcal{P}\mu'\|.$$

A sequence of finite economies $\{F^k\}_{k\in\mathbb{N}}$, $F^k = [\eta^k, S^k]$ converges to a continuum economy $E = [\eta, S]$ if the empirical distribution of types η^k converges to η in the weak sense, and the vectors of capacity per student S^k converge to S. Given a stable matching of a continuum economy μ, and a stable matching of a finite economy $\tilde{\mu}$, we define

$$d(\tilde{\mu}, \mu) = \sup_P \|P - \mathcal{P}\mu\|$$

over all vectors P with $\tilde{\mathcal{M}}P = \tilde{\mu}$. The sequence of stable matchings $\{\tilde{\mu}^k\}_{k\in\mathbb{N}}$ with respect to finite economies F converges to stable matching μ of continuum economy E if $d(\tilde{\mu}^k, \mu)$ converges to 0.

Finally, we will show that the set of stable matchings of large finite economies becomes small under certain conditions. To state this, we define the diameter of the set of stable matchings of a finite economy F as $\sup\{\|P - P'\| : P$ and P' are market clearing cutoffs of $F\}$.

4. Main Results: Convergence and Uniqueness

We are now ready to state the main results of the paper. The first result shows that, typically, continuum economies have a unique stable matching. Note that, in this section, since we consider sequences of economies, we will explicitly denote the dependence of demand functions on measures as $D(\cdot|\eta)$, and on economies as $D(\cdot|E)$ or $D(\cdot|F)$. We begin by defining a notion of smoothness of measures.

Definition 3. The distrubution of student types η is *regular* if the image under $D(\cdot|\eta)$ of the closure of the set

$$\{P \in [0,1]^C : D(\cdot|\eta) \text{ is not continuosly differentiable at } P\}$$

has Lebesgue measure 0.

[22] Weak convergence of measures is defined as the integrals $\int f d\eta^k$ converging to $\int f d\eta$ for every bounded continuous function $f : \Theta \to \mathbb{R}$. In analysis, this is usually termed weak-* convergence.

This definition is very general, and includes cases where η does not have a density, and where it has points where demand is not differentiable. While the condition is somewhat technical, it is always satisfied for example if $D(\cdot|\eta)$ is continuously differentiable, or if η admits a continuous density.

The next result gives conditions for the continuum model to have a unique stable matching.

Theorem 1. *Consider an economy $E = [\eta, S]$.*

(1) *If η has full support, then E has a unique stable matching.*

(2) *If η is any regular distribution, then for almost every vector of capacities S with $\sum_c S_c < 1$ the economy E has a unique stable matching.*

The theorem has two parts. First, it shows that, whenever η has full support, a limit economy has a unique stable matching. Therefore, whenever the set of students is rich enough, an economy has a unique stable matching.[23] Moreover, it shows that, even if the full support assumption does not hold, in a very general setting for almost every S there exists a unique stable matching. Therefore, the typical case is for the continuum model to have a unique stable matching, with supply and demand uniquely clearing the market.

Proof Sketch. Here we outline the main ideas in the proof, which is deferred to Appendix B. The core of the argument employs tools from differentiable topology, an approach pioneered in general equilibrium theory by the classic paper of Debreu (1970).[24] Moreover, the proof uses two results, developed in Appendix A, extending classic results of matching theory to the continuum model. The first is the lattice theorem, which guarantees that for any economy E the set of market clearing cutoffs is a complete lattice. In particular, this implies that there exist smallest and largest vectors of market clearing cutoffs. In the proof we will denote these cutoffs P^- and P^+. The other result is the rural hospitals theorem, which guarantees that the measure of unmatched students in any two stable matchings is the same.

Part (1).

[23]The assumption of full support may be weakened to, for all $P \leq P', P \neq P'$, the set

$$\{\theta \in \Theta : e^\theta \not< P \text{ and } e^\theta < P'\}$$

having positive η measure. For details see the working paper version of Azevedo (2011). This assumption is satisfied, for example, in the case of perfectly correlated college preferences. That is, when for all θ in the support of η and any $c, c' \in C$ we have $e^\theta_c = e^\theta_{c'}$.

[24]For an overview of the literature spurred by this seminal paper see Mas-Colell (1990). Our argument is quite different from that used by Gretsky et al. (1999), who establish that in the continuum assignment model, which has transferable utility, generic economies have a unique Walrasian equilibrium price vector. The argument in the latter paper is based on the social gains function, which is not defined in a setting with only ordinal preferences.

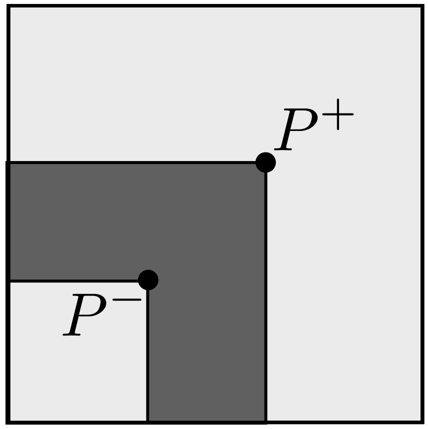

FIGURE 4. The Figure illustrates the Proof of Theorem 1. The shaded area corresponds scores in the set $\{e^\theta \in [0,1]^C : e^\theta < P^+, e^\theta \not< P^-\}$, which is used in the proof of Theorem 1 Part (1). Students who find all colleges acceptable and have scores in this set are matched under P^- but are unmatched under P^+.

We now consider the case where $P^+ > 0$, and defer the general case to the Appendix. Note that the set of unmatched students at P^+ contains the set of unmatched students at P^-, and their difference contains the set

$$\{\theta \in \Theta : e^\theta < P^+, e^\theta \not< P^-\}.$$

By the rural hospitals theorem, the mass of unmatched agents at P^+ and P^- must be the same, and therefore this set must have η measure 0. Since η has full support, this implies that $P^- = P^+$, and therefore there is a unique stable matching (Figure 4).

Part (2).

For simplicity, consider the case where the function $D(P|\eta)$ is continuously differentiable. The general case is covered in the Appendix.

The proof is based on Sard's Theorem, from differential topology.[25] The theorem states that, given a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$, we have that, for almost every $S_0 \in \mathbb{R}^n$, the derivative $\partial f(P_0)$ is nonsingular at every solution P_0 of $f(P_0) = S_0$. The intuition for this result is easy to see in one dimension. The theorem says that, if we randomly perturb the graph of a function with a small vertical translation, all roots have a non-zero derivative with probability 1.

The key step in the proof is the following claim. Consider S such that the economy $E = [\eta, S]$ has more than one stable matching. Then the derivative matrix $\partial D(P^*|\eta)$ is not invertible, for at least one market clearing cutoff P^* of E. To see why this is true, note that, by the lattice theorem, E has a smallest and a largest market clearing cutoff, $P^- \leq P^+$. For simplicity, we restrict attention to the case where $0 < P^- < P^+$, and

[25]See Guillemin and Pollack (1974); Milnor (1997).

defer the general case to the Appendix. For any P in the cube $[P^-, P^+]$, the measure of unmatched students

$$1 - \sum_c D_c(P|\eta) \tag{4.1}$$

must be higher than the measure of unmatched students at P^-, but lower than the measure at P^+. However, by the rural hospitals theorem, this measure is the same at P^- and P^+. Therefore, the expression (4.1) must be constant in the cube $[P^-, P^+]$. This implies that the derivative matrix of D at P^- satisfies

$$\sum_c \partial D_c(P^-|\eta) = 0.$$

Therefore, the matrix of derivatives $\partial D(P^-|\eta)$ is not invertible, proving the claim.

The result then follows from this claim and Sard's Theorem. The claim implies that, for S such that $\sum_c S_c < 1$, the set of S for which $[\eta, S]$ has multiple stable matchings is contained in the set of S where $\partial D(P^*|\eta)$ is not invertible for some market clearing cutoff P^*. Sard's Theorem implies that this latter set has measure 0, since market clearing cutoffs are roots of the market clearing equation $D(P|\eta) = S$. This completes the proof.

□

The next theorem establishes the link between the continuum model and the standard discrete Gale and Shapley model. It shows that when an economy E has a unique stable matching, which is the generic case, (1) it corresponds to the limit of stable matchings of approximating finite economies, (2) approximating finite economies have a small set of stable matchings, and (3) the unique stable matching varies continuously with the fundamentals of the economy.

Theorem 2. *Assume that the continuum economy E admits a unique stable matching μ. We then have*

(1) *For any sequence of stable matchings $\{\tilde{\mu}^k\}_{k\in\mathbb{N}}$ of finite economies in a sequence $\{F^k\}_{k\in\mathbb{N}}$ converging to E, we have that $\tilde{\mu}^k$ converges to μ.*

(2) *Moreover, the diameter of the set of stable matchings of $\{F^k\}_{k\in\mathbb{N}}$ converges to 0.*

(3) *The stable matching correspondence is continuous at E within the set of continuum economies $\mathcal{E}$.*

The theorem considers continuum economies E with a unique stable matching, which Theorem 1 guarantees is the typical case.

Part (1) justifies using the simple continuum model as an approximation of the classic Gale and Shapley (1962) discrete model. Formally, the unique stable matching of the limit economy is the limit of any sequence of stable matchings of approximating finite economies. We emphasize that, for a sequence of finite economies $\{F^k\}_{k\in\mathbb{N}}$ to converge to a limit economy E, it is necessary that the empirical distribution of student types converges. Therefore, the economies F^k have an increasing number of students, and a fixed number of colleges.

Part (2) states that the diameter of the set of stable matchings of any such sequence of approximating finite economies converges to 0. This means that, as economies in the sequence become sufficiently large, the set of stable matchings becomes small. More precisely, even if such an economy has several stable matchings, cutoffs are very similar in any stable matching. To a first approximation, supply and demand clear large matching markets uniquely.

Finally, Part (3) states that the unique stable matching of E varies continuously in the set of all continuum economies $\mathcal{E}$. That is, the stable matching varies continuously with the fundamentals of economy E. The result shows that, in applications, comparative statics can be derived from supply and demand equations, much like Debreu's (1970) result in general equilibrium. Moreover, Part (3) is of significance for studies that use data and simulations to inform market design (Abdulkadiroglu et al. 2009; Budish and Cantillon 2012). It implies that, in large matching markets, the conclusions of such simulations are not sensitive to small changes in the fundamentals.

The proof of Theorem 2 Part (1) is based on the observation that if a sequence of economies F^k converges to an economy E, then its demand functions converge. With this observation it is possible to prove that any sequence of stable matchings of $\{F^k\}_{k\in\mathbb{N}}$ converges to a stable matching of E. Since E has a unique stable matching, any such sequence must converge to the same stable matching. Part (2) then follows from Part (1). As for Part (3), the argument is similar to that in Part (1). Uniqueness plays an important role in Theorem 2. In Section 5.5 we give a knife edge example of an economy with multiple stable matchings, and where Theorem 2 fails. Moreover, we give a result showing that in a large set of cases where uniqueness fails, the set of stable matchings may change discontinuously with small changes in the fundamentals, and that none of the conclusions of Theorem 2 hold.

5. Applications and Extensions

5.1. **Comparative Statics and a Price-Theoretic Analysis of Competition and School Quality.** This section considers the classic question of whether competition between public schools improves school quality, which besides being interesting on its on, illustrates the derivation of comparative statics in the continuum framework. Hatfield et al. (2011a) consider an important aspect of this problem, namely whether competition for the best students gives schools incentives to improve.[26] They study the incentives

[26]The effect of competition on the provision of services by public schools, and local government services in general, is a classic topic in the economics of education and the public sector. Tiebout (1956) has pointed out that competition between locations allows agents to sort efficiently into places that offer

for public schools to invest in quality in a city where there is school choice, so that schools compete for students, using the standard discrete Gale and Shapley framework. They show that, in large markets, the incentives for schools to invest in quality are nonnegative, but are silent about their magnitude, and to what types of investments schools pursue.[27] To address these issues, we approach the problem from a price-theoretic perspective.

Consider a city with a number of public schools $c = 1, \cdots, C$, each with capacity S_c. Students and schools are matched according to a stable matching. This is a stark description of the institutional arrangements in cities like New York, where a centralized clearinghouse assigns students using a stable matching mechanism. Students are denoted as i, in a set of students I. Note that I is a set of students, distinct from the set Θ of student types. Schools' preferences over students are given by scores e^i_c. We assume that the vectors e^i are distributed according to a distribution function $G(\cdot)$ in $[0,1]^C$, with a continuous density $g > 0$.

Students' preferences depend on the quality $\delta_c \in \mathbb{R}$ of each school. δ_c should be interpreted as a vertical quality measure, in that all students prefer higher δ_c. However, different students may be affected differently by δ_c. So, for example, if δ_c measures the quality of a school's calculus course, then students of high academic caliber, or with a focus in math, will be more sensitive to changes in δ_c. One of the advantages of our approach is that it predicts which groups of students a school would like to target with improvements in quality. Student i has utility $u^i_c(\delta_c) > 0$ of attending school c, increasing in δ_c, and utility 0 of being unmatched. The measure of students who are indifferent between two schools is 0, for any value of δ. Given δ, preferences induce a distribution η_δ over student types Θ, which we assume to be have a density $f_\delta > 0$, smooth in δ and θ.

Under these assumptions, given δ, there exists a unique stable matching μ_δ. Let $P^*(\delta)$ be the unique associated market clearing cutoffs. Dependence on δ will be omitted when there is no risk of confusion.

For concreteness, we define the aggregate quality of a school's entering class as

$$Q_c(\delta) = \int_{\mu_\delta(c)} e^\theta_c d\eta_\delta(\theta)$$

the level of public services they prefer. With respect to schools, the literature has emphasized the importance of competition and choice to private efficiency of allocations, and spillover effects (see Hoxby, 2000 and references therein). More closely related to Hatfield et al. (2011a) are papers that consider whether competition gives school administrators incentives to perform better, such as Hoxby's (1999) model of moral hazard where families may move, and school districts compete for resources.

[27]In contrast, Hatfield et al. (2011a) provide sharp results comparing different school choice mechanisms, which we do not pursue as the present paper deals exclusively with stable matchings. In our view, this highlights how the standard discrete Gale and Shapley framework and the continuum model are complementary, each being useful for addressing different questions.

That is, the integral of scores e_c^θ over all students matched to the school. We consider how a school's quality δ_c affects the quality of its entering class Q_c. The motivation is that, following Hatfield et al. (2011a), if schools are concerned about Q_c,[28] then a direct link between δ_c an Q_c gives school administrators incentives to improve quality δ_c. Note that, since we are not performing an equilibrium analysis, it is not assumed that the quality of the entering class Q_c is the sole objective of a school. Even if schools have complex objectives, the impact of δ_c on Q_c isolates how investments benefit schools purely on the dimension of competing for a strong entering class. That is, the selection benefits of investment, driven by schools competing for students as opposed to being assigned a fixed entering class.

The effect of a school investing in quality can be written in terms of average characteristics of students who are marginally choosing or being chosen by schools, much like the effect of a demand shift in markets with prices is a function of characteristics of marginal consumers (Spence, 1975; Veiga and Weyl, 2012). To highlight the intuition behind the effect of investing in quality, we define the following quantities.

- The number N_c of additional students attracted to school c by a marginal increase in quality:
$$N_c \equiv \frac{d}{d\delta_c} D_c(P)|_{P=P^*(\delta)} = \int_{\{\theta:D^\theta(P^*(\delta))=c\}} \frac{d}{d\delta_c} f_\delta(\theta)d\theta.$$
- The average quality of the attracted students:
$$\bar{e}_c \equiv \int_{\{\theta:D^\theta(P^*(\delta))=c\}} e_c^\theta \cdot \frac{d}{d\delta_c} f_\delta(\theta)d\theta/N_c.$$
- The set of students who are marginally accepted to school c' and would go to school c otherwise:
$$\tilde{M}_{c'c} \equiv \{\theta : c' \succ^\theta c, P_{c'} = e_{c'}^\theta, P_c \leq e_c^\theta, P_{c''} > e_{c''}^\theta \forall c'' \neq c' : c'' \succ^\theta c\}.$$
- The number of student in this margin, and their average scores:
$$\begin{aligned} M_{c'c} &\equiv \int_{\tilde{M}_{c'c}} f_\delta(\theta)d\theta \\ \bar{P}_{c'c} &\equiv E[e_c^\theta|\theta \in \tilde{M}_{c'c}]. \end{aligned}$$

The effect of school quality δ_c on the quality of the entering class Q_c is as follows.

Proposition 1. *Assume that $P^*(\delta) > 0$, and that P is differentiable in δ_c. Then the quality of the entering class Q_c is differentiable in school quality δ_c, and its derivative*

[28]There is evidence that schools in NYC are concerned about the quality of their incoming classes, as many schools used to withhold capacity to game the allocation system used previously to the Abdulkadiroglu et al. (2005b) redesign of the match.

can be decomposed as

$$\frac{dQ_c}{d\delta_c} = \underbrace{[\bar{e}_c - P_c^*] \cdot N_c}_{\textit{Direct Effect}} - \underbrace{\sum_{c' \neq c} [\bar{P}_{c'c} - P_c^*] \cdot M_{c'c} \cdot (-\frac{dP_{c'}^*}{d\delta_c})}_{\textit{Market Power Effect}} . \tag{5.1}$$

The direct effect term is weakly positive, always giving incentives to invest in quality. The market power terms increase (decrease) the incentives to invest in quality if an increase (decrease) in the quality of school c increases the market clearing cutoff of school c', that is $dP_{c'}^*(\delta)/d\delta_c > 0$ (< 0).

The proposition states that the effect of an increase in quality can be decomposed in two terms. The direct effect is the increase in quality, holding cutoffs P fixed, due to students with $e_c \geq P_c^*$ choosing school c with higher frequency. Note that this term is proportional to $\bar{e}_c - P_c^*$. Since the total number of students that the school is matched to is fixed at S_c, the gain is only a change in composition. As the school attracts more students with average score $\bar{e}_c$ it must give up marginal students with scores P_c^*. The change in quality $\bar{e}_c - P_c^*$ is multiplied by N_c, the number of students who change their choices.

The market power effect measures how much the school loses due to its higher quality decreasing the equilibrium cutoffs of other schools. It is (the sum over all other schools c' of) the product of the change in cutoffs of the other school $(-\frac{dP_{c'}^*}{d\delta_c})$, times the quantity of students in the margin that change schools due to a small change in cutoffs, $M_{c'c}$, times the difference in the average quality of these students and the quality of a marginal student $\bar{P}_{c'c} - P_c^*$. The market power effect from school c' has the same sign as $dP_{c'}^*/d\delta_c$. It reduces the incentives to invest in quality if increasing δ_c reduces the selectivity of school c'. However, it can be positive in the counterintuitive case where improving the quality of school c increases the selectivity of school c'. The latter case is only possible if $C \geq 3$.[29] The intuition for the direction of the market power effect is that improvements in quality help if they induce competing schools to become more selective, but harm in the more intuitive case where improving quality makes other schools less selective, and therefore compete more aggressively for students.

[29]To see this, write the aggregate demand function conditional on δ as $D(P, \delta)$. Then $D(P^*(\delta), \delta) = S$. By the implicit function theorem, we have $\partial_\delta P^* = -(\partial_P D)^{-1} \cdot \partial_\delta D$. If $C = 2$, solving this system implies $dP_{c'}^*/d\delta_c \leq 0$ for $c \neq c'$. With $C = 3$, an example of $dP_{c'}^*/d\delta_c > 0$ for $c \neq c'$ is given by

$$\partial_P D = \begin{pmatrix} -10 & 1 & 1 \\ 4 & -10 & 1 \\ 4 & 1 & -10 \end{pmatrix}, \partial_{\delta_1} D = (10, -9, -1).$$

In this example the effect of increasing the quality of college 1 on cutoffs is $\partial_{\delta_1} P^* = (.98, -.49, .24)$, so that the cutoff of college 3 goes up with an increase in δ_1. The intuition is that an increase in quality of college 1 takes more students from college 2 than college 3, and the decrease in the selectivity of college 2 induces college 3 to become more selective.

Hatfield et al.'s (2011a) main result is that, in a large thick market, where each school comprises a negligible fraction of the market, the incentives to invest in quality are weakly positive. Within our framework this can be interpreted as saying that, in such markets, the market power term becomes small, and therefore $dQ_c/d\delta_c \geq 0$.

Note that the decomposition of incentives in equation (5.1) gives conditions where schools have muted incentives to invest in quality improvements for lower ranked students. If δ_c is a dimension of quality such that $df_\delta(\theta)/d\delta_c \approx 0$ unless $e_c^\theta \leq P_c$ or $e_c^\theta \approx P_c$, then the direct effect

$$[\bar{e}_c - P_c^*] \cdot N_c \approx 0.$$

Consider the case where the effect of the quality of school c on the cutoffs of other schools is either small, as in a large market, or has the intuitive sign $dP_{c'}^*/d\delta_c \leq 0$. Then the small direct effect and weakly negative market power effect imply $dQ_c/d\delta_c \leq 0$. Therefore, by allowing schools to compete, school choice gives incentives to invest in improvements benefiting the best students, but not the marginal accepted students. An example would be that a school has incentives to invest in a better calculus teacher, and assigning counselor time to advise students in applying to top colleges; and at the same time small or negative incentives to improve the quality of classes for lower ranked students, or invest counselor time in helping students with low grades. The logic of this result is that, since the quantity of students S_c that are matched to school c is fixed, for every student of score e_c that a school gains by improving quality it must shed a marginal student with score P_c^*. The direct effect can only be profitable if $e_c - P_c^*$ is appreciably greater than 0. The argument is completed by the observation that the market power term is weakly negative if $dP_{c'}^*/d\delta_c \leq 0$. Note that marginal students with scores $e_c \approx P_c^*$ are not necessarily "bad". For an elite high school, cutoffs P_c^* are high, in the sense that a type e_c student is very desirable.[30] Yet, since changes in quality only shift the composition of an entering class, it is still the case that the incentives to invest in attracting such students is small. Another way to frame this discussion is that the only scenario where the incentives to invest in marginal students may be positive is when a school does have market power, in the sense that it can affect the cutoffs of other schools, and for at least one of these other schools $dP_{c'}^*/d\delta_c > 0$.

The model yields an additional distortion. Even though quality affects $u_c^i(\delta_c)$ for all students, schools are only concerned with the impact on students who are indifferent between different schools, as equation (5.1) only depends on changes in f_δ. This is the familiar Spence (1975) distortion of a quality setting monopolist. Its manifestation in our setting is that schools' investment decisions take into account marginal but not inframarginal students.

[30]For example, Stuyvesant High School's SAT scores are in the 99.9^{th} percentile in the state of New York (Abdulkadiroglu et al., 2011).

Finally, if we assume that schools are symmetrically differentiated,[31] it is possible to gain further intuition on the market power effect. If the function $f_\delta(\theta)$ is symmetric over schools, and all $S_c = S_{c'}, \delta_c = \delta_{c'}$, then the market power term reduces to

$$-\frac{M_{c'c}}{M_{c\emptyset} + C \cdot M_{c'c}} \cdot [\bar{P}_{c'c} - P_c^*] \cdot N_c,$$

where $M_{c\emptyset} = \int_{\{\theta : e_c^\theta = P_c, e_{c'}^\theta < P_{c'} \forall c' \neq c\}} f_\delta(\theta) d\theta$ is the $C-1$ dimensional mass of agents who are marginally accepted to school c and not accepted to any other schools. In the symmetric case, the market power effect is negative, and proportional to the quality wedge $\bar{P}_{c'c} - P_c^*$, times the amount of students that school c attracts with improvements in quality N_c. Ceteris paribus, the absolute value of the market power effect grows with $M_{c'c}$, the mass of students on the margin between school c' and c. These are the students that school c may lose to c' if c' competes more aggressively. The absolute value of the market power effect also decreases with the number of schools C, and holding fixed the other quantities it converges to 0 as the number of schools grows. The expression suggests conditions under which competition reduces the incentives for schools to invest in quality improvements for marginal applicants. This is the case when a small number of schools compete for densely populated margins $\tilde{M}_{cc'}$. An example would be a city with a small number of elite schools, that compete mostly with each other for the best students, but are horizontally differentiated, so that many students are in the margins $\tilde{M}_{cc'}$.

This effect might help explain puzzling findings from regression discontinuity studies of elite schools. Dobbie and Fryer (2011) and Abdulkadiroglu et al. (2011) find that marginally accepted students to the top three exam schools in Boston and New York do not dot attain higher SAT scores, despite the better peers and large amount of resources invested in these schools. This is consistent with the prediction that competition gives elite schools incentives to compete for the best students, but not to invest in improvements that benefit marginal students. At the same time, marginally rejected students at elite schools are likely to be among the best students in the schools they eventually go to. Therefore, the non-elite schools have incentives to tilt investments towards these students, which helps to explain the absence of a large difference in outcomes.

The analysis in this section could be extended in a number of ways. If the model specified costs for schools to invest, and possibly more complex objectives, it would be possible to derive first order conditions for equilibrium play of schools. By specifying social welfare, the equilibrium conditions could be compared with optimization by a social planner. As the goal of this section is simply to illustrate the derivation of comparative statics in the continuum framework, in the interest of space we leave these extensions for future research, and discuss related applications in the conclusion. We

[31] We would like to thank Glen Weyl for the suggestion to consider the symmetric case.

do note that the expression derived in Proposition 1 is valid for general demand shocks in matching markets, so that the methodology can be readily applied to other markets. Moreover, this type of comparative static leads to straightforward equilibrium analysis in a market where firms are assumed to play strategically, as illustrated in Azevedo (2011).

5.2. **Random Economies and Convergence Rates.** This section extends the convergence results to randomly generated finite economies. The results also give bounds on the speed of convergence of the set of stable matchings. Many mechanisms used in practical market design explicitly incorporate randomly generated preferences, so that the results imply new characterizations of the asymptotics of such mechanisms, which we explore in Section 5.4.

We begin this section bounding the difference between market clearing cutoffs in a continuum economy and in a finite approximation.

Proposition 2. *Assume that the continuum economy $E = [\eta, S]$ admits a unique stable matching μ, and $\sum_c S_c < 1$. Let P^* be the associated market clearing cutoff, and assume $D(\cdot|\eta)$ is C^1, and $\partial D(P^*)$ is invertible. Then there exists $\alpha \geq 0$ such that, for any finite economy $F = [\eta^F, S^F]$, we have*

$$\sup\{||P^F - P^*|| : P^F \text{ is a market clearing cutoff of } F\}$$
$$\leq \alpha \cdot (\sup_{P \in [0,1]^C} ||D(P|\eta) - D(P|\eta^F)|| + ||S - S^F||).$$

The proposition shows that the distance between market clearing cutoffs of a continuum economy and a discrete approximation is of the same order of magnitude as the distance between the associated vectors of capacities, plus the difference between the demand functions. Therefore, if the distance between the empirical distribution of types η^F and η is small, and the distance between the supply vectors S^F and S is small, the continuum model is guaranteed to provide a good approximation for finite economies.

We now extend the convergence in Theorem 2 to economies where agents are randomly drawn, with types independently and identically distributed. The following proposition implies not only convergence of the sets of stable matchings, but also a strong bound on the speed of convergence. For a real number x we denote by $[x]$ the nearest integer to x.

Proposition 3. *Assume that the continuum economy $E = [\eta, S]$ admits a unique stable matching μ, and $\sum_c S_c < 1$. Let $F^k = [\eta^k, S^k]$ be a randomly drawn finite economy, with k students drawn independently according to η and the vector of capacity per student S^k defined as $S^k k = [Sk]$. Let $\{\tilde{\mu}^k\}_{k \in \mathbb{N}}$ be a sequence of random variables, such that each $\tilde{\mu}^k$ is a stable matching of F^k. We have the following results.*

(1) F^k converges almost surely to E, and $\tilde{\mu}^k$ converges almost surely to μ.

Moreover, convergence is exponentially fast in the following sense. Assume that $D(\cdot|\eta)$ is C^1, and $\partial D(P^*)$ is invertible. Let P^* be the unique market clearing cutoff of E, and fix $\epsilon > 0$. We then have that:

(2) There exist constants $\alpha, \beta > 0$ such that probability that F^k has a market clearing cutoff P^k with $\|P^k - P^*\| \geq \epsilon$ is bounded by

$$\Pr\{F^k \textit{ has a market clearing cutoff } P^k \textit{ with } \|P^k - P^*\| > \epsilon\} \leq \alpha \cdot e^{-\beta k}.$$

(3) Moreover, if η can be represented by a continuous density, let the G^k be fraction of students in economy F^k that receives a match different from that in the limit economy, that is, $D^\theta(P^k) \neq D^\theta(P^*)$ for some market clearing cutoff P^k of F^k . Then G^k converges to 0 almost surely, and there exist $\alpha', \beta' > 0$ such that the probability that $G^k > \epsilon$ is bounded by

$$\Pr\{G^k > \epsilon\} \leq \alpha' \cdot e^{-\beta' k}.$$

The first part of the proposition says that the stable matchings of the randomly drawn economies converge almost surely to stable matchings of the limit approximation. This justifies using the continuum model as an approximation of the discrete model in settings where preference are random, such as in mechanisms that rely on tie-breaking lotteries.

The second part of the proposition gives bounds on how fast convergence takes place. Given $\epsilon > 0$, the Corollary guarantees that the probability that market clearing cutoffs in F^k deviate from those in E by more than ϵ converges to 0 exponentially. Moreover, it guarantees that the fraction of students that may receive different matches in the continuum and finite economy is lower than ϵ with probability converging to 1 exponentially.

The proof of Proposition 2 starts by observing that the excess demand function $z(\cdot|E) = D(\cdot|E) - S$ must be bounded away from 0 outside of a neighborhood of P^*. It then uses the approximation of z by its derivative to bound the distance of market clearing cutoffs of economy F^k to P^*. The proof of Proposition 3 uses the Glivenko-Cantelli Theorem to show almost sure convergence of the empirical distributions of types η^k to η. Theorem 2 then guarantees almost sure convergence of stable matchings $\tilde{\mu}^k$ to μ. The exponential bounds follow from Proposition 3, and from the Vapnik and Chervonenkis (1971) bounds from computational learning theory, that guarantee fast convergence of $D(P|F^k)$ to $D(P|E)$, uniformly in P.

5.3. **Comparative Statics in Large Finite Markets.** One of the advantages of the continuum model is that comparative statics can be derived using standard techniques. However, the applicability of the model depends on the comparative statics results extending to actual finite markets. The following proposition guarantees that this is the case.

Proposition 4. *Consider two limit economies E, E', with unique market clearing cutoffs P, P'. Let $\{F^k\}_{k\in\mathbb{N}}, \{F'^k\}_{k\in\mathbb{N}}$ be sequences of finite economies with $F^k \to E, F'^k \to E'$. Then there exists k_0 such that for all $k \geq k_0$ and any pair of market clearing cutoffs P^k of F^k and P'^k of F'^k, if $P_c > P'_c$ then $P^k > P'^k$, and if $P_c < P'_c$ then $P^k < P'^k$.*

The proposition considers continuum economies E, E' with unique stable matchings. It shows that if the market clearing cutoffs are ordered in a particular way, then the sets of market clearing cutoffs of approximating discrete economies are strongly ordered in the same way. Therefore, even though discrete economies may have several stable matchings, setwise comparative statics must be the same as in the continuum model.

5.4. **Market Design Applications.** We now apply our results to derive novel results in market design. Since many matching and assignment mechanisms use lotteries to break ties, our results on convergence of random economies readily imply asymptotic characterizations and large market properties of these mechanisms. Specifically, we give a simple derivation of results by Che and Kojima (2010) for the canonical random serial dictatorship mechanism, and generalize them with novel results for a state-of-the art mechanism used in real school choice systems.

5.4.1. *The Random Serial Dictatorship Mechanism.* The assignment problem consists of allocating indivisible objects to a set of agents. No transfers of a numeraire or any other commodity are possible. The most well-known solution to the assignment problem is the random serial dictatorship (RSD) mechanism. In the RSD mechanism, agents are first ordered randomly by a lottery. They then take turns picking their favorite object, out of the ones that are left. Recently, Che and Kojima (2010) have characterized the asymptotic limit of the RSD mechanism.[32] In their model, the number of object types is fixed, and the number of agents and copies of each object grows. Their main result is that RSD is asymptotically equivalent to the probabilistic serial mechanism proposed by Bogomolnaia and Moulin (2001). This is a particular case of our results, as the serial dictatorship mechanism is equivalent to deferred acceptance when all colleges have the same ranking over students. This section formalizes this point.

In the assignment problem there are C object types $c = 1, 2, \ldots, C$. An instance of the assignment problem is given by $AP = (k, m, S)$, where k is the number of agents, m is a vector with $m_\succ$ representing the fraction of agents with preferences $\succ$ for each $\succ \in \mathcal{R}$, and S a vector with S_c being the number of copies of object c available per capita. An allocation specifies for each agent $i \in \{1, 2, \cdots, k\}$ a probability $x^i(c|AP)$ of

[32]This asymptotically characterization has been generalized by Liu and Pycia (2011) to any uniform randomization over Pareto efficient mechanisms under an equicontinuity condition.

receiving each object c. Since we will only consider allocations that treat ex ante equal agents equally, we denote by $x^{\succ}(c|AP)$ the probability of an agent with preferences $\succ$ receiving object c, for all preferences $\succ$ present in the assignment problem.

We can describe RSD as a particular case of the deferred acceptance mechanism where all colleges have the same preferences. First, we give agents priorities based on a lottery l, generating a random finite college admissions problem $F(AP, l)$, where agents correspond to students, and colleges to objects. Formally, given assignment problem AP, randomly assign each agent i a single lottery number l^i uniformly in $[0, 1]$, that gives her score in all colleges (that is, objects) of $e^i_c = l$. Associate with this agent a student type $\theta_i = (\succ^i, e^i)$. This induces a random discrete economy $F(AP, l)$ as in the previous section. That is, as l is a random variable, $F(AP, l)$ is a random finite economy, and for particular draws of l it is a finite economy. For almost every draw of l the economy $F(AP, l)$ has strict preferences. Each agent i's allocation $x^i(c|AP)$ under RSD is then equal to the probability of receiving object c in her allocation in $F(AP, l)$.

Consider now a sequence of finite assignment problems $\{AP^k\}_{k\in\mathbb{N}}$, $AP^k = (k, m^k, S^k)_{k\in\mathbb{N}}$. Assume (m^k, S^k) converges to some (m, S) with $S > 0, m > 0$. Let each l^k be a lottery consisting of k draws, one for each agent, uniformly distributed in $[0, 1]^k$. For each k, the assignment problem and the lottery induce a random economy $F(AP^k, l^k)$.

Note that the finite economies $F(AP^k, l^k)$ converge almost surely to a continuum economy E with a vector S of quotas, a mass $m_{\succ}$ of agents with each preference list $\succ$, and scores e^{θ} uniformly distributed along the diagonal of $[0, 1]^C$. This limit economy has a unique market clearing cutoff $P(m, S)$ (see footnote 23). By Proposition 3, cutoffs in large finite economies are converging almost surely to $P(m, S)$. We have the following characterization of the limit of the RSD mechanism.

Proposition 5. *Under the RSD mechanism the probability $x^{\succ}(c|AP^k)$ that an agent with preferences $\succ$ will receive object c converges to*

$$\int_{l\in[0,1]} \mathbf{1}_{(c=\arg\max_{\succ}\{c\in C|P_c(m,S)\le l\})} dl.$$

That is, the cutoffs of the limit economy describe the limit allocation of the RSD mechanism. In the limit, agents are given a lottery number uniformly drawn between 0 and 1, and receive their favorite object out of the ones with cutoffs below the lottery number. Inspection of the market clearing equations shows that cutoffs correspond to 1 minus the times where objects run out in the probabilistic serial mechanism. This yields the Che and Kojima (2010) result on the asymptotic equivalence of RSD and the probabilistic serial mechanism.

5.4.2. *School Choice Mechanisms.* We now derive new results for deferred acceptance mechanisms used by actual clearinghouses that allocate seats in public schools to students. These mechanisms generalize RSD, as in school choice some students are given priority to certain schools. While dealing with priorities would be difficult using combinatorial techiniques as in Che and Kojima (2010), it is straightforward to generalize the analysis based on Proposition 3.

The school choice problem consists of assigning seats in public schools to students, while observing priorities some students may have to certain schools. It differs from the assignment problem because schools give priorities to subsets of students. It differs from the classic college admissions problem in that often schools are indifferent between large sets of students (Abdulkadiroglu and Sönmez 2003). For example, a school may give priority to students living within its walking zone, but treat all students within a priority class equally. In Boston and NYC, the clearinghouses that assign seats in public schools to students were recently redesigned by academic economists (Abdulkadiroglu et al. 2005a,b). The chosen mechanism was deferred acceptance with single tie-breaking (DA-STB). DA-STB first orders all students using a single lottery, which is used to break indifferences in the schools' priorities, generating a college admissions problem with strict preferences. It then runs the student-proposing deferred acceptance algorithm given the refined preferences (Abdulkadiroglu et al. 2009; Kesten and Ünver 2010).

We can use our framework to derive the asymptotics of the DA-STB mechanism. Fix a set of schools $C = \{1, \ldots, C\}$. Students are described as $i = (\succ^i, e^i)$ given by a strict preference list $\succ^i$ and a vector of scores e^i. However, to incorporate the idea that schools only have very coarse priorities, corresponding to a small number of priority classes, we assume that all e^i_c are integers in $\{0, 1, 2, \ldots, \bar{e}\}$ for $\bar{e} \geq 0$. Therefore, the set of possible student types is finite. We denote by $\bar{\Theta}$ the set of possible types. A school choice problem $SC = (k, m, S)$ is given by a number of students k, a fraction m_i of students of each of the finite types $\bar{\theta} \in \bar{\Theta}$, and a vector of capacity per capita of each school S.

We can describe the DA-STB mechanism as first breaking indifferences through a lottery l, which generates a finite college admissions model $F(SC, l)$, and then giving each student the student-proposing deferred acceptance allocation. Assume each student $i \in \{0, 1, \cdots, k\}$ receives a lottery number l^i independently uniformly distributed in $[0, 1]$. The student's refined score in each school is given by her priority, given by her type, plus lottery number, $e^{\theta_i}_c = e^i_c + l_i$.[33] The refined type is defined as $\theta_i = (\succ^i, e^{\theta_i})$. Therefore, the lottery yields a randomly generated finite economy $F(SC, l)$, as defined in Corollary 3. The DA-STB mechanism then assigns each student i in F to her match

[33]Note that in this finite economy scores are in the set $[0, \bar{e}+1]^C$, and not $[0, 1]^C$ as we defined before. It is straightforward to extend the model to this setting.

in the unique student-optimal stable matching. For each type $\bar{\theta} \in \bar{\Theta}$ in the school choice problem, denote by $x^{\bar{\theta}}_{DA-STB}(c|SC)$ the probability that a student with type $\bar{\theta}$ receives school c, if type $\bar{\theta}$ is present in the economy.

Consider now a sequence of school choice problems $SC^k = (k, m^k, S^k)$, each with $k \to \infty$ students. Problem k has a fraction $m^k_{\bar{\theta}}$ of students of each type, and school c has capacity S^k_c per student. Assume (m^k, S^k) converges to some (m, S) with $S > 0, m > 0$.

Analogously to the assignment problem, as the number of agents grows, the aggregate randomness generated by the lottery disappears. The randomly generated economies $F(SC^k, l^k)$ are converging almost surely to a limit economy, given as follows. For each of the possible types $\bar{\theta} \in \bar{\Theta}$, let the measure $\eta_{\bar{\theta}}$ over Θ be uniformly distributed in the line segment $\succ^{\bar{\theta}} \times [e^{\bar{\theta}}, e^{\bar{\theta}} + (1, 1, \cdots, 1)]$, with total mass 1. Let $\eta = \sum_{\bar{\theta} \in \bar{\Theta}} m_{\bar{\theta}} \cdot \eta_{\bar{\theta}}$. The limit continuum economy is given by $E = [\eta, S]$. We have the following generalization of the result in the previous section.

Proposition 6. *Assume the limit economy E has a unique market clearing cutoff $P(m, S)$. Then the probability $x^{\bar{\theta}}_{DA-STB}(c|SC^k)$ that DA-STB assigns a student with type $\bar{\theta} \in \bar{\Theta}$ to school c converges to*

$$\int_{l \in [0,1]} \mathbf{1}_{(c = \arg\max_{\succ} \{c \in C \,|\, P_c(m,S) \leq e^{\theta}_c + l\})} dl.$$

Moreover, the realized fraction of agents of type $\bar{\theta}$ that are assigned to school c converges almost surely to this value.

The proposition says that the asymptotic limit of the DA-STB allocation can be described using cutoffs. The intuition is that, after tie-breaks, a discrete economy with a large number of students is very similar to a continuum economy where students have lottery numbers uniformly distributed in $[0, 1]$. The main limitation of the proposition is that it requires the continuum economy to have a unique market clearing cutoff. Although we know that this is valid for generic vectors of capacities S, example 2 in Supplementary Appendix F shows that it is not always the case.

The second part of the result shows that as the market grows the aggregate randomness of the DA-STB mechanism disappears. Although the mechanism depends on random lottery draws, the fraction of agents with the same priority and preferences going to each school converges almost surely to that in the asymptotic limit. Therefore, while the allocation of an individual agent depends on the lottery, the aggregate allocation is unlikely to change with different draws. This limit result is consistent with data from the New York City match. Abdulkadiroglu et al. (2009) report that in multiple runs of the algorithm the average number of applicants assigned to their first choice is 32,105.3, with a standard deviation of only 62.2. The proposition predicts that in increasingly large markets, with a similar distribution of preferences and seats per capita, this standard deviation divided by the total number of students (in the NYC data $62.2/78,728 \approx 0.0008$) converges to 0.

The proposition has an important consequence for the efficiency of DA-STB. Che and Kojima (2010) show that, while the RSD mechanism is ordinally inefficient, the magnitude of this inefficiency goes to 0 as the number of agents grows. Similarly, the DA-STB mechanism is ex-Post inefficient, having a positive probability of its outcome being Pareto dominated by other stable matchings (Erdil and Ergin 2008). Examples show that, unlike RSD, this inefficiency *does not* go away in a large market. Indeed, we give an example in Supplementary Appendix F where the probability that DA-STB produces a Pareto dominated outcome converges to 1 as the market grows.

Finally, the proposition generalizes the result in the previous section, that describes the asymptotic limit of the RSD mechanism. RSD corresponds to DA-STB in the case where all students have equal priorities. Therefore, the market clearing equations provide a unified way to understand asymptotics of RSD, the probabilistic serial mechanism, and DA-STB. Moreover, one could easily consider other ways in which the lottery l is drawn, and derive asymptotics of other mechanisms, such as deferred acceptance with multiple tie-breaking discussed by Abdulkadiroglu et al. (2009).

5.5. **Markets with Multiple Stable Matchings.** Section 4 shows that almost every continuum economy has a unique stable matching. Here we provide an example of a continuum economy with multiple stable matchings.[34] This shows that existence of a unique stable matching cannot be guaranteed in general. Moreover, we show that when a continuum economy has multiple stable matchings, very generally none of them are robust to small perturbations of fundamentals. This implies that the conclusions of Theorem 2, linking the discrete and continuum models, are not valid for general economies.

Example 1. (Multiple Stable Matchings)

There are two colleges $c = 1, 2$ with equal capacity $S_1 = S_2 = 1/2$. Students differ in their height, which is uniformly distributed between 0 and 1. While college 1 prefers taller students, college 2 prefers shorter students. We let $e_1^\theta = \theta$'s height while $e_2^\theta = 1 - \theta$'s height. The preferences of the students are uncorrelated with their height, and half of the students prefer each college.

Under these assumptions, $P = (0, 0)$ clears the market, with demand 1/2 for each college. Likewise, setting $P = (1/2, 1/2)$ corresponds to a stable matching, with the taller half of the population going to college 1, and the shorter half to college 2. These are respectively the student-optimal and college-optimal stable matchings. For P_1 and

[34]We would like to thank Ted Bergstrom for suggesting this example.

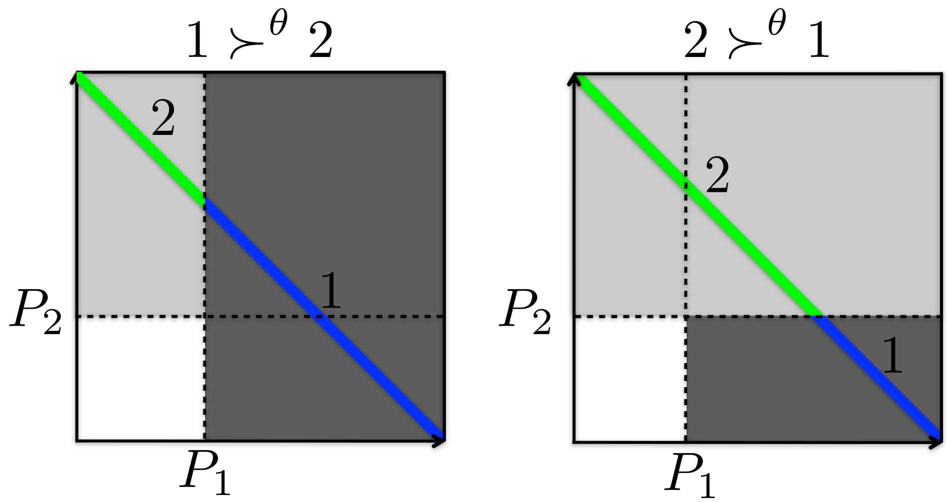

FIGURE 5. The figure depicts the distribution of types and the demand function in Example 1. Agent types are uniformly distributed over the solid lines. At cutoffs $(P_1, P_2) \in [0, 1/2]^2$, agent types in the region of the solid lines labeled 1 demand college 1 and agent types in the region labeled 2 demand college 2. This implies the formula for demand in equation (5.2), and in particular for any p with $0 \leq p \leq 1/2$ the vector of cutoffs (p, p) clears the market.

P_2 in $[0, 1/2]$, the demand functions are

$$D_c(P) = 1/2 - (P_c - P_{c'}). \tag{5.2}$$

Inspecting these equations, one can see that any cutoff vector in the segment $[(0, 0), (1/2, 1/2)]$ corresponds to a stable matching.

Note that none of these stable matchings is robust to small perturbations of fundamentals. Consider adding a small amount of capacity to each college. If this is done, at least one of the colleges must be in excess supply, and have a cutoff of 0 in equilibrium. In turn, this implies that the other college will have a cutoff of 0. Therefore, the only stable matching would correspond to $P = (0, 0)$. Likewise, in an economy where each college had slightly smaller capacity than 1/2, any market clearing cutoff involves cutoffs greater than 1/2. Otherwise, the demand equation (5.2) would imply that there is excess demand for at least one of the colleges.

The following proposition generalizes the example. It shows that, when the set of stable matchings is large, none of the stable matchings are robust to small perturbations. The statement uses the fact, proven in Appendix A, that for any economy E there exists a smallest and a largest market clearing cutoff, in the sense of the usual partial ordering of $\mathbb{R}^C$.

Proposition 7. ***(Non Robustness)*** *Consider an economy E with more than one stable matching and $\sum_c S_c < 1$. Let P be one of its market clearing cutoffs. Assume P*

is either strictly larger than the smallest market clearing cutoff P^-, or strictly smaller than the largest P^+. Let N be a sufficiently small neighborhood of P. Then there exists a sequence of economies E^k converging to E without any market clearing cutoffs in N.

Proof. Suppose $P > P^-$; the case $P < P^+$ is analogous. Assume N is small enough such that all points $P' \in N$ satisfy $P' > P^-$. Denote $E = [\eta, S]$, and let $E^k = [\eta, S^k]$, where $S^k_c = S_c + 1/Ck$. Consider a sequence $\{P^k\}_{k\in\mathbb{N}}$ of market clearing cutoffs of E^k. Then

$$\sum_{c\in C} D_c(P^k|\eta) = \frac{1}{k} + \sum S_c.$$

Note that, for all points P' in N,

$$\sum_{c\in C} D_c(P'|\eta) \leq \sum_{c\in C} D_c(P^-|\eta) = \sum S_c < \sum S^k_c.$$

However, for large enough k, $\sum S^k_c < 1$, which means that for any market clearing cutoff P^k of E^k we must have $D(P^k|\eta) = S^k_c$, and therefore there are no market clearing cutoffs in N. □

6. Conclusion

This paper proposes a new model of matching markets with a large number of agents on one side. The model admits complex heterogeneous preferences, as in the Gale and Shapley (1962) framework. At the same time, it allows for straightforward derivation of comparative statics, as stable matchings are the solution to a set of supply and demand equations. We show that the model corresponds to the limit of the standard Gale and Shapley model, that stable matchings are essentially unique in such economies, and apply the model to derive asymptotics of mechanisms used in practical market design, and to quantify the incentives for schools to invest in quality when they compete for students. In these closing remarks, we highlight four points that were not addressed in the analysis.

First, analysis of matching markets has typically taken one of two polar perspectives: either focusing on assortative matching, where rich comparative statics can be derived, or using models based on Gale and Shapley, where a limited set of such results are possible. We view our approach as a middle ground that complements these analyses. This is illustrated by our school competition application, where we derive key comparative statics as functions of the distribution of preferences and competitive structure in a market. In general, with the continuum model it is possible to use standard price-theoretic arguments in markets where agents have more than one dimension of heterogeneity, and assortativeness is not a good approximation.

Second, despite the importance of strategic firm behavior in industrial organization, there is very little work on how firms behave strategically in matching markets. One of the advantages of the continuum model is that it permits tractable analysis of games where firms can set prices, quantities, quality, or other product dimensions, or make ex ante investments. For example, it is straightforward to extend our analysis of quality investments between schools to a game, so that one could compare Nash equilibrium to the choices of a social planner. More generally, several models in industrial organization are two stage games where firsm first make a strategic choice, and outcomes are given by market clearing. Examples of the strategic variable include capacity (the Cournot model), prices (the Bertrand model or the empirically dominant Nash in prices competition with differentiated products), or a supply function (Klemperer and Meyer, 1989). In any of these settings, the comparative statics of the continuum model imply simple first-order conditions that characterize equilibrium behavior. The tractability of this approach is illustrated in Azevedo (2011), with a simple model of quantity competition. A promising avenue of future research is to consider such two-stage specifications, where firms make strategic choices and the allocation is given by stability, as both different assumptions on the strategy space and specific topics in industrial organization remain unexplored in matching markets.

Third, a number of recent papers have empirically estimated matching models with heterogeneous preferences, using different methodologies. Bajari and Fox (2013) use the inequalities from the stability condition to gauge the efficiency of FCC auctions. Ho (2009) considers a strategic bargaining game as opposed to a frictionless stability notion to study insurer-hospital networks. In an analysis of venture capital firms Sørensen (2007) imposes restrictions on preferences so that the Gale and Shapley model has a unique stable matching, and his structural model has a well defined likelihood function which he uses for estimation. Agarwal (2012) assumes vertical preferences on one side of the market, and exploits information from doctors matched to the same hospital to identify and estimate a model of the medical match. A natural extension of our model is to specify an empirical model for preferences. Since stable matchings are unique in our framework, such a model implies, given parameters, a distribution of colleges to which an agent is matched conditional on her observable characteristics.[35] Such an approach requires assuming that the market is large enough to be well approximated by our continuum model. It would be interesting to examine conditions under which

[35]For example, let Z_{ic} be a vector of student i and college c observable characteristics. A simple model has a continuum mass of students with some distribution over observables, and preferences given by $u^i_c = f(Z_{ic}, \epsilon_i, \xi_c, \alpha)$, and colleges have preferences given by $\pi^i_c = g(Z_{ic}, \epsilon_i, \xi_c, \beta)$, where ϵ_i and ξ_c are random variables representing unobserved characteristics, and α and β parameters to be estimated. If it is assumed for example that, given a distribution of ϵ_i conditional on Z_{ic}, $u^i_c$and π^i_c have a non-atomic distribution with full support in $[0,1]^{2C}$ for any fixed ξ, α, and β, then there is a s a unique stable matching, and for each vector of observables Z_{ic} there is a well-defined probability of student i being matched to each college c.

such models can be credibly identified and estimated. Moreover, it is important to understand, in markets where agents on one side match with many agents, the advantages and disadvantages of using such a model for empirical work, versus the alternatives in the literature.

Fourth, in recent years the matching literature has explored frameworks more general than the Gale and Shapley model, on which we focus. These include non-responsive preferences, many-to-many matching, markets with more than two sides, and externalities between agents.[36] In these models, the very existence of stable matchings depends on complex restrictions on preferences, such as substitutability.[37] It would be interesting to understand to what extent the continuum of traders assumption obviates the need for such restrictions.[38]

The common theme in our analysis is applying basic ideas from competitive equilibrium to matchings markets. This core idea permeates our characterization of stable matchings in terms of supply and demand equations, the decomposition of the effect of improving school quality in a direct and a market power effect, and characterizing the asymptotics of school choice mechanisms in terms of cutoffs. We hope this underlying idea will prove useful in the analysis of other problems, and broaden the applicability of Gale and Shapley's (1962) notion of stability, yielding insights in specific markets where Becker's (1973) assumptions of vertical preferences and assortative matching do not hold.

[36]See respectively Kelso and Crawford (1982); Hatfield and Milgrom (2005) Echenique and Oviedo (2006); Roth (1984), Ostrovsky (2008); Hatfield et al. (2011b), and Sasaki and Toda (1996); Epple and Romano (1998).

[37]See Hatfield and Milgrom (2005); Hatfield and Kojima (2010) and Hatfield et al. (2011b).

[38]See Ashlagi et al. (2011) and Kojima et al. (2010) for existence with stable matchings with complementarities in large markets. In general equilibrium, Starr (1969) demonstrates the existence of approximate competitive equilibria in large markets without convex preferences. See also Azevedo et al. (Forthcoming), who show that, in general equilibrium with indivisible commodities and quasilinear preferences, the continuum of traders assumption guarantees existence without the usual assumption of gross substitutes preferences made in the literature (Gul and Stacchetti, 1999).

Appendix

The Appendix contains ommited proofs, as well as additional results which are used to derive the results in the text. Appendix A extends some results of classic matching theory to the continuum model. It proves convergence of the analogue of the Gale and Shapley deferred acceptance algorithm, existence of a stable matching, the lattice theorem, and the rural hospitals theorem. It also contains a proof of the continuum supply and demand lemma. Appendix B derives our main results. Appendix C collects additional omitted proofs.

Appendix A. Preliminary Results

We begin the analysis by deriving some basic properties of the set of stable matchings in the continuum model. Besides being of independent interest, they will be useful in the derivation of the main results. Throughout this section we fix a continuum economy $E = [\eta, S]$, and omit dependence on E, η, and S in the notation.

A.1. **Existence of a stable matching.** We begin by proving the existence of a stable matching. Following the classic proof by Gale and Shapley (1962), we do so by defining the continuum analogue of the Gale and Shapley algorithm, and proving that it converges to a stable matching.

The continuum version of the Gale and Shapley student proposing algorithm is defined as follows. The state of the algorithm at round k is a (not necessarily stable) matching μ^k, a list $r^k(\theta)$ of colleges that have rejected each student θ so far, and a vector x^k of cutoffs. The algorithm starts with the matching μ^0 where all students unassigned, no rejections $r^0(\cdot) \equiv \emptyset$, and $x^0_c \equiv 0$. In each round, the state is updated as follows.

- Step 1: Each student that is unassigned at μ^k is tentatively assigned to her favorite college that hasn't rejected her yet, if there are any.
- Step 2: Each college rejects all students strictly below the minimum threshold score $x_c^{k+1} \geq x_c^k$ such that the measure of students assigned to it is smaller or equal to S_c. μ^{k+1} is defined by this matching, and r^{k+1} updated with the corresponding rejections.

We have that, although the algorithm does not necessarily finish in a finite number of steps, the tentative assignments converge to a stable matching.

Proposition A1. ***(Convergence of the Deferred Acceptance Algorithm)*** *The student-proposing deferred acceptance algorithm converges pointwise to a stable matching.*

Proof. To see that the algorithm converges, note that each student can only be rejected at most C times. Consequently, for every student there exists k high enough such that in all rounds of the algorithm past k she is assigned the same college or matched to herself, so the pointwise limit exists. Therefore, there exists a function $\mu : \Theta \to C \cup \{\Theta\}$ that is the pointwise limit $\mu(\theta) = \lim_{k\to\infty} \mu^k(\theta)$. We may extend μ to C by setting $\mu(c) = \{\theta \in \Theta : \mu(\theta) = c\}$. To see that the limit μ is a matching, we have to prove that the measure of students assigned to each college in the limit is not greater than its capacity. At each round k of the algorithm, let R^k be the measure of rejected students. Again, because no student can be rejected more than C times, we have $R^k \to 0$. Moreover, it must be the case that $\eta(\mu(c)) \leq \eta(\mu^k(c)) + R_k \leq S_c + R^k$ for every k. Therefore, $\eta(\mu(c)) \leq S_c$.

Note that μ satisfies the consistency conditions 1-3 for a stable matching. Condition 4 (right continuity) follows from the fact that sets of rejected students are always of the form $e^\theta_c < x$. Therefore, μ is a matching.

The proof that the matching μ is stable follows Gale and Shapley (1962). Assume by contradiction that (θ, c) is a blocking pair. If $\eta(\mu(c)) < S_c$, then c does not reject any students during the algorithm, which contradicts (θ, c) being a blocking pair. This implies that there is θ' in $\mu(c)$ with $e^{\theta'}_c < e^\theta_c$. At some round k of the algorithm, both types θ and θ' are already matched to their final outcomes. However, since θ was rejected by c in an earlier round, it must be that $x^k_c > e^\theta_c$. Therefore, $e^{\theta'}_c$ would have to be rejected at round k, which is a contradiction. □

The proposition shows that the traditional algorithm for finding stable matchings works in the continuum model, although the algorithm converges without necessarily finishing in a finite number of steps. An immediate corollary of this proposition is that stable matchings always exist.[39]

Corollary A1. ***(Existence)*** *There exists at least one stable matching.*

A.2. **The Supply and Demand Lemma.** We prove the continuum supply and demand Lemma 1.

Proof. **(Lemma 1)** Let μ be a stable matching, and $P = \mathcal{P}\mu$. Consider a student θ with $\mu(\theta) = c$. By definition of the operator $\mathcal{P}$, $P_c \leq e^\theta_c$. Consider a college c' that θ prefers over c. By right continuity, there is a student $\theta_+ = (\succ^\theta, e^{\theta+})$ with slightly higher scores than θ that is matched to c and prefers c'. By stability of μ all the students that are matched to c' have higher c' scores than θ_+, so $P_{c'} \geq e^{\theta+}_{c'} > e^\theta_{c'}$. Following the argument for all colleges that θ prefers to c, we see that there are no colleges that are better than

[39]Although we chose to follow Gale and Shapley's (1962) classic existence proof closely, it is possible to give a shorter existence proof using Tarski's fixed point theorem. We follow these lines in the extension to matching with contracts.

c and that θ can afford at cutoffs P. Therefore, c is better than any other college that θ can afford, so $D^\theta(P) = \mu(\theta)$. This implies that no college is over-demanded given P, and that $\mathcal{MP}\mu = \mu$. To conclude that P is a market clearing cutoff, note that if $\eta(\mu(c)) < S_c$ stability implies that a student whose first choice is c and has score at c of zero is matched to c. Therefore, $P_c = 0$.

To prove the other direction of the lemma, let P be a market clearing cutoff, and $\mu = \mathcal{M}P$. By the definition of $D^\theta(P)$, μ is right-continuous and measurable. Because P is a market clearing cutoff, μ respects capacity constraints. It respects the consistency conditions to be a matching by definition. To show that μ is stable, consider any potential blocking pair (θ, c) with $\mu(\theta) \prec^\theta c$. Since θ does not demand c (i.e., $\mu(\theta) = D^\theta(P) \neq c$), it must be that $P_c > e^\theta_c$, so $P_c > 0$ and c has no empty seats. For any type θ' such that $\theta' \in \mu(c)$, we have that $e^{\theta'}_c \geq P_c > e^\theta_c$, and therefore (θ, c) is not a blocking pair. Thus, μ is stable.

We now show that $\mathcal{PM}$ is the indentity. Let $P' = \mathcal{P}\mu$. If $\mu(\theta) = c$, then $e^\theta_c \geq P_c$. Therefore,

$$P'_c = (\mathcal{P}\mu)_c = \inf_{\theta \in \mu(c)} e^\theta_c \geq P_c.$$

However, if θ is a student with $e^\theta_c = P_c$ whose favorite college is c, then $\mu(\theta) = D^\theta(P) = c$. Therefore $P'_c \leq P_c$. These two inequalities imply that $P' = P$, and therefore $\mathcal{PM}P = P$. $\square$

A.3. **Lattice Theorem and Rural Hospitals Theorem.** Consider the sup ($\vee$) and inf ($\wedge$) operators on $\mathbb{R}^n$ as lattice operators on cutoffs. That is, given two vectors of cutoffs

$$(P \vee P')_c = \sup\{P_c, P'_c\}.$$

More generally, given an arbitrary set of cutoffs $X \subseteq 2^{([0,1]^C)}$, we define the sup and inf operators analogously. That is

$$(\vee X)_c = \sup_{P \in X} P_c.$$

We then have that the set of market clearing cutoffs forms a complete lattice with respect to these operators.

Theorem A1. ***(Lattice Theorem)*** *The set of market clearing cutoffs is a complete lattice under* $\vee, \wedge$.

Proof. First note that the set of market clearing cutoffs is nonempty. Now, consider two market clearing cutoffs P and P', and let $P^+ = P \vee P'$. Take a college c, and assume without loss of generality that $P_c \leq P'_c$. By the definition of demand, we must have that $D_c(P^+) \geq D_c(P')$, as $P^+_c = P'_c$ and the cutoffs of other colleges are higher under P^+. In addition, if $P'_c > 0$, then $D_c(P^+) \geq S_c \geq D_c(P)$. Moreover, if $P'_c = 0$, then $P_c = P'_c$, and $D_c(P^+) \geq D_c(P)$. Either way, we have that

$$D_c(P^+) \geq \max\{D_c(P), D_c(P')\}.$$

Moreover, the demand for staying unmatched $1 - \sum_{c \in C} D_c(\cdot)$ must at least as large under P^+ than under P or P'. Because demand for staying unmatched plus for all colleges always sums to 1, we have that, for all colleges, $D_c(P^+) = D_c(P) = D_c(P')$. In particular, P^+ is a market clearing cutoff. The proof for the inf operator is analogous. Moreover, it follows by induction that the sup and inf of any finite set of market clearing cutoffs is a market clearing cutoff. This establishes that the set of market clearing cutoffs is a lattice.

Last, we show that the lattice is a complete lattice. That is, that the sup of any arbitrary, and possibly infinite, set of market clearing cutoffs $\mathbf{P} \subseteq \mathbb{R}^C$ is a market clearing cutoff. Every subset of $\mathbb{R}^C$ has a sup, so that the vector $\vee\mathbf{P}$ is well-defined. Moreover, there exists a sequence of finite sets $\mathbf{P}_k \subseteq \mathbf{P}$ such that the vectors $\vee\mathbf{P}_k \in \mathbb{R}^C$ converge to $\vee\mathbf{P}$ as k grows. By the first part of the proof, each $\vee\mathbf{P}_k$ is a market clearing cutoff. Since the demand function is continuous,[40] the set of market clearing cutoffs is closed. It follows that $\vee\mathbf{P} = \lim_{k\to\infty} \vee\mathbf{P}_k$ is a market clearing cutoff. The proof for the inf operator is analogous. □

This theorem imposes a strict structure in the set of stable matchings. It differs from the classic Conway lattice theorem in the discrete setting (Knuth 1976), as the set of stable matchings forms a lattice with respect to the operation of taking the sup of the associated cutoff vectors. In the discrete model, where the sup of two matchings is defined as the matching where each student gets her favorite college in each of the matchings. Such a proposition does not carry over to the continuum model. As a direct corollary of the proof we have the following.

Theorem A2. ***(Rural Hospitals Theorem)*** *The measure of students matched to each college is the same in any stable matching. Furthermore, if a college does not fill its capacity, it is matched to the same set of students in every stable matching, except for a set of students with η measure* 0*.*

Proof. The first part was proved in the proof of Theorem A1, where we showed that, for any pair of market clearing cutoffs, aggregate demand is the same for all colleges. To see the second part, consider two stable matchings μ and μ'. Let $P = \mathcal{P}\mu$, $P' = \mathcal{P}\mu'$. Let $P^+ = P \vee P'$ and $\mu^+ = \mathcal{M}(P^+)$. Consider now a college c such that $\eta(\mu(c)) < S_c$. Therefore $0 = P_c = P'_c = \max\{P_c, P'_c\} = P^+_c = 0$. By the gross substitutes property of demand we have that $\mu(c) \subseteq \mu^+(c)$ and $\mu'(c) \subseteq \mu^+(c)$. By the

[40]The fact that demand is continuous follows from Lemma B1 below, which does not rely on the lattice theorem. To see this, take the sequences $\{S^k\}_{k\in\mathbb{N}}$ and $\{\eta^k\}_{k\in\mathbb{N}}$ in the statement of the lemma to be constant.

first part of the theorem we know that the measure of $\mu(c)$, $\mu'(c)$, and $\mu^+(c)$ are the same. Therefore, $\eta(\mu^+(c)\backslash\mu(c)) = 0$. Consequently, $\eta(\mu(c)\backslash\mu'(c)) \leq \eta(\mu^+(c)\backslash\mu'(c)) = 0$. Using a symmetric argument we get that $\eta(\mu'(c)\backslash\mu(c)) = 0$, completing the proof. □

This result implies that a hospital that does not fill its quota in one stable matching does not fill its quota in any other stable matching. Moreover, the measure of unmatched students is the same in every stable matching.

Appendix B. Main Results

B.1. **Uniqueness.** We can now prove Theorem 1. We denote the excess demand given a vector of cutoffs P and an economy $E = [\eta, S]$ by

$$z(P|E) = D(P|\eta) - S.$$

Proof. (Theorem 1)

Part (1):

By the lattice theorem, E has smallest and greatest market clearing cutoffs $P^- \leq P^+$, and corresponding stable matchings μ^-, μ^+. In the text, we prove that there is a unique stable matching when $P^+ > 0$. We now consider the general case, where it may be that for some colleges $P_c^+ = 0$. Let $C^+ = \{c \in C : P_c^+ \neq P_c^-\}$. In particular, for all colleges in C^+ we have $P_c^+ > 0$. Let $C^0 = C\backslash C^+$. Note that, since for all colleges $c \in C^0$ we have $P_c^+ = P_c^-$, and for all colleges c in C^+ we have $P_c^+ > P_c^-$, we have that

$$\{\theta \in \Theta : \mu^+(\theta) \in C^+\} \subseteq \{\theta \in \Theta : \mu^-(\theta) \in C^+\}.$$

By the rural hospitals theorem, the difference between these two sets must have measure 0. That is

$$\eta(\{\theta \in \Theta : \mu^-(\theta) \in C^+\}\backslash\{\theta \in \Theta : \mu^+(\theta) \in C^+\}) = 0.$$

Let $\succ^+$ be a fixed preference relation that ranks all colleges in C^+ higher than those in C^0. Then the set in the above equation must contain all students with preference $\succ^+$ and scores $P_c^- \leq e_c^\theta < P_c^+$ for all $c \in C^+$. That is,

$$\begin{aligned}&\{(\succ^+, e^\theta) \in \Theta : P_c^- \leq e_c^\theta < P_c^+ \text{ for all } c \in C^+\} \\ &\subseteq \{\theta \in \Theta : \mu^-(\theta) \in C^+\}\backslash\{\theta \in \Theta : \mu^+(\theta) \in C^+\}.\end{aligned}$$

Therefore, the measure of this set must be 0:

$$\eta(\{(\succ^+, e^\theta) \in \Theta : P_c^- \leq e_c^\theta < P_c^+ \text{ for all } c \in C^+\}) = 0.$$

By the full support assumption, and since $P_c^- < P_c^+$ for all c in C^+, this can only be the case if C^+ is the empty set. Since $P_c^- = P_c^+$ for $c \in C^0$, we have that $P^- = P^+$, and therefore there exists a unique vector of market clearing cutoffs.

Part (2):

The basic idea of the proof is similar to the particular case considered in the main text, with some care being necessary due to the added generality. We begin by noting that, as a consequence of Sard's Theorem,[41] and our definition of a regular distribution of types, for almost all vectors S, at all market clearing cutoffs of economy $[\eta, S]$ the demand function is continuously differentiable with a nonsingular derivative.

Claim B1. For almost every $S \in \mathbb{R}_+^C$ with $\sum_c S_c < 1$, at every market clearing cutoff P^* of $[\eta, S]$, demand is continuously differentiable and the derivative matrix $\partial D(P^*|\eta)$ is invertible.

Proof of Claim B1. Note that, for all S such that $\sum_c S_c < 1$, all market clearing cutoffs P^* of $[\eta, S]$ satisfy $D(P^*|\eta) = S$. Define the closure of the set of points where the demand function is not differentiable as

$$NDP = \text{closure}(\{P \in [0,1]^C : D(\cdot|\eta) \text{ is not continuosuly differentiable at } P\}).$$

Note that, by the definition of a regular distribution of types η, the image of NDP under $D(\cdot|\eta)$ has measure 0. In particular, for almost every S, demand at every associated market clearing cutoff is continuously differentiable.

Moreover, restricted to the open set $[0,1]^C \backslash NDP$, the demand function is continuously differentiable. Consequently, by Sard's theorem, the set of critical values of $D(\cdot|\eta)$ restricted to $[0,1]^C \backslash NDP$ has measure 0. That is, for almost all S, there are no vectors P in $[0,1]^C \backslash NDP$ such that $D(P|\eta) = S$ and $\partial D(P|\eta)$ is singular. Taken together, these two observations imply that, for almost all S, demand at associated market clearing cutoffs is both continuously differentiable and has an invertible derivative matrix. □

The key step in the proof is the following claim, which establishes that economies with multiple stable matchings must have at least one market clearing cutoff where demand is either not differentiable, or has a singular derivative.

Claim B2. Consider S such that $E = [\eta, S]$ has more than one stable matching. Then there exists at least one market clearing cutoff P^* of E where either demand is not differentiable, or the derivative matrix $\partial D(P^*|E)$ of the demand function is singular.

Proof of Claim B2. If there is at least one market clearing cutoff of E where demand is not differentiable, we are done. Consider now the case where demand is differentiable at all market clearing cutoffs of E.

[41] Sard's theorem can be stated as follows. Let $f : X \to \mathbb{R}^C$ be a continuously differentiable function from an open set $X \subseteq \mathbb{R}^C$ into $\mathbb{R}^C$. A critical point of f is $x \in X$ such that $\partial f(x)$ is a singular matrix. A critical value is any $y \in \mathbb{R}^C$ that is the image of a critical point. Sard's theorem states that the set of critical values of f has Lebesgue measure 0. See Guillemin and Pollack (1974); Milnor (1997), and the intuitive discussion of Sard's theorem in Section 4.

By the lattice theorem, economy E has smallest and largest market clearing cutoffs, with $P^- \leq P^+$. Let

$$C^+ = \{c : P^-_c < P^+_c\}.$$

C^+ is nonempty, due to the assumption that E has more than one market clearing cutoff. Let F be the subspace of $\mathbb{R}^C$ where all coordinates corresponding to colleges not in C^+ are zero, that is

$$F = \{v \in \mathbb{R}^C : v_c = 0 \text{ for all } c \notin C^+\}.$$

Consider $P \in [P^-, P^+]$. For any college $c \notin C^+$ we have $P^+_c = P^-_c = P_c$, and therefore

$$D_c(P^+|\eta) \geq D_c(P|\eta) \geq D_c(P^-|\eta).$$

By the rural hospitals theorem, $D_c(P^-|\eta) = D_c(P^+|\eta)$, and therefore $D_c(\cdot|\eta)$ is constant on the cube $[P^-, P^+]$. In particular, for any $c \in C^+$ and $c' \notin C^+$ we have that

$$\partial_c D_{c'}(P^-|\eta) = 0. \tag{B.1}$$

That is, the derivative matrix ∂D takes the subspace F into itself.

In addition, for all $P \in [P^-, P^+]$, it follows from the definition of demand that

$$\sum_{c' \in C^+} D_{c'}(P^-|\eta) \geq \sum_{c' \in C^+} D_{c'}(P|\eta) \geq \sum_{c' \in C^+} D_{c'}(P^+|\eta).$$

By the rural hospitals theorem, we have that $D_{c'}(P^-|\eta) = D_{c'}(P^+|\eta)$ for all $c' \in C^+$, and therefore $\sum_{c' \in C^+} D_{c'}(P|\eta)$ is constant on the cube $[P^-, P^+]$. This implies that

$$\sum_{c' \in C^+} \partial_c D_{c'}(P^-|\eta) = 0$$

for all $c, c' \in C^+$. Consequently, the linear transformation $\partial D(P^-|\eta)$ restricted to the subspace F is not invertible. Since $\partial D(P^-|\eta)$ takes F into itself, we have that $\partial D(P^-|\eta)$ is not invertible, proving the claim.

□

The result now follows from Claims B1 and B2. Take S such that $\sum_c S_c < 1$ and economy $E = [\eta, S]$ has more than one stable matching. By Claim B2, in at least one of the market clearing cutoffs of E, demand is either non-differentiable, or has a singular derivative. However, Claim B1 shows that this only holds for a measure 0 set of vectors S. Therefore, the set of vectors S such that $\sum_c S_c < 1$ and there is more than one stable matching has measure 0, completing the proof. □

B.2. **Continuity and convergence.**

B.2.1. *Continuity Within $\mathcal{E}$.* This section establishes that the stable matching correspondence is continuous around an economy $E \in \mathcal{E}$ with a unique stable matching. That is, that if a continuum economy has a unique stable matching, it varies continuously with the fundamentals.

Note that, by our definition of convergence, we have that if the sequence of continuum economies $\{E^k\}_{k\in\mathbb{N}}$ converges to a continuum economy E, then the functions $z(\cdot|E^k)$ converge pointwise to $z(\cdot|E)$. Moreover, using the assumption that firms' indifference curves have measure 0 at E, we have that, if we simultaneously take convergent sequences of cutoffs and economies, then the associated excess demand converges to excess demand in the limit. This is formalized in the following lemma.

Lemma B1. *Consider a continuum economy $E = [\eta, S]$, a vector of cutoffs P and a sequence of cutoffs $\{P^k\}_{k\in\mathbb{N}}$ converging to P. If $\{\eta^k\}_{k\in\mathbb{N}}$ converges to η in the weak-* sense and $\{S^k\}_{k\in\mathbb{N}}$ converges to S then*

$$z(P^k|[\eta^k, S^k]) = D(P^k|\eta^k) - S^k$$

converges to $z(P|E)$.

Proof. Let G^k be the set

$$G^k = \cup_c\{\theta \in \Theta : \|e^\theta_c - P_c\| \leq \sup_{k'\geq k} \|P^{k'}_c - P_c\|\}.$$

The set

$$\cap_k G^k = \cup_c\{\theta \in \Theta : e^\theta_c = P_c\},$$

has η-measure 0 by the strict preferences assumption 1. Since the G^k are nested, we have that $\eta(G^k)$ converges to 0 as $k \to \infty$.

Now take $\epsilon > 0$. There exists k_0 such that for all $k \geq k_0$ we have $\eta(G^k) < \epsilon/4$. Since the measures η^k converge to η in the weak sense, we may assume also that $\eta^k(G^{k_0}) < \epsilon/2$. Since the G^k are nested, this implies $\eta^k(G^k) < \epsilon/2$ for all $k \geq k_0$. Note that $D^\theta(P)$ and $D^\theta(P^k)$ may only differ for $\theta \in G^k$. We have that

$$\|D(P|\eta) - D(P^k|\eta^k)\| \leq \|D(P|\eta) - D(P|\eta^k)\| + \|D(P|\eta^k) - D(P^k|\eta^k)\|.$$

As η^k converges to η, we may take k_0 large enough so that the first term is less than $\epsilon/2$. Moreover, since the measure $\eta(G^k) < \epsilon/2$, we have that for all $k > k_0$ the second term is less than $\epsilon/2$. Therefore, the above difference is less than ϵ, completing the proof. □

Note that this lemma immediately implies the following:

Lemma B2. *Consider a continuum economy $E = [\eta, S]$, a vector of cutoffs P a sequence of cutoffs $\{P^k\}_{k\in\mathbb{N}}$ converging to P, and a sequence of continuum economies $\{E^k\}_{k\in\mathbb{N}}$ converging to E. We have that $z(P^k|E^k)$ converges to $z(P|E)$.*

Using the lemma, we show that the stable matching correspondence is upper hemicontinuous.

Proposition B1. ***(Upper Hemicontinuity)*** *The stable matching correspondence is upper hemicontinuous*

Proof. Consider a sequence $\{E^k, P^k\}_{k\in\mathbb{N}}$ of continuum economies and associated market clearing cutoffs, with $E^k \to E$ and $P^k \to P$, for some continuum economy E and vector of cutoffs P. We have $z(P|E) = \lim_{k\to\infty} z(P^k, E^k) \leq 0$. If $P_c > 0$, for large enough k we must have $P_c^k > 0$ so that $z_c(P|E) = \lim_{k\to\infty} z_c(P^k, E^k) = 0$. Therefore, P is a market clearing cutoff of E. □

With uniqueness, continuity also follows easily.

Lemma B3. ***(Continuity)*** *Let E be a continuum economy with a unique stable matching. Then the stable matching correspondence is continuous at E.*

Proof. Let P be the unique market clearing cutoff of E. Consider a sequence $\{E^k, P^k\}_{k\in\mathbb{N}}$ of economies and associated market clearing cutoffs, with $E^k \to E$. Assume, by contradiction that P^k does not converge to P. Then P^k has a convergent subsequence that converges to another point $P' \in [0,1]^C$, with $P' \neq P$. By the previous proposition, P' must be a market clearing cutoff of E, contradicting the fact that P is the unique market clearing cutoff of E. □

B.2.2. *Convergence of Finite Economics.* We now consider the relationships between the stable matchings of a continuum economy, and stable matchings of a sequence of discrete economies that converge to it. The argument follows similar lines as that for convergence of a sequence of continuum economies in the preceding Subsection.

For finite economies F, we define the excess demand function as in the continuous case:

$$z(P|F) = D(P|F) - S.$$

Note that, with this definition, P is a market clearing cutoff for finite economy F iff $z(P|F) \leq 0$, with $z_c(P|F) = 0$ for all colleges c such that $P_c > 0$.

From Lemma B1 we immediately obtain the following result.

Lemma B4. *Consider a limit economy E, a sequence of cutoffs $\{P^k\}_{k\in\mathbb{N}}$ converging to P, and a sequence of finite economies $\{F^k\}_{k\in\mathbb{N}}$ converging to E. We then have that $z(P^k|F^k)$ converges to $z(P|E)$.*

This lemma then implies the following upper hemicontinuity property.

Proposition B2. ***(Convergence)*** *Let E be a continuum economy, and $\{F^k, P^k\}_{k\in\mathbb{N}}$ a sequence of discrete economies and associated market clearing cutoffs, with $F^k \to E$ and $P^k \to P$. Then P is a market clearing cutoff of E.*

Proof. **(Proposition B2)** We have $z(P|E) = \lim_{k\to\infty} z(P^k|F^k) \leq 0$. If $P_c > 0$, then $P_c^k > 0$ for large enough k, and we have $z_c(P|E) = \lim_{k\to\infty} z_c(P^k|F^k) = 0$. □

When the continuum economy has a unique stable matching, we can prove the stronger result below.

Lemma B5. ***(Convergence with uniqueness)*** *Let E be a continuum economy with a unique market clearing cutoff P, and $\{F^k, P^k\}_{k\in\mathbb{N}}$ a sequence of discrete economies and associated market clearing cutoffs, with $F^k \to E$. Then $P^k \to P$.*

Proof. **(Lemma B5)** To reach a contradiction, assume that P^k does not converge to P. Then P^k has a convergent subsequence that converges to another point $P' \in [0,1]^C$, with $P' \neq P$. By Proposition B2, P' is a market clearing cutoff of E. Therefore, we have that $P' \neq P$ is a market clearing cutoff, a contradiction with P being the unique market clearing cutoff of E. □

B.2.3. *Proof of Theorem 2.* Theorem 2 follows from the previous results.

Proof. **(Theorem 2)** Part (3) follows from Lemma B3 and Part (1) follows from Lemma B2. As for Part (2), note first that given an economy F^k the set of market clearing cutoffs is compact, which follows from the definition of market clearing cutoffs, and continuity of the demand function. Therefore, there exist market clearing cutoffs P^k and P'^k of F^k such that the diameter of F^k is $\|P^k - P'^k\|$. However, by Part (1), both sequences $\{P^k\}_{k\in\mathbb{N}}$ and $\{P'^k\}_{k\in\mathbb{N}}$ are converging to P, and therefore the diameter of F^k is converging to 0. □

Appendix C. Additional Proofs

This section collects proofs that were omitted in the text. We begin by establishing main proposition for the school competition application in Section 5.1.

Proof. (Proposition 1)

Aggregate quality is defined as

$$\begin{aligned} Q_c(\delta) &= \int_{\mu_\delta(c)} e_c^\theta d\eta_\delta(\theta) \\ &= \int_{\{\theta: D^\theta(P^*(\delta))=c\}} e_c^\theta \cdot f_\delta(\theta) d\theta. \end{aligned}$$

By Leibniz's rule, Q_c is differentiable in δ_c, and the derivative is given by

$$\begin{aligned} \frac{dQ_c(\delta)}{d\delta_c} = & \int_{\{\theta: D^\theta(P^*(\delta))=c\}} e_c^\theta \cdot \tfrac{d}{d\delta_c} f_\delta(\theta) d\theta \\ + & \sum_{c'\neq c} \tfrac{dP^*_{c'}}{d\delta_c} \cdot M_{c'c} \cdot \bar{P}_{c'c} \\ - & \tfrac{dP^*_c}{d\delta_c} \cdot [M_{c\emptyset} + \sum_{c'\neq c} M_{cc'}] \cdot P^*_c. \end{aligned} \tag{C.1}$$

The first term is the integral of the derivative of the integrand, and the last two terms the change in the integral due to the integration region $\{\theta : D^\theta(P^*(\delta)) = c\}$ changing with δ_c. The terms in the second line are the changes due to changes in the cutoffs $P^*_{c'}$, the students that school c gains (or loses) because school c' becomes more (less) selective. The quantity of these students is $\frac{dP^*_{c'}}{d\delta_c} \cdot M_{c'c}$, and their average quality $\bar{P}_{c'c}$. The last line is the term representing the students lost due to school c raising its cutoff P_c. These students number $[M_{c\emptyset} + \sum_{c' \neq c} M_{c'c}]$, and have average quality P^*_c. Note that, since the total number of students admited at school c is constant and equal to S_c, we have

$$
\begin{aligned}
0 = & \int_{\{\theta: D^\theta(P^*(\delta))=c\}} \tfrac{d}{d\delta_c} f_\delta(\theta) d\theta \\
+ & \quad \sum_{c' \neq c} \tfrac{dP^*_{c'}}{d\delta_c} \cdot M_{c'c} \\
- & \tfrac{dP^*_c}{d\delta_c} \cdot [M_{c\emptyset} + \sum_{c' \neq c} M_{cc'}].
\end{aligned}
$$

Therefore, if we substitute $\frac{dP^*_c}{d\delta_c} \cdot [M_{c\emptyset} + \sum_{c' \neq c} M_{cc'}]$ in equation (C.1) we have

$$
\begin{aligned}
\frac{dQ_c(\delta)}{d\delta_c} = & \int_{\{\theta: D^\theta(P^*(\delta))=c\}} [e^\theta_c - P^*_c] \cdot \tfrac{d}{d\delta_c} f_\delta(\theta) d\theta \\
+ & \quad \sum_{c' \neq c} \tfrac{dP^*_{c'}}{d\delta_c} \cdot M_{c'c} \cdot [\bar{P}_{c'c} - P^*_c].
\end{aligned}
$$

The term in the second line is the market power effect as defined in the text. That the term in the first line equals the expression in Proposition 1 follows from the definition of N_c and $\bar{e}_c$.

To see that the direct effect is positive, note that by definition $\bar{e}_c \geq P^*(\delta)$, and since $u^i_c(\delta)$ is increasing in δ_c we have $N_c \geq 0$. □

We now provide the derivation of the market power effect in Section 5.1 when schools are symmetrically differentiated.

Additional details on Section 5.1.

In Section 5.1 we gave a formula for the market power effect when the function $f_\delta(\theta)$ is symmetric over all schools, and schools choose the same level of quality. This formula follows from substituting an expression for $dP^*/d\delta_c$ in the formula for the market power term. To obtain the formula for $dP^*/d\delta_c$, we start from the point δ where all $\delta_c = \delta_{c'}$. In this case, all $P^*_c(\delta) = P^*_{c'}(\delta)$. If school c changes δ_c, the the cutoff $P^*_c(\delta)$ of school c will change. The cutoffs of the other schools will change, but all other schools $c' \neq c$ will have the same cutoff $P^*_{c'}(\delta)$. We denote $D_c(P_c, P_{c'}|\delta)$ for the demand for school c, and $D_{c'}(P_c, P_{c'}|\delta)$ for the demand for each other school under these cutoffs. Applying the implicit function theorem to the system of two equations

$$
\begin{aligned}
D_c(P_c, P_{c'}|\delta) &= S_c \\
D_{c'}(P_c, P_{c'}|\delta) &= S_c
\end{aligned}
$$

we get

$$\begin{aligned}
\frac{d}{dP_c}D_c \cdot \frac{dP_c}{d\delta_c} + \frac{d}{dP_{c'}}D_c \cdot \frac{dP_{c'}}{d\delta_c} + \frac{d}{d\delta_c}D_c &= 0 \\
\frac{d}{dP_c}D_{c'} \cdot \frac{dP_c}{d\delta_c} + \frac{d}{dP_{c'}}D_{c'} \cdot \frac{dP_{c'}}{d\delta_c} + \frac{d}{d\delta_c}D_{c'} &= 0.
\end{aligned}$$

Substituting the derivative of the demand function as a function of the mass of agents on the margins $\tilde{M}_{cc'}$, the system becomes

$$\begin{aligned}
-[M_{c\emptyset} + (C-1)M_{cc'}] \cdot \frac{dP_c}{d\delta_c} + [(C-1)M_{c'c}] \cdot \frac{dP_{c'}}{d\delta_c} + \frac{d}{d\delta_c}D_c &= 0 \\
[M_{cc'}] \cdot \frac{dP_c}{d\delta_c} - [M_{c'\emptyset} + M_{c'c}] \cdot \frac{dP_{c'}}{d\delta_c} + \frac{d}{d\delta_c}D_{c'} &= 0.
\end{aligned}$$

Due to the symmetry of the problem, $M_{cc'} = M_{c'c}$, $M_{c\emptyset} = M_{c'\emptyset}$, and $\frac{d}{d\delta_c}D_c = -(C-1)\frac{d}{d\delta_c}D_{c'}$. The formula in the text then follows from solving the system.

Next, we establish the bound for how close the market clearing cutoffs of a discrete economy are to those in a continuum economy with a similar distribution of student types and supply of seats per capita.

Proof. (Proposition 2) Note that since $\sum_c S_c < 1$, market clearing cutoffs satisfy $z(P|E) = 0$. In what follows we always take α to be large enough such that, for any finite economy F such that the bound in the proposition has any content (that is, the right side is less than one), $\sum_c S_c^F < 1$. This guarantees that market clearing cutoffs in such an economy must satisfy $z(P|F) = 0$.

The proof begins by showing that at economy E, cutoffs P that are far from the market clearing cutoff P^* have large excess demands, in the sense that their norm is bounded below by a multiple of the distance to the market clearing cutoff P^*. Formally, let $B^\epsilon = \{P \in [0,1]^C : \|P - P^*\| < \epsilon\}$. Let

$$\begin{aligned}
P^\epsilon &= \arg\min_{P \notin B^\epsilon} ||z(P|E)|| \text{ and} \\
M^\epsilon &= \min_{P \notin B^\epsilon} ||z(P|E)||.
\end{aligned}$$

Note that, due to the continuity of the demand function, both P^ϵ and M^ϵ are well defined. Moreover, P^ϵ may be a set of values, in the case of multiple minima. In what follows, we will take a single-valued selection from this set, so that P^ϵ represents one of the minima. With this convention, $M^\epsilon = \|z(P^\epsilon|E)\|$. We will now show that there exists $\alpha > 0$ such that for all $0 < \epsilon \leq 1$

$$M^\epsilon \geq \frac{1}{\alpha} \cdot \epsilon.$$

To see this, note that since $D(\cdot|E)$ is C^1, we have that

$$z(P|E) = D(P|E) - S = \partial D(P^*|E) \cdot (P - P^*) + g(P - P^*), \tag{C.2}$$

where the continuous function $g(\cdot)$ satisfies that for any $\epsilon' > 0$, there exists $\delta > 0$ such that, for all $P \in B^{\delta}$,

$$\frac{\|g(P - P^*)\|}{\|P - P^*\|} < \epsilon'.$$

Since $\partial_P D(P^*|E)$ is nonsingular, we may take $A > 0$ such that

$$||\partial_P D(P^*|E) \cdot v|| \geq 2A \cdot ||v||, \tag{C.3}$$

for any vector $v \in \mathbb{R}^C$.

By the property of $g(\cdot)$ above, with $\epsilon' = A$, we may take $0 < \epsilon_0 \leq 1$ such that

$$\frac{||g(P - P^*)||}{||P - P^*||} < A \tag{C.4}$$

for all $P \in B^{\epsilon_0}$. Therefore, for all $P \in B^{\epsilon_0}$ we have

$$\begin{aligned}
||z(P|E)|| &= ||\partial D(P^*|E) \cdot (P - P^*) + g(P - P^*)|| \\
&\geq ||\partial D(P^*|E) \cdot (P - P^*)|| - ||g(P - P^*)|| \\
&\geq 2A \cdot ||P - P^*|| - ||\frac{g(P - P^*)}{||P - P^*||}|| \cdot ||P - P^*|| \\
&\geq 2A \cdot ||P - P^*|| - A \cdot ||P - P^*|| \\
&= A \cdot ||P - P^*||.
\end{aligned}$$

The first equality follows from the derivative formula for excess demand in equation (C.2). The inequality in the second line follows from the triangle inequality. The inequality in the third line follows from the bound in inequality (C.3) for the left term, and algebra for the right term. The inequality in the fourth line is a consequence of applying the bound in inequality (C.4) to the right term. Finally, the last line follows from subtracting the right term from the left term. The above reasoning establishes that for all $P \in B^{\epsilon_0}$ excess demand is bounded from below by

$$\|z(P|E)\| \geq A \cdot \|P - P^*\|,$$

which is linear in $\|P - P^*\|$. In particular, this implies that, for all $0 < \epsilon < \epsilon_0$, we have

$$M^{\epsilon} \geq \min\{A \cdot \epsilon, M^{\epsilon_0}\}. \tag{C.5}$$

We will now use this bound to obtain a bound that is valid for all $0 < \epsilon \leq 1$. Since E has a unique stable matching we have that $M^{\epsilon_0} > 0$. Take $\alpha > 0$ such that

$$\frac{1}{\alpha} = \min\{A, M^{\epsilon_0}\}.$$

Therefore, if $0 < \epsilon < M^{\epsilon_0}/A$ we have $M^{\epsilon} \geq \min\{A \cdot \epsilon, M^{\epsilon_0}\} = A \cdot \epsilon \geq \frac{1}{\alpha} \cdot \epsilon$. If $M^{\epsilon_0}/A \leq \epsilon \leq 1$, then $M^{\epsilon} \geq \min\{A \cdot \epsilon, M^{\epsilon_0}\} = M^{\epsilon_0} \geq \frac{1}{\alpha} \geq \frac{1}{\alpha}\epsilon$. Either way, we have the desired bound

$$M^{\epsilon} \geq \frac{1}{\alpha} \cdot \epsilon \tag{C.6}$$

for all $0 < \epsilon \leq 1$.

We now prove the proposition. If P^F is a market clearing vector of the finite economy F then

$$\|z(P^F|E) - z(P^F|F)\| = \|z(P^F|E)\| \geq \frac{1}{\alpha} \cdot \|P^F - P^*\|.$$

The first equality follows from excess demand at a strictly positive market clearing cutoff being 0, and the second by the bound for M^{ϵ} in inequality (C.6). Moreover, by the triangle inequality we have that

$$\begin{aligned} ||z(P^F|E) - z(P^F|F)|| &\leq ||D(P^F|\eta) - D(P^F|\eta^F)|| + ||S - S^F|| \\ &\leq \sup_{P \in [0,1]^C} ||D(P|\eta) - D(P|\eta^F)|| + ||S - S^F||. \end{aligned}$$

Combining these two inequalities we obtain the desired bound

$$\|P^F - P^*\| \leq \alpha \cdot (\sup_{P \in [0,1]^C} \|D(P|\eta) - D(P|\eta^F)\| + \|S - S^F\|).$$

□

Using this bound, we can prove the results on sequences of randomly drawn finite economies.

Proof. (Proposition 3)

Part (1): Almost sure convergence.

First we show that that the sequence of random economies $\{F^k\}_{k \in \mathbb{N}}$ converges to E almost surely. It is true by assumption that S^k converges to S. Moreover, by the Glivenko-Cantelli Theorem, the realized measure η^k converges to η in the weak-* topology almost surely. Therefore, by definition of convergence, we have that F^k converges to E almost surely. This implies, by Theorem 2, that μ^k converges to μ.

Part (2): Bound on $\|P^* - P^k\|$.

By Proposition 2, we may take $\alpha_0 \geq 0$ such that, for all k, realization of the discrete economy F^k, and market clearing cutoff P^k of F^k,

$$\|P^k - P^*\| \leq \alpha_0 \cdot (\sup_{P \in [0,1]^C} \|D(P|\eta) - D(P|\eta^k)\| + \|S - S^k\|). \tag{C.7}$$

Let the agents in finite economy F^k be $\theta^{1,k}, \theta^{2,k}, \cdots, \theta^{k,k}$. The demand function at economy F^k is the random variable

$$D_c(P|\eta^k) = \sum_{i=1,\cdots,k} 1_{\theta^{i,k} \in \{\theta \in \Theta : D^{\theta}(P) = c\}} / k.$$

That is, D_c are similar to empirical distribution functions, measuring the fraction of agents $\theta^{i,k}$ whose types are in the set $\{\theta \in \Theta : D^\theta(P) = c\}$. By the Vapnik-Chervonenkis Theorem,[42] there exists exists α_1 such that the probability

$$\Pr\{\sup_{P\in[0,1]^C} |D(P|\eta^k) - D(P|\eta)| > \epsilon/2\alpha_0\} \leq \alpha_1 \cdot \exp(-\frac{k}{8}(\frac{\epsilon}{2\alpha_0})^2).$$

Note that this bound is uniform in P.

If this is the case, by equation (C.7), the distance of all market clearing cutoffs P^k of F^k is bounded by

$$\begin{aligned} ||P^k - P^*|| &\leq \alpha_0\epsilon/2\alpha_0 + \alpha_0 \cdot ||S - S^k|| \\ &\leq \epsilon/2 + \alpha_0/k. \end{aligned}$$

Therefore, for $k \geq k_0 \equiv 2\alpha_0/\epsilon$,

$$\|P^k - P^*\| \leq \epsilon.$$

This implies that there exist $\alpha \geq 0$ and $\beta > 0$ such that the probability that F^k has any market clearing cutoffs with $|P^k - P^*| > \epsilon$ is lower than $\alpha e^{-\beta k}$, for any $k \geq k_0$. Moreover, we may take α such that the bound is only informative for $k \geq k_0$, so that the bound holds for all k as in the proposition statement. This completes the proof.

Part (3): Bound on G^k.

Let $\bar{f}$ be the supremum of the density of η. Denote the set of agents with scores which have at least one coordinate close to P^*_c as

$$\bar{\Theta} = \{\theta \in \Theta : \exists c \in C : \|e^\theta_c - P^*_c\| \leq \epsilon/4C\bar{f}\}.$$

The η measure of the set $\bar{\Theta}$ is bounded by

$$\eta(\bar{\Theta}) \leq 2C\bar{f} \cdot (\epsilon/4C\bar{f}) = \epsilon/2.$$

Let the agents in finite economy F^k be $\theta^{1,k}, \theta^{2,k}, \cdots, \theta^{k,k}$. The fraction of agents in economy F^k that have types in $\bar{\Theta}$ is given by the random variable

$$\tilde{G}^k = \sum_{i=1,\cdots,k} 1_{\theta^{i,k}\in\bar{\Theta}}/k.$$

By the Vapnik-Chervonenkis Theorem, in the argument of Part (2), we could have taken the constants $\alpha' \geq 0$ and $\beta' > 0$ in a way that the probability that both the fraction of agents with types in $\bar{\Theta}$ differs from the expected number $\eta(\bar{\Theta}) \leq \epsilon/2$ by more than $\epsilon/2$

[42]See Theorem 12.5 in Devroye et al. (1996) p. 197. As remarked in p. 198, the bound given in p. 197 is looser than the bound originally established by Vapnik and Chervonenkis (1971), which we use. The simple proof given in Devroye et al. (1996) follows the lines of Pollard (1984). The theorem can be proven using Hoeffding's inequality, and generalizes the Dvoretzky et al. (1956) inequality to the multidimensional case, and to arbitrary classes of measurable sets, not only sets of the form $\{x \in \mathbb{R}^n : x \leq \bar{x}\}$. The important requirement for the theorem to apply in our setting is that the Vapnik-Chervonenkis dimension of the class of sets $\{\theta \in \Theta : D^\theta(P) = c\}$ is finite.

is lower than $\alpha' e^{-\beta' k}/2$, and the probability

$$\Pr\{\sup_{P\in[0,1]^C} \|D(P|\eta^k) - D(P|\eta)\| > \epsilon/2\alpha_0\} \leq \alpha' e^{-\beta' k}/2.$$

If neither event happens, then $\tilde{G}^k \leq \epsilon/2 + \epsilon/2 = \epsilon$. Moreover, whenever this is the case all agents θ matched to a college different than $D^\theta(P^*)$ must be in $\bar{\Theta}$, so that $G^k \leq \tilde{G}^k \leq \epsilon$. The probability that neither event happens is at least $1 - \alpha' e^{-\beta' k}/2 - \alpha' e^{-\beta' k}/2 = 1 - \alpha' e^{-\beta' k}$. Therefore, the probability that $G^k > \epsilon$ is bounded above by $\alpha' e^{-\beta' k}$, as desired. □

We now consider the robustness result showing that comparative statics of the continuum model can be used to derive setwise comparisons for the discrete Gale and Shapley model.

Proof. (Proposition 4) Consider the case where $P_c > P'_c$, the other case is analogous. Let $\epsilon = \|P_c - P'_c\|$. By Theorem 2 Part (3) the diameters of the set of stable matchings of economies in the sequences F^k and F'^k converge to 0. Therefore, we may take k_0 large enough such that for any market clearing cutoffs P^k, P'^k of F^k, F'^k we have

$$\begin{aligned} \|P^k - P\| &< \epsilon/2 \\ \|P'^k - P\| &< \epsilon/2. \end{aligned}$$

By the triangle inequality we then have

$$P^k_c > P_c - \epsilon/2 \geq P'_c + \epsilon/2 > P'^k_c.$$

□

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Supplementary Appendix (Not for Publication)

The supplementary appendix contains additional results discussed in the paper. Appendix D extends the basic model to allow for flexible wages and contracts. Appendix E discusses the connection between pre-matchings and cutoffs, and Appendix F gives an example of a school choice problem where DA-STB is inefficient.

Appendix D. Matching with Flexible Wages and Contracts

In many markets, agents negotiate not only who matches with whom, but also wages and other contractual terms. When hiring faculty most universities negotiate both in wages and teaching load. Firms that supply or demand a given production input may negotiate, besides the price, terms like quality or timeliness of the deliveries. This section extends the continuum model to include these possibilities. Remarkably, it is still the case that stable matchings have the simple cutoff structure described above. The extension permits the comparison of different market institutions, such as personalized versus uniform wages.

D.1. **The Setting.** Following the standard terminology, we now consider a set of **doctor types** $\theta \in \Theta$ distributed according to a measure η, a finite set of **hospitals** $h \in H$, with H also denoting the number of hospitals, and a set of **contracts** X. η is defined over a σ-algebra Σ^{Θ}. Each contract x in X specifies

$$x = (\theta_x, h_x, w_x).$$

That is, a doctor θ_x, a hospital h_x, and other terms of the contract w_x. In addition, the set of contracts is assumed to contain an empty contract $\emptyset \in X$, which corresponds to being unmatched. A case of particular interest, which we later return to, is when w is a wage, and agents have quasilinear preferences.

A **matching** is a function $\mu : \Theta \cup H \to X \cup 2^X$, such that

(1) For all $\theta \in \Theta$: $\mu(\theta) \in \{x : x = \emptyset \text{ or } \theta_x = \theta\}$.
(2) For all $h \in H$: $\mu(h) \subseteq \{x : h_x = h\}$, the set $\{\theta_x : x \in \mu(h)\}$ is measurable, and $\eta(\{\theta_x : x \in \mu(h)\}) \leq S_h$.
(3) If $h_{\mu(\theta)} = h$ then $\mu(\theta) \in \mu(h)$, and if for some $x \in \mu(h)$ we have $\theta_x = \theta$, then $\mu(\theta) = x$.

That is, a matching associates each doctor (hospital) to a (set of) contract(s) that contains it, or to the empty contract. In addition, each doctor can be assigned to at most one contract. We say that a hospital h and a doctor θ are matched at μ if $h_{\mu(\theta)} = h$. Moreover, hospitals must be matched to a set of doctors of measure not exceeding its capacity S_h. Finally, (3) is a consistency condition that a doctor is matched to a hospital iff the hospital is matched to the doctor.

Models of matching with contracts have been proposed by Kelso and Crawford (1982); Hatfield and Milgrom (2005).[43] Those papers define stable matchings with respect to preferences of firms over sets of contracts. We focus on a simpler model, where stability is defined with respect to preferences of firms over single contracts. This corresponds to the approach that focuses on responsive preferences in the college admissions problem. This restriction considerably simplifies the exposition, as the same arguments used in the previous sections may be applied. In what follows, we assume that hospitals have **preferences** over single contracts that contain it, and the empty contract, and agents have preferences over contracts that contain them and over being unmatched.

Assume that doctors' preferences can be expressed by a **utility function** $u^\theta(x)$, and hospitals' by a utility function $\pi_h(x)$. The utility of being unmatched is normalized to 0. In the continuum model, we impose some restrictions on preferences and the set of available contracts. Let X_h^θ be the set of contracts that contain both a hospital h and a doctor θ.

Assumption D1. ***(Regularity Conditions)***

- **(Compactness)** There exists $M > 0$ such that, for any doctor-hospital pair θ, h, the set
$$\{(u^\theta(x), \pi_h(x)) | x \in X_h^\theta\}$$
is a compact subset of $[0, M]^2$.
- **(No Redundancy)** Given θ, h, no contract in $x \in X_h^\theta$ weakly Pareto dominates, nor has the same payoffs as another contract $x' \in X_h^\theta$.
- **(Completeness)** Given a hospital $h \in H$ and $0 < p \le M$, there exists a doctor θ such that $X_{h'}^\theta = \emptyset$ for $h' \neq h$, and
$$\sup_{x:u^\theta(x)>0} \pi_h(x) = \max_{x:u^\theta(x)\ge 0} \pi_h(x) = p.$$
- **(Measurability)** Given any Lebesgue measurable set K in $\mathbb{R}^2$ and $h \in H$, the σ-algebra Σ^Θ contains all sets of the form
$$\{\theta \in \Theta | K = \{(u^\theta(x), \pi_h(x)) | x \in X_h^\theta\}\}.$$

The assumptions accomodate many cases of interest, such as the transferable utility model considered below. The conditions make the analysis based on cutoffs tractable for the following reasons. The compactness and no redundancy assumptions guarantee that, for any hospital-doctor pair, there always exists a unique contract that is optimal

[43]See Sönmez and Switzer (Forthcoming); Sönmez (Forthcoming) for applications of these models to real-life market design problems.

for the doctor subject to giving the hospital a minimum level of profits. Completeness guarantees that there is always a marginal doctor type that is not matched to a hospital, even if this doctor type is not in the support of the distribution of types. The measurability condition guarantees that the measure η over the set of doctors is sufficiently rich so that a matching defined by cutoffs is measurable.

We are now ready to define stable matchings. Note that, since we only consider contracts yielding non-negative payoffs to both parties, we do not have to worry about individual rationality. A doctor-hospital pair θ, h is said to **block** a matching μ if there is a contract $x = (\theta, h, w)$ that θ prefers over $\mu(\theta)$, that is $u^\theta(x) > u^\theta(\mu(\theta))$, and either (i) hospital h does not fill its capacity $\eta(\{\theta_{x'} : x' \in \mu(h)\}) < S_h$, and h prefers x to the empty contract, $\pi_h(x) > 0$, or (ii) h is matched to a contract $x' \in \mu(h)$ which it likes strictly less than contract x, that is $\pi_h(x') < \pi_h(x)$.

Definition 4. A matching μ is **stable** if it has no blocking pairs.

D.2. **Cutoffs.** We now show that, within our matching with contracts framework, the allocation of doctors to hospitals is determined by an H-dimensional vector of cutoffs. It is convenient to think of cutoffs as the marginal value of capacity at each hospital – how much utility the hospital would gain from a small increase in capacity. Cutoffs are numbers $P_h \in [0, M]$, and a vector of cutoffs $P \in [0, M]^H$.

Denote an agent's maximum utility of working for a hospital h and providing the hospital with utility of at least P_h as

$$\begin{aligned} \bar{u}_h^\theta(P) &= \sup u^\theta(x) \\ \text{s.t.} & \quad x \in X_h^\theta \\ & \quad \pi_h(x) \geq P_h. \end{aligned}$$

We refer to this as the **reservation utility**[44] that hospital h offers doctor θ. Note that the reservation utility may be $-\infty$ if the feasible set $\{x \in X_h^\theta : \pi_x(x) \geq P_h\}$ is empty. Moreover, whenever this sup is finite, it is attained by some contract x, due to the compactness assumption. We define $\bar{u}_\emptyset^\theta(\cdot) \equiv 0$.

Now define a doctor's demand as the hospital that offers her the highest reservation utility given a vector of cutoffs. Note that doctors demand hospitals, and not contracts. The **demand** of a doctor θ given a vector of cutoffs P is

$$D^\theta(P) = \arg \max_{H \cup \{\emptyset\}} \bar{u}_h^\theta(P),$$

Demand may not be uniquely defined, as an agent may have the same reservation utility in more than one hospital.

[44]Note that, even though formally $\bar{u}_h^\theta(P)$ depends on the entire vector P, it is constant on the coordinates P_{-h}.

Henceforth, we will make the following assumption, which guarantees that demand is uniquely defined for almost all doctors.

Assumption D2. ***(Strict Preferences)*** *For any cutoff vector $P \in [0, M]^C$, and hospital h, the following sets have η-measure* 0*:*

- The set of doctors with $\bar{u}_h^\theta(P) = \bar{u}_{h'}^\theta(P) > 0$ for some hospital $h' \neq h$.
- The set of doctors for which $\bar{u}_h^\theta(P) = 0$.
- The set of doctors for which $\bar{u}_h^\theta(P)$ is not continuous at P.

The first two requirements ask that, for any vector of cutoffs P, the set of doctors who are indifferent between the best offers of two hospitals, or of a hospital and being unmatched, has measure 0. This is true if there is sufficient heterogeneity of preferences in the population, with types having a non-atomic distribution, hence why this is termed a strict preferences assumption. The third condition is that, at a fixed P, reservation utility varies continuously for almost all doctors. The intuition is that, since reservation utility is decreasing in P_h, it can be discontinuous for at most a countable number of values of P_h. The assumption is that there is sufficient heterogeneity among doctors such that these discontinuities coincide for only a measure 0 set of doctors.

From now on, we fix a measurable selection from the demand correspondence, so that it is a function. The **aggregate demand** for a hospital is defined as

$$D_h(P) = \eta(\{\theta \in \Theta : D^\theta(P) = h\}).$$

The aggregate demand vector is defined as $D(P) = \{D_h(P)\}_{h \in H}$. Note that $D_h(P)$ does not depend on the arbitrarily defined demand of agents which are indifferent between more than one hospital, by the strict preferences assumption. Furthermore, demand is continuous in P, as shown by the following claim.

Claim D1. $D(P)$ is continuous in P

Proof. Take an arbitrary vector of cutoffs P_0 and constant $\epsilon > 0$. To establish continuity we will show that there there exists $\delta_\epsilon > 0$ such that $\|D(P) - D(P_0)\| < \epsilon$ for any P with $\|P - P_0\| \leq \delta_\epsilon$. To see this, define for any $\delta > 0$ the set

$$\begin{aligned}\Theta_\delta = \{\theta \in \Theta : \ & |\bar{u}_h^\theta(P) - \bar{u}_h^\theta(P_0)| < \epsilon/2, \text{ for all } h \text{ and } P \text{ with } \|P - P_0\| < \delta, \\ & |\bar{u}_{h'}^\theta(P_0) - \bar{u}_h^\theta(P_0)| > \delta, \text{ for all hospitals } h, h', \text{ and} \\ & |\bar{u}_\emptyset^\theta(P_0) - \bar{u}_h^\theta(P_0)| > \delta, \text{ for all hospitals } h\}.\end{aligned}$$

Note that the intersection of all such sets is contained in the following set:

$$\begin{aligned}\cap_{\delta > 0}\Theta_\delta \subseteq \{\theta \in \Theta : \ & \bar{u}_h^\theta \text{ is continuous at } P_0 \text{for all } h, \\ & \bar{u}_{h'}^\theta(P_0) \neq \bar{u}_h^\theta(P_0), \text{ for all hospitals } h, h', \text{ and} \\ & \bar{u}_\emptyset^\theta(P_0) \neq \bar{u}_h^\theta(P_0), \text{ for all hospitals } h\}.\end{aligned}$$

Moreover, by Assumption D2, this latter set has measure 0. Since the Θ_δ are nested, we can take δ_ϵ small enough such that $\eta(\Theta_{\delta_\epsilon}) < \epsilon/2$, and $\delta_\epsilon < \epsilon/2$.

To complete the proof, we will show that, for any P such that $\|P - P_0\| < \delta_\epsilon$, we have $\|D(P) - D(P_0)\| < \epsilon$. To see this, consider $\theta \in \Theta_{\delta_\epsilon}$. If $D^\theta(P_0) \neq \emptyset$, let $h = D^\theta(P_0)$. Then, for any $h' \neq h$,

$$\bar{u}^\theta_{h'}(P) < \bar{u}^\theta_{h'}(P_0) + \epsilon/2 < \bar{u}^\theta_h(P_0) - \epsilon/2 + \epsilon/2 = \bar{u}^\theta_h(P).$$

The first inequality follows from the first condition in the definition of Θ_{δ_ϵ}. The second inequality follows from $\bar{u}^\theta_h(P_0) > \bar{u}^\theta_h(P)$, from the second condition in the definition of Θ_{δ_ϵ}, and the fact that $\delta_\epsilon < \epsilon/2$. This argument, and an analogous argument with $\emptyset$ instead of h', implies that $D^\theta(P) = D^\theta(P_0)$. Likewise, an analogous argument holds when $D^\theta(P_0) = \emptyset$. Therefore, for all $\theta \in \Theta_{\delta_\epsilon}$, we have $D^\theta(P) = D^\theta(P_0)$. Since $\eta(\Theta_{\delta_\epsilon}) < \epsilon$, we have that $\|D(P) - D(P_0)\| < \epsilon$, as desired.

A **market clearing cutoff** is defined exactly as in Definition 2. Given a stable matching μ, let $P = \mathcal{P}\mu$ be given by

$$P_h = \inf\{\pi_h(x) | x \in \mu(h)\},$$

if $\eta(\mu(h)) = S_h$ and $P_h = 0$ otherwise. Given a market clearing cutoff P, we define $\mu = \mathcal{M}P$ as follows. Consider first a doctor θ. If $D^\theta(P) = \emptyset$, then θ is unmatched: $\mu(\theta) = \emptyset$. If $D^\theta(P) = h \in H$, then $\mu(\theta)$ is defined as the contract that gives the highest payoff to h conditional on θ not having a better offer elsewhere. Formally,

$$\begin{aligned} \mu(\theta) \quad &= \quad \arg\max_{x \in X^\theta_h} \pi_h(x) \\ &\text{s.t.} \quad u^\theta(x) \geq \bar{u}^\theta_{h'}(P) \text{ for all } h' \neq h, \end{aligned} \tag{D.1}$$

Note that $\mu(\theta)$ is uniquely defined, by the compactness and no redundancy assumptions. Since we defined $\mu(\theta)$ for all doctors, we can uniquely define it for each hospital as

$$\mu(h) = \{\mu(\theta) : \theta \in \Theta \text{ and } h_{\mu(\theta)} = h\}.$$

We have the following extension of the supply and demand lemma. □

Lemma 3. ***(Supply and Demand Lemma with Contracts)*** *If μ is a stable matching, then $\mathcal{P}\mu$ is a market clearing cutoff, and if P is a market clearing cutoff then $\mathcal{M}P$ is a stable matching.*

Proof. **Part 1.** If μ is a stable matching, then $P = \mathcal{P}\mu$ is a market clearing cutoff.

We begin by proving a claim that will be used in the proof.

Claim D2. For almost all θ and $h \in H \cup \{\phi\}$ such that $\mu(\theta) = h$, we have $D^\theta(P) = h$.

Proof. Assume, to reach a contradiction, that $D^\theta(P) = h' \neq h$ for a positive measure of doctors. By the definition of demand, we have that, for any such doctor θ,

$\bar{u}^{\theta}_{h'}(P) \geq \max_{h'' \neq h'} \bar{u}^{\theta}_{h''}(P)$. Moreover, from the strict preferences assumption, there exists a positive mass of doctor types θ such that $\bar{u}^{\theta}_{h'}(P) > \max_{h'' \neq h'} \bar{u}^{\theta}_{h''}(P)$, and the functions $\bar{u}^{\theta}_{h''}(P)$ for each $h'' \in H \cup \{\emptyset\}$ are continuous at P. Let θ_0 be one such doctor. By the definition of $\bar{u}^{\theta_0}_{c}$ we have that $u^{\theta_0}(\mu(\theta_0)) \leq \bar{u}^{\theta_0}_{h}(P)$. Consequently, there exists a contract $x \in X^{\theta_0}_{h'}$ such that

$$\begin{aligned} u^{\theta_0}(x) &> u^{\theta_0}(\mu(\theta_0)) \\ \pi_{h'}(x) &> P_{h'}. \end{aligned}$$

We now show that this implies that θ_0 and h' block the matching μ. By definition of $\mathcal{P}$, and the completeness assumption, there exist contracts in $\mu(h')$ giving hospital h' payoffs arbitrarily close to $P_{h'}$. Therefore, there exists a contract $x' \in \mu(h')$ with $\pi_{h'}(x') < \pi_{h'}(x)$, so that h' and θ_0 block μ. This contradicts the fact that μ is stable. $\square$

This claim implies that $D(P) \leq S$. To prove that P is a market clearing cutoff, we only have to show that for any h such that $P_h > 0$, we have $D(P) = S$. To see this, note that, by the completeness assumption, there exists a doctor θ who may only contract with hospital h, and such that

$$\max_{x : u^{\theta}(x) \geq 0} \pi_h(x) < P_h,$$

and there exists a contract $x \in X^{\theta}_{h}$ with

$$\begin{aligned} \pi_h(x) &> 0 \\ u^{\theta}(x) &> 0. \end{aligned}$$

Note that, by the definition of $\mathcal{P}$, θ is not matched to h at μ, as h is only matched to contracts that yield utility of at least P_h. Therefore, θ is unmatched at μ, that is $\mu(\theta) = \emptyset$. Since μ is stable, θ and h cannot be a blocking pair, and therefore h must be matched to a mass S_h of doctors, that is

$$\eta(\{\theta : h_{\mu(\theta)} = h\}) = S_h.$$

Using the claim proved above, we have that

$$D_h(P) = \eta(\{\theta : h_{\mu(\theta)} = h\}) = S_h,$$

completing the proof of Part 1.

Part 2. If P is a vector of market clearing cutoffs, then $\mu = \mathcal{M}P$ is a stable matching.

First note that μ is a matching. It satisfies the requirement that doctors are matched to hospitals, that hospitals are matched to sets of doctors, and the consistency requirements by definition. Since $D^{\theta}(P)$ is a measurable selection from the demand correspondence, it satisfies the requirement that hospitals are matched to measurable sets of doctors. And finally, it satisfies that each hospital is matched to a set of doctors not exceeding its capacity, because P being a market clearing cutoff implies $D(P) \leq S$.

We now show that μ is stable. The proof uses the following claim.

Claim D3. For all $h \in H$ and $x \in \mu(h)$, we have $\pi_h(x) \geq P_h$.

Proof. Let $\theta = \theta_x$. By definition of $\mathcal{M}$,

$$\bar{u}^\theta_h(P) \geq \max_{h' \in H \cup \{\emptyset\}} \bar{u}^\theta_{h'}(P).$$

Therefore, by definition of $\bar{u}^\theta_h$, there exists $x' \in X^\theta_h$ such that

$$\begin{aligned} u^\theta(x') &\geq \max_{h' \in H \cup \{\emptyset\}} \bar{u}^\theta_{h'}(P) \\ \pi_h(x') &\geq P_h. \end{aligned}$$

Moreover, by the definition of $\mathcal{M}$, we have $\pi_h(x) \geq \pi_h(x')$. Consequently, $\pi_h(x) \geq P_h$, as desired. □

To see that μ is stable, note that, by the no redundancy assumptions, no contracts are Pareto dominated, so that there can only be blocking pairs formed of agents who are not matched to each other. Consider a pair θ, h, who are not matched at μ. We will show they cannot form a blocking pair. First note that, by the definition of $\mathcal{M}$,

$$u^\theta(\mu(\theta)) \geq \bar{u}^\theta_h(P). \tag{D.2}$$

Consider now the case where $D_h(P) < S_h$. Therefore, $P_h = 0$. This implies that

$$\bar{u}^\theta_h(P) = \max_{x \in X^\theta_h} u^\theta(x).$$

Equation (D.2) then implies that

$$u^\theta(\mu(\theta)) \geq \max_{x \in X^\theta_h} u^\theta(x),$$

and therefore θ and h are not a blocking pair.

Finally consider the case where $D_h(P) = S_h$. The definition of $\mathcal{M}$ then implies that the mass of doctors matched to h at μ equals S_c. By Claim D3, for all contracts $x \in \mu(h)$, we have $\pi_h(x) \geq P_h$. If there exists $x' \in X^\theta_h$ such that $\pi_h(x') \geq P_h$, we then have that $u^\theta(x') \leq \bar{u}^\theta_h(P)$. Therefore, by equation (D.2), we have that $u^\theta(x') \leq u^\theta(\mu(\theta))$. Consequently, θ and h are not a blocking pair. This completes the proof. □

Note that, in the matching with contracts setting, there is no longer a bijection between market clearing cutoffs and stable matchings. This happens for two reasons, one substantial and one technical. The substantial reason is that the contract terms w are not uniquely determined by cutoffs, as there is room for doctors and hospitals to share the surplus of relationships in different ways, without violating stability. The technical reason is that we have not imposed a condition akin to right continuity in the model from Section 2, which precludes multiplicities of stable matchings that differ in measure 0 sets.

D.3. **Existence.** To establish the existence of a stable matching, we must modify the previous argument, which used the deferred acceptance algorithm. One simple modification is using a version of the algorithm that Biró (2007) terms a "score limit algorithm", which calculates a stable matching by progressively increasing cutoffs to clear the market. A straightforward application of Tarski's fixed point theorem gives us existence in this case.

Proposition 8. *A stable matching with contracts always exists.*

Proof. Consider the operator $T : [0, M]^H \to [0, M]^H$ defined by $P' = TP$ is the smallest solution $P' \in [0, M]^H$ to the system of inequalities

$$D_h(P'_h, P_{-h}) \leq S_h.$$

We will show that this operator has a fixed point, and that this fixed point is a market clearing cutoff.[45]

First note that, by the continuity of D_h, and since $D_h(M, P_h) = 0$, the smallest solution to this equation is well-defined. Therefore, T is well-defined. Moreover, since $D_h(P'_h, P_{-h})$ is weakly increasing in P_{-h} and weakly decreasing in P'_h we have that T is weakly increasing in P. We know that T takes the cube $[0, M]^H$ into itself, by definition. By Tarski's fixed point theorem, T has a fixed point.

It only remains to show that every fixed point P^* of T is a market clearing cutoff. By definition of T we have that $D_h(P^*) \leq S_h$ for all hospitals $h \in H$, so that demand for no hospital exceeds supply. Consider h such that $P^*_h > 0$. By definition of T we have that

$$D_h(P_h, P^*_{-h}) > S_h$$

for any $P_h < P^*_h$. By continuity of demand we have that $D_h(P^*) \geq S_h$, which combined with the fact that demand does not exceed supply implies that $D_h(P^*) = S_h$, and therefore P^* is a market clearing cutoff. □

D.4. **The Quasilinear Case.** A particularly interesting case of the model is when contracts only specify a wage w, and preferences are quasilinear. That is, the utility of a contract $x = (\theta, h, w)$ is

$$\begin{aligned} u^\theta(x) &= u^\theta_h + w \\ \pi_h(x) &= \pi^\theta_h - w. \end{aligned}$$

[45] A cutoff limit algorithm can be described as starting with cutoffs of 0, and succesively applying the operator T. What T does in each step is raising the cutoff of each hospital just enough to clear the market for the hospital given the cutoffs of other hospitals.

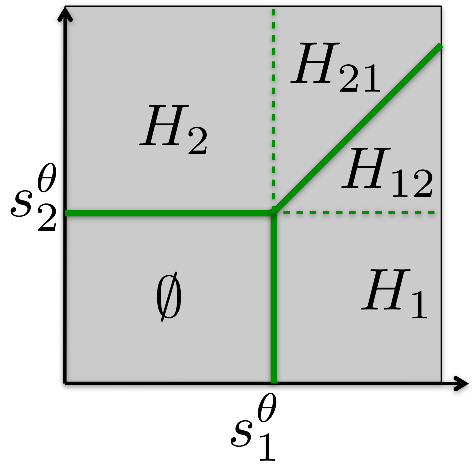

FIGURE 6. A matching with transferable utility with two hospitals. The square represents the set of possible surplus vectors s^θ. Doctors in regions H_1 and H_{12} are matched to hospital 1, and doctors in regions H_2 and H_{21} to hospital 2.

and contracts include all possible values of w, such that these values are in $[0, M]$. Define the **surplus** of a doctor-hospital pair as

$$s_h^\theta = u_h^\theta + \pi_h^\theta.$$

We assume that M is large enough so that, for all θ in the support of η we have $0 \leq s_i^\theta \leq M$, so that doctors and hospitals may freely divide the surplus of a relationship. We assume moreover that assumptions D1 and D2hold. Denote a model satisfying the above properties as a matching with contracts model with quasilinear preferences. From the definition of reservation utility we get that, for all doctors in the support of η,

$$\bar{u}_h^\theta(P) = s_h^\theta - P_h.$$

Therefore, in any stable matching, doctors are sorted into the hospitals where $s_h^\theta - P_h$ is the highest, subject to it being positive. One immediate consequence is that doctors *do not go* necessarily to the hospital where they generate the largest surplus s_h^θ. If $P_h \neq P_{h'}$, it may be the case that $s_h^\theta > s_{h'}^\theta$, but doctor θ is assigned to h'. However, the allocation of doctors to hospitals does maximize the total surplus generated in the economy, given the capacity constraints (see Azevedo (2011) Appendix A.2). Figure 6 plots a stable matching in an economy with two hospitals.

Let the distribution of surplus vectors s^θ be η_s. We then have the following uniqueness result.

Proposition 9. *Consider a matching with contracts model with quasilinear preferences. If η_S has full support over $[0, M]^C$ then there is a unique vector of market clearing cutoffs.*

Proof. We begin by showing that the set of market clearing cutoffs is a lattice.

Claim D4. The set of market clearing cutoffs is a complete lattice.

Proof. Define the operator $T : [0, M]^H \to [0, M]^H$ as follows. Let $TP = P'$, with P'_h being the solution to

$$D_h(P'_h, P_{-h}) = S_h \tag{D.3}$$

if such a solution $P'_h \in [0, M]$ exists, and 0 otherwise.[46] We will show that set of fixed points of T equals the set of market clearing cutoffs.

Note that, since $D_h(M, P_{-h}) = 0$ and demand is continuous, if a solution p to $D(p, P_{-h})$ does not exist, then $D(p, P_{-h}) < S_h$ for all $p \in [0, M]$, and therefore $P'_h = 0$ and $D(P'_h, P_{-h}) < S_h$. We will use this observation to show that the fixed points of T correspond to market clearing cutoffs.

Consider a fixed point P^* of T. For a given h, since $TP^* = P^*$, either equation (D.3) has a solution, and we have $D_h(P^*) = S_h$, or the equation has no solutions, in which case $D_h(P^*) < S_h$ and $P^*_h = 0$. Therefore, P^* is a market clearing cutoff.

Consider now a market clearing cutoff P^*. For any hospital $h \in H$, if $P^*_h > 0$, we have that $D_h(P^*) = S_h$, so that $(TP^*)_h = P^*_h$. If $P^*_h = 0$ we have that either the market clears exactly, $D_h(P^*) = S_h$, in which case $(TP^*)_h = P^*_h$, or that h is in excess supply, $S_h > D_h(P^*) \geq D_h(p, P^*_{-h})$ for all $p \in [0, M]$, and therefore $(TP^*)_h = 0 = P^*_h$. Since this holds for all hospitals, P^* is a fixed point.

Now that we have established that the set of fixed points of T equals the set of market clearing cutoffs, we can show that this set if a lattice. To see this, note that T is weakly increasing in P, and takes $[0, M]$ in itself. Therefore, by Tarski's Theorem, the set of fixed points is a non-empty complete lattice. □

Let P^- and P^+ be the smallest and largest market clearing cutoffs. Let H^+ be the subset of hospitals for which $P^+_h > P^-_h$. That is

$$H^+ = \{h \in H : P^+_h > P^-_h\}.$$

In particular, for all $h \in H^+$ we have $P^+_h > 0$.

If H^+ is empty, then $P^- = P^+$, and we are done. Assume henceforth that H^+ is nonempty. Since both P^- and P^+ are market clearing cutoffs, we have that

$$\sum_{h \in H^+} D_h(P^-) \leq \sum_{h \in H^+} S_h = \sum_{h \in H^+} D_h(P^+).$$

[46]Note that, if the solution to the equation defining P'_h exists, it is unique, as the full support assumption implies that the left side is strictly decreasing in P'_h.

However, since $P_h^- = P_h^+$ for $h \notin H^+$, and $P_h^- < P_h^+$ for all $h \in H^+$ we have that

$$\sum_{h \in H^+} D_h(P^-) \geq \sum_{h \in H^+} D_h(P^+).$$

Therefore,

$$\sum_{h \in H^+} D_h(P^-) = \sum_{h \in H^+} D_h(P^+).$$

Under the assumption that the support of η_s is the set $[0, M]^H$, this can only be true if $P^- = P^+$, completing the proof. $\square$

The proposition guarantees that the allocation of doctors to hospital is unique, up to a measure 0 set of doctors. However, the stable matching is not unique, as wages are not uniquely determined by stability. The intuition is that, in a stable matching, a hospital may offer a doctor any wage such that the doctor's utility is above that in her next best choice, and the hospital's gain from the relationship above its reservation value of capacity. Therefore, in general the surplus s_h^θ can be divided in different ways without compromising stability.

Appendix E. Relationship to Pre-Matchings

This section clarifies the connection and differences between the cutoffs approach to stable matchings, and the approach using pre-matchings, proposed by Adachi (2000) and Echenique and Oviedo (2004, 2006).

The pre-matchings approach is fundamentally different from the cutoffs approach. The key advantage of the pre-matchings approach is that stable matchings can be analyzed as fixed points of a monotone operator in the set of pre-matchings. Although the set of pre-matchings is much larger set than the set of matchings, this result is useful for deriving general lattice theoretic results, due to the monotonicity of the operator. Echenique and Oviedo (2006) use the fixed point result to establish results on the existence and structure of the set of stable matchings in models with very general preferences and many to many matching. This is in constrast to the cutoff approach, as the set of cutoffs is much smaller than the set of matchings,[47] and the usefulness of cutoffs is in deriving asymptotic results and simplified models for comparative statics.

Nevertheless, in this section we show that there is an important connection between the two approaches. Namely, we show that, when colleges have responsive preferences over students, the intermediate steps of the monotone operator used by Echenique and Oviedo (2006) can be written in terms of cutoffs. With this observation, we can show that, in the case of responsive preferences, our result that $\tilde{\mathcal{M}}\tilde{\mathcal{P}}$ restricted to the set

[47] Formally, in the continuum model the set of all matchings is infinite dimensional, while the set of cutoffs has dimension equal to the number of colleges. The formal definition below clarifies how the set of pre-matchings is much larger than the set of all matchings.

of stable matchings is the identity is closely related to the result that fixed points of the monotone operator proposed by Adachi (2000); Echenique and Oviedo (2006) correspond to stable matchings. While this does not imply the cutoff lemma, nor the Echenique and Oviedo (2006) results, which are valid for more general preferences, this connection may be useful to extend the supply and demand approach to generalized matchings models.

E.1. **Formal Definition of Pre-matchings and the Fixed Point Operator.** Consider a finite set C of colleges c and finite set $\tilde{\Theta}$ of students θ .

Definition 5. A **prematching** $\nu = (\nu^C, \nu^{\tilde{\Theta}})$ is a pair of functions

$$\begin{aligned} \nu^C : C &\longrightarrow 2^{\tilde{\Theta}} \\ \nu^{\tilde{\Theta}} : \tilde{\Theta} &\longrightarrow C \cup \Theta, \end{aligned}$$

such that, for all $\theta \in \tilde{\Theta}$, $\nu^{\tilde{\Theta}}(\theta) \in C$ or $\nu^{\tilde{\Theta}}(\theta) = \theta$.

That is, a prematching ν specifies a set of students $\nu^C(c)$ matched to each college c, and a college or remaining unmatched $\nu^{\tilde{\Theta}}(\theta)$ to each student θ. There is no consistency requirement between ν^C and $\nu^{\tilde{\Theta}}$. We denote the set of pre-matchings by $\mathcal{V} = \mathcal{V}^C \times \mathcal{V}^{\tilde{\Theta}}$, where $\mathcal{V}^C$ and $\mathcal{V}^{\tilde{\Theta}}$ are the set of functions that may be the first and second coordinates of a prematching. Intuitively, the set of pre-matchings can be thought of as being a larger set than the set of matchings, as either coordinate has sufficient information to fully specify a matching.

We now follow Echenique and Oviedo (2006) in defining the operator T over the set of pre-matchings, with fixed points corresponding to stable matchings. The operator is defined in terms of choice functions, which they define using preference relations of agents over sets of match partners. For each college c we consider a **choice function** with, for any $A \subseteq \tilde{\Theta}$, $Ch(c, A)$ denoting the college's preferred subset of A. Likewise, for each student we consider a **choice function** $Ch(\theta, A)$ which picks a college in A or θ, out of any subset $A \subseteq C$, depending on the student's preferences. Given a prematching ν, define

$$\begin{aligned} U(c,\nu) &= \{\theta \in \tilde{\Theta} : c \in Ch(\theta, \nu^{\tilde{\Theta}}(\theta) \cup \{c\})\}, \text{ and} \\ V(\theta,\nu) &= \{c \in C : \theta \in Ch(c, \nu^C(c) \cup \{\theta\})\}. \end{aligned}$$

That is, a student is in $U(c,\nu)$ if he would choose c over his assignment under $\nu^{\tilde{\Theta}}$, and a college is in $V(\theta,\nu)$ if θ would be one of its chosen students out of $v^C(c) \cup \{\theta\}$. The **fixed point operator** $T : \mathcal{V} \to \mathcal{V}$ is is defined by

$$\begin{aligned} (Tv)(\theta) &= Ch(\theta, V(\theta,\nu)) \\ (Tv)(c) &= Ch(c, U(c,\nu)). \end{aligned}$$

E.2. **Definition of the Fixed Point Operator with Cutoffs in the Responsive Preferences Case.** We now return to the model of Section 3.1, where students have preferences $\succ^\theta$ over colleges, and colleges have preferences over individual students (given by e_c), and a capacity S_c.[48]

In this setting, the choice maps have a simple definition.[49] Namely, if $|A| \leq S_c$ then $Ch(c, A) = A$ and if $|A| > S_c$ then $Ch(c, A)$ the subset of the S_c students in A with the highest scores. Likewise, $Ch(\theta, A)$ is student θ's preferred college in A if A is nonempty, or θ otherwise.

To define the operator T using cutoffs, we must define cutoffs on both sides of the market. A vector of cutoffs on the college side, denoted P^C, is defined as in the main text, and the set of all such cutoffs is $\mathbb{P}^C \equiv [0,1]^C$. A vector of cutoff in the student side is denoted $P^{\tilde{\Theta}}$. For each student θ, the cutoff $P^{\tilde{\Theta}}_\theta \in C$ or $P^{\tilde{\Theta}}_\theta = \theta$. Intuitively, the cutoff $P^{\tilde{\Theta}}_\theta$ denotes the least preferred college that student θ is willing to match with. The set of student side cutoffs is denoted as $\mathbb{P}^{\tilde{\Theta}} \equiv \times_{\tilde{\Theta}} C$.

We now define the following operators. The first two extend the operators $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{M}}$ to pre-matchings, and the last two are the mirror images of $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{M}}$ on the other side of the market.

- Define $\mathcal{P}^C : \mathcal{V}^C \longrightarrow \mathbb{P}^C$ as, for each college c, the S_c-highest score of a student in $\nu^C(c)$, denoted as
$$\left(\mathcal{P}^C(\nu^C)\right)_c = \{e^\theta_c : \theta \in v^C(c)\}_{(S_c)},$$
with the convention the the S_c-highest score is 0 when there are less than S_c students in $\nu^C(c)$. Notice that $\mathcal{P}^C$ coincides with $\tilde{\mathcal{P}}$ when ν^C corresponds to a stable matching.[50]
- Define $\mathcal{M}^{\tilde{\Theta}} : \mathbb{P}^C \longrightarrow \mathcal{V}^{\tilde{\Theta}}$ in similar fashion as $\tilde{\mathcal{M}}$ was defined in the main text,
$$(\mathcal{M}^{\tilde{\Theta}}(P^C))(\theta) = D^\theta(P^C).$$
- Define $\mathcal{P}^{\tilde{\Theta}} : \mathcal{V}^{\tilde{\Theta}} \longrightarrow \mathbb{P}^{\tilde{\Theta}}$ as
$$(\mathcal{P}^{\tilde{\Theta}}(\nu^{\tilde{\Theta}}))_\theta = \nu^{\tilde{\Theta}}(\theta).$$
- Define $\mathcal{M}^C : \mathbb{P}^{\tilde{\Theta}} \longrightarrow \mathcal{V}^C$ as
$$(\mathcal{M}^C(P^{\tilde{\Theta}}))(c) = Ch(c, \{\theta : c \succeq^\theta P^{\tilde{\Theta}}(\theta)\}).$$

The following Claim clarifies how the map T can be written in terms of cutoffs.

[48]The college preferences over individual students can be extended to responsive preferences over sets of students.
[49]See Echenique and Oviedo (2006) p.240 for how the choice function is defined given preferences.
[50]Formally, $\mathcal{P}^C(\nu^C) = \tilde{\mathcal{P}}\tilde{\mu}$ when for all $c \in C$ we have $\mu(c) = \nu^C(c)$ and $\#\nu^C(c) \leq S_c$.

Claim E1. For any prematching $\nu \in \mathcal{V}$ we have

$$T\nu = (\mathcal{M}^C \mathcal{P}^{\tilde{\Theta}} \nu^{\tilde{\Theta}}, \mathcal{M}^{\tilde{\Theta}} \mathcal{P}^C \nu^C).$$

Proof. We begin with the student side. Consider a student θ. Notice that, by definition of the set $V(\theta, \nu)$ we have

$$V(\theta, \nu) = \{c \in C : \theta \in Ch(c, \nu^C(c) \cup \{\theta\})\}.$$

Moreover, by the definition of the choice function, θ is chosen by college c if and only if e_c^θ is at least as high as the S_c-highest score at college c among the students in $\nu^C(c)$. Therefore, we have

$$V(\theta, \nu) = \{c \in C : \, e_c^\theta \geq \left(\mathcal{P}^C(\nu^C)\right)_c\}.$$

Using this equation, we can write the student side of the operator T as

$$\begin{aligned}
(T\nu)(\theta) &= Ch(\theta, V(\theta, \nu)) \\
&= Ch(\theta, \{c \in C : \, e_c^\theta \geq \left(\mathcal{P}^C(\nu^C)\right)_c\}) \\
&= D^\theta(\mathcal{P}^C(\nu^C)).
\end{aligned}$$

By the definition of $\mathcal{M}^{\tilde{\Theta}}$, this equals $\mathcal{M}^{\tilde{\Theta}} \mathcal{P}^C(\nu^C)$ as desired.

Consider now the college side. Fix a college c. By definition of the set $U(c, \nu)$ we have

$$U(c, \nu) = \{\theta \in \tilde{\Theta} : c \in Ch(\theta, \nu^{\tilde{\Theta}}(\theta) \cup \{c\})\}.$$

Note that $c \in Ch(\theta, \nu^{\tilde{\Theta}}(\theta) \cup \{c\})$ if and only if $c = Ch(\theta, \nu^{\tilde{\Theta}}(\theta) \cup \{c\})$, as students match to at most a single college. This is the case if and only if $c \succeq^\theta \nu^{\tilde{\Theta}}(\theta)$. Moreover, by definition of $\mathcal{P}^{\tilde{\Theta}}$ we have $(\mathcal{P}^{\tilde{\Theta}}(\nu^{\tilde{\Theta}}))_\theta = \nu^{\tilde{\Theta}}(\theta)$. Therefore,

$$U(c, \nu) = \{\theta \in \tilde{\Theta} : c \succeq^\theta (\mathcal{P}^{\tilde{\Theta}}(\nu^{\tilde{\Theta}}))_\theta\}.$$

Consider now the operator T. We have

$$\begin{aligned}
(T\nu)(c) &= Ch(c, U(c, \nu)) \\
&= Ch(c, \{\theta \in \tilde{\Theta} : c \succeq^\theta (\mathcal{P}^{\tilde{\Theta}}(\nu^{\tilde{\Theta}}))_\theta\}).
\end{aligned}$$

By the definition of the operator $\mathcal{M}^C$, we have

$$(T\nu)(c) = (\mathcal{M}^C(\mathcal{P}^{\tilde{\Theta}}(v^{\tilde{\Theta}})))(c),$$

completing the proof. □

We now discuss the relationship between our results and those of Echenique and Oviedo (2004, 2006) and Adachi (2000). We say that a matching μ is **associated** with a prematching $g(\mu) = \nu$ by letting each coordinate be the matching restricted to colleges or students, i.e., $\nu = (\mu|_C, \mu|_{\tilde{\Theta}})$. Note that the function g is injective. If $g^{-1}(\nu) \neq \emptyset$ we say that prematching ν is **associated** with matching $g^{-1}(\nu)$. Adachi's main result is that a prematching v is a fixed point of T iff it is associated with a stable matching.

First note that Adachi result is distinct from the fact that stable matchings are associated with market clearing cutoffs. Since cutoffs are only an intermediate step in the operator T, the fact that the fixed points of T correspond to stable matchings is distinct from the relationship between market clearing cutoffs and stable matchings.

However, the fixed point result is closely related to our result that $\tilde{\mathcal{M}}\tilde{\mathcal{P}}$ restricted to stable matchings is the identity map. With an analogous argument to the one we used to prove that result, it is possible to prove that, given a stable matching μ, the operator $\mathcal{M}^C\mathcal{P}^{\tilde{\Theta}}$ (or $\mathcal{M}^{\tilde{\Theta}}\mathcal{P}^C$) takes $\mu|_{\tilde{\Theta}}$ into $\mu|_C$ (or $\mu|_C$ into $\mu|_{\tilde{\Theta}}$). By Claim E1, this is equivalent to Adachi's result that pre-matchings corresponding to stable matchings are fixed points of T. Therefore, we could have derived Adachi's fixed point result using cutoffs, or proven the result about $\tilde{\mathcal{M}}\tilde{\mathcal{P}}$ using Adachi's result. Note however that this argument is only valid with responsive preferences, so that we cannot use this argument to establish the more general results of Echenique and Oviedo (2004, 2006).

Appendix F. A Large Market Where Deferred Acceptance with Single Tie-Breaking Is Inefficient

This section presents a simple example that shows that the deferred acceptance with single tie-breaking mechanism (see Section 5.4) can produce Pareto dominated outcomes for a large share of the students with high probability, even in a large market.[51]

Example 2. (School Choice)

A city has two schools $c = 1, 2$ with the same capacity. Students have priorities to schools according to the walk zones where they live in. Half of the students live in the walk zone of each school. In this example, the grass is always greener on the other side, so that students always prefer the school to which they don't have priority. The city uses the DA-STB mechanism. To break ties, the city gives each student a single lottery number l uniformly distributed in $[0, 1]$. The student's score is

$$l + I(\theta \text{ is in } c\text{'s walk zone}).$$

In the continuum economy induced by the DA-STB mechanism, there is a mass $1/2$ of students in the walk zone of each school, and $S_1 = S_2 = 1/2$. Figure 7 depicts the distribution of students in the economy.

We now analyze the stable matches in this continuum economy. Note that market clearing cutoffs must be in $[0, 1]$, as the mass of students with priority to each school

[51]The example is a continuum version of an example used by Erdil and Ergin (2008) to show a shortcoming of deferred acceptance with single tie-breaking: it may produce matchings which are ex post inefficient with respect to the true preferences, before the tie-breaking, being dominated by other stable matchings.

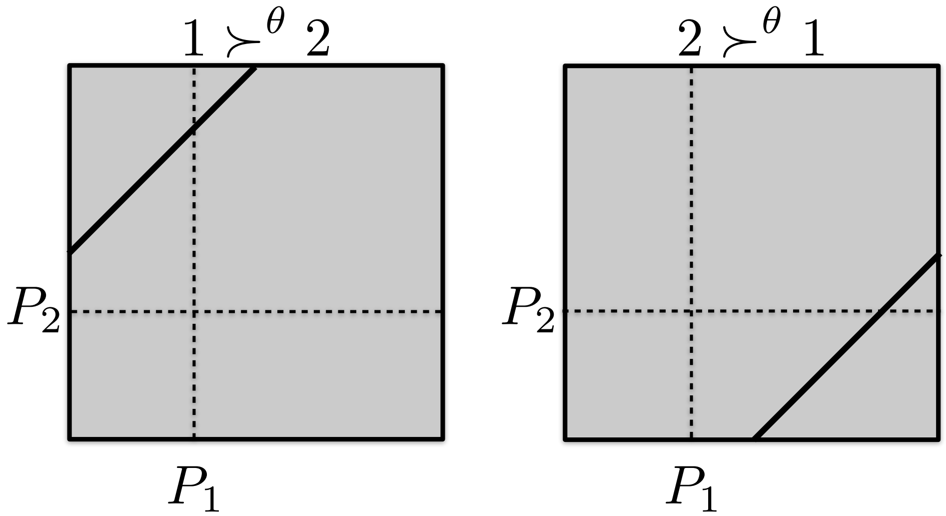

FIGURE 7. The distribution of student types in Example 2. The unit mass of students is uniformly distributed over the solid lines. The left square represents students in the walk zone of school 2, and the right square students with priority to school 1. The dashed lines represents one of an infinite number of possible vectors of market clearing cutoffs $P_1 = P_2$.

is only large enough to exactly fill each school. Consequently, the market clearing equations can be written, for $0 \leq P \leq (1,1)$, as

$$\begin{aligned} 1 &= 2 \quad S_1 = (1 - P_1) + P_2 \\ 1 &= 2 \quad S_2 = (1 - P_2) + P_1. \end{aligned}$$

The first equation describes demand for school 1. $1 - P_1/2$ students in the walk zone of 2 are able to afford it, and that is the first term. Moreover, an additional $P_2/2$ students in the walk zone of 1 would rather go to 2, but don't have high enough lottery number, so they have to stay in school 1. The market clearing equation for school 2 is the same.

These equations are equivalent to

$$P_1 = P_2.$$

Hence, any point in the line $\{P = (x,x) | x \in [0,1]\}$ is a market clearing cutoff - the lattice of stable matchings has infinite points, ranging from a student-optimal stable matching, $P = (0,0)$ to a school-optimal stable matching $P = (1,1)$.

Now consider a slightly different continuum economy, that has a small mass of agents that have no priority, so that the new mass has e^{θ} uniformly distributed in $[(0,0),(1,1)]$. It's easy to see that in that case the unique stable matching is $P = (1,1)$. Therefore, adding this small mass undoes all stable matchings of the original continuum economy, except for $P = (1,1)$. In addition, it is also possible to find perturbations that undo the school-optimal stable matching $P = (1,1)$. If we add a small amount ϵ of capacity to school 1, the unique stable matching is $P = (0, 0)$. And if we reduce the capacity of school 1 by $\epsilon/2$, the unique stable matching is $P = (1+\epsilon, 1)$, which is close to $P = (1, 1)$.

Consider now a school choice problem where, in addition to these students, there is a small mass of students that have no priority, and prefer school 1. By the argument in the above paragraph, the only stable matching of the continuum economy induced by the new school choice problem is one where $P = (1, 1)$. Therefore, this school choice problem leads to agents attending the school to which they have priority with probability close to 1. However, it is a stable allocation for all students with priorities to attend the school they prefer. Therefore, DA-STB produces outcomes that are Pareto dominated for many students relative to a stable allocation. This is in contrast with the result by Che and Kojima (2010), who show that in the case without priorities the RSD mechanism is approximately ordinally efficient in a large market.