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Problem Generation

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Motivation

Problems similar to a given problem.

- Avoid copyright issues
- Prevent cheating in MOOCs (Unsynchronized instruction)

Problems of a given difficulty level & concept usage.

- Generate progressions
- Generate personalized workflows

Content Classification

Various subject domains

Arithmetic, Algebra, Logic, Programming, Language Learning, …

Can be classified into Procedural or Conceptual.

Procedural

Mathematical Procedures (Addition, Long division, GCD/LCM, Gaussian Elimination)

Algorithmic Procedures (BFS, insertion sort, regular expression -> automaton)

Conceptual

Proofs (Algebraic theorems, Natural deduction, Non-regularity)

Constructions (Geometric ruler/compass, Automata constructions, Algorithms)

Key Ideas

Procedural Content

- Test input generation techniques

Conceptual Content

Template based guess and verify

Symbolic methods (solution generation in reverse)

Addition Procedure

Concept
Single digit addition
Multiple digit w/o carry
Single carry
Two single carries
Double carry
Triple carry
Extra digit in i/p & new digit in o/p

```
Add(int array A, int array B)
  ℓ := Max(Len(A), Len(B));
  for i=0 to ℓ-1                          ▷ Loop over digits (L)
    if (i ≥ Len(A)) t := B[i];             ▷ Different # of digits (D)
    else if (i ≥ Len(B)) t := A[i];        ▷ Different # of digits (D)
    else t:=A[i]+B[i];
    if (C[i] == 1) t:=t+1;                 ▷ Carry from prev. step (C)
    if (t > 9) {R[i]:=t-10; C[i + 1]:=1;}
    else R[i] := t;
  if (C[ℓ] == 1) R[ℓ] := 1;                ▷ Extra digit in output (E)
```

CHI 2013: Erik Andersen, Sumit Gulwani, Zoran Popovic.

Addition Procedure

Concept	Trace Characteristic
Single digit addition	L
Multiple digit w/o carry	LL+
Single carry	L* (LC) L*
Two single carries	L* (LC) L+ (LC) L*
Double carry	L* (LCLC) L*
Triple carry	L* (LCLCLCLC) L*
Extra digit in i/p & new digit in o/p	L* CLDCE

```
Add(int array A, int array B)
  ℓ := Max(Len(A), Len(B));
  for i=0 to ℓ-1                          ▷ Loop over digits (L)
    if (i ≥ Len(A)) t := B[i];            ▷ Different # of digits (D)
    else if (i ≥ Len(B)) t := A[i];       ▷ Different # of digits (D)
    else t:=A[i]+B[i];
    if (C[i] == 1) t:=t+1;                ▷ Carry from prev. step (C)
    if (t > 9) {R[i]:=t-10; C[i + 1]:=1;}
    else R[i] := t;
  if (C[ℓ] == 1) R[ℓ] := 1;               ▷ Extra digit in output (E)
```

Addition Procedure

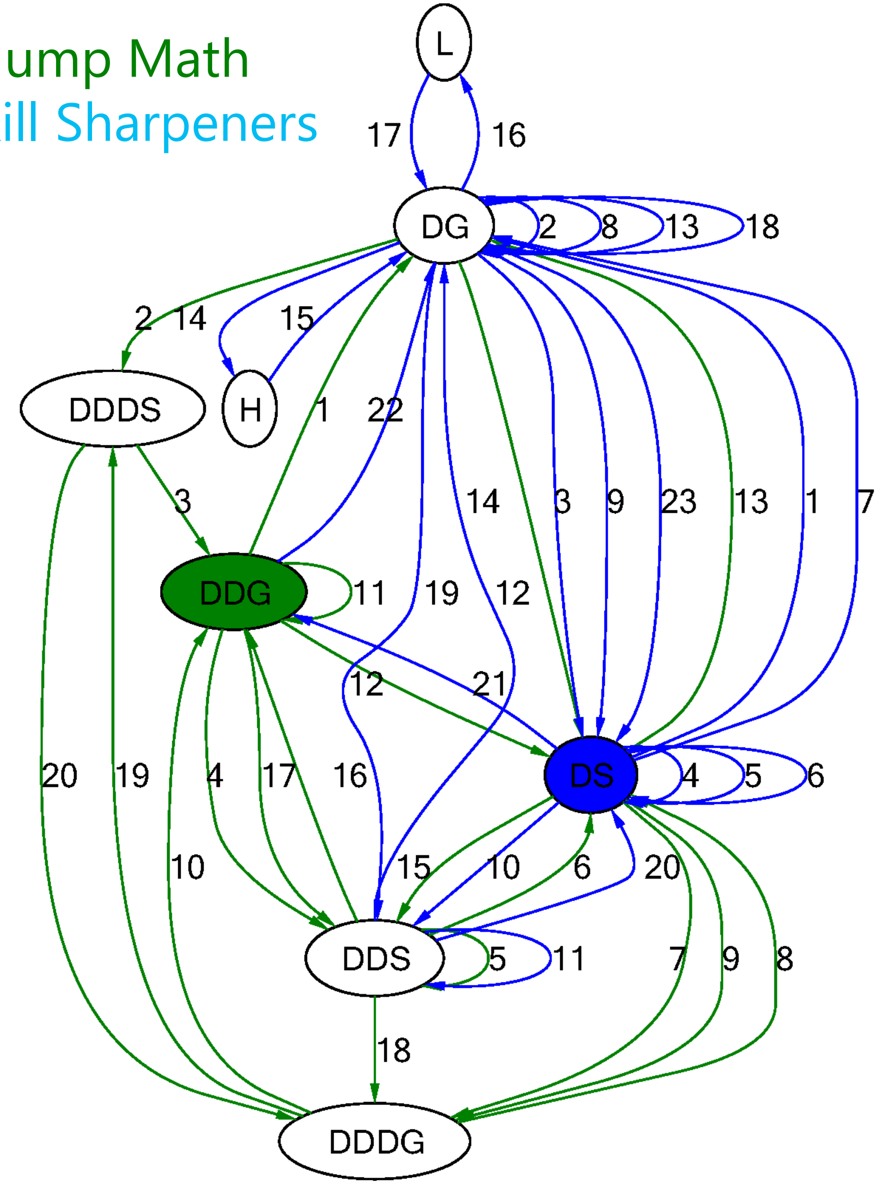
Concept	Trace Characteristic	Sample Input
Single digit addition	L	3+2
Multiple digit w/o carry	LL+	1234 +8765
Single carry	L* (LC) L*	1234 + 8757
Two single carries	L* (LC) L+ (LC) L*	1234 + 8857
Double carry	L* (LCLC) L*	1234 + 8667
Triple carry	L* (LCLCLCLC) L*	1234 + 8767
Extra digit in i/p & new digit in o/p	L* CLDCE	9234 + 900

```
Add(int array A, int array B)
  ℓ := Max(Len(A), Len(B));
  for i=0 to ℓ-1                          ▷ Loop over digits (L)
    if (i ≥ Len(A)) t := B[i];            ▷ Different # of digits (D)
    else if (i ≥ Len(B)) t := A[i];       ▷ Different # of digits (D)
    else t:=A[i]+B[i];
    if (C[i] == 1) t:=t+1;                ▷ Carry from prev. step (C)
    if (t > 9) {R[i]:=t-10; C[i+1]:=1;}
    else R[i] := t;
  if (C[ℓ] == 1) R[ℓ] := 1;               ▷ Extra digit in output (E)
```

Comparing Progressions: Integer Comparison

Green Progression: Jump Math
Blue Progression: Skill Sharpeners

```
procedure COMPARE(a, b)
    if len(a) > len(b) then
        return more                        ▷ More digits (H)
    else if len(a) < len(b) then
        return less                        ▷ Fewer digits (L)
    end if
    for i ← 0, len(a) − 1 do               ▷ For each digit (D)
        if a_i > b_i then
            return more                    ▷ Digit is larger (G)
        else if a_i < b_i then
            return less                    ▷ Digit is smaller (S)
        end if
    end for
    return equal                           ▷ Equal (E)
end procedure
```



Green Progression moves (quickly) into involved problems that involve comparing more digits.
But it ignore an entire class of levels (H and L)!

Key Ideas

Procedural Content

Test input generation techniques

Conceptual Content

- Template based guess and verify

Symbolic methods (solution generation in reverse)

Trigonometry Problems

Example Problem: $(\sec x + \cos x)(\sec x - \cos x) = \tan^2 x + \sin^2 x$

Query: $(T_1(x) \pm T_2(x))(T_3(x) \pm T_4(x)) = T_5^2(x) \pm T_6^2(x)$

New problems generated:

$$(\csc x + \cos x)(\csc x - \cos x) = \cot^2 x + \sin^2 x$$
$$(\csc x - \sin x)(\csc x + \sin x) = \cot^2 x + \cos^2 x$$
$$(\sec x + \sin x)(\sec x - \sin x) = \tan^2 x + \cos^2 x$$
$$\vdots$$

Trigonometry Problems

Example Problem: $(\sec x + \cos x)(\sec x - \cos x) = \tan^2 x + \sin^2 x$

Query: $(T_1(x) \pm T_2(x))(T_3(x) \pm T_4(x)) = T_5^2(x) \pm T_6^2(x)$

New problems generated:

$$(\csc x + \cos x)(\csc x - \cos x) = \cot^2 x + \sin^2 x$$
$$(\csc x - \sin x)(\csc x + \sin x) = \cot^2 x + \cos^2 x$$
$$(\sec x + \sin x)(\sec x - \sin x) = \tan^2 x + \cos^2 x$$
$$\vdots$$
$$(\tan x + \sin x)(\tan x - \sin x) = \tan^2 x - \sin^2 x$$
$$(\csc x + \cos x)(\csc x - \cos x) = \csc^2 x - \cos^2 x$$
$$\vdots$$

Trigonometry Problems

Example Problem: $(\sec x + \cos x)(\sec x - \cos x) = \tan^2 x + \sin^2 x$

Query: $(T_1(x) \pm T_2(x))\,(T_3(x) \pm T_4(x)) = T_5^2(x) \pm T_6^2(x), \qquad T_1 \neq T_5$

New problems generated:

$$(\csc x + \cos x)(\csc x - \cos x) = \cot^2 x + \sin^2 x$$
$$(\csc x - \sin x)(\csc x + \sin x) = \cot^2 x + \cos^2 x$$
$$(\sec x + \sin x)(\sec x - \sin x) = \tan^2 x + \cos^2 x$$
$$\vdots$$
$$(\tan x + \sin x)(\tan x - \sin x) = \tan^2 x - \sin^2 x$$
$$(\csc x + \cos x)(\csc x - \cos x) = \csc^2 x - \cos^2 x$$
$$\vdots$$

Limits/Series Problems

Example Problem: $\lim_{n\to\infty} \sum_{i=0}^{n} \frac{2i^2+i+1}{5^i} = \frac{5}{2}$

Query: $\lim_{n\to\infty} \sum_{i=0}^{n} \frac{C_0 i^2 + C_1 i + C_2}{{C_3}^i} = \frac{C_4}{C_5}$ $\quad C_0 \neq 0 \wedge \gcd(C_0, C_1, C_2) = \gcd(C_4, C_5) = 1$

New problems generated:

$\lim_{n\to\infty} \sum_{i=0}^{n} \frac{3i^2+2i+1}{7^i} = \frac{7}{3}$

$\lim_{n\to\infty} \sum_{i=0}^{n} \frac{3i^2+3i+1}{4^i} = 4$

$\lim_{n\to\infty} \sum_{i=0}^{n} \frac{i^2}{3^i} = \frac{3}{2}$

$\lim_{n\to\infty} \sum_{i=0}^{n} \frac{5i^2+3i+3}{6^i} = 6$

Integration Problems

Example Problem: $\int (\csc x)\,(\csc x - \cot x)\,dx = \csc x - \cot x$

Query: $\int T_0(x)(T_1(x) \pm T_2(x))dx = T_4(x) \pm T_5(x), \quad T_1 \neq T_2 \wedge T_4 \neq T_5$

New problems generated:

$$\int (\tan x)\,(\cos x + \sec x)\,dx = \sec x - \cos x$$

$$\int (\sec x)\,(\tan x + \sec x)\,dx = \sec x + \cot x$$

$$\int (\cot x)\,(\sin x + \csc x)\,dx = \sin x - \csc x$$

Determinant Problems

Example Problem: $$\begin{vmatrix} (x+y)^2 & zx & zy \\ zx & (y+z)^2 & xy \\ yz & xy & (z+x)^2 \end{vmatrix} = 2xyz\,(x+y+z)^3$$

Query: $$\begin{vmatrix} F_0(x,y,z) & F_1(x,y,z) & F_2(x,y,z) \\ F_3(x,y,z) & F_4(x,y,z) & F_5(x,y,z) \\ F_6(x,y,z) & F_7(x,y,z) & F_8(x,y,z) \end{vmatrix} = C_{10}F_9(x,y,z)$$

$$F_i := F_j[x \to y; y \to z; z \to x] \text{ where } (i,j) \in \{(4,0),(8,4),(5,1),\dots\}$$

New problems generated:

$$\begin{vmatrix} y^2 & x^2 & (y+x)^2 \\ (z+y)^2 & z^2 & y^2 \\ z^2 & (x+z)^2 & x^2 \end{vmatrix} = 2(xy+yz+zx)^3$$

$$\begin{vmatrix} yz+y^2 & xy & xy \\ yz & zx+z^2 & yz \\ zx & zx & xy+x^2 \end{vmatrix} = 4x^2y^2z^2$$

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Sentence Completion

1. The principal characterized his pupils as ________ because they were pampered and spoiled by their indulgent parents.

2. The commentator characterized the electorate as ________ because it was unpredictable and given to constantly shifting moods.

(a) cosseted
(b) disingenuous
(c) corrosive
(d) laconic
(e) mercurial

One of the above problems is a real SAT problem, while the other one was automatically generated!

Key Ideas

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➢ Symbolic methods (solution generation in reverse)

Natural Deduction

Prove that: $x_1 \vee (x_2 \wedge x_3)$
and $x_1 \rightarrow x_4$
and $x_4 \rightarrow x_5$

implies $x_2 \vee x_5$

Inference Rule	Premises	Conclusion
Modus Ponens (MP)	$p \rightarrow q,\ p$	q
Hypothetical Syllogism (HS)	$p \rightarrow q,\ q \rightarrow p$	$p \rightarrow r$
Disjunctive Syllogism (DS)	$p \vee q,\ \neg p$	q
Simplification (Simp)	$p \wedge q$	q

Replacement Rule	Proposition	Equiv. Proposition
Distribution	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
Double Negation	p	$\neg\neg p$
Implication	$p \rightarrow q$	$\neg p \vee q$
Equivalence	$p \equiv q$	$(p \rightarrow q) \wedge (q \rightarrow p)$

Similar Problem Generation

Similar Problems = those that have a minimal proof with the same sequence of inference rules as used by a minimal proof of given problem.

Premise 1	Premise 2	Premise 3	Conclusion
$x_1 \lor (x_2 \land x_3)$	$x_1 \rightarrow x_4$	$x_4 \rightarrow x_5$	$x_2 \lor x_5$

⬇ Similar Problems

Premise 1	Premise 2	Premise 3	Conclusion
$x_1 \equiv x_2$	$x_3 \rightarrow \neg x_2$	$(x_4 \rightarrow x_5) \rightarrow x_3$	$x_1 \rightarrow (x_y \land \neg x_5)$
$x_1 \land (x_2 \rightarrow x_3)$	$(x_1 \lor x_4) \rightarrow \neg x_5$	$x_2 \lor x_5$	$(x_1 \lor x_4) \land \neg x_5$
$(x_1 \lor x_2) \rightarrow x_3$	$x_3 \rightarrow (x_1 \land x_4)$	$(x_1 \land x_4) \rightarrow x_5$	$x_1 \rightarrow x_5$
$(x_1 \rightarrow x_2) \rightarrow x_3$	$x_3 \rightarrow \neg x_4$	$x_1 \lor (x_5 \lor x_4)$	$x_5 \lor (x_2 \rightarrow x_1)$
$x_1 \rightarrow (x_2 \land x_3)$	$x_4 \rightarrow \neg x_2$	$(x_3 \equiv x_5) \rightarrow x_4$	$x_1 \rightarrow (x_3 \equiv \neg x_5)$

Parameterized Problem Generation

Parameters:

# of premises = 3, Size of propositions ≤ 4
# of variables = 3, # of inference steps = 2
Inference rules = { DS, HS }

Parameterized Problems

Premise 1	Premise 2	Premise 3	Conclusion
$(x_1 \rightarrow x_3) \rightarrow x_2$	$x_2 \rightarrow x_3$	$\neg x_3$	$x_1 \wedge \neg x_3$
$x_3 \rightarrow x_1$	$(x_3 \equiv x_1) \rightarrow x_2$	$\neg x_2$	$x_1 \wedge \neg x_3$
$(x_1 \equiv x_3) \vee (x_1 \equiv x_2)$	$(x_1 \equiv x_2) \rightarrow x_3$	$\neg x_3$	$x_1 \equiv x_3$
$x_1 \equiv \neg x_3$	$x_2 \vee x_1$	$x_3 \rightarrow \neg x_2$	$x_1 \wedge \neg x_3$
$x_3 \rightarrow x_1$	$x_1 \rightarrow (x_2 \wedge x_3)$	$x_3 \rightarrow \neg x_2$	$\neg x_3$

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- ➢ Symbolic methods (solution generation in reverse)

Board Games

Game Rules

Allows variants of classic games.

Tic-Tac-Toe
4*4
Row/Column only

Expertise level of players

Allows leveling the playing field.

Conclusion

Improve Education

Make it personalized + interactive

Various aspects

Problem Generation

Solution Generation

Feedback Generation

Inter-disciplinary research area

Synthesis/Search techniques

Natural language processing (for word problems)

Machine learning (for analytics)