# Peaches, Lemons, and Cookies: Designing Auction Markets with Dispersed Information*

Ittai Abraham †, Susan Athey ‡, Moshe Babaioff §, Michael D. Grubb ¶

First version: January 3, 2011

May 16, 2014

**Abstract**

This paper studies the role of ex-ante information asymmetries in second price, common value auctions. Motivated by information structures that arise commonly in applications such as online advertising, we seek to understand what types of information asymmetries lead to substantial reductions in revenue for the auctioneer. One application of our results concerns online advertising auctions in the presence of "cookies," which allow individual advertisers to recognize advertising opportunities for users who, for example, are customers of their websites. Cookies create substantial information asymmetries both ex ante and at the interim stage, when advertisers form their beliefs. The paper proceeds by first introducing a new refinement, which we call "tremble robust equilibrium" (TRE), which overcomes the problem of multiplicity of equilibria in many domains of interest. Second, we consider a special information structure, where only one bidder has access to superior information, and show that the seller's revenue in the unique TRE is equal to the expected value of the object conditional on the lowest possible signal, no matter how unlikely it is that this signal is realized. In the third part of the paper, we study the case where multiple bidders may be informed, providing additional characterizations of the impact of information structure on revenue. Finally, we consider revenue maximization in a richer setting with a private component to valuations.

*The paper was accepted for presentation at ACM-EC 2013 and a one page abstract appeared in its proceedings. We thank Gleb Romanyuk for exceptional research assistance. We thank Nageeb Ali and Preston McAfee for careful reading and valuable suggestions. We also thank seminar audiences at Harvard, Stanford GSB, Northwestern, NBER, NYCE Day, UC Berkeley, University of Michigan, Cornell, Microsoft Research, Hebrew University, AEA 2011 Meetings, and Tel Aviv International Workshop on Game Theory for useful feedback.

†Microsoft Research, ittaia@microsoft.com

‡Stanford University and Microsoft Research, athey@stanford.edu

§Microsoft Research, moshe@microsoft.com

¶Boston College, michael.grubb@bc.edu

# 1 Introduction

At least since Milgrom and Weber (1982a)'s classic paper, economists have studied the role of information revelation in the design of common value auctions. Milgrom and Weber (1982a)'s linkage principle shows that the auctioneer typically benefits by releasing information publicly to all bidders. In many important classes of applications, however, the information revelation problem is more subtle. The auctioneer may not be able to directly observe and release information, but rather has the option to allow bidders to assess information on their own. The auctioneer may not be able to verify whether and how bidders exercise this option, and the content of the information remains the private information of bidders.

This type of problem arises in the classic examples of common value auctions, auctions for natural resources such as oil and timber: in principle, the auctioneer can either limit or facilitate access to bidders seeking to perform seismic surveys or cruise tracts of timber. But there are many other applications as well. In used car auctions, the auctioneer has some control over the type and extent of inspections potential buyers may perform. In internet car auctions, some buyers may be local and have the ability to inspect a car in person; the seller can choose whether to allow this or not. In auctions for financial assets, some bidders may have access to better information about the assets. In all of these cases, some bidders may not be able to directly verify whether other bidders have access to superior information in a particular auction.

This paper develops new theoretical results about the impact of the information structure on revenue in common value auctions, focusing on situations where there may be strong asymmetries of information at either the ex ante (before bidders observe their signals) or the interim stage (after observing their signals). The primary motivating application for our study is online advertising. A growing trend in display advertisement is that advertisers are targeting display ads with increasing sophistication by tracking web surfers using cookies (Helft and Vega 2010). Cookies placed on users' computers by specific web sites can be used to match a user with information such as the user's order history with an online retailer, their recent history of airline searches on a travel website, or their browsing and clicking behavior across a network of online publishers (such as publishers on the same advertising network). A critically important difference between the two cases is the source of information. In the case of search advertising or traditional display advertising, the publisher has as much or more information than the advertiser, and the information is disclosed to all ad buyers symmetrically. In cookie-based display-advertising, however, ad buyers bring their own private information collected via cookies stored on web surfers' computers.

Although there are a variety of mechanisms for selling display advertising, auctions are a leading method, especially for "remnant" inventory, and it is in these markets that cookies potentially play a very important role. For example, Google's ad exchange is currently described as a second-price auction that takes place in real time: that is, at the moment an internet user views a page on an internet publisher, a call is made to the ad exchange, bidders on the exchange instantaneously view information provided by the exchange about the publisher and the user as well as any cookies they may have for the individual user, and based on that information, place a bid. The cookie is only meaningful to the bidder if it belongs directly to the bidder (e.g., Amazon.com may have a cookie on the machines of regular customers), or if the bidder has purchased access to specific cookies from a third-party information broker. Cookie-based bidding potentially makes display auctions inherently asymmetric at both the ex ante and the interim stage. At the ex ante stage, bidders may vary greatly in their likelihood of holding informative cookies, both because popular websites have more opportunities to track visitors and because different sites vary in the sophistication of their tracking technologies. At the interim stage, for a particular impression, a bidder who has a cookie has a substantial information advantage relative to those who do not.

If cookies only provided advertisers private-value information, then increasing sophistication in the prevalence and use of cookies by advertisers would present ad inventory sellers a two-way trade-off between better matching of advertisements with impressions and reduced competition in thinner markets (Levin and Milgrom 2010). In such a private value setting, Board (2009) shows that irrespective of such asymmetry, more cookies and more targeting always increase second-price auction revenue as long as the market is sufficiently thick. However, cookies undoubtedly also contain substantial common value information. (For instance, when one bidder has a cookie which identifies an impression as due to a web-bot rather than a human, the impression is of zero value to all bidders.) As a result, the inherent asymmetry created by cookies can lead to cream skimming or lemons avoidance by informationally advantaged bidders, with potentially dire consequences for seller revenues.[1]

Thus, a designer of online advertising markets (or other markets with similar informational issues) faces an interesting set of market design problems. One question is whether the market should encourage or discourage the use of cookies, and how the performance of the market will be affected by increases in the prevalence of cookies. This is within the control of the market designer: in display advertising, it is up to the marketplace to determine how products are defined. All advertising opportunities from a given publisher can be grouped together, for example. Google's ad exchange reportedly does not support revealing all possible cookies. A second market design question concerns the allocation problem: if an auction is to be used, what format performs best? Both first and second price auctions are used in the industry. There are a number of other design questions, as well, including whether reserve prices, entry fees, or other modifications to a basic auction should be considered.

In order to understand the market design tradeoffs involved in an environment with these

[1] Cream skimming refers to buying up the best inventory, while lemons avoidance refers to avoiding the worst inventory.

kinds of information asymmetries, the first part of our paper specifies a model of pure common-value second-price auctions. Perhaps surprisingly, the existing literature leaves a number of questions open. For example, while it is well known that the presence of an informationally-advantaged bidder will substantially reduce seller revenues in a sealed-bid first-price auction (FPA) for an item with common value (Milgrom and Weber 1982b, Engelbrecht-Wiggans, Milgrom and Weber 1983, Hendricks and Porter 1988), substantially less is known about the same issue in the context of second-price auctions. One of the main impediments to progress has been the well known multiplicity of Bayesian Nash Equilibria in second-price common-value auctions (Milgrom 1981). As a consequence, little is known about what types of information structures lead to more or less severe reductions in revenue.

In order to address the multiplicity problem, we begin by suggesting a new refinement, *tremble robust equilibrium*.[2] Tremble robust equilibrium (TRE) selects only Bayesian Nash Equilibria that are near to an equilibrium (in undominated bids) of a perturbed game in which a random bidder enters with vanishingly small probability $\varepsilon$ and then bids smoothly over the support of valuations. In addition to capturing an aspect of the real-world uncertainty faced by bidders in the kinds of applications we are interested in, we argue that this refinement has a number of attractive properties. In many cases, this refinement selects a unique equilibrium. We study a model with discrete signals, and in this setting, when bidders are ex ante symmetric TRE selects the analog of the symmetric equilibrium studied by Milgrom and Weber (1982a) in a setting with continuous signals. Moreover, it rules out intuitively unappealing equilibria in which uninformed bidders bid aggressively because they can rely on others to set fair prices.

We then proceed to analyze a number of special cases of common value second price auctions using the TRE refinement. To develop some intuition about our main results, consider first a very simple example of an information structure in a common value auction. Only one bidder uses cookie tracking (that is, only one bidder is privately informed), and the bidder can only determine the presence or absence of the cookie: that is, the informed bidder has a binary signal which either takes on the value {no-cookie} or {cookie}. The other bidders cannot assess the existence of the cookie for a particular impression (though they know the overall information structure, including the probability of cookies). Apart from the restriction to a binary rather than continuous signal, this corresponds to the setting of informational advantage studied by Milgrom and Weber (1982b) in first-price auctions.

We show that for this simple information structure, there is a unique TRE in the second-price auction, one with intuitive appeal. We are then able to address some interesting comparative statics questions about when, and why, different kinds of information asymmetries can have dramatically different impacts on revenue.

Consider two cases within this simple information structure. In the first case, cookies

[2] In Section 2 we discuss some standard refinements and explain why they do not adequately address the multiplicity problem in common value SPA.

identify “peaches,” or high-value impressions. This is perhaps the most natural assumption - someone who has been to an advertiser’s website before is more likely to be an active internet shopper than a random web surfer. In the second case, cookies identify “lemons,” or low-value impressions. This might occur if a prior visit indicates the surfer is in fact a web-bot and not a real person. In both cases, information is otherwise symmetric across bidders, and the value of the impression is common to all bidders.

At first, it might seem that for both the “lemons” and “peaches” cases, there could be dire consequences for revenue, due to the extreme adverse selection: one bidder has strictly better information than the others. However, the surprising result is that in the “peaches” case, revenue loss from asymmetric information (relative to the case where both agents are uninformed) is minimal. In contrast, in the “lemons” case, revenue collapses to the value of the “lemons,” even if the probability that an impression has a cookie is arbitrarily close to zero. This result contrasts with that of Engelbrecht-Wiggans et al. (1983), which when applied to our model shows that the revenue losses from a first-price auction should be proportional to the fraction of impressions that have cookies for both lemons and peaches. Putting our results together with Engelbrecht-Wiggans et al. (1983), it follows that a first-price auction will perform substantially better in an environment where one bidder has access to relatively rare cookies for “lemons.”[3]

We next generalize the information structure, allowing for cookies (signals) to be more richly informative, and the informed bidder to have a signal drawn from a finite set. We show that there is a *unique* TRE of the second-price auction. In that TRE the informed bidder is bidding his posterior given his signal, while every uninformed bidder bids the object’s expected value conditional on the worst signal being realized (the signal of the informed bidder with lowest posterior), and the revenue equals this bid. Thus the TRE is one in which, like Akerlof’s (1970) classic “market for lemons,” uninformed buyers are not willing to pay more than their lowest possible value. To understand the role of the refinement, we note that there are Bayesian Nash equilibria with higher revenues, in which a single uninformed bidder bids more aggressively and relies on the informed bidder to bid his expected value and set a “fair” price. Our refinement rules these equilibria out because, in a nearby perturbed auction with a random bidder, aggressive bids by an uninformed bidder would sometimes win at a high price set by the random bidder rather than a fair price set by the informed bidder and hence be unprofitable.

Beyond the initial case of a single informed bidder, we extend our analysis to the general case of multiple informed bidders with richer signal structures. First, we consider domains in which multiple informed bidders receive discrete signals but restrict attention to information

[3]We find that first price auction revenues are always higher than second price auction revenues when only one bidder is informed. However, the difference is on the order of $\epsilon^2$ when the informed bidder has access to relatively rare cookies for “peaches”that arrive at rate $\epsilon$.

structures that satisfy the *strong-high-signal* property, which we define. The strong high signal property is sufficient to ensure the existence of a *unique* TRE, as in the single informed bidder case. Second, we consider all monotonic domains with two bidders, each with a binary-signal (where each signal is either low or high). For any such domain we characterize a TRE and prove it is *unique* (the paper's most technically demanding result).

Our analysis of the second-price auction between two informed bidders with cookies encompasses two special cases. The first is that in which one bidder never receives a cookie - or that only one bidder is informed. The second is that in which bidders are symmetric ex ante. These are variations of the polar extremes of ex ante asymmetry and symmetry studied respectively by Milgrom and Weber (1982b) (for first price auctions) and Milgrom and Weber (1982a) (for multiple auction formats). As already stated, the first-price auction has higher revenue under extreme asymmetry when only one bidder is informed. Focusing on the symmetric equilibrium of the SPA when bidders are ex ante symmetric, Milgrom and Weber (1982a) show that the SPA has higher revenue than the FPA. Since, in our setting, the TRE refinement selects the symmetric equilibrium when bidders are symmetric ex ante, the same result applies.[4] Thus the revenue ranking between first and second price auctions is reversed by sufficient ex ante asymmetry.

Taken together, our findings show that common value second price auctions can be vulnerable to low-revenue outcomes when bidders are asymmetric ex ante. Moreover, low-revenue outcomes are associated with particular forms of asymmetry. It is not sufficient for a bidder to be ex ante better informed than another for revenues to suffer substantially. Rather, the key vulnerability of the second price auction is to ex ante asymmetry with respect to information about lemons. In contrast, the distinction between information about lemons and peaches appears unimportant for first price auction revenues (meaning that they are likely to be a good alternative to the second price auction when the likelihood of discovering lemons is asymmetric across bidders).

So far, we have focused mainly on the costs of information asymmetry, while suppressing any benefit. In the last section of the paper, we extend the model beyond pure common values. We show that the problems created by information asymmetry remain and we suggest alternative mechanism designs that extract most of the possible revenue.

## 2 The Solution Concept

This paper seeks to understand how revenues in a common value second price auction depend on the structure of information held by bidders. A serious challenge to comparing revenues across different information structures is that for any given information structure there are

[4]Milgrom and Weber's (1982a) result is proved for continuous signals, but the authors point out in footnote 15 that it is true more generally.

typically many different equilibria with widely different revenues (Milgrom 1981).

A common approach in the literature with symmetric bidders is to focus on the symmetric equilibrium. As shown by Milgrom and Weber (1982a) and Matthews (1984), this selects the equilibrium in which each bidder bids the object's expected value conditional on the highest signal of competing bidders being equal to her own. This excludes extreme equilibria such as one in which one bidder bids an object's maximum value and all other bidders bid zero. Unfortunately it is not clear how the symmetry refinement can be extended to asymmetric environments of the type we are interested in, or why symmetry should be expected in equilibrium.[5]

Consider the following simple "lemon or peach" scenario. A common value good is equally likely to be a peach (with value $P$) or a lemon (with value $L < P$). There are two bidders in a second price auction. One is perfectly informed about the value of the good, while the other only knows the prior. What bidding strategies and revenues should we expect?

Nash equilibrium provides no prediction about revenue beyond an upper bound of the full surplus. It is an equilibrium for the informed bidder to bid his value and the uninformed bidder to bid $P$, which results in full surplus extraction. However, it is also an equilibrium for the uninformed bidder to bid $10P$ and the informed bidder to bid $L$, earning revenue $L$. There are no symmetric equilibria to focus on.

A natural refinement is to restrict attention to Nash equilibria in which bidders only use undominated bids. For such strategies, bids are always between $L$ and $P$. Notice that unlike in the private value model, agents do *not* necessarily have a dominant strategy in a common value second price auction. Indeed, in the scenario described above the informed agent has a dominant strategy (to bid the value given his signal), while the uninformed agent does not.[6]

Thus ruling out dominated bids restricts the informed bidder to use her dominant strategy and bid her value. However the only restriction placed on the uninformed bidder is that he not bid less than $L$ or more than $P$. Revenue could be anywhere between $L$ and the full surplus.

The uninformed bidder faces a severe adverse selection problem: for any bid less than $P$ she only wins lemons. Our intuition is that this adverse selection problem makes the equilibrium in which the uninformed bids $L$ most plausible. The reason higher bids can be equilibrium strategies is that the informed bidder always sets the price. The uninformed can bid above $L$ safe in the knowledge that the price will always be set fairly at the item's value. The model implicitly ignores the fact that the real world is a risky and uncertain place and that bidding above $L$ exposes the uninformed bidder to the possibility of overpaying for a lemon without

[5] Klemperer (1998) argues that with *almost common values* all reasonable equilibria are extremely asymmetric.

[6] To see that, observe that for any two bids $b_1$ and $b_2$ such that $P \geq b_1 > b_2 \geq L$ there exist two strategies of the informed agent such that for one strategy the utility from $b_1$ is higher, while for the other strategy the utility from $b_2$ is higher. Bidding $b_1$ is superior to bidding $b_2$ when the informed is bidding $(b_1 + b_2)/2$ when the value is $P$, and bidding $L$ when the value is $L$. On the other hand bidding $b_2$ is superior to bidding $b_1$ when the informed is bidding $(b_1 + b_2)/2$ when the value is $L$, and bidding $L$ when the value is $P$ (handing out the good items to the other bidder).

any possible benefit of winning a cheap peach.

Now consider perturbing the game by adding, with some small probability $\epsilon > 0$, a non-strategic bidder who bids randomly between $L$ and $P$ using a "nice" distribution (having full support and continuous density between $L$ and $P$).[7] The purpose is to make the game "noisy" to eliminate unreasonable equilibria by ensuring that the underlying adverse selection problem in the game has consequences. Given that the informed bidder bids the value, the presence of a random bidder means that $L$ is the only undominated bid for an uninformed bidder. The informed bidder ensures that the uninformed bidder can never win the object at a discount below value. However the random bidder ensures that any bid above $L$ risks overpaying for a low value object when the random bidder sets the price. Thus bidding above $L$ leads to a strictly negative payoff. We observe that by adding noise a *unique* strategy profile and revenue is predicted.

Motivated by the this example, we want to consider only Nash equilibria that are nearby to Nash equilibria (in undominated bids) of games perturbed with an $\varepsilon$ probability of an additional random bidder. In the spirit of other perturbation based refinements, such as trembling-hand perfection, we identify Nash equilibria that are nearby by considering the limit as $\varepsilon$ goes to zero. Therefore, we define a *Tremble Robust Equilibrium (TRE)* to be a Nash equilibrium that is the limit, as $\epsilon$ goes to zero, of a series of Nash equilibria (using undominated bids) of each modification of the original game in which another "random" bidder is added with small probability $\epsilon$. The random bidder bids a random value drawn from a distribution with continuous and positive density over the "relevant" values. The formal definition of this new refinement is presented in Section 3.2. Moreover, if there is a profile of strategies that is a Nash Equilibrium in *any* such small perturbation of the original game, we call it a *strong Tremble Robust Equilibrium.*

In our lemons or peach example with one informed and one uninformed bidder, the unique TRE is a strong TRE and predicts that the informed bidder bids the value while the uninformed bidder bids the value of a lemon.

## 2.1 Related Refinements

**Perturbation Based Refinements** It is natural to ask how TRE compares to Selten's (1975) Trembling-Hand Perfect Equilibrium. Consider our preceding peach or lemon example with one informed and one uninformed bidder. In Appendix D we show that two extensions by Simon and Stinchcombe (1995) of trembling-hand perfection to infinite games (which we

[7] In an analysis of the generalized second price (GSP) auction for sponsored search with independent valuations and complete information, Hashimoto (2010) proposes to refine the set of equilibria by adding a non-strategic random bidder that participates in the auction with small probability. Edelman, Ostrovsky and Schwarz (2007) and Varian (2007) have shown that GSP has an envy-free efficient equilibrium; the main result of Hashimoto (2010) is that this equilibrium does not survive the refinement.

adjust to incomplete information) are too permissive: they make the same revenue prediction as Nash equilibrium. Revenues could be anywhere between the value of a lemon and the full surplus. On the other hand, in the same setting, if we restrict the tremble of the informed agent to be *independent* of his signal then in the unique trembling-hand perfect equilibrium predicts that the uninformed agent bids the unconditional expected value of the item, contrary to our expectation.

Recent work by Parreiras (2006), Cheng and Tan (2007), Larson (2009), and Kempe, Syrgkanis and Tardos (2013) introduce perturbations to select a unique equilibrium in two-bidder auctions with continuously distributed signals. Parreiras (2006) and Kempe et al. (2013) perturb the auction format by assuming that winning bidders pay their own bid rather than the second highest with probability $\varepsilon$ (Parreiras (2006) focuses on the limit as $\varepsilon$ goes to zero). Cheng and Tan (2007) and Larson (2009) introduce private value perturbations to the common value environment and take the limit as these perturbations go to zero. In contrast to TRE, the equilibrium selected is sensitive to assumptions about the distributions of the vanishing perturbations.[8]

Our finding that sufficient ex ante asymmetry favors first price auctions over second price auctions (reversing Milgrom and Weber's (1982a) result from the symmetric case) is similar to Cheng and Tan's (2007) result that ex ante asymmetry favors first-price auctions but contrasts with Parreiras's (2006) and Kempe et al.'s (2013) findings that Milgrom and Weber's (1982a) first and second-price auction revenue ranking result is robust to asymmetry.

**Iterated Deletion of Dominated Strategies** An alternative approach taken in the literature that has been applied to auctions with more than two bidders is to select equilibria that survive iterated deletion of dominated strategies. Harstad and Levin (1985) consider the case in which the first order-statistic of bidders' signals is a sufficient statistic for the object's value in the Milgrom and Weber (1982a) setting with symmetric bidders and continuously distributed signals. For this case, Harstad and Levin (1985) shows that iterated deletion of dominated strategies uniquely selects the symmetric Milgrom and Weber (1982a) equilibrium. Einy, Haimanko, Orzach and Sela (2002) consider the case of asymmetric bidders and discrete signals with finite support. They show that if the information structure is *connected* then iterated deletion of dominated strategies selects a set of *sophisticated equilibria* with a unique Pareto-dominant (from bidders' perspective) equilibrium.

Malueg and Orzach (2009) apply Einy et al.'s (2002) refinement in two examples and Malueg and Orzach (2011) apply it to the special case of two-bidder auctions with *connected*

[8] Cheng and Tan (2007) assume private value perturbations are perfectly correlated with common value signals and are symmetric across bidders. The symmetry of perturbations (across asymmetric bidders) selects a unique equilibrium. Larson (2009) allows for asymmetric perturbations which are assumed to be independent of common value signals and shows that the equilibrium selected depends on the ratio of the standard deviations of the two bidders' private value perturbations.

and *overlapping* information partitions. For a particular one-parameter family of common-value distributions, Malueg and Orzach (2011) find that distributions with sufficiently thin left tails yield lower revenues in second-price auctions than in first-price auctions.

Einy et al.'s (2002) result applies to our lemon or peach example with one informed and one uninformed bidder, as this can be represented as a connected domain. Iterated deletion of dominated strategies is unhelpful on its own: the uninformed bidder may still bid anywhere between the value of a lemon and a peach. However, the Pareto dominant equilibrium for the bidders is that in which the uninformed bidder bids the value of a lemon. This is the equilibrium we believe to be natural and coincides with the unique TRE.

The primary drawback to Einy et al.'s (2002) approach is that the required assumptions on the information structure are very restrictive. For instance, we show in Appendix E.1 that Einy et al.'s (2002) connectedness property is strictly more restrictive than our *strong-high-signal* property. Moreover, connectedness rules out many interesting settings such as our model of two bidders with binary signals in which neither bidder is perfectly informed. In contrast, our TRE refinement selects a unique equilibrium in this setting.

# 3 The Model and Tremble Robust Equilibrium

We start by presenting our model followed by the refinement.

## 3.1 The Model

An auctioneer is offering an indivisible good to a set $N$ of $n$ potential buyers. Let $\Omega$ be the set of states of the world (possibly infinite). There is a commonly known prior distribution $H \in \Delta(\Omega)$ over states of the world.

Let $\omega \in \Omega$ be the realized state of the world, which is not observed by the buyers. The value of the item to agent $i$ when the state of the world is $\omega$ is $v_i(\omega) \geq 0$. Each buyer $i$ gets a signal about the state of the world $\omega$ from a finite set of signals $S_i$. For every state $\omega \in \Omega$ and buyer $i$, there is a commonly known distribution over signals $d_i(\omega) \in \Delta(S_i)$. Each buyer $i$ gets a private signal $s_i \in S_i$, sampled from $d_i(\omega)$. Signal $s_i \in S_i$ for agent $i$ is *feasible* if agent $i$ receives signal $s_i$ with positive probability, and the vector of signals $s = (s_1, s_2, \ldots, s_n) \in S_1 \times S_2 \times \ldots \times S_n$ is feasible if it is realized with positive probability. Without loss of generality, we assume that for every $i$, every signal $s_i \in S_i$ is feasible. We denote the set of feasible signal vectors by $S$.

When buyer $i$ realizes signal $s_i$, we denote his updated expected value of the good by $v_i(s_i) = E[v_i(\omega)|s_i]$. Similarly, we denote the posterior expected value given signal vector $s$ by $v_i(s) = E[v_i(\omega)|s]$.

**Definition 1.** *Given a linear order over $S_j$ for every $j$, we say that $s \leq s'$ if for every $j$ it holds that $s_j \leq s'_j$ according to the linear order on $S_j$. A domain is* monotonic *if under some* *linear order over each $S_j$ for every agent $i$ and two feasible vectors of signals $s$ and $s'$ such that $s \leq s'$ it holds that $v_i(s) \leq v_i(s')$.*

Let $T_j \subseteq S_j$ be a set of signals for buyer $j$, and let $T = T_1 \times T_2 \times \ldots \times T_n$ be a product of such subsets, one for each buyer. We say that $T$ is feasible if some $t = (t_1, t_2, \ldots, t_n) \in T$ is feasible ($t$ is realized with positive probability). For $T$ that is feasible let $v_i(T)$ be the expected value that agent $i$ has for the good, conditional on the signal $s_j$ of each buyer $j$ being from $T_j$.

A pure strategy $\mu_i$ for agent $i$ is a mapping from his signal to his bid: $\mu_i : S_i \to \Re_+$, that is $\mu_i(s_i) \in \Re_+$. A mixed strategy $\mu_i$ for agent $i$ is a mapping from his signal to a distribution over non-negative bids.

## 3.2 Tremble Robust Equilibrium

We define the TRE refinement in the context of any auction game and in Section 4 apply it to a common value second price auction.

The refinement is based on the restriction to undominated bids and the addition of a random bidder that bids according to a standard distribution. To define a standard distribution, let $v_{min}$ and $v_{max}$ be the infimum and supremum undominated bids for any bidder $i$ and signal $s_i \in S_i$.[9]

**Definition 2.** *We say that a distribution $R$ is* standard *if the support of $R$ is $[v_{min}, v_{max}]$ (the "relevant" values), $R$ is continuous, strictly increasing and differentiable, and its density $r$ is continuous and positive on the interval.*

Consider an auction and the game $\lambda$ that is induced by the auction. We next define the game perturbed by the addition of a random bidder.

**Definition 3.** *For a standard distribution $R$ and $\epsilon > 0$ define $\lambda(\epsilon, R)$ to be the game induced by $\lambda$ with the following modification: with probability $\epsilon$ there is an additional bidder submitting a bid $b$ sampled according to $R$. We call $\lambda(\epsilon, R)$ an $(\epsilon, R)$-*tremble *of the game $\lambda$.*

Alternatively, one can think of the $(\epsilon, R)$-tremble of the game $\lambda$ as a game with $n+1$ agents, the $n$ original agents and a random bidder that bids $v_{min}$ with probability $1-\epsilon$ and bids according to $R$ with probability $\epsilon$. The unconditional distribution according to which the random bidder is bidding is denoted by $\hat{R}$ and is defined as $\hat{R}(x) = 1 - \epsilon + \epsilon \cdot R(x)$ for all $x > v_{min}$. The density of $\hat{R}(x)$ for all $x > v_{min}$ is $\hat{r}(x) = \epsilon \cdot r(x)$.

Let $\mu_i$ be a strategy of agent $i$. A strategy maps the signal of the agent to a distribution over bids. The strategy is a pure strategy if for every signal the mapping is to a single bid. Let $\mu$ be a vector of strategies, one for each agent.

[9] If $B_i(s_i)$ is the set of undominated bids for bidder $i$ with signal $s_i$ then $v_{min} = \min_{i \in \{1,\ldots,N\}} \min_{s_i \in S_i} \inf B_i(s_i)$ and $v_{min} = \max_{i \in \{1,\ldots,N\}} \max_{s_i \in S_i} \sup B_i(s_i)$.

**Definition 4.** *(i) A (pure or mixed) Nash equilibrium $\mu$ is a* Tremble Robust Equilibrium (TRE) *of the game $\lambda$ if there exists a standard distribution $R$ and a sequence of positive numbers $\{\epsilon_j\}_{j=1}^{\infty}$ such that*

1. *$lim_{j\to\infty}\epsilon_j = 0$.*
2. *$\mu^{\epsilon_j}$ is a (pure or mixed) Nash equilibrium of the game $\lambda(\epsilon_j, R)$, the $(\epsilon_j, R)$-tremble of the game $\lambda$, for every $\epsilon_j$.*
3. *for every agent $i \in N$ and signal $s_i \in S_i$, bidders do not submit dominated bids and $\{\mu_i^{\epsilon_j}(s_i)\}_{j=1}^{\infty}$ converges in distribution to $\mu_i(s_i)$.*

*(ii) $\mu$ is a* strong Tremble Robust Equilibrium *if it is a TRE and, in addition, for the decreasing sequence $\{\epsilon_j\}_{j=1}^{\infty}$ satisfying (1) and (2) above, there exists $k$ such that for every $j > k$ in (2) it holds that $\mu^{\epsilon_j} = \mu$.*

# 4 Common Value SPA Auction

In this section we consider the restriction of the above model to the common value case and study the second price auction. When we talk about the *second price auction (SPA) game* we refer to the game induced by a second price auction with a random tie breaking rule. In the common value model, the state of the world determines a common value of the good to all buyers such that $v_i(\omega) = v(\omega)$ for some function $v(\omega)$ and every bidder $i$. To characterize undominated bids in the common value SPA, let

$$\begin{aligned} v_{min}(s_i) &= \min\{v(s_i, s_{-i})|s_{-i} \text{ such that } (s_i, s_{-i}) \in S\} \text{ and} \\ v_{max}(s_i) &= \max\{v(s_i, s_{-i})|s_{-i} \text{ such that } (s_i, s_{-i}) \in S\} \end{aligned}$$

be the minimal and maximal possible values for $i$ conditional on his signal $s_i$. Undominated bids lie within the interval $[v_{min}(s_i), v_{max}(s_i)]$. Moreover, $v_{min} = \min_{s\in S}\{v(s)\}$ and $v_{max} = \max_{s\in S}\{v(s)\}$ are the minimal and maximal possible values conditional on any feasible signal profile, respectively.

## 4.1 Only One Informed Bidder

We first describe the important special case in which only an *informed buyer $I$* has some information about the value, while all others are *uninformed buyers* (always receiving a null signal). In a slight abuse of notation, below we drop the subscript "$I$" from the informed bidder's signal $s_I$, feasible signal set $S_I$, and interim expected value $v(s_I)$ because only the informed bidder's signal matters. Theorem 5 characterizes a strong TRE in pure strategies and shows that it is the unique TRE.

**Theorem 5.** *In any domain with one informed buyer and any number of uninformed buyers, the unique TRE of the SPA game is the profile of strategies $\mu$ in which:*

1. *the informed buyer bids $b_I(s) = v(s)$.*
2. *each of the uninformed buyers bids the informed bidders minimum possible expected value, $b_U = \min_{\hat{s} \in S} v(\hat{s}) = v_{min}$.*

*Moreover, this profile is a strong TRE in pure strategies.*

*Proof.* To show that $\mu$ is a strong TRE of the SPA game it is sufficient to show that it is a pure NE in any $(\epsilon, R)$-tremble of the game. This is true as the strategy of the informed bidder is dominant, being a best response to any strategies of the uninformed bidders. Additionally, the strategy of any uninformed bidder is a best response to the dominant strategy played by the informed bidder and the strategies of the other uninformed bidders (it gives 0 utility and no strategy gives positive utility). Finally, $\mu$ is trivially a pure strategy profile.

Next we show that it is the unique TRE. Clearly the strategy of the informed bidder is the unique strategy in undominated bids (even among mixed strategies) as for any signal his bid is the unique bid that dominates any other bid. For any uninformed bidder, bidding below $b_U$ is dominated by bidding $b_U$, while bidding above $b_U$ cannot be a best response to the unique strategy of the informed bidder in any $(\epsilon, R)$-tremble of the game (thus will not be a NE in any $(\epsilon, R)$-tremble of the game). □

We stress that the strategy of the uninformed is independent of the probability of the informed buyer receiving the signal that generates the lowest expectation $v_{min}$: even a tiny (but positive) probability of receiving a signal is sufficient to cause the uninformed buyers to bid so low.

The following corollary is immediate from Theorem 5, it shows that the revenue of the SPA with only one informed bidder in the unique TRE is as low as it can be with undominated bids.

**Corollary 6.** *In the unique TRE of the SPA game with one informed buyer and any number of uninformed buyers, the expected revenue of the auctioneer is $R = v_{min}$.*

We point out the connection to Akerlof's (1970) lemons market problem: In both cases uninformed buyers pay no more than the value of a "lemon" and similar adverse selection phenomena drive both results. Yet, we note that in the SPA with common values, adverse selection by itself does *not* necessarily imply low bids by uninformed bidders in every Nash equilibrium: it is a NE for the informed agent to bid according to his signal while the uninformed agent bids *any* value $X$ (as any bid results in zero utility to the uninformed agent). In particular, the uninformed agent *is able* to win the highest quality items in NE (unlike in the market for lemons) because the informed agent is not the seller. Thus, multiplicity of NE as well as the ability of the uninformed party to bid high and win high quality items in NE make the common value SPA somewhat different than the markets for lemons. Our TRE refinement selects an equilibrium in the spirit of the market for lemons, by selecting a *unique* NE for which uninformed buyers indeed do bid the value of lemons.

We next discuss the implications of these results for the seller's revenue in display advertisement common-value SPA with ex ante asymmetric information.

**Example 7.** *Impressions in display ad auctions have various qualities (values) dependent on the likelihood of the user to be influenced by the ad to buy some product. Assume that there are two qualities (common value for an impression), low (L for Lemon) and high (P for Peach), that is* $\Omega = \{L, P\}$*. A peach is more valuable than a lemon, that is* $v(P) > v(L)$*. The commonly known prior is that with probability* $p \in (0,1)$ *the impression is a peach* $P$*, and with probability* $1-p$ *it is a lemon. Fix any small* $\hat{\epsilon} > 0$*. We consider the follow two possible information structures.*

- *In the case that the informed buyer is* $\hat{\epsilon}$-informed about peaches *the set of signals for the informed is* $S = \{D, S_P\}$*. Conditional on the quality being high* ($\omega = P$) *the informed buyer gets signal* $S_P$ *with probability* $\hat{\epsilon}$*, otherwise he gets the default signal* $D$*.*
- *In the case that the informed buyer is* $\hat{\epsilon}$-informed about lemons *the set of signals for the informed is* $S = \{D, S_L\}$*. Conditional on the quality being low* ($\omega = L$) *the informed buyer gets signal* $S_L$ *with probability* $\hat{\epsilon}$*, otherwise he gets the default signal* $D$*.*

Although in both cases the informed buyer has a tiny probability ($\hat{\epsilon} > 0$) of knowing the exact quality, there is a substantial difference in the revenue the seller gets in the SPA in the unique TRE, as implied by Corollary 6. For an informed buyer that is $\hat{\epsilon}$-informed about peaches the revenue is the value conditional on the default signal, and it converges to the unconditional expectation $E[v(\omega)]$ when $\hat{\epsilon}$ converges to 0. In contrast, for an informed buyer that is $\hat{\epsilon}$-informed about lemons the revenue collapses to $v(L)$, for any $\hat{\epsilon} > 0$. We summarize these corollaries in the following easy to prove observation.

**Observation 8.** *In the case that the informed buyer is* $\hat{\epsilon}$-informed about peaches *the revenue of the seller in the unique TRE is* $R^{SPA}_{peaches} = \frac{E[v(\omega)] - \hat{\epsilon} v(P) p}{1 - \hat{\epsilon} p}$*, which converges to the unconditional expectation* $E[v(\omega)]$ *when* $\hat{\epsilon}$ *converges to* 0*.*

*In the case that the informed buyer is* $\hat{\epsilon}$-informed about lemons *the revenue of the seller in the unique TRE is* $R^{SPA}_{lemons} = v(L)$*, for any* $\hat{\epsilon} > 0$*.*

Note that this revenue collapse result with a buyer that is $\hat{\epsilon}$-informed about lemons extends to the case that the informed bidder also sometimes gets other signals (e.g. a signal about a peach), as long as he has positive probability of getting a signal about a lemon.

### 4.1.1 FPA vs. SPA

By comparing the SPA revenue result in Corollary 6 with the FPA revenue result in Theorem 4 of Engelbrecht-Wiggans et al. (1983), it is straightforward to show that FPA revenues are always higher than SPA revenues when only one bidder is informed.

**Corollary 9.** *Consider any common value domain with $n$ agents, $n-1$ of them are uninformed, and the last agent is informed with any information structure. In any such domain the revenue of the first price auction is strictly higher than the revenue in the unique TRE of the second price auction game.*

*Proof.* Recall that $v(s)$ is the informed bidder's interim expected value conditional on $s$ and that $v_{min}$ is the minimum such value. Let $x = v(s)$ and $F$ be the cumulative distribution function of $x$. According to Corollary 6, SPA revenue equals $v_{min}$. According to Theorem 4 of Engelbrecht-Wiggans et al. (1983), FPA revenue is

$$\int_0^\infty (1 - F(x))^2 dx,$$

which can be re-written as $v_{min} + \int_{v_{min}}^\infty (1 - F(x))^2 dx$. For an informed bidder, $F(v_{min}) < 1$ so this is clearly strictly more than $v_{min}$. □

This result clearly implies that for both information structures we consider in Example 7 the revenue of the FPA is larger than the revenue of the unique TRE of the SPA game. For that example we can compute the revenue differences exactly. In Appendix A.1 we we use the Engelbrecht-Wiggans et al.'s (1983) revenue result for FPA and show that in both the case that the informed is $\hat{\epsilon}$-informed about lemons and the case he is $\hat{\epsilon}$-informed about peaches, the revenue of the FPA is

$$R^{FPA} = E\left[v\left(\omega\right)\right] - \hat{\epsilon}p(1-p)\left(v(P) - v(L)\right).$$

Notice that the revenue loss is proportional to $\hat{\epsilon}$, the arrival rate of cookies, regardless of whether cookies contain information about lemons or about peaches. Thus while FPA revenues are always higher than SPA revenues, the difference is substantial only when cookies identify lemons. In particular, loss in revenue from using a SPA rather than a FPA is proportional to $\hat{\epsilon}^2$ when cookies identify peaches:

$$R^{FPA}_{peaches} - R^{SPA}_{peaches} = \hat{\epsilon}^2 \frac{p^2(1-p)}{1-\hat{\epsilon}p}\left(v(P) - v(L)\right).$$

However, when cookies identify lemons, the loss is

$$R^{FPA}_{lemons} - R^{SPA}_{lemons} = \left(1 - \hat{\epsilon}(1-p)\right) p \left(v(P) - v(L)\right),$$

or approximately $p\,(v(P) - v(L))$ when $\hat{\epsilon}$ is small.

We have seen that for both the case that the informed agent is $\hat{\epsilon}$-informed about lemons and the case that the informed agent is $\hat{\epsilon}$-informed about peaches, revenue of FPA does not collapse (does not tend to $v_{min}$ for small epsilon). We next show that this is implied by a much more general observation. We observe that the revenue of FPA can be bounded from below, independent of the information structure. In Appendix A.2 we prove the following proposition.

**Proposition 10.** *Consider any common value domain with items of value in* $[0, 1]$ *and expected value of* $E = E[v]$*. Assume that there are* $n$ *agents,* $n - 1$ *of them are uninformed. For any information structure for the informed agent the revenue of the FPA is at least* $(E[v])^2$*.*

Consider the case that items can have very low value (say 0) and that the expected value $E[v]$ is some positive constant $E$. This observation, in particular, says that the revenue of the FPA does not collapse to zero no matter what the information structure is, in contrast to the revenue of SPA in the unique TRE, which can be arbitrarily small if the informed agent has positive probability of getting a signal with posterior close to zero (like in the case he is $\hat{\epsilon}$-informed about lemons).

We also observed that the revenue of the FPA is continuous in $F(x)$, thus a small change in the information structure of the informed agent implies a small change in the revenue of the seller. This is in contrast to the SPA revenue, which by our result can change dramatically due to a small change in the information structure. This is exactly the case when all agents are uninformed and one of them becomes $\hat{\epsilon}$-informed about lemons. This small change in information structure greatly affects the revenue of the SPA.

## 4.2 Many Agents, each with Finitely Many Signals

We begin this section by recursively defining the *strong high signal* property of information structures in order to generalize the single informed agent result in Theorem 5 to richer domains in which multiple agents have informative signals. Next, Theorem 12 states this section's main result: The strong-high-signal property is sufficient to ensure existence of a *unique* TRE. In particular, it implies that the profile of strategies in which each agent bids the posterior given his signal and the *worst* feasible combination of signals of the others is a strong TRE in pure strategies and the unique TRE. Finally, Observation 14 shows that a strong TRE in pure strategies might not exist if the property is violated.

In every domain satisfying the strong-high-signal property, there exists a signal $s_i$ for some agent $i$ such that $v(s_i) = v_{max}$. Such a signal is strong in the sense that it is a sufficient statistic for the value. (Conditional on $s_i$, other signals $s_{j \neq i}$ are uninformative.) Such a signal is also high in the sense that no other information set could lead to a higher expected value. Moreover, if we condition on that signal *not* being realized and consider the domain with that restriction, that domain also satisfies the condition.[10] (A domain in which all agents are uninformed satisfies the strong-high-signal property.)

**Definition 11.** *Consider a common value domain with $n$ agents, each with finitely many signals. We say that such a domain satisfies the* strong-high-signal *property if: (1) For some agent $i$ and signal $s_i \in S_i$ it holds that $v(s_i) = v_{max}$, and (2) if we consider the same domain but restricted to the case that agent $i$ does* not *receive the signal $s_i$, if that restricted domain contains any feasible vector of signals then it also satisfies the strong-high-signal property.*

Any domain with one informed agent satisfies the strong-high-signal property, as at each point one can take the signal with the highest posterior value for the informed agent and recursively remove it. An example for a slightly more interesting domain that satisfies the strong-high-signal property is any monotonic domain with two agents, each agent $i$ has a binary signal in $\{L_i, H_i\}$ (where $H_i$ is the higher signal), and for which it holds that $v(H_1, H_2) = v(H_1) \geq v(H_2)$. The assumption $v(H_1, H_2) = v(H_1)$ implies $H_1$ is a strong high signal satisfying $v(H_1) = v_{max}$. To prove that the property holds we only need to consider the domain restricted to agent 1 receiving $L_1$. But that domain clearly satisfies the property as it has at most one informed agent (agent 2). We discuss other domains that satisfy the strong-high-signal property, including those with a simple partition structure, later in this section and the next.

We next state the main result of this section, its proof is in Appendix B.1.

**Theorem 12.** *Consider a common value domain with $n$ agents, each with finitely many signals, in which the strong-high-signal property holds. Let $\mu$ be the profile of strategies in which agent $i$ with signal $s_i \in S_i$ bids $v_{min}(s_i)$. Then $\mu$ is the* unique *TRE and moreover, $\mu$ is a strong TRE in pure strategies.*

This theorem has significant implications regarding the revenue of the seller in the unique TRE. In this TRE each bidder bids the posterior given his signal and the *worst* feasible combination of signals of the others, which can be much lower than the interim valuation given only the bidder's signal. For the special case of only one informed agent, the revenue equals to the lowest posterior of the informed (Corollary 6), which can be significantly lower than the expected value of the item.

The result of Theorem 12 clearly applies to the case that only one agent is informed, in this case $\mu$ is a strong TRE in pure strategies that is exactly the one described by Theorem 5. The theorem also applies to the case of monotonic domain with two agents with binary signals when $v(H_1, H_2) = v(H_1) \geq v(H_2)$. In this case the profile $\mu$ is the profile in which agent 1 getting signal $H_1$ bids $v(H_1)$, agent 2 getting signal $H_2$ bids $v(L_1, H_2)$, and each agent bids $v(L_1, L_2)$ given his low signal.

[10]By definition, $v_{max}$ is a function of the domain. When we remove a high signal, its value falls.

Another family of domains for which Theorem 12 applies is the family of *connected domains* which are studied by Einy et al. (2002). Connected domains are defined as follows.

**Definition 13.** *A domain is called a* connected domain *if the following conditions hold. Each agent $i$ has a partition $\Pi_i$ of the set of states $\Omega$, and his signal is the element of the partition that includes the realized state. The information partition $\Pi_i$ of bidder $i$ is* connected *(with respect to the common value $v$) if every $\pi_i \in \Pi_i$ has the property that, when $\omega_1, \omega_2 \in \pi_i$ and $v(\omega_1) \leq v(\omega_2)$ then every $\omega \in \Omega$ with $v(\omega_1) \leq v(\omega) \leq v(\omega_2)$ is necessarily in $\pi_i$. A common-value domain is* connected *(with respect to the common value) if the information partition $\Pi_i$ is connected for every agent $i$.*

In Appendix E.1 we show that any connected domain satisfies the strong-high-signal property, thus Theorem 12 applies to any connected domain. Moreover, we observe that for connected domains the profile $\mu$ is exactly the profile of strategies pointed out by Einy et al. (2002) (the single "sophisticated equilibrium" that Pareto-dominates the rest in terms of bidders resulting utilities). We note that while connected domains allow multiple agents to have multiple signals each, there are some simple domains, even ones with a single informed bidder, that are not connected. In Appendix E.1 we also present a simple domain that satisfies the strong-high-signal property (thus Theorem 12 applies) but is not connected, and also is not equivalent to any connect domain (thus the result of Einy et al. does not apply).

### The strong-high-signal property

The following observation follows from the uniqueness result presented in Theorem 19 for any domain covered by that theorem. It implies that if the strong-high-signal property is violated, the result presented in Theorem 12 can fail.

**Observation 14.** *There exists a domain for which the strong-high-signal property does not hold and for which there does not exist a strong TRE in pure strategies.*

### 4.2.1 Generalizing "Lemons and Peaches" to $n$ agents

In this section, Propositions 16 and 18 generalize the revenue result about a single agent, with either lemons or peaches information, to many agents each getting a signal from a finite set (signals are not necessarily binary).

Consider a monotonic common value domain with items of value in $[0, 1]$ and expected value of $E[v(\omega)]$. Assume that there are $n$ agents, each receiving a signal $s_i$ from an ordered, finite set of signals $S_i$. Let $L_i$ and $H_i$ denote the lowest and highest signals of agent $i$, respectively. Assume that the domain satisfies the conditions of Theorem 12.

We define an agent $i$ to be slightly *informed about peaches* if his *non-peaches* signal $L_i$ occurs with probability close to 1.

**Definition 15.** *Fix any $\epsilon_i \geq 0$. Agent $i$ is $\epsilon_i$-*informed about peaches*, if $Pr[s_i \neq L_i] = \sum_{s_i \in S_i \setminus \{L_i\}} Pr[s_i] \leq \epsilon_i$.*

If all $n$ agents are slightly informed about peaches, then SPA revenue in the unique TRE is high (close to social welfare, which is $E[v(\omega)]$).

**Proposition 16.** *Fix any non-negative constants $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$. Consider any monotonic domain for which (1) $v(\omega) \in [0,1]$, (2) the strong-high-signal property holds, and (3) every agent $i \in \{1, 2, \ldots, n\}$ is $\epsilon_i$-informed about peaches. In the unique TRE $\mu$ (as defined in Theorem 12) SPA revenue is at least*

$$E[v(\omega)] - \sum_{j=1}^{n} \epsilon_j$$

Proofs of Propositions 16 and 18 are in Appendix B.2. We next define what it means to be slightly *informed about lemons.*

**Definition 17.** *Fix any $\epsilon_i > 0$. Agent $i$ is $\epsilon_i$-*informed about lemons*, if*

- $0 < Pr[s_i \neq H_i] = \sum_{s_i \in S_i \setminus \{H_i\}} Pr[s_i] < \epsilon_i$.
- *for any $s_i \in S_i \setminus \{H_i\}$, if $(s_i, s_{-i})$ is feasible then $v(s_i, s_{-i}) < \epsilon_i$.*

Informally, the first assumption states that all lemon signals (not $H_i$, the *non-lemon* signal) for agent $i$ are rare, the probability that any of them is realized is at most $\epsilon_i$. The second assumption states that when $i$ receives any one of his lemons signals it actually indicates that the value of the item, even in the best case, is very low (at most $\epsilon_i$).

Proposition 18 shows that when some agents are slightly informed about peaches and the rest of the agents are slightly informed about lemons, revenue will be very low (as long as some non-degeneracy conditions are satisfied).

**Proposition 18.** *Fix $n \geq i$ and positive constants $\epsilon_1, \epsilon_2, \ldots, \epsilon_i$. Consider any monotonic domain with $n$ bidders for which: (1) $v(\omega) \in [0,1]$, (2) the strong-high-signal property holds, (3) each agent $j \in \{1, 2, \ldots, i-1\}$ is $\epsilon_j$-informed about peaches, (4) agent $i$ is $\epsilon_i$-informed about lemons, and (5) the following non-degeneracy conditions hold:*

- *For any $j < i$ the signal $H_i$ does not imply $L_j$ (alternatively, $(L_j, s_i, s_{-\{i,j\}})$ is feasible for some $s_i \neq H_i$ and some $s_{-\{i,j\}}$).*
- *For any $j > i$ and any signal $s_j \in S_j$, $(s_j, s_i, s_{-\{i,j\}})$ is feasible for some $s_i \neq H_i$ and some $s_{-\{i,j\}}$.*

*Then SPA revenue in the unique TRE $\mu$ (as defined in Theorem 12), is at most*

$$\epsilon_i + \sum_{j=1}^{i} \epsilon_j$$

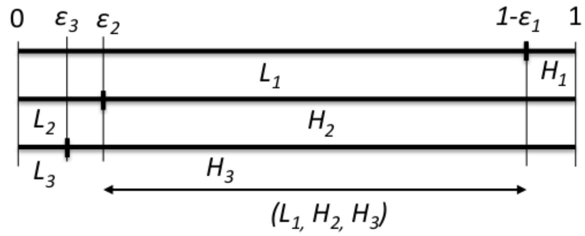


Figure 1: A simple example illustrating Proposition 18

Proposition 18 applies to the domain illustrated in Figure 1. The item's value $v$ is sampled uniformly from $[0, 1]$. Each agent $j$ has a different threshold $t_j$: he gets signal $H_j$ if $v \geq t_j$ and signal $L_j$ otherwise. It holds that $0 < t_3 = \epsilon_3 < t_2 = \epsilon_2 < t_1 = 1 - \epsilon_1 < 1$. Agent 1 is $\epsilon_1$ informed about peaches, while agents 2 and 3 are $\epsilon_2$ and $\epsilon_3$ informed about lemons, respectively. It is easy to verify that the non-degeneracy conditions hold.[11] Proposition 18 applies for $i = 2$ and implies that the revenue is at most $\epsilon_1 + 2\epsilon_2$ by the following argument. As illustrated in Figure 1, the signal profile $(L_1, H_2, H_3)$ occurs if the value is between $\epsilon_2$ and $1 - \epsilon_1$, an event that occurs with probability $1 - (\epsilon_1 + \epsilon_2)$. Therefore, a combination of signals other than $(L_1, H_2, H_3)$ happen with probability $\epsilon_1 + \epsilon_2$ and as $v \leq 1$ it contributes at most $\epsilon_1 + \epsilon_2$ to the expected revenue. The signal combination $(L_1, H_2, H_3)$ occurs with probability smaller than 1. While the bid of agent 2 in that case is high (almost 1/2), both agent 1 and 3 bid at most $\epsilon_2$ with signals $L_1$ and $H_3$, respectively (as they never win when agent 2 gets signal $H_2$). The contribution to the expected revenue in this case is thus bounded by $\epsilon_2$. We conclude that the total revenue is at most $(\epsilon_1 + \epsilon_2) + \epsilon_2$.

The example in Figure 1 can be generalized to allow for many agents and many signals for each, as follows. The item's value $v$ is sampled uniformly from $[0, 1]$. Each agent $j$ has an increasing list of $k_j + 1$ thresholds satisfying $0 = t_j^0 < t_j^1 < t_j^3 < \ldots < t_j^{k_j} = 1$, and his signal indicates the interval between two consecutive thresholds of his that includes the realized value. Fix $i \leq n$. The condition that every agent $j < i$ is $\epsilon_j$-informed about peaches is satisfied when $t_j^1 > 1 - \epsilon_j$. The condition that agent $i$ is $\epsilon_i$-informed about lemons is satisfied when $t_i^{k_i - 1} < \epsilon_i$. Every agent $j > i$ is also $\epsilon_i$-informed about lemons when $t_i^{k_i-1} > t_j^{k_j-1}$. For such an agent $j$, the value conditional on his best signal is not as high as the value conditional on $i$'s best signal (this captures the second non-degeneracy condition). It is easy to verify that the first non-degeneracy condition is satisfied for any such a domain. Proposition 18 states that the revenue is at most $\epsilon_i + \sum_{j=1}^{i} \epsilon_j$. The seller's revenue is low even though with high probability (at least $1 - \sum_{j=1}^{i} \epsilon_j$) agent $i$ gets signal $H_i$ and is bidding relatively high (at least $(1 - \epsilon_i - \max_{j<i} \epsilon_j)/2$). The revenue is low as all other agents are bidding low (at most $\epsilon_i$) and thus the second highest bid is also low.

[11]Non-degeneracy fails when $\epsilon_3 = \epsilon_2$ as the combination $(L_1, L_2, H_3)$ becomes infeasible. It is easy to see that in this case the result fails as, on the likely profile $(L_1, H_2, H_3)$, both agent 2 and 3 are bidding high.

## 4.3 Two Agents, Each with a Binary Signal

When more than one agent is partially informed about the state of the world and the strong-high-signal property is violated, the situation becomes more complicated. In this section we characterize the unique TRE for any monotonic domain with two bidders who receive binary signals.

Let $\{L_i, H_i\}$ be the low and high signals, respectively, of agent $i \in \{1, 2\}$. With some abuse of notation we will also use $H_i$ to denote the event that the signal of agent $i$ was realized to $H_i$, and similarly for $L_i$. We denote $V_{LL} = v(L_1, L_2)$, $V_{HH} = v(H_1, H_2)$, $V_{HL} = v(H_1, L_2)$, $V_{LH} = v(L_1, H_2)$, and normalize $V_{LL} = 0$. We assume that (1) the domain is monotonic, that is $V_{LH}, V_{HL} \in [V_{LL}, V_{HH}]$, (2) both bidders have information,[12] that is $Pr[H_1], Pr[H_2] \in (0, 1)$, and (3) the strong-high-signal property fails,[13] that is $Pr[H_1, H_2] > 0$ and $V_{HH} > \max\{v(H_1), v(H_2)\}$.

Our assumptions $V_{HH} > \max\{v(H_1), v(H_2)\}$ and $V_{LH}, V_{HL} \geq V_{LL} = 0$ imply $V_{HH} > Pr[H_1, L_2](V_{HH} - V_{HL}) > 0$ and $V_{HH} > Pr[L_1, H_2](V_{HH} - V_{LH}) > 0$. We label agents 1 and 2 such that:

$$0 < Pr[H_1, L_2](V_{HH} - V_{HL}) \leq Pr[L_1, H_2](V_{HH} - V_{LH}) < V_{HH}. \tag{1}$$

**Theorem 19.** *Consider any monotonic domain with two bidders, each with a binary signal, satisfying Equation (1), $Pr[H_1, H_2] > 0$, and $Pr[H_1], Pr[H_2] \in (0, 1)$. The unique TRE of the SPA game is the profile of strategies $\mu$ in which:*

- *Every bidder $i$ bids $V_{LL} = 0$ when getting signal $L_i$.*
- *Bidder 1 with signal $H_1$ always bids $V_{HH}$.*
- *Bidder 2 with signal $H_2$*
  - *bids $V_{HH}$ with probability $\frac{Pr[H_1,L_2]}{Pr[L_1,H_2]} \cdot \frac{V_{HH}-V_{HL}}{V_{HH}-V_{LH}}$, and*
  - *bids $V_{LH}$ with the remaining probability.*

Theorem 19 shows that equation (1) identifies bidder 1 as the *strong* or more aggressive bidder and bidder 2 as the *weak* bidder. We provide intuition in our sketch of the proof below. First, however, an immediate corollary is a prediction about seller revenue in the unique TRE of the game.

[12] If $Pr[H_i] \in \{0, 1\}$ then that bidder $i$ is uninformed and the results of Section 4.1 apply.

[13] Theorem 12 applies if the strong-high-signal property holds. If $Pr[H_1, H_2] = 0$ then each agent $i$ knows the value exactly when getting signal $H_i$. Theorem 12 applies and in the unique TRE each agent $i$ bids $v(H_i)$ when getting $H_i$, and $V_{LL}$ otherwise. If $V_{HH} = \max\{v(H_1), v(H_2)\}$ then (labeling agents such that $v(H_1) \geq v(H_2)$) $V_{HH} = v(H_1)$ and agent 1 knows the value exactly when getting signal $H_1$. Theorem 12 applies and in the unique TRE agent 1 bids $v(H_1)$ given signal $H_1$, agent 2 bids $V_{LH}$ given signal $H_2$, and both agents bid $V_{LL}$ given a low signal.

**Corollary 20.** *The seller's expected revenue under the unique profile predicted by Theorem 19 is*

$$(1 - Pr[H_1, H_2]) \cdot V_{LL} + Pr[H_1, H_2] \cdot \left( V_{LH} + (V_{HH} - V_{LH}) \cdot \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \cdot \frac{V_{HH} - V_{HL}}{V_{HH} - V_{LH}} \right)$$

Revenue is $V_{LL}$ unless both bidders receive a high signal, in which case revenue is less than $V_{HH}$. As a result, expected revenue can be an arbitrarily small fraction of welfare. That is the case, for example, when $V_{HL} = V_{LL} = 0$ and $Pr[H_1, L_2]$ tends to 0. Additionally, for the special case in which the value is zero unless both agents receive a high signal ($V_{HH} = 1$ while $V_{HL} = V_{LH} = V_{LL} = 0$), the revenue of the seller is a $\frac{Pr[H_1,L_2]}{Pr[L_1,H_2]}$ fraction of the welfare.

A particular interesting special case for which Theorem 19 holds is the ax ante symmetric case, $V_{HL} = V_{LH} < V_{HH}$ and $Pr[H_1, L_2] = Pr[L_1, H_2] > 0$. In this case the unique TRE is symmetric, predicting that both agents bid $V_{HH}$ given a high signal but bid $V_{LL} = 0$ otherwise (a pure strategy). Thus when bidders are symmetric ex ante, our TRE refinement selects the symmetric equilibrium studied by Milgrom and Weber (1982a) and others. Hence, Milgrom and Weber's (1982a) result ranking second-price auction revenue higher than first-price auction revenue applies.[14] Comparing this to the result in Section 4.1 that first-price auction revenue is always higher than second-price auction revenue when only one bidder is informed illustrates that the revenue ranking depends on the level of ex ante asymmetry. While second-price auctions dominate under symmetric conditions, first-price auctions generate more revenue in sufficiently asymmetric settings.

Given ex ante asymmetry, the unique TRE identified by Theorem 19 is in mixed strategies (agent 2 is mixing between bidding $V_{LL}$ and bidding $V_{HH}$, both with positive probability) and we conclude that there is no pure TRE. Moreover, it is easy to see that the unique TRE is not a strong TRE, as one can observe that in any $(\epsilon, R)$-tremble of the game agent 2 has negative utility by bidding $V_{HH}$, while bidding 0 ensures 0 utility.

## Sketch of the Proof of Theorem 19

Next we sketch the proof of Theorem 19 and provide intuition for the result. For the complete proof see Appendix F.2.

Fix any standard distribution $R$ and $\epsilon > 0$ and let $\lambda(\epsilon, R)$ be the $(\epsilon, R)$-tremble of the game. In the $(\epsilon, R)$-tremble of the game the random bidder enters the auction with small probability $\epsilon > 0$ and is bidding according to a standard distribution $R$ (its support is $[V_{LL}, V_{HH}]$).

The proof relies on two results. (1) First, we show that in each of the games $\lambda(\epsilon, R)$ a mixed NE $\mu^\epsilon$ exists (Lemma 26). (2) Second, we show that the limit of any sequence of NE $\mu^\epsilon$ in the games $\{\lambda(\epsilon, R)\}_\epsilon$ must converge to $\mu$ as $\epsilon$ goes to zero. As $\mu$ is a NE of the original

[14] Milgrom and Weber's (1982a) result is proved for continuous signals, but the authors point out in footnote 15 that it is true more generally.

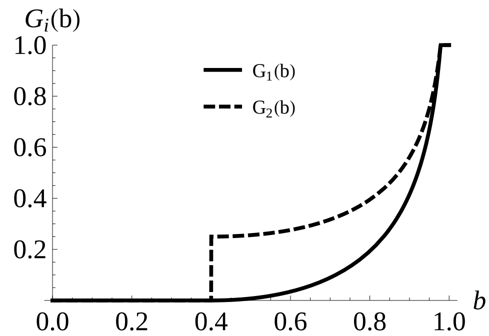


Figure 2: A example of the bidding CDFs for the two bidders when getting their high signals. In this example bidder 1 is the stronger bidder, and in the unique NE of the $\lambda(\epsilon, R)$, most of the probability mass of $G_1$ is close to 1. Bidder 2 has an atom at $V_{LH}$, and the rest of the remaining probability mass of $G_2$ is also close to 1.

game, these two result imply that it is the *unique* TRE.

We defer the first result to the appendix and next present the high level arguments for the second result. We first observe that if bidders never submit dominated bids, bidder $i \in \{1, 2\}$ that receives signal $L_i$ must not bid outside the interval $[V_{LL}, v(L_i, H_j)]$, while bidder $j$ that receives signal $H_j$ must bid at least $v(L_i, H_j)$. As a result, a bidder $i$ with signal $L_i$ will never bid above $V_{LL}$ because doing so means paying at least the item's value (a lower bound for $j$'s bid) and risks overpaying if the random bidder sets the price. Thus a bidder $i$ with signal $L_i$ always bids exactly $V_{LL}$.

Turning to bidder $i$'s strategy given the high signal $H_i$, we first establish notation to describe the bidding strategies. Recall that $\mu^\epsilon$ denotes a NE of the tremble $\lambda(\epsilon, R)$, and define $G_i = \mu_i^\epsilon(H_i)$ to be the cumulative distribution of bidder $i$'s bids conditional on her receiving the signal $H_i$. When it exists, we denote the derivative of $G_i$ by $g_i$.

We show that bidding strategies conditional on high signals must fall into one of two cases. In both cases, the weak bidder (bidder 2) with signal $H_2$ bids an atom at $V_{LH}$ (except in the special case of symmetric bidders in which there are no atoms). Moreover, in both cases, both bidders mix continuously over the interval $(b_{min}, b_{max})$ for some $b_{max} > b_{min} \geq V_{LH}$ and there are no bids outside $[V_{LH}, b_{max}]$. In the first case (illustrated in Figure 2), there is no gap in bidding as $V_{LH} = b_{min}$ and there are no atoms in the strong bidder's bid distribution. In the second case, there is a gap in bidding between $V_{LH}$ and an atom in the strong bidder's bid distribution at $b_{min} > V_{LH}$.

In the second case, the strong bidder's atom serves to keep bidder 2 with signal $H_2$ indifferent between bidding $V_{LH}$ and bidding just above $b_{min}$. It is just the right size to provide a benefit for bidding above $V_{LH}$ equal to the additional cost associated with overpaying due to a random bid falling between $V_{LH}$ and $b_{min}$ when the strong bidder has the low signal $L_1$. This cost goes to zero as $\epsilon$ goes to zero and the random bidder vanishes. Thus the strong bidder's atom at $b_{min}$ also vanishes as $\epsilon$ goes to zero, and is not part of the TRE.

The remainder of the result follows from considering bidder first-order conditions which apply over the interval $(b_{min}, b_{max})$ where both bidders mix continuously. In this interval, if bidder $i$ has signal $H_i$, his bid $b$ could be pivotal in one of three ways. First, a bid $b$ could tie bidder $j$ and beat the random bidder (an event with density $\Pr[H_j|H_i]\, g_j(b)\, \hat{R}(b)$), leading to a gain of $(V_{HH} - b)$. Second, a bid b could tie the random bidder and beat bidder $j$ with a high signal $H_j$ (an event with density $\Pr[H_j|H_i]\, \hat{r}(b)\, G_j(b)$), again leading to a gain of $(V_{HH} - b)$. Third, a bid b could tie the random bidder and beat bidder $j$ with a low signal $L_j$ (an event with density $\Pr[L_j|H_i]\, \hat{r}(b)\, G_j(b)$), leading to loss from overpayment of $(b - E[V \mid H_i, L_j])$. The first-order condition for $b$ to be an optimal bid requires that these expected gains and losses from a slight bid change are equal so there is no benefit to raising or lowering the bid:

$$\Pr[H_j|H_i]\left(\hat{r}(b)\, G_j(b) + \hat{R}(b)\, g_j(b)\right)(V_{HH} - b) = \Pr[L_j|H_i]\, \hat{r}(b)\,(b - E[V \mid H_i, L_j]) \tag{2}$$

In the limit as $\varepsilon$ goes to zero and the random bidder vanishes, a bid is only pivotal if it ties the strategic bidder. Thus the right-hand side of the first-order condition in equation (2) goes to zero and all bids in $(b_{\min}, b_{\max}]$ must approach $v_{HH}$. Note that this implies that, in the limit as $\varepsilon$ goes to zero, the strong bidder 1 bids $V_{HH}$ with probability 1. This follows because bidder 1's atom at $b_{min}$ vanishes so that in the limit all her bids fall in $(b_{\min}, b_{\max}]$. To determine the probability bidder 2 bids $V_{HH}$, we solve the differential equation given in equation (2) to find $G_2(b)$ for $\varepsilon > 0$ and take the limit of $1 - G_2(V_{LH})$ as $\varepsilon$ goes to zero (see the appendix for details).

Next, we provide an informal intuition for the size of bidder 2's atom at $V_{HH}$. In the limit as $\varepsilon$ tends to zero, all bidding mass in $(b_{\min}, b_{\max}]$ approaches $v_{HH}$. Thus, in the limit bidder $j$ bids $V_{HH}$ conditional on $H_j$ with probability $\lim_{\varepsilon\to 0}\int_{b_{min}}^{b_{max}} g_j(b|H_j)db$. Moreover, as bidder 1 has an atom of size 1, bidder 2's atom is equal to the ratio of the atoms:

$$\Pr(b_2 = V_{HH}|H_2) = \frac{\lim_{\varepsilon\to 0}\int_{b_{min}}^{b_{max}} g_2(b)db}{\lim_{\varepsilon\to 0}\int_{b_{min}}^{b_{max}} g_1(b)db} = \lim_{\varepsilon\to 0}\frac{\int_{b_{min}}^{b_{max}} g_2(b)db}{\int_{b_{min}}^{b_{max}} g_1(b)db}. \tag{3}$$

The second equality above relies on the fact that $\lim_{\varepsilon\to 0}\int_{b_{min}}^{b_{max}} g_1(b)db = 1 > 0$.

We solve the first-order condition from equation (2) for the bid density $g_j(b)$ and present the solution in equation (4). This characterizes the bid density of bidder $j$ required for $i$ with signal $H_i$ to bid $b$:

$$g_j(b|H_j) = \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{\hat{r}(b)}{\hat{R}(b)} \cdot \frac{b - E[V \mid H_i, L_j]}{V_{HH} - b} - \frac{\hat{r}(b)}{\hat{R}(b)} \cdot G_j(b|H_j). \tag{4}$$

The first term in equation (4) is proportional the ratio of $i$'s potential loss from overpaying when bidder $j$ has a low signal $L_j$ to $i$'s potential gain from winning when bidder $j$ has a high signal $H_i$. The second term is $\mathcal{O}(\varepsilon)$ and hence unimportant for small $\varepsilon$. Substituting this expression into equation (3), while omitting $\mathcal{O}(\varepsilon)$ terms and cancelling $\hat{r}(b)/\hat{R}(b)$, gives bidder 2's atom at $V_{HH}$:

$$\Pr(b_2 = V_{HH}|H_2) = \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \cdot \lim_{\varepsilon \to 0} \frac{\int_{b_{min}}^{b_{max}} \frac{b - V_{HL}}{V_{HH} - b} db}{\int_{b_{min}}^{b_{max}} \frac{b - V_{LH}}{V_{HH} - b} db} = \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \cdot \frac{V_{HH} - V_{HL}}{V_{HH} - V_{LH}}. \quad (5)$$

The second equality above follows from the fact that $\lim_{\varepsilon \to 0} b_{max} = V_{HH}$ and a result shown in Lemma 56 in the appendix. Thus, bidder 2's atom at $V_{HH}$ is proportional to the potential overpayment by bidder 1 bidding $V_{HH}$ when bidder 2 has a low signal to the potential overpayment by bidder 2 bidding $V_{HH}$ when bidder 1 has a low signal. Finally, bidder 2's atom at $V_{LH}$ has complementary probability $1 - \frac{Pr[H_1,L_2]}{Pr[L_1,H_2]} \cdot \frac{V_{HH}-V_{HL}}{V_{HH}-V_{LH}}$.

# 5 Discussion: Mechanism Design

The previous section shows that in the common value model the revenue of the SPA may be only a small fraction of total welfare. In this section we consider how to maximize the seller's revenue.

In the common value model there is a trivial mechanism that is ex-ante individually rational and maximizes both welfare and revenue: offer the first buyer a take-it-or-leave-it offer to buy the item for the price equal to the unconditional expected value of the item.

Unfortunately, this trivial mechanism does not extend to cases with a private component to the item's value. For example, in online advertising markets it is reasonable to assume that an informed buyer (advertiser) that has an accurate signal about the user (from a cookie on the user's machine) can tailor a specific advertisement to the specific user, generating some additional value over the common value created by placing a generic advertisement that is not user specific.

This motivates us to consider the following generalization of the model with a single informed bidder, in which the informed bidder is also *advantaged*. In this model there are $n$ potential buyers. One random buyer $i$ is informed about the state of the world (gets a signal $s_i \in S_i$), while the others are uninformed.[15] Signals are ordered by the expected common value to an uninformed bidder. Given the maximal signal $s_{max}$, the value for the informed bidder is larger than the common value by a bonus $B > 0$. For other signals there is no bonus.[16]

Let $E$ be the unconditional expected value of the item to an uninformed bidder, $L$ be the

[15]McAfee (2011) considers a related pure common value model in which the probability of being informed is independent across bidders.

[16]In this extension the advantaged bidder is not known ex ante. In contrast, the literature on *almost-common-value* auctions assumes that one bidder is known ex ante to value the object slightly more than other bidders (Bikhchandani 1988, Avery and Kagel 1997, Klemperer 1998, Bulow, Huang and Klemperer 1999, Levin and Kagel 2005).

expected value of the item conditional on the lowest signal $s_{min}$, and $p_{max}$ be the probability of the highest signal $s_{max}$. The expected social welfare when the realized informed bidder always gets the item is $E + p_{max}B$. In this model selling the item ax-ante to a fixed agent at his expected value will generate revenue of $E + \frac{p_{max}B}{n}$, which can be significantly lower than the maximal social welfare.

The unique TRE of the second price auction in this scenario is efficient. Yet, one can easily extend Theorem 5 to this model and see that for any realized informed bidder the unique TRE in this model is exactly the same as the one described by the theorem (with the adjustment that the informed bidder with signal $s_{max}$ bids his value that includes the bonus). Thus revenues may fall far short of capturing total surplus.

Nevertheless, using an auction entry fee, we can build a mechanism that is ex-ante individually rational, is socially efficient, and extracts (almost) the entire welfare as revenue. This is true in the mechanism's *unique* TRE, as we explain below.

The mechanism has two stages. First bidders choose whether to pay an auction entry fee. Second, those who have paid the entry fee compete in a SPA. Theorem 5 (and its extension to this model) predicts a *unique* TRE in the SPA subgame. The payment in the SPA is always $L$. The SPA entry fee is set to be slightly less than the expected utility that an agent gets by participating, assuming all agents participate in the SPA and bid according to the *unique* TRE in that subgame. Thus the entry price is set to be slightly less than $(E + p_{max}B - L)/n$.

As TRE provides a *unique* prediction to the outcome of the second stage, agents have a *unique* rational decision when facing the entry decision, and they choose to pay the entry fee. Thus, in the *unique* subgame-perfect-equilibrium that uses the TRE refinement, agents will all choose to pay the entry fee and the SPA allocation will be socially efficient. Although the revenue in the SPA is low, essentially the entire expected utility an agent gets from this auction is charged as entry fee. The revenue from entry would be (almost) $n(E + p_{max}B - L)/n = E + p_{max}B - L$, while the revenue in the SPA would be $L$. Thus the total revenue is (almost) the social welfare $E + p_{max}B$.

The above mechanism can only be used when both seller and agents can reasonably predict the outcome of the SPA that takes place at the second stage, for which the unique TRE prediction is potentially helpful. The mechanism can be extended to any other scenario in which a uniqueness result can be proven about the outcome of the SPA game under some solution concept.

### Interim Individually Rational Mechanism

While the entry fee mechanism is ex ante individually rational, it is not interim individually rational. We next design an interim individually rational mechanism for this setting, when the informed player has only two signals $s_{min}$ and $s_{max}$. Our mechanism is dominant strategy incentive compatible. Let $L$ be the value conditional on $s_{min}$ and $P + B$ be the value of the advantaged bidder conditional on $s_{max}$.

While our model is not one of independent private values, it is sufficiently close that it seems useful to consider the optimal auction when each player's value is sampled independently and identically from the following distribution: the value is $L$ with probability $1 - 1/n$, and $P + B$ with probability $p_{max} = 1/n$. For this instance, Myerson's optimal auction is to have some reserve price $r$ and some floor price $f$. If some bidders bid at least $r$ then we run a second price auction with reserve $r$, otherwise we randomly choose a winner among those who bid at least $f$ and charge the winner $f$.[17]

In our advantaged bidder model, we propose using this mechanism with $f = L$ and $r = P + B - z$, where $z = (P + B - f)/n$ is the expected utility of agent $i$ bidding $f$ given signal $s_{max}$ (conditional on every other agent $j$ bidding $f$). The revenue obtained is $(1 - p_{max})f + p_{max}r$. Note that this is at least $(1 - 1/n)-$fraction of the efficient social welfare which is $(1 - p_{max})L + p_{max}(P + B)$.

# 6 Conclusion

This paper analyzes the impact of information asymmetries in common value auctions, in an environment that captures key features of real world markets such as online advertising. In these environments, bidders may have access to qualitatively different information; for example, some bidders may have access to signals which occasionally reveal that a user is a "robot" rather than a real person, or that an asset has no value. We show that in second-price auctions, this type of information structure has severe negative consequences for revenue. From a market design perspective, this suggests that auctioneers should think carefully about enabling such information structures, and they may also consider first-price auctions as an alternative to second-price auctions, which are widely used in applications such as online advertising.

# References

**Ahlswede, R. and D. Daykin**, "An inequality for weights of two families of sets, their unions and intersections.," *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 1979, *93*, 183–185.

**Akerlof, George A.**, "The Market for "Lemons": Quality Uncertainty and the Market Mechanism," *The Quarterly Journal of Economics*, 1970, *84* (3), 488–500.

[17]This mechanism is similar to the *but-it-now or take-a-chance* mechanism proposed by Celis, Lewis, Mobius and Nazerzadeh (2012).

**Avery, Christopher and John H. Kagel**, "Second-Price Auctions with Asymmetric Payoffs: An Experimental Investigation," *Journal of Economics and Management Strategy*, 1997, *6* (3), 573–603.

**Bikhchandani, Sushil**, "Reputation in repeated second-price auctions," *Journal of Economic Theory*, 1988, *46* (1), 97–119.

**Billingsley, Patrick**, *Probability & Measure*, Wiley-Interscience, 1995.

**Board, Simon**, "Revealing information in auctions: the allocation effect," *Journal of Economic Theory*, 2009, *38* (1), 125–135.

**Bulow, Jeremy, Ming Huang, and Paul Klemperer**, "Toeholds and Takeovers," *Journal of Political Economy*, 1999, *107* (3), 427–454.

**Celis, Elisa L., Gregory Lewis, Markus M. Mobius, and Hamid Nazerzadeh**, "Buy-it-now or Take-a-chance: Price Discrimination through Randomized Auctions," 2012.

**Cheng, Harrison and Guofu Tan**, "Asymmetric common value auctions with applications to auctions with resale," *Unpublished manuscript*, 2007.

**Edelman, Benjamin, Michael Ostrovsky, and Michael Schwarz**, "Internet Advertising and the Generalized Second-Price Auction: Selling Billions of Dollars Worth of Keywords," *American Economic Review*, March 2007, *97* (1), 242–259.

**Einy, Ezra, Ori Haimanko, Ram Orzach, and Aner Sela**, "Dominance solvability of second-price auctions with differential information," *Journal of Mathematical Economics*, 2002, *37* (3), 247–258.

**Engelbrecht-Wiggans, Richard, Paul R. Milgrom, and Robert J. Weber**, "Competitive bidding and proprietary information," *Journal of Mathematical Economics*, 1983, *11* (2), 161–169.

**Harstad, Ronald M. and Dan Levin**, "A Class of Dominance Solvable Common-Value Auctions," *The Review of Economic Studies*, 1985, *52* (3), 525–528.

**Hashimoto, T.**, "Equilibrium Selection, Inefficiency, and Instability in Internet Advertising Auctions," in "Sixth Ad Auctions Workshop" 2010.

**Helft, Miguel and Tanzina Vega**, "Retargeting Ads Follow Surfers to Other Sites," *The New York Times*, August 30 2010, *The Front Page*, A1. `http://www.nytimes.com/2010/08/30/technology/30adstalk.html`.

**Hendricks, Kenneth and Robert H. Porter**, "An Empirical Study of an Auction with Asymmetric Information," *The American Economic Review*, 1988, *78* (5), 865–883.

**Kempe, David, Vasilis Syrgkanis, and Eva Tardos**, "Information Asymmetries in Common-Value Auctions with Discrete Signals," *SSRN eLibrary*, 2013.

**Klemperer, Paul**, "Auctions with almost common values: The 'Wallet Game' and its applications," *European Economic Review*, 1998, *42* (3-5), 757–769.

**Larson, Nathan**, "Private value perturbations and informational advantage in common value auctions," *Games and Economic Behavior*, 2009, *65* (2), 430–460.

**Levin, Dan and John H. Kagel**, "Almost common values auctions revisited," *European Economic Review*, 2005, *49* (5), 1125–1136.

**Levin, Jonathan and Paul Milgrom**, "Online Advertising: Heterogeneity and Conflation in Market Design," *American Economic Review*, 2010, *100* (2), 603–07.

**Malueg, David A. and Ram Orzach**, "Revenue comparison in common-value auctions: Two examples," *Economics Letters*, 2009, *105* (2), 177–180.

___ **and** ___ , "Equilibrium and revenue in a family of common-value first-price auctions with differential information," *International Journal of Game Theory*, 2011.

**Matthews, Steven A.**, "Information Acquisition in Discriminatory Auctions," in M. Boyer and R. Kihlstrom, eds., *Bayesian Models in Economic Theory*, Amsterdam: North-Holland, 1984, pp. 181–207.

**McAfee, R. Preston**, "The Design of Advertising Exchanges," *Review of Industrial Organization*, 2011, *39* (3), 169–185.

**Milgrom, Paul R.**, "Rational Expectations, Information Acquisition, and Competitive Bidding," *Econometrica*, 1981, *49* (4), 921–943.

___ **and Robert J. Weber**, "A Theory of Auctions and Competitive Bidding," *Econometrica*, 1982, *50* (5), 1089–1122.

___ **and** ___ , "The value of information in a sealed-bid auction," *Journal of Mathematical Economics*, 1982, *10* (1), 105–114.

**Parreiras, Sérgio O.**, "Affiliated Common Value Auctions with Differential Information: The Two Bidder Case," *Contributions to Theoretical Economics*, 2006, *6* (1), Article 12. `http://www.bepress.com/bejte/contributions/vol6/iss1/art12`.

**Selten, R.**, "Reexamination of the perfectness concept for equilibrium points in extensive games," *International Journal of Game Theory*, 1975, *4*, 25–55.

**Simon, Leo K and Maxwell B Stinchcombe**, "Equilibrium Refinement for Infinite Normal-Form Games," *Econometrica*, November 1995, *63* (6), 1421–43.

**Varian, Hal R.**, "Position auctions," *International Journal of Industrial Organization*, December 2007, *25* (6), 1163–1178.

# Suggested Online Appendix

# A One Informed Agent

## A.1 FPA Revenue in Example 7

Let $h = E\left[v\left(w\right)|s\right]$ be the informed bidder's interim value given signal $s$, and $F$ be its cumulative distribution. Then by Engelbrecht-Wiggans et al.'s (1983) Theorem 4, FPA revenue is $\int_0^\infty \left(1 - F\left(h\right)\right)^2 dh$. First consider the peaches case. Let $E^- = \frac{E[v(w)] - \hat{\epsilon} p v(P)}{1 - \hat{\epsilon} p}$ be the posterior given the signal $\emptyset$. As shown in the main text, $h \in \{E^-, v\left(P\right)\}$ and $\Pr\left(h = v\left(P\right)\right) = \hat{\epsilon} p$. Therefore

$$F_{peaches}\left(h\right) = \begin{cases} 0, & h < E^- \\ 1 - \hat{\epsilon} p, & h \in \left[E^-, v\left(P\right)\right) \\ 1, & h \geq v\left(P\right) \end{cases},$$

and hence

$$R^{FPA}_{peaches} = \left(1 - \left(\hat{\epsilon} p\right)^2\right) \frac{E\left[v\left(w\right)\right] - \hat{\epsilon} p v\left(P\right)}{1 - \hat{\epsilon} p} + \left(\hat{\epsilon} p\right)^2 v\left(P\right) = E\left[v\left(w\right)\right] - \hat{\epsilon} p \left(1 - p\right)\left(v\left(P\right) - v\left(L\right)\right).$$

Second, consider the lemons case. Let $E^+ = \frac{E[v(w)] - \hat{\epsilon}(1-p) v(L)}{1 - \hat{\epsilon}(1-p)}$ be the posterior given the signal $\emptyset$. Now, $h \in \{v\left(L\right), E^+\}$ and $\Pr\left(h = v\left(L\right)\right) = \hat{\epsilon}\left(1 - p\right)$. Therefore

$$F_{lemons}\left(h\right) = \begin{cases} 0, & h < v\left(L\right) \\ \hat{\epsilon}(1-p), & h \in \left[v\left(L\right), E^+\right) \\ 1, & h \geq E^+ \end{cases},$$

and hence

$$\begin{aligned} R^{FPA}_{lemons} &= v(L) + \left(1 - \hat{\epsilon}\left(1 - p\right)\right)^2 \left(E^+ - v(L)\right) \\ &= E\left[v\left(w\right)\right] - \hat{\epsilon} p \left(1 - p\right)\left(v\left(P\right) - v\left(L\right)\right). \end{aligned}$$

## A.2 Bounding the FPA Revenue from Below

In this section we prove Proposition 10.

*Proof.* (of Proposition 10) Define the informed bidder's interim expected value conditional on receiving signal $s$ as $h(s) = E[v(\omega)|s]$. Further, let $F$ be the cumulative distribution function of $h$. Note that as items have value in $[0, 1]$, $h \in [0, 1]$ and $F(1) = 1$. According to Theorem 4 of Engelbrecht-Wiggans et al. (1983), FPA revenue is

$$\int_0^1 (1 - F(h))^2 dh$$

and the informed agent expected profit is

$$\int_0^1 F(h)(1-F(h))dh$$

Note that the revenue and the informed agent's profit sum up to $E$, the expected value of the item (the social welfare). To bound the revenue from below we bound the informed agent's profit from above. We use the following result due to Ahlswede and Daykin (1979).

**Lemma 21.** *If, for* 4 *non-negative functions* $g_1, g_2, g_3, g_4$ *mapping* $\mathbb{R} \to \mathbb{R}$*, the following holds:*

$$\textit{for all } x, y \in \mathbb{R},\ \ g_1(\max(x,y)) \cdot g_2(\min(x,y)) \geq g_3(x) \cdot g_4(y),$$

*then it follows that*

$$\int_a^b g_1(t)dt \cdot \int_a^b g_2(t)dt \geq \int_a^b g_3(t)dt \int_a^b g_4(t)dt.$$

We apply this lemma by setting

$$g_1(t) = F(t),\ \ g_2(t) = 1-F(t),\ \ g_3(t) = F(t)\cdot(1-F(t)),\ \ g_4(t) = 1.$$

Monotonicity of $F$ implies that the conditions of the lemma hold. Indeed, if $x'' > x'$,

$$F(x'')\cdot(1-F(x')) \geq F(x'')\cdot(1-F(x''))$$

and

$$F(x'')\cdot(1-F(x')) \geq F(x')\cdot(1-F(x'))$$

Then, it follows that

$$E\cdot(1-E) = \int_0^1 F(t)dt \cdot \int_0^1 (1-F(t))dt \geq \int_0^1 F(t)(1-F(t))dt.$$

As the revenue equals to the welfare minus the informed agent's profit we conclude that the revenue is bounded from above by $E^2$:

$$\int_0^1 (1-F(h))^2 dh = E - \int_0^1 F(t)(1-F(t))dt \leq E^2$$

$\square$

# B Many Agents, each with Finitely Many Signals

## B.1 Proof of Theorem 12

We prove Theorem 12 by induction. We introduce a lemma and an observation that are applied at each induction step. Before introducing the lemma we make two simple observations about the implications of the strong-high-signal property, and define an order on all signals.

**Preliminaries:** First, we note that if $s_i$ is a strong high signal ($v(s_i) = v_{max}$) two properties follow immediately:

- $v(s') \le v(s_i, s_{-i})$ for any feasible signal profile $s' \in S$.
- $v(s_i, s'_{-i}) = v(s_i, s_{-i})$ for any $s'_{-i} \in S_{-i}$ such that $(s_i, s'_{-i})$ is feasible.

The first property is equivalent to $v(s_i) = v_{max}$ given the definition of $v_{max}$. The second property follows from it. If there were some $v(s_i, s'_{-i}) < v(s_i, s_{-i})$ then we know that $v(s_i, s'_{-i}) < v_{max}$ by definition of $v_{max}$. But if there is positive probability that $v < v_{max}$ (ie $(s_i, s'_{-i})$ is feasible, as assumed) then $v(s_i) < v_{max}$, a contradiction. These properties are used in the proof of Lemma 22 when we invoke the strong-high-signal property.

Second, we define a natural binary relation between signals using the relation between the lower bounds they place on the expected value. We say that signal $s_i$ of bidder $i$ is *weakly lower* than signal $s_j$ of bidder $j$ if $v_{min}(s_i) \le v_{min}(s_j)$, and is *strictly higher* than signal $s_j$ of bidder $j$ if $v_{min}(s_i) > v_{min}(s_j)$.

**A Lemma** The next lemma is a major step in showing that bidder $i$ with signal $s_i$ does not bid *above* $v_{min}(s_i)$.

**Lemma 22.** *Fix a signal $s_j$ received by bidder $j$ and any strategy profile $\eta$ in which every bidder $i$ with signal $s_i$ strictly higher than $s_j$ ($v_{min}(s_i) > v_{min}(s_j)$) bids $v_{min}(s_i)$ with probability 1.*

1. *If $\eta$ is a NE of the tremble $\lambda(\epsilon, R)$ with $\epsilon > 0$, then no bidder $i$ with signal $s_i$ (including bidder $j$ with signal $s_j$) weakly lower than $s_j$ bids strictly above $v_{min}(s_j)$.*
2. *In the original game $\lambda$ and in any tremble $\lambda(\epsilon, R)$ for $\epsilon > 0$, in any NE the utility of bidder $j$ with signal $s_j$ from bidding $v_{min}(s_j)$ is at least as high as his utility from any higher bid.*

*Proof.* **Proof of part (1)**: Let $\bar{b}_i(s_i)$ be the supremum bid by bidder $i$ with signal $s_i$. Let $\bar{b}$ be the maximum supremum bid among signals weakly lower than $s_j$:

$$\bar{b} \equiv \max_{i \in N, s_i \in S_i} \{\bar{b}_i(s_i) : v_{min}(s_i) \le v_{min}(s_j)\}.$$

Suppose $\eta$ is a NE of the tremble $\lambda(\epsilon, R)$ but $\bar{b} > v_{min}(s_j)$. Let $\delta > 0$ be sufficiently small such that (1) $v_{min}(s_j) < \bar{b} - \delta$ and (2) for any bidder $i$ and signal $s_i$ if $\bar{b}_i(s_i) < \bar{b}$ implies $\bar{b}_i(s_i) < \bar{b} - \delta$. With positive probability, no signal strictly higher than $s_j$ is realized and the high bid falls in the interval $(\bar{b} - \delta, \bar{b}]$. Therefore at least one bidder $k$ with a signal $s_k$ that satisfies $\bar{b}_k(s_k) = \bar{b}$ and $v_{min}(s_k) \leq v_{min}(s_j)$ (possibly $k = j$ and $s_k = s_j$) wins with positive probability with a bid in the interval $(\bar{b} - \delta, \bar{b}]$.

Fix any bid $b \in (\bar{b} - \delta, \bar{b}]$ that wins with positive probability conditional on being placed by bidder $k$ with signal $s_k$. We show below that for bidder $k$ with signal $s_k$, bidding $v_{min}(s_j)$ is strictly more profitable than bidding $b$. Because bidder $k$ with signal $s_k$ makes such bids with positive probability, this contradicts $\eta$ being a NE. The argument follows below.

Consider a particular realization in which bidder $k$ receives signal $s_k$. Let $b_{-k}^{max}$ be the highest realized bid of bidders other than $k$ (including the random bidder). Further, let bidder $i$ be the bidder who has the highest realized signal $s_i$ and his bid be $b_i$. (If there are multiple bidders whose signals tie for the highest then choose any $i$ from the set.)

Now compare $k$'s outcome from bidding $b$ rather than $v_{min}(s_j)$. If $b_{-k}^{max} < v_{min}(s_j)$ or $b_{-k}^{max} > b$ then $k$'s outcome and payoff are unchanged by bidding $b$ rather than $v_{min}(s_j)$. However, if $b_{-k}^{max} \in [v_{min}(s_j), b]$ then $k$ wins and pays $b_{-k}^{max}$ by bidding $b$ at some cases were he was losing by bidding $v_{min}(s_j)$. Consider three cases. **First**, suppose that some bidder with a signal strictly higher than $s_j$ is bidding $b_{-k}^{max}$. Then by assumption $b_{-k}^{max} = b_i = v_{min}(s_i)$ and by the strong-high-signal property (SHSP) $v(s) = v_{min}(s_i)$. Thus the additional win is priced at its value and does not change $k$'s payoff. **Second**, suppose $b_{-k}^{max} = v_{min}(s_j)$. Then by SHSP $v(s) \leq v_{min}(s_j)$ and the additional win is priced at or above its value and weakly reduces $k$'s payoff. **Third**, suppose $b_{-k}^{max} \in (v_{min}(s_j), b]$ and it is not the bid of a bidder with a strictly higher signal. If no signal strictly higher than $s_j$ is realized, then by SHSP $v(s) \leq v_{min}(s_j)$. If at least one signal strictly higher than $s_j$ is realized, then by assumption $b_{-k}^{max} > b_i = v_{min}(s_i)$ and by SHSP $v(s) = v_{min}(s_i)$. In either case, the additional win must be priced strictly above its value ($b_{-k}^{max} > v(s)$) and strictly reduces $k$'s payoff.

The preceding paragraph shows that for any realization, bidding $b$ yields a weakly lower payoff for $k$ than bidding $v_{min}(s_j)$ and in the third case yields a strictly lower payoff. The third case occurs with positive probability in any tremble $\lambda(\epsilon, R)$ with $\epsilon > 0$. Therefore bidding $b$ rather than $v_{min}(s_j)$ strictly reduces $k$'s expected payoff ex ante.

**Proof of part (2)**: In the proof of part (1) above, we showed that (for any realization) bidding $b$ yields a weakly lower payoff for bidder $k$ with signal $s_k$ than bidding $v_{min}(s_j)$. The same argument can be repeated under the assumptions of part (2) to show that bidding $v_{min}(s_j)$ is *weakly* better than any higher bid for bidder $j$ with signal $s_j$. Note that we do not claim a strict payoff ranking because in profile $\eta$ bidder $j$ (unlike bidder $k$) might win with zero probability at both bids. $\square$

**An Observation** We next observe that bidder $i$ with signal $s_i$ that only submits undominated bids never bids below $v_{min}(s_i)$.

**Observation 23.** *In the original game $\lambda$ and in any tremble $\lambda(\epsilon, R)$ for $\epsilon > 0$, for bidder $i$ with signal $s_i$ bidding $v_{min}(s_i)$ weakly dominates bidding any amount $b_i < v_{min}(s_i)$.*[18]

**The Proof** We combine this observation with Lemma 22 to prove Theorem 12.

*Proof.* (of Theorem 12) Fix any strict linear order on the signals that is consistent with the order of lower bounds they place on the expected value. That is, fix an arbitrary order satisfying that for every $s_i$ and $s_j$, if $v_{min}(s_i) > v_{min}(s_j)$ then $s_i$ is ranked higher than $s_j$.

The proof proceeds by induction. The base case considers the highest signal according to the fixed order. Suppose the highest signal is bidder $i$'s signal $s_i$. By Observation 23 bidding $v_{min}(s_i)$ dominates any lower bid for bidder $i$ with signal $s_i$. Moreover, SHSP implies that for the highest signal $s_i$, for any $s_{-i} \in S_{-i}$ such that $(s_i, s_{-i})$ is feasible, it holds that $v_{min}(s_i) = v(s_i, s_{-i})$. Thus $v_{min}(s_i) = v_{max}(s_i)$ and therefore in any tremble $\lambda(\epsilon, R)$ in which the bid of agent $i$ with signal $s_i$ belongs to $[v_{min}(s_i), v_{max}(s_i)]$ it holds that the bid must be $v_{min}(s_i) = v_{max}(s_i)$. Moreover, bidding $v_{min}(s_i)$ is a dominant strategy for bidder $i$ with signal $s_i$ in the original game $\lambda$ and any tremble $\lambda(\epsilon, R)$.

We move to the induction step. Consider the $l^{th}$ highest signal, which is $s_j$ received by bidder $j$. Assume that every bidder $i$ with strictly higher signal $s_i$ (that is, $v_{min}(s_i) > v_{min}(s_j)$) bids $v_{min}(s_i)$ with probability 1. Observation 23 and claim (2) of Lemma 22 imply that it is a best response for bidder $j$ with signal $s_j$ to bid $v_{min}(s_j)$ in the original game $\lambda$ and the tremble $\lambda(\epsilon, R)$. Moreover, Observation 23 and claim (1) of Lemma 22 imply that this is the *unique* best response in any NE in undominated bids of any tremble $\lambda(\epsilon, R)$, for $\epsilon > 0$.

Proceeding by induction through all signals shows that the pure strategy profile $\mu$ is a Nash equilibrium both in the original game $\lambda$ and in any tremble $\lambda(\epsilon, R)$ with $\epsilon > 0$. Moreover, it is the unique Nash equilibrium in undominated bids in any tremble $\lambda(\epsilon, R)$ with $\epsilon > 0$. The theorem follows directly. $\square$

## B.2 Generalizing "Lemons and Peaches" to $n$ agents

We present the proof of Proposition 16.

*Proof.* (of Proposition 16) If $\sum_{i=1}^{n} \epsilon_i \geq 1$ the claim follows trivially. Items have values in $[0, 1]$ and thus $E[v(\omega)] \leq 1$, this implies that $E[v(\omega)] - \sum_{j=1}^{n} \epsilon_j \leq 0$, and the claim about the revenue clearly holds as revenue is non-negative since every bid is non-negative. We next assume that $\sum_{i=1}^{n} \epsilon_i < 1$.

[18] It is trivial to come up with strategies for the other bidders for which bid $v_{min}(s_i)$ gives strictly higher utility than bid $b_i$.

Let $L = (L_1, L_2, \dots, L_n)$ be the combination of signals in which each agent $i$ gets signals $L_i$. Observe that $Pr[not\ L] \leq \sum_{i=1}^{n} Pr[s_i \neq L_i] \leq \sum_{i=1}^{n} \epsilon_i$ as every agent $i$ is $\epsilon_i$-informed about peaches, thus $Pr[L] \geq 1 - \sum_{j=1}^{n} \epsilon_j > 0$ which means that $L$ is feasible. As the domain is monotonic and $L_i$ is the lowest signal for agent $i$, for every feasible $s$ it holds that $v(L) \leq v(s)$. This implies that $v_{min}(s_i) \geq v(L)$ for every agent $i$ and signal $s_i \in S_i$.

As all bids are at least $v(L)$, the revenue is at least $v(L)$, thus it is sufficient to show that $v(L) \geq E[v(\omega)] - \sum_{j=1}^{n} \epsilon_j$.

Observe that

$$E[v(\omega)] = v(L) \cdot Pr[L] + v(not\ L) \cdot Pr[not\ L]$$

Which implies that

$$v(L) = \frac{E[v(\omega)] - v(not\ L) \cdot Pr[not\ L]}{Pr[L]} \geq E[v(\omega)] - Pr[not\ L] \geq E[v(\omega)] - \sum_{i=1}^{n} \epsilon_i$$

since $0 < Pr[L] \leq 1$, $v(not\ L) \leq 1$ (as for any $\omega$ it holds that $v(\omega) \in [0,1]$), and $Pr[not\ L] < \sum_{i=1}^{n} \epsilon_i$. □

Next, we present the proof of Proposition 18.

*Proof.* (of Proposition 18 ) If $\sum_{i=1}^{n} \epsilon_i \geq 1$ the claim follows trivially. Items have values in $[0,1]$ and thus all bids are at most 1, which implies that the revenue is at most 1. We next assume that $\sum_{i=1}^{n} \epsilon_i < 1$.

Since each $j < i$ is $\epsilon_j$-informed about peaches it holds that

$$Pr[L_1, L_2, \dots, L_{i-1}] \geq 1 - \sum_{j=1}^{i-1} \epsilon_j$$

Now, since $i$ is $\epsilon_i$-informed about lemons it holds that $Pr[H_i] \geq 1 - \epsilon_i$, and thus

$$Pr[L_1, L_2, \dots, L_{i-1}, H_i] \geq Pr[L_1, L_2, \dots, L_{i-1}] + Pr[H_i] - 1 \geq 1 - \sum_{j=1}^{i} \epsilon_j > 0$$

The revenue obtained when the signals of agents $1, 2, \dots, i$ are *not* realized to $(L_1, L_2, \dots, L_{i-1}, H_i)$ is at most the maximal value of any item, which is 1, and that happens with probability at most $\sum_{j=1}^{i} \epsilon_j$. Thus this case contributes at most $\sum_{j=1}^{i} \epsilon_j$ to the expected revenue.

We next bound the revenue obtained when the signals of agents $1, 2, \dots, i$ are realized to $(L_1, L_2, \dots, L_{i-1}, H_i)$, and event that happens with probability at most 1. To prove the claim it is sufficient to show that the maximal of the bids of all agents other than $i$ is at most $\epsilon_i$, since this upper bounds the revenue. We first bound the bid $\mu_j(L_j)$ of any agent $j < i$ when getting signal $L_j$. By the first non-degeneracy assumption $(L_j, s_i, s_{-\{i,j\}})$ is feasible for some $s_i \neq H_i$

and some $s_{-\{i,j\}}$. As agent $i$ is $\epsilon_i$-informed about lemons it holds that $v(L_j, s_i, s_{-\{i,j\}}) \leq \epsilon_i$ and thus $\mu_j(L_j) \leq \epsilon_i$.

We next bound the bid $v_{min}(s_j)$ of any agent $j > i$ when getting any signal $s_j \in S_j$. By the second non-degeneracy assumption $(s_j, s_i, s_{-\{i,j\}})$ is feasible for some $s_i \neq H_i$ and some $s_{-\{i,j\}}$. As agent $i$ is $\epsilon_i$-informed about lemons it holds that $v(s_j, s_i, s_{-\{i,j\}}) \leq \epsilon_i$ and thus $v_{min}(s_j) \leq \epsilon_i$. We have shown that when the signals of agents $1, 2, \ldots, i$ are realized to $(L_1, L_2, \ldots, L_{i-1}, H_i)$ the maximal of the bids of all agents other than $i$ is at most $\epsilon_i$, thus the revenue in this case is bounded by $\epsilon_i$, and the claim follows. $\square$

# C Two Agents, Each with a Binary Signal

## C.1 Outline of the Proof of Theorem 19

We next present an outline of the proof of Theorem 19 along with four lemmas that we use to prove the result. The proof of these lemmas appears in Appendix F.

*Proof outline:*

Fix any standard distribution $R$ and $\epsilon > 0$ and let $\lambda(\epsilon, R)$ be the $(\epsilon, R)$-tremble of the game. In the $(\epsilon, R)$-tremble of the game the random bidder arrives to the auction with small probability $\epsilon > 0$ and is bidding according to a standard distribution $R$ (its support is $[V_{LL}, V_{HH}]$).

To prove Theorem 19, we begin by developing a series of necessary conditions that any NE $\mu^\epsilon$ of the tremble $\lambda(\epsilon, R)$ must satisfy. These are summarized in Lemmas 24 and 25 presented below. Next, we show that (for sufficiently small $\epsilon$) a (mixed) NE of the tremble $\lambda(\epsilon, R)$ exists (Lemma 26). This existence result implies that for any standard distribution $R$, there exists a sequence of $\epsilon$ converging to zero and an associated sequence of NE $\{\mu^\epsilon\}$ corresponding to the trembles $\lambda(\epsilon, R)$. The final step is to use the necessary conditions developed in Lemmas 24 and 25 to show that the limit of any such sequence $\{\mu^\epsilon\}$ must converge to $\mu$ as $\epsilon$ goes to zero (Lemma 27). It then follows that $\mu$ is the *unique* TRE.

Below, we present the four Lemmas 24-27. To simplify notation we normalize $V_{LL} = 0$ and $V_{HH} = 1$ and denote $v_1 = V_{HL}$ and $v_2 = V_{LH}$. Moreover, for a given $\mu^\epsilon$, we define the following notation. Let $\underline{b}_i = \inf\{b : G_i(b) > 0\}$ and $\bar{b}_i = \inf\{x : G_i(x) = 1\}$ for agent $i \in \{1, 2\}$. Define $\underline{b} = \min\{\underline{b}_1, \underline{b}_2\}$, $b_{min} = \max\{\underline{b}_1, \underline{b}_2\}$ and $b_{max} = \max\{\bar{b}_i, \bar{b}_j\}$. Note that when bidders never submit dominated bids by definition it holds that $1 \geq b_{max} \geq b_{min} \geq \underline{b} \geq 0$.

We start with some necessary conditions that any NE $\mu^\epsilon$ in a fixed $\lambda(\epsilon, R)$ must satisfy.

**Lemma 24.** *At $\mu^\epsilon$ the following must hold.*

1. *For some $j \in \{1, 2\}$ it holds that $\underline{b} = \underline{b}_j = v_j$ and $b_{min} = \underline{b}_i \geq v_i$ for $i \neq j$.*
2. *Both $G_1$ and $G_2$ are continuous and strictly increasing on $(b_{min}, b_{max})$. It holds that $G_1(b_{max}) = G_2(b_{max}) = 1$. Moreover, if $b_{max} > b_{min}$ then $b_{max} = \bar{b}_1 = \bar{b}_2$.*

3. *For every bidder* $i \in \{1,2\}$ *it holds that* $G_i(b) = 0$ *for every* $b \in [0, \underline{b})$*, and* $G_i(b) = G_i(\underline{b})$ *for every* $b \in [\underline{b}, b_{min})$.

4. *If* $b_{min} = \underline{b}$ *then* $\underline{b} = \max\{v_1, v_2\}$*. Additionally, if* $v_1 = v_2$ *then* $b_{min} = \underline{b} = v_1 = v_2$ *and no bidder has any atom anywhere. If* $v_i > v_j$ *then* $b_{min} = \underline{b} = v_i$ *and* $i$ *has an atom at* $\underline{b}$*, while* $j$ *has no atoms.*

5. *If* $b_{min} > \underline{b}$ *then for one agent, say* $j$*, it holds that* $\underline{b} = \underline{b}_j = v_j$*. Bidder* $j$ *has an atom at* $v_j$ *and bidder* $i \neq j$ *has an atom at*

$$b_{min} = b_i^*(G_j(v_j)) = \frac{\Pr[H_j|H_i]G_j(v_j) + v_i \Pr[L_j|H_i]}{\Pr[H_j|H_i]G_j(v_j) + \Pr[L_j|H_i]} > \max\{v_i, v_j\} \tag{6}$$

   *and* $b_{min}$ *satisfies* $b_{min} \leq v(H_i)$*, and* $b_{min} = v(H_i)$ *if and only if* $G_j(v_j) = 1$.

   *It also holds that either*

   - $b_{max} = b_{min}$*, in this case* $G_i(b_{min}) = 1$*,* $G_j(v_j) = 1$ *(*$j$ *always bids* $v_j$*,* $i$ *always bids* $b_{min}$*). Or*
   - $b_{max} > b_{min}$*,* $G_i(b_{min}) > 0$ *and*

$$G_i(b_{min}) = \frac{\Pr\left[L_i|H_j\right]}{\Pr\left[H_i|H_j\right]} \frac{\int_{v_j}^{b_{min}} (x - v_j)\, \hat{r}(x)\, dx}{\hat{R}(b_{min})(1 - b_{min})} \tag{7}$$

Building on the preceding necessary conditions that apply for all $\epsilon$, the next result gives tighter necessary conditions for NE in the tremble $\lambda(\epsilon, R)$ when $\epsilon$ is sufficiently small. To develop the result we first apply the first-order conditions for optimal bidding over the interval $(b_{min}, b_{max})$ to characterize bid distributions above $b_{min}$. Next, we show that for sufficiently small $\epsilon$ it holds that $b_{max} > b_{min}$ (ruling out the case $b_{max} = b_{min}$ allowed for in Lemma 24). Finally, we complete the proof by more tightly characterizing the size and placement of atoms at the bottom of bidders' bid distributions.

**Lemma 25.** *If* $\epsilon$ *is small enough at* $\mu^\epsilon$ *the following must hold. There must exist* $b_{min}$ *and* $b_{max}$ *such that* $1 > b_{max} > b_{min} \geq 0$ *and:*

- *The two bidders are symmetric (*$Pr[H_1, L_2] = Pr[L_1, H_2]$ *and* $v_1 = v_2$*) if and only if* $b_{min} = \underline{b} = v_1 = v_2$ *and* $G_1(b_{min}) = G_2(b_{min}) = 0$ *(no atoms).*

- *If* $Pr[H_1, L_2](1 - v_1) = Pr[L_1, H_2](1 - v_2)$ *but the bidders are not symmetric, and it holds that* $v_1 > v_2$ *and* $Pr[H_1, L_2] < Pr[L_1, H_2]$*, then bidder* 1 *has an atom at* $b_{min} = \underline{b}_1$ *of size* $G_1(b_{min}) > 0$*, and bidder* 2 *has an atom at* $v_2 = \underline{b}_2 = \underline{b} < b_{min}$ *of size* $G_2(v_2) > 0$*. It holds that*

$$b_{min} = b_1^*(G_2(v_2)) = \frac{\Pr[H_2|H_1]G_2(v_2) + v_1 \Pr[L_2|H_1]}{\Pr[H_2|H_1]G_2(v_2) + \Pr[L_2|H_1]} > \max\{v_1, v_2\} \tag{8}$$

$$G_1(b_{min}) = \frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \frac{\int_{v_2}^{b_{min}} (x - v_2)\, \hat{r}\,(x)\, dx}{\hat{R}\,(b_{min})\,(1 - b_{min})} \tag{9}$$

$$G_2(v_2) = \frac{\hat{R}(b_{max})}{\hat{R}(b_{min})} - \left( \frac{\hat{R}(b_{max})}{\hat{R}(b_{min})} - G_1(b_{min}) \right) \cdot \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \cdot \frac{\int_{b_{min}}^{b_{max}} \frac{x - v_1}{1-x} r(x) dx}{\int_{b_{min}}^{b_{max}} \frac{x - v_2}{1-x} r(x) dx} \tag{10}$$

- *Assume* $Pr[H_1, L_2](1 - v_1) < Pr[L_1, H_2](1 - v_2)$*. Then either*
  - $b_{min} = \underline{b}$*, bidder* 1 *has no atom (*$G_1(b_{min}) = 0$*) and bidder* 2 *has an atom at* $\underline{b} = \underline{b}_2 = v_2 \geq v_1$ *of size* $G_2(v_2) > 0$ *specified by Equation (10), or*
  - $b_{min} > \underline{b}$*, bidder* 1 *has an atom at* $b_{min} = \underline{b}_1$ *specified by Equation (8), its size* $G_1(b_{min}) > 0$ *is specified by Equation (9), and bidder* 2 *has an atom at* $v_2 = \underline{b}_2 = \underline{b} < b_{min}$ *of size* $G_2(v_2) > 0$ *specified by Equation (10).*

*Moreover, it always hold that*

$$G_1(b) = \begin{cases} 0 & \text{if } 0 \leq b < b_{min}; \\ \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{\epsilon}{\hat{R}(b)} \cdot \int_{b_{min}}^{b} \frac{x - v_2}{1-x} r(x) dx + G_1(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)} & \text{if } b_{min} \leq b \leq b_{max}; \\ 1 & \text{if } b_{max} < b \leq 1. \end{cases} \tag{11}$$

*and*

$$G_2(b) = \begin{cases} 0 & \text{if } 0 \leq b < v_2; \\ G_2(v_2) & \text{if } v_2 \leq b < b_{min}; \\ \frac{Pr[L_2|H_1]}{Pr[H_2|H_1]} \cdot \frac{\epsilon}{\hat{R}(b)} \int_{b_{min}}^{b} \frac{x - v_1}{1-x} \cdot r(x) dx + G_2(v_2) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)} & \text{if } b_{min} \leq b \leq b_{max}; \\ 1 & \text{if } b_{max} < b \leq 1. \end{cases} \tag{12}$$

We next show that, fixing any standard distribution $R$ (such as the uniform distribution), for sufficiently small $\epsilon$ there exists a NE in the tremble $\lambda(\epsilon, R)$ satisfying the necessary conditions identified in Lemma 25. We prove existence separately for three sets of parameter values. For symmetric bidders, we show the existence of an equilibrium with no atoms (case 1). For asymmetric bidders we show the existence of either a one-atom (case 2) or a two-atom (case 3).

In each case, the proof involves three steps. First we show existence of parameters $b_{min}$, $b_{max}$, $G_1(b_{min})$, and $G_2(v_2)$ that satisfy the necessary conditions in Lemma 25. Second, we show that, for the chosen parameters, $G_1$ and $G_2$ are well defined distributions (non-decreasing, and satisfying $G_1(0) = G_2(0) = 0$ and $G_1(1) = G_2(1) = 1$). Third we show that the constructed bid distributions are best responses. By construction, bidder $i \in \{1, 2\}$ is indifferent to all bids in the support of his bid distribution and we show that every bid outside the support gives weakly lower utility.

**Lemma 26.** *Fix any standard distribution $R$. For every small enough $\epsilon > 0$ there exists a mixed NE $\mu^\epsilon$ in the game $\lambda(\epsilon, R)$.*

The final step is to show that any sequence of NE $\{\mu^\epsilon\}$ in the trembles $\lambda(\epsilon, R)$ converges to $\mu$ as $\epsilon$ goes to zero. The result, stated in Lemma 27, is proved by considering the implication of the necessary conditions identified in Lemma 25 as $\epsilon$ goes to zero. In particular, we prove a sequence of four claims about bid distributions in the limit as $\epsilon$ goes to zero given conditions from Lemma 25. (1) We show that $\lim_{\epsilon \to 0} b_{max} = 1$ by evaluating equations (11)-(12) at $b = b_{max}$ and imposing $G_1(b_{max}) = G_1(b_{max}) = 1$. (2) From equation (9), we show that bidder 1's atom at $b_{min}$ (if it exists at all) vanishes as $\epsilon$ goes to zero. (3) From equation (10), we show that bidder 2's atom at $v_2$ goes to $1 - \frac{Pr[H_1,L_2]}{Pr[L_1,H_2]} \cdot \frac{V_{HH}-V_{HL}}{V_{HH}-V_{LH}}$. (4) Finally, we use equations (11)-(12) to show that all the bidding mass above each bidder's infimum bid goes to 1. Thus, in the limit, bidder 1 is bidding 1 with probability 1, while bidder 2 is bidding 1 with probability $\frac{Pr[H_1,L_2]}{Pr[L_1,H_2]} \cdot \frac{V_{HH}-V_{HL}}{V_{HH}-V_{LH}}$, as we need to show.

**Lemma 27.** *Fix a standard distribution $R$ and a sequence of $\epsilon$ converging to zero. The associated sequence of NE $\{\mu^\epsilon\}$ in the trembles $\lambda(\epsilon, R)$ converges to the NE $\mu$ in the original game $\lambda$.*

# D Multiplicity of Equilibria under Perfect Equilibrium

Considering refinements for our game, one natural candidate is Selten's (1975) Tremble Hand Perfect Equilibrium (PE). In this section we show that in our common value SPA with asymmetric information, PE does *not* provide the natural unique prediction one would expect in the most basic setting with two agents: one informed agent with a binary signal, and one uninformed agent. Note that in this setting there is a unique TRE and it is a strong TRE in pure strategies. In this natural equilibrium, the informed bids his posterior value while the uninformed bids to match the lowest possible bid of the informed.

Formally, consider the setting with two agents, one informed agent with a binary signal, and one uninformed agent. Assume that the common value is 0 conditional on the informed low signal, and 1 conditional on his high signal. Each signal is realized with probability 1/2. Each agent's action space (bid space) is the set $[0, 1]$ (an infinite set). In the unique TRE, the informed bids 0 on the low signal and 1 on the high signal, while the uninformed always bids 0.

We note that PE is usually defined for *finite normal form games* while our game is a game of incomplete information with infinite strategy spaces (finite type spaces but infinite action spaces). The adaptation of the solution concept to incomplete information is relatively straightforward. The move to infinite games is more delicate and we discuss two adaptations that were suggested in Simon and Stinchcombe (1995) (extending these adaptations to the incomplete information setting) and show that neither provide a unique prediction.

We start by presenting Simon and Stinchcombe's (1995) reformulation Selten's (1975) Tremble Hand Perfect Equilibrium (PE) for finite (normal form) games with complete information. Let $N$ be a finite set of agents. For agent $i \in N$ let $A_i$ be a finite set of pure actions, and let $A = \times_{i\in N} A_i$. Let $\Delta_i$ (resp. $\Delta_i^{fs}$) be the set of probability distributions (resp. full support probability distributions) on $A_i$. Let $\Delta = \times_{i\in N}\Delta_i$ and $\Delta^{fs} = \times_{i\in N}\Delta_i^{fs}$. For $\mu \in \Delta$, let $Br_i(\mu_{-i})$ denote $i$'s set of mixed-strategy best-responses to the vector of strategies of the others $\mu_{-i}$.

**Definition 28.** *( Selten (1975)) Consider a finite game. Fix $\epsilon > 0$. A vector $\mu^\epsilon = (\mu_i^\epsilon)_{i\in N}$ in $\Delta^{fs}$ is an $\epsilon$-*Perfect Equilibrium *if for each agent $i \in N$ it holds that*[19]

$$d_i(\mu_i^\epsilon, Br_i(\mu_{-i}^\epsilon)) < \epsilon$$

*where $d_i(\mu_i, \nu_i) = \sum_{a_i\in A_i} |\mu_i(a_i) - \nu_i(a_i)|$.*

*A vector $\mu = (\mu_i)_{i\in N}$ in $\Delta$ is a* Perfect Equilibrium *if there exists an infinite sequence of positive numbers $\epsilon_1, \epsilon_2, \ldots$ which converges to $0$ such that (1) for each $j$, $\mu^{\epsilon_j}$ is an $\epsilon_j$-Perfect Equilibrium and (2) for every $i \in N$ it holds that $\mu_i^{\epsilon_j}$ converges in distribution to $\mu_i$ when $j$ goes to infinity.*

Loosely speaking, for a finite (normal form) game a Perfect Equilibrium is a limit, as $\epsilon$ goes to 0, of a sequence of full support strategy vectors, each element of such a vector is $\epsilon$ close to being a best response to the other agent's strategies in that element of the sequence of strategy vectors.

We next discuss two adaptations, suggested in (Simon and Stinchcombe 1995), of PE to infinite games. The first is called *"limit-of-finite"* which considers the limit of a sequence of strategies in a sequence of finite games, in each game only a finite subset of actions is allowed and every player's strategy has full support. The distance from every action to the set of allowed actions goes to zero and the sequence of strategies converges to the "limit-of-finite". The second is called *strong perfect equilibrium* which looks directly at the infinite game and requires strictly positive mass to every nonempty open subset and the sequence of strategies converges to the strong perfect equilibrium.

Next, we adjust these concepts to games with incomplete information, finite types spaces but infinite action spaces, and show that neither predict a unique equilibrium in the simple setting discussed above.[20]

[19]Informally, his strategy is at most $\epsilon$ away from being a best response.

[20]We note that with tremble that is *independent* of the signal of the informed agent, such multiplicity of equilibria result cannot be proven. Yet, the unique equlibirum that is the result of any such tremble is *not* the one we would

## D.1 Limit of Finite Games

We next define the notion of *limit-of-finite* Perfect Equilibrium for games with incomplete information, finite types spaces but infinite action spaces. The approach is to define perfect equilibrium as the limit of $\epsilon$-perfect equilibria for sequences of successively larger (more refined) finite games.

Let $N$ be a finite set of agents. For agent $i \in N$ let $T_i$ be a finite set of types for agent $i$. Assume that the agents have a common prior over types. Let $A_i$ be a compact (infinite) set of actions. Let $B_i$ be a nonempty finite subset of $A_i$, and let $B = \times_{i \in N} B_i$. For such a $B_i$, let $\Delta_i(B_i)$ (resp. $\Delta_i^{fs}(B_i)$) be the set of probability distributions (resp. full support probability distributions) on $B_i$.

A $B_i$-supported mixed strategy $\mu_i(B_i)$ for agent $i$ is a mapping from his type $t_i$ to an element of $\Delta_i(B_i)$. For a profile of mixed strategies $\mu(B) = (\mu_i(B_i))_{i \in N}$, agent $i$ and type $t_i \in T_i$, let $Br_i^{t_i}(B_i, \mu_{-i})$ denote $i$'s set of $B_i$-supported mixed-strategy best-responses to the vector of strategies of the others $\mu_{-i}(B_{-i})$ (with respect to the given prior and the utility functions) when his type is $t_i$.

**Definition 29.** *Consider a game with incomplete information, finite types spaces but infinite action spaces. Fix $\epsilon > 0$ and $\delta > 0$. For each agent $i \in N$ let $B_i^\delta$ denote a finite subset of $A_i$ within (distance) $\delta$ of $A_i$. A vector $\mu^{(\epsilon,\delta)} = (\mu_i^{(\epsilon,\delta)})_{i \in N}$ such that for each $i$ and $t_i \in T_i$ it holds that $\mu_i^{(\epsilon,\delta)}(t_i) \in \Delta_i^{fs}(B_i^\delta)$ is an $(\epsilon, \delta)$-*Perfect Equilibrium *if for each agent $i \in N$ and type $t_i \in T_i$ it holds that*

$$d_i^\delta(\mu_i^{(\epsilon,\delta)}(t_i), Br_i^{t_i}(B_i^\delta, \mu_{-i}^{(\epsilon,\delta)})) < \epsilon$$

*where $d_i^\delta(\mu_i, \nu_i) = \sum_{a_i \in B_i^\delta} |\mu_i(a_i) - \nu_i(a_i)|$.*

*A vector $\mu = (\mu_i)_{i \in N}$ is a* limit-of-finite Perfect Equilibrium *if there exists two infinite sequences of positive numbers $\epsilon_1, \epsilon_2, \ldots$ and $\delta_1, \delta_2, \ldots$ both converging to 0 such that (1) for each $j$, $\mu^{(\epsilon_j,\delta_j)}$ is an $(\epsilon_j, \delta_j)$-Perfect Equilibrium and (2) for every $i \in N$ and $t_i \in T_i$ it holds that $\mu_i^{(\epsilon_j,\delta_j)}(t_i)$ converges in distribution to $\mu_i(t_i)$ when $j$ goes to infinity.*

We next show that there are multiple strong PE in the infinite game with one informed agent with a binary signal and one uninformed agent.

**Proposition 30.** *Consider the infinite game with one informed agent with a binary signal and one uninformed agent as defined above. For any $y \in (0, 1)$, the following is a (pure strategy) limit-of-finite perfect equilibrium in this infinite game: The informed bids according to his*

expect. In the same setting of an item of a common value 0 or 1, with equal probability, and two agents, one perfectly informed and one uninformed, we observe the following. For any tremble of the informed that is independent of the informed agent's signal, the best response of the uninformed agent is to bid the unconditional expectation (half) as this is the value of the item conditional on winning in the case the informed trembles (and if he does not, the uninformed agent just pays the exact value of the item if winning, as the price is set by the informed agent).

*dominant strategy (his posterior: 0 on low signal, 1 on high signal), while the uninformed always bids $y$.*

*Proof.* Consider the following natural way to make our game finite by discretizing the bids: fix a large natural number $m$ and only allow bids of the form $k/m$ for $k \in \{0, 1, \dots, m\}$. Note that as $m$ grows to infinity the distance between any bid $y$ and such a set of bids decreases to zero.

Fix $\epsilon > 0$ that is small enough. Fix $m$ that is large enough and fix $k_0 \in \{1, \dots, m-1\}$ such that $(k_0+1)/m$ has minimal distance to $y$ out of all bids of form $k/m$. To prove the claim we present a profile of strategies with full support over the discrete set of bids that is close to the profile in which the informed bids according to his dominant strategy while the uninformed always bids $y$. The strategies that we build have an atom of size at least $1-\epsilon$ on the specified bids. For the informed with low signal, the probability on every bid other than 0 is proportional to $\epsilon^2$, while for the informed with high signal the probability of every bid other than 1 is proportional to $\epsilon^3$, except for $k_0/m$ for which he assigns probability of about $\epsilon$. This motivates the uninformed to bid $(k_0+1)/m$, right above this "gift" given by the informed bidder with high signal, and we show that such a bid is his best response. We next define the strategies formally.

The informed agent with low signal is bidding 0 with probability $1-\epsilon^2$, and for any $k \in \{1, \dots, m\}$ he bids $k/m$ with probability $\epsilon^2/m$. The informed agent with high signal is bidding 1 with probability $1-\epsilon$. He bids $k_0/m$ with probability $\epsilon - \epsilon^3$, and for any $k \in \{0, \dots, m-1\}$ such that $k \neq k_0$, he bids $k/m$ with probability $\epsilon^3/(m-1)$.

The uninformed agent is bidding $(k_0+1)/m$ with probability $1-\epsilon$, and for any $k \in \{0, \dots, m\}$ such that $k \neq k_0+1$ he bids $k/m$ with probability $\epsilon/m$.

The informed agent has a dominant strategy to bid his posterior value, and his strategy is clearly $\epsilon$ close to that strategy. It remains to show that the strategy of the uninformed is $\epsilon$ close to his best response (to the strategy of the informed). We claim that if $\epsilon$ is small enough the best response of the uninformed to the strategy of the informed is to bid $(k_0+1)/m$ with probability 1. Indeed, consider any bid $j/m$:

- If $j = k_0+1$ then the informed has positive utility as when the value is high he has utility of at least $1/m$ with probability at least $(\epsilon - \epsilon^3)$. When the value is low his loss is at most $(k_0+1)/m$ and this happens only with probability at most $\epsilon^2$. For small enough $\epsilon$ the loss will be smaller than the gain.
- If $j = 0$ then the uninformed has utility 0.
- If $0 < j < k_0$ then the uninformed wins item of value 1 with probability at most $j\epsilon^3/(2 \cdot (m-1))$ (as the quality is high with probability 1/2 and in such case he only wins if the informed is bidding below him), thus his expected value is at most $j\epsilon^3/(2 \cdot (m-1))$. On the other hand his expected payment is at least $(1/4) \cdot (\epsilon^2/m) \cdot (1/m)$ (in case it is

low value he pays at least $1/m$ with probability $(1/2) \cdot (\epsilon^2/m)$ - the probability of the other bidding $1/m$ and tie is broken in favor of him). Thus his expected utility is at most $j\epsilon^3/(2 \cdot (m-1)) - \epsilon^2/4m^2$ which is negative for small enough $\epsilon > 0$.

- If $j = k_0$ then we claim that this bid is dominated by bidding $(k_0+1)/m$. Due to random tie breaking the bid of $k_0/m$ only wins half of the times when the value is high and the informed is also bidding $k_0/m$. By increasing his bid to $(k_0+1)/m$ the uninformed will always win in this case. The affect of this change is linear in $\epsilon$. The negative effect due to winning more when the informed gets the low signal is only of the order of $\epsilon^2$, thus for small enough $\epsilon$ it will be smaller.
- If $j > k_0 + 1$ then we claim that this bid is dominated by bidding $(j-1)/m$. This follow since the probability of winning high value items decreases by order of $\epsilon^3$, while the probability of not paying for low value items decreases by order of $\epsilon^2$.

□

Note that the proof of the proposition shows that PE does not provide a unique prediction even if we consider finite discrete action spaces. This seems to indicate that the problem with PE (with respect to our setting) is deeper than just its extension to games with infinite action spaces.

## D.2 Strong Perfect Equilibrium

We next define the notion of *strong* Perfect Equilibrium for games with incomplete information, finite types spaces but infinite action spaces. Let $N$ be a finite set of agents. For agent $i \in N$ let $T_i$ be a finite set of types for agent $i$. Assume that the agents have a common prior over types. Let $A_i$ be a compact (infinite) set of actions. Let $\Delta_i$ be the set of probability measures on $A_i$, while $\Delta_i^{fs}$ be the set of probability measures on $A_i$ assigning strictly positive mass to every nonempty open subset of $A_i$. We measure the distance between two measures $\mu, \nu$ on an infinite actions space using the following metric:

$$\rho(\mu, \nu) = \sup\{|\mu(B) - \nu(B)| \ : B \ measurable\}$$

A mixed strategy $\mu_i$ for agent $i$ is a mapping from his type $t_i \in T_i$ to an element of $\Delta_i$. For a profile of mixed strategies $\mu = (\mu_i)_{i \in N}$ agent $i$ and type $t_i \in T_i$, let $Br_i^{t_i}(\mu_{-i})$ denote $i$'s set of mixed-strategy best-responses to the vector of strategies of the others $\mu_{-i}$ (with respect to the given prior and the utility functions) when his type is $t_i$.

**Definition 31.** *Consider a game with incomplete information, finite types spaces but infinite action spaces. Fix $\epsilon > 0$. A vector $\mu^\epsilon = (\mu_i^\epsilon)_{i \in N}$ such that for each $i$ and $t_i \in T_i$ it holds that*

$\mu_i(t_i) \in \Delta_i^{fs}$ *is a* strong $\epsilon$-Perfect Equilibrium *if for each agent* $i \in N$ *and type* $t_i \in T_i$ *it holds that*

$$\rho_i(\mu_i^\epsilon(t_i), Br_i^{t_i}(\mu_{-i}^\epsilon)) < \epsilon$$

*A vector* $\mu = (\mu_i)_{i \in N}$ *is a* strong Perfect Equilibrium *if there exists an infinite sequence of positive numbers* $\epsilon_1, \epsilon_2, \ldots$ *which converges to* 0 *such that (1) for each* $j$, $\mu^{\epsilon_j}$ *is a strong* $\epsilon_j$*-Perfect Equilibrium and (2) for every* $i \in N$ *and* $t_i \in T_i$ *it holds that* $\mu_i^{\epsilon_j}(t_i)$ *converges in distribution to* $\mu_i(t_i)$ *when* $j$ *goes to infinity.*

We next show that there are multiple strong PE in the infinite game with one informed agent with a binary signal and one uninformed agent. The construction of the strategies in the next proposition is very similar to the one in Proposition 30.

**Proposition 32.** *Consider the infinite game with one informed agent with a binary signal and one uninformed agent as defined above. For any* $y \in (0,1)$*, the following is a (pure strategy) strong perfect equilibrium in this infinite game: The informed bids according to his dominant strategy (his posterior:* 0 *on low signal,* 1 *on high signal), while the uninformed always bids* $y$*.*

*Proof.* Fix some $y \in (0,1)$. Consider the following tremble for a given $\epsilon > 0$ that is small enough.

The informed agent with low signal is bidding with CDF $F_L(x) = 1 - \epsilon^2 + x\epsilon^2$ for $x \in [0,1]$. (He bids 0 with probability $1-\epsilon^2$ or uniformly between 0 and 1 with probability $\epsilon^2$.)

The informed agent with high signal is bidding with CDF $F_H$: For $x \in [0, y-\epsilon]$ it holds that $F_H(x) = x\epsilon^3$. For $x \in (y-\epsilon, y]$ it holds that $F_H(x) = F_H(y-\epsilon) + (x-y+\epsilon)(1-\epsilon^2)$. For $x \in (y,1)$ it holds that $F_H(x) = F_H(y) + (x-y)\epsilon^3$, and finally, $F_H(1) = 1$. (He bids 1 with probability $1-\epsilon+\epsilon^4$, uniformly between $y-\epsilon$ and $y$ with probability $\epsilon-\epsilon^3$, and uniformly over all other bids in $[0,1]$ with the remaining probability $\epsilon^3(1-\epsilon)$.)

The uninformed agent is bidding with CDF $G$: For $x \in [0,y)$ it holds that $G(x) = x\epsilon$. For $x = y$ it holds that $G(x) = G(y) = y\epsilon + 1 - \epsilon$. For $x \in (y,1]$ it holds that $G(x) = G(y) + (x-y)\epsilon$. (He bids $y$ with probability $1-\epsilon$ or uniformly between 0 and 1 with probability $\epsilon$.)

Clearly these strategies have full support and their limit as $\epsilon$ goes to 0 is as required.

The informed agent has a dominant strategy to bid his posterior value, and his strategy is clearly $\epsilon$ close to that strategy. It remains to show that the strategy of the uninformed is $\epsilon$ close to his best response (to the strategy of the informed). We claim that if $\epsilon$ is small enough the best response of the uninformed to the strategy of the informed is to bid $y$ with probability 1. Indeed, consider any bid $z$:

- If $z = 0$ then the agent has utility 0.
- If $z = y$ then for small enough $\epsilon > 0$ the agent has positive utility. Indeed his expected gain from high value items is at least $1/2 \cdot F_H(y)(1-y) = (\epsilon - \epsilon^3(1-y+\epsilon))(1-y)/2 \geq c\epsilon$

for some constant $c > 0$ (for small enough $\epsilon > 0$), while his expected loss from low value items is at most $1/2 \cdot (1 - F_L(0))y \leq (y/2)\epsilon^2 \leq \epsilon^2$.

- If $0 < z < y$ then for small enough $\epsilon > 0$ it holds that $0 < z < y - \epsilon$. Moreover, for small enough $\epsilon > 0$ the agent has negative utility. Indeed his expected gain is at most $1/2 \cdot F_H(z) \cdot 1 \leq z\epsilon^3$, while his expected loss is at least $1/2 \cdot (F_L(z) - F_L(z/2)) \cdot z/2 \geq z^2\epsilon^2/4$.
- If $z > y$ then for small enough $\epsilon > 0$ the agent can increase his utility by bidding $y$ instead of bidding $z$. Indeed his expected loss of value by bidding $y$ instead of $z$ is at most $1/2 \cdot (F_H(z) - F_H(y)) \cdot 1 = (y - z)\epsilon^3/2$, while his expected reduction in payment is at least $1/2 \cdot (F_L(z) - F_L(y)) \cdot y \geq (z - y)\epsilon^2/2$.

□

# E Many Agents, each with Finitely Many Signals

## E.1 Relation to the work of Einy et al.(2002)

Einy et al. (2002) study common value second price auction in domains that are *connected*. For *connected domains* Einy et al. consider the concept of *sophisticated equilibrium*, which makes successive rounds of dominated strategy eliminations. This process might result in multiple equilibria and that paper points out a single sophisticated equilibrium that Pareto-dominates the rest in terms of bidders resulting utilities, and it is also the only sophisticated equilibrium that guarantees every bidder non-negative utility. Moreover, this is the only sophisticated equilibrium that survives the elimination process if an uninformed bidder is added to the domain.

In this section we observe that Theorem 12 applies to any connected domain, as any such domain satisfies the strong-high-signal property. Moreover, we observe that for connected domains the TRE of Theorem 12 is exactly the one pointed out by Einy et al. (2002). Finally, we show that some domain that satisfy the strong-high-signal property are not connected. Some obvious such domains are monotonic domains in which the mapping from the state of the world to signals is not deterministic (yet they still satisfy the strong-high-signal property), but we also present examples of domains in which the mapping is deterministic yet they are not connected and for which Theorem 12 applies.

Before formally presenting connected domains we present an example due to Einy et al. (2002) and the TRE we (as well as Einy et al.) pick for that domain.

**Example 33.** *Assume that there are two buyers and four states of the world $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, with $v(\omega_i) = i$ and states all are equally probable ( $H(\omega_i) = 1/4$ for all $i \in \{1, 2, 3, 4\}$ ). If the state is $\omega_1$ then agent 1 gets the signal $L_1$, otherwise he gets $H_1$. If the state is $\omega_4$ then* *agent 2 gets the signal $H_2$, otherwise he gets $L_2$. In $\mu$ ,the TRE of Theorem 12 it holds that $\mu_2(H_2) = v(H_1, H_2) = 4$, $\mu_1(H_1) = v(H_1, L_2) = 2.5$ and $\mu_1(L_1) = \mu_2(L_2) = v(L_1, L_2) = 1$.*

We next define connected domains.

**Definition 34.** *A domain is a* connected domain *if the following hold. Each agent $i$ has a partition $\Pi_i$ of the state of nature and his signal is the element of the partition that include the realized state. The information partition $\Pi_i$ of bidder $i$ is* connected *(with respect to the common value $v$) if every $\pi_i \in \Pi_i$ has the property that, when $\omega_1, \omega_2 \in \pi_i$ and $v(\omega_1) \leq v(\omega_2)$ then every $\omega \in \Omega$ with $v(\omega_1) \leq v(\omega) \leq v(\omega_2)$ is necessarily in $\pi_i$. A common-value domain is* connected *(with respect to the common value) if for every agent $i$ his information partition $\Pi_i$ is connected.*

**Lemma 35.** *Every connected domain satisfies that strong-high-signal property.*

*Proof.* Let $\Pi^*$ be the coarsest partition of $\Omega$ that refines the partition $\Pi_j$ for every agent $j$. Let $\sigma$ denote an element of $\Pi^*$. Let $v(\sigma)$ denote the expected value of the item conditional on $\sigma$. We prove the claim by induction on the number of elements in $\Pi^*$. If this number is 1 the claim trivially holds as the domain in which no agent gets any information satisfies the property by definition.

Assume that we have proven the claim for every $\Pi^*$ of size smaller than $k$, we prove the claim for $\Pi^*$ of size $k$. Consider that element $\sigma$ of $\Pi^*$ such that $v(\sigma)$ is maximal. There must exist an agent $i$ and signal $s_i$ such that $s_i$ implies $\sigma$, otherwise $\Pi^*$ is not the coarsest refinement. There is only one combination of signals that has value $v(\sigma)$, in that combination each agent gets the best signal (the one with the highest value conditional on the signal). Now, as the domain is connected it holds that $v(\sigma) > v(t)$ for every combination of signals $t$. This implies that $s_i$ has the required properties from the top signal at a domain that satisfies the strong-high-signal property. Removing this signal creates another connected domain, and its coarsest partition has only $k - 1$ elements, so by the induction hypothesis it satisfies the strong-high-signal property. We conclude that the original domain satisfies the strong-high-signal property as we need to show. □

**Proposition 36.** *For every connected domain the TRE of Theorem 12 is exactly the same as the unique sophisticated equilibrium picked by Einy et al. (2002) (the sophisticated equilibrium that survives the elimination process if an uninformed bidder is added to the domain).*

*Proof.* Einy et al. show that unique sophisticated equilibrium that they pick can be computed as follows. One can look at $\Pi^*$, the coarsest partition of $\Omega$ that refines the partition $\Pi_j$ for every agent $j$. Let $\sigma$ denote an element of $\Pi^*$. Let $v(\sigma)$ denote the expected value of the item conditional on $\sigma$. An order over elements $\sigma_1, \sigma_2 \in \Pi^*$ is naturally defined by the order on the corresponding values $v(\sigma_1)$ and $v(\sigma_2)$. For agent $j$ with signal $\pi_j \in \Pi_j$ the bid is defined to $\min_{\sigma \in \pi_j} v(\sigma)$. An equivalent definition is that agent $j$ with signal $\pi_j \in \Pi_j$ bids $min\{v(\pi_j, \pi_{-j}) | \pi_{-j} \in S_{-j} \text{ and } (\pi_j, \pi_{-j}) \text{ is feasible}\}$, which is exactly $\mu_j(s_j)$ as defined in Theorem 12. □

We next show that there are domains that are not connected yet satisfy the strong-high-signal property. This implies that Theorem 12 applies to a strict superset of the domains that are handled by Einy et al. (2002). We start with a simple example with only one informed bidder.

**Example 37.** *Consider a domain with two buyers and three states of the world $\Omega = \{\omega_1, \omega_2, \omega_3\}$, with $v(\omega_1) = 0$, $v(\omega_2) = 4$, $v(\omega_3) = 10$ and all states are equally probable ($H(\omega_i) = 1/3$ for all $i \in \{1, 2, 3\}$). If the state is $\omega_1$ or $\omega_3$ then agent 1 gets the signal $H_1$, otherwise he gets $L_1$. Agent 2 is not informed at all. This example is covered by Theorem 12 and moreover it is covered by Theorem 5. Yet, this domain is not connected, as signal $H_1$ of agent 1 indicates that the state is $\omega_1$ or $\omega_3$ and does not include $\omega_2$.*

We also present an example with more than one informed bidder, in this example there are 2 agents and each has a binary signal.

**Example 38.** *Assume that there are two buyers and four states of the world $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ with $v(\omega_1) = 0$, $v(\omega_2) = 4$, $v(\omega_3) = 6$, $v(\omega_4) = 10$, and all states are equally probable ($H(\omega_i) = 1/4$ for all $i \in \{1, 2, 3, 4\}$). If the state is $\omega_4$ then agent 1 gets the signal $H_1$, otherwise he gets $L_1$. If the state is $\omega_1$ or $\omega_3$ then agent 2 gets the signal $L_2$, otherwise he gets $H_2$. (note that this is not connected as $\omega_2$ does not belong to $L_2$). In the TRE $\mu$ of Theorem 12 it holds that $\mu_1(H_1) = v(H_1, H_2) = 10$, $\mu_2(H_2) = v(L_1, H_2) = 4$ and $\mu_1(L_1) = \mu_2(L_2) = v(L_1, L_2) = 3$.*

While Example 37 presents a very simple domain that is not connected, it is clear that there exists a different representation of the states of the world for which a domain with exactly the same signal structure and posteriors, is indeed connected. In this new representation each state corresponds to one of the informed agent's signals and the value corresponds to the posterior value given that signal. That is, we can define $\Omega' = \{\omega'_1, \omega'_2\}$, with $v(\omega'_1) = 5$, $v(\omega'_2) = 4$, and the probabilities are $H(\omega'_1) = 2/3$ and $H(\omega'_2) = 1/3$. If the state is $\omega'_1$ then agent 1 gets the signal $H_1$, otherwise he gets $L_1$. Agent 2 is not informed at all. Clearly under the new representation the domain is connected, and the domain is equivalent to the original domain.

One might wonder if *any* domain that satisfies the strong-high-signal property can be transformed to an equivalent connect domain. We next show that this is not the case, presenting a domain that satisfies the property and cannot be represented by a connect domain. This shows that Theorem 12 applies to domains that do not have a representation as connected domains.

The domain we consider is the domain presented in Example 38, with $v(\omega_2)$ assigned a value of 2 instead of 4. Clearly in a connected domain that is equivalent to that domain it must be the case that signals $H_1$ and $H_2$ are both received for some subset of states of the world such that for each such state the value is at least as high as the value if signal $H_1$ is not received. Now connectivity for $H_2$ implies that $v(L_1) \geq v(L_2)$ which does not hold for the domain we are considering.

# F Two Agents, Each with a Binary Signal

In this section we present a complete proof of Theorem 19. We define notation as it is first used throughout the appendix. For those reading nonlinearly, please refer to the notation summary in Section F.5 Table 1.

## F.1 Proof of Lemma 24 (Necessary conditions part I)

In this section we use $i$ to denote a bidder, either bidder 1 or 2. When we want to refer to the other bidder we use $j$ to denote that bidder, and assume that $j \neq i$. To simplify the notation we denote $v_1 = V_{HL}$ and $v_2 = V_{LH}$ and (without loss of generality) normalize $V_{LL} = 0$ and $V_{HH} = 1$. We assume that $0 < Pr[H_1, L_2](1 - v_1) \leq Pr[L_1, H_2](1 - v_2) < 1$, and that in case of equality $v_1 \geq v_2$. Note that this implies that $\min\{Pr[H_1, L_2], Pr[L_1, H_2]\} > 0$.

Let $R$ be a standard distribution and fix some $\epsilon > 0$. Consider a NE $\mu^\epsilon$ of the $(\epsilon, R)$-tremble of the game $\lambda$. We first show that if bidders never submit dominated bids bidder $i \in \{1, 2\}$ that receives signal $L_i$ must bid $V_{LL} = 0$.

**Lemma 39.** *At $\mu^\epsilon$ the following must hold. For each bidder $i \in \{1, 2\}$ it holds that: (1) Bidder $i$ with signal $L_i$ always bids $V_{LL} = 0$. (2) Bidder $i$ with signal $H_i$ always bids at least $v_i$.*

*Proof.* By assumption, bidders do not make weakly dominated bids. Therefore, bidder $i$ bids at least 0 given signal $L_i$ and at least $v(H_i, L_j)$ given signal $H_i$. Similarly, bidder $i$ bids no more than $v(L_i, H_j)$ given signal $L_i$ and no more than 1 given signal $H_i$. Bidder 1 with signal $L_1$ cannot bid $b \in (0, v_2)$ because she would only win when bidder 2 has a low signal and the value is zero but she would pay a positive amount due to the random bidder. Increasing the bid to $v_2$ incurs the same losses conditional on $L_2$ as bidding just below $v_2$ and earns zero conditional on $H_2$ because any wins are priced at their value $v_2$. Therefore bidder 1 must bid 0 given a low signal, and the same is true for bidder 2 by similar logic. $\square$

Given this lemma we focus in the rest of the proof at the bidding of each bidder $i$ given his high signal $H_i$. (e.g. if we say that some bid "is optimal for $i$" we mean to say that this bid "is optimal for $i$ with signal $H_i$"). We define $G_i$ to be the cumulative distribution function of bidder $i$'s bids conditional on $i$ having signal $H_i$, that is $G_i = \mu^\epsilon(H_i)$. We say that $G$ has an *atom* at $b$ if $G$ is discontinuous at $b$. We define $G^-(b) = \sup_{x<b} G(x)$ to be the left-hand limit

of $G$ evaluated at $b$. We say that a bid $b$ of bidder $j$ is *optimal* (or is in the *support*[21] if the utility from that bid (given the other agent's strategy and the random bidder) is at least as high as with any other bid.

**Lemma 40.** *At $\mu^\epsilon$ the following must hold. Assume that $G_j$ is discontinuous at $b < 1$ ($j$ has an atom at $b$), then $\exists \delta > 0$ such that bidding in the interval $(b-\delta, b]$ is not optimal for $i$ as it is strictly dominated by bidding $b+\delta$.*

*Proof.* Let $\Delta$ be the discrete increase in $G_j$ at $b$. For $\delta > 0$ small enough bidding $b+\delta$ is strictly better than bidding in $(b-\delta, b]$ as the probability of winning increases by at least $\Delta/2$ (moving from $b$ to $b+\delta$ means always winning against the atom instead of tie-breaking), while the increase in payment when winning low value items tends to 0 as $\delta$ go to zero (as the random bidder is bidding continuously). □

Define $v_j^{win}(b)$ to be the expected value of the items $j$ gets, conditional on winning with bid $b$. Then, for any $b > 0$,

$$v_j^{win}(b) = \frac{\Pr[H_i \mid H_j]\left(G_i^-(b) + G_i(b)\right)/2 + \Pr[L_i \mid H_j]\, v_j}{\Pr[H_i \mid H_j]\left(G_i^-(b) + G_i(b)\right)/2 + \Pr[L_i \mid H_j]}. \tag{13}$$

(In this expression, $\left(G_i^-(b) + G_i(b)\right)/2$ is the probability that $j$ wins with bid $b$ given $H_i$, accounting for the fact that there is a tie with probability $\left(G_i(b) - G_i^-(b)\right)$ that is broken $50-50$.)

**Lemma 41.** *At $\mu^\epsilon$ the following must hold. If $b \in [0,1)$ is an optimal bid of bidder $j$ then $b \geq v_j^{win}(b)$.*

*Proof.* If $v_j^{win}(b) = v_j$ the claim follows from $b \geq v_j$ (Lemma 39 ).

Now assume in contradiction that $b < v_j^{win}(b) \leq 1$ and that $v_j^{win}(b) > v_j$. It must hold that $G_i(b) > 0$, since $G_i(b) = 0$ implies $v_j^{win}(b) = v_j$. If $i$ has an atom at $b < 1$ then $b$ is not optimal for $j$ by Lemma 40, contradicting our assumption that $b$ is optimal for $j$. Thus, bidder $i$ does not have an atom at $b$.

We show that for $\delta > 0$ that is small enough ($\delta < \max 1 - b, v_j^{win}(b) - b$), bidding $b+\delta$ gives higher utility. To show that the bid $b+\delta$ gives higher utility than $b$, we consider the difference in utility due to such an increase in the bid. There are two cases: first, if bidder $j$ wins with $b+\delta$ but would have lost with $b$ due to a bid of $i$ in $(b, b+\delta)$ then $j$ wins an item of value 1 and pays at most $b+\delta < 1$, having positive utility. Second, if $i$ was not bidding in $(b, b+\delta)$ but the random bidders does, by bidding $b+\delta$ bidder $j$ is now winning items with expected value at least $v_j^{win}(b)$ (which is non-decreasing by inspection of equation (13)) and paying at

[21] Note that this is a slight misuse of the term "support", as the standard notion refers to a closed set but the boundary points of that set might not be optimal.

most $b+\delta$. As $\delta < v_j^{win}(b) - b$ the expected value from such a win is positive. Moreover, this second event happens with strictly positive probability because the random bidder is bidding continuously over $[0,1]$ and given no atoms of bidder $i$ at $b>0$ bidder $i$ bids less than $b$ with probability $Pr[L_i|H_j] + Pr[H_i|H_j]G_i(b) > 0$. We conclude that such an increase in bid strictly increases the utility. $\square$

Let $\Pi_i(b_i)$ be the expected profit for bidder $i$ conditional on signal $H_i$ and bid $b_i$. Then, for any $b>0$,

$$\begin{aligned}\Pi_i(b_i) &= \Pr[L_j|H_i]\,\hat{R}(b_i)\,(v_i - E[b_r|b_r<b_i]) \\ &\quad + \Pr[H_j|H_i]\,\hat{R}(b_i)\,G_j^-(b_i)\,(1 - E[\max\{b_r,b_j\}\,|\,\max\{b_r,b_j\}<b_i]) \\ &\quad + \Pr[H_j|H_i]\,\hat{R}(b_i)\left(G_j(b_i) - G_j^-(b_i)\right)(1-b_i)/2\end{aligned}$$

The first term handles the case that $j$ receives the signal $L_j$, in this case he bids $V_{LL}=0$ and the price is set by the random bidder. The second and third terms handle the case that $j$ receives the signal $H_j$. The second term is for the case that $b_j < b_i$, while the third handles the case that $b_j = b_i$.

Noting that

$$\begin{aligned}&\hat{R}(b_i)\,G_j^-(b_i)\,E[\max\{b_r,b_j\}\,|\,\max\{b_r,b_j\}<b_i] \\ = \;& \hat{R}(b_i)\,G_j^-(b_i)\int_0^{b_i}\left(1 - \frac{\hat{R}(x)\,G_j(x)}{\hat{R}(b_i)\,G_j^-(b_i)}\right)dx = b_i\hat{R}(b_i)\,G_j^-(b_i) - \int_0^{b_i}\hat{R}(x)\,G_j(x)\,dx,\end{aligned}$$

profits may be written more explicitly as

$$\begin{aligned}\Pi_i(b_i) &= \Pr[L_j|H_i]\int_0^{b_i}(v_i - x)\,\hat{r}(x)dx \\ &\quad + \Pr[H_j|H_i]\,\hat{R}(b_i)\,G_j^-(b_i)\,(1-b_i) + \int_0^{b_i}\hat{R}(x)\,G_j(x)\,dx \\ &\quad + \Pr[H_j|H_i]\,\hat{R}(b_i)\left(G_j(b_i) - G_j^-(b_i)\right)(1-b_i)/2\end{aligned} \tag{14}$$

Where $G_i(b)$ is differentiable, the derivative of $\Pi_i(b_i)$ with respect to $b_i$ is then

$$\frac{d\Pi_i(b_i)}{db_i} = \Pr[L_j|H_i]\,\hat{r}(b_i)\,(v_i - b_i) + \Pr[H_j|H_i]\left(\hat{r}(b_i)\,G_j(b_i) + \hat{R}(b_i)\,g_j(b_i)\right)(1-b_i) \tag{15}$$

The next result follows from equation (15) evaluated over an interval for which $g_j(b)$ is zero.

**Lemma 42.** *At $\mu^\epsilon$ the following must hold. For $1 \geq b^+ > b^- \geq 0$ suppose that $G_j(b^-) =$*

$G_j^-(b^+)$ *($j$ does not bid on $(b^-, b^+)$). Let* $\Gamma = G_j(b^-)$. *Let*

$$b_i^*(\Gamma) = \frac{\Pr[H_j|H_i]\,\Gamma + \Pr[L_j|H_i]\,v_i}{\Pr[H_j|H_i]\,\Gamma + \Pr[L_j|H_i]} \tag{16}$$

***(1)*** *If $b_i^*(\Gamma) \in (b^-, b^+]$ then $b_i^*(\Gamma)$ strictly dominates any other bid by $i$ in $[b^-, b^+)$.*

***(2)*** *If $b_i^*(\Gamma) \notin (b^-, b^+]$ then all bids $b \in (b^-, b^+]$ are strictly sub optimal for $i$. Moreover, if $b_i^*(\Gamma) \leq b^-$, then $i$'s payoff is strictly decreasing in $b$ over $(b^-, b^+)$.*

***(3)*** *If $i$ has an optimal bid $b \in (b^-, b^+]$ it holds that $b = b_i^*(\Gamma)$.*

*Proof.* $G_j(b)$ is constant over $(b^-, b^+)$ and thus $g_j(b) = 0$ for every $b \in (b^-, b^+)$. Therefore $\Pi_i(b_i)$ is continuous and differentiable in $b_i$ for every $b_i \in (b^-, b^+)$. Moreover, since $g_j(b_i)$ is zero for any such $b_i$, the derivative with respect to $b_i$ is

$$\frac{d\Pi_i(b_i)}{db_i} = \hat{r}(b_i)\big(\Pr[H_j|H_i]\,\Gamma\,(1 - b_i) + \Pr[L_j|H_i]\,(v_i - b_i)\big)$$

As we assume that $1 > \Pr[L_j|H_i] > 0$ and $\hat{r}(b_i) > 0$ and it holds that $\Pr[H_j|H_i] \geq 0$, this function of $b_i$ is not identically 0. The function has a unique 0 at $b_i^*(\Gamma)$, it is positive for $b_i < b_i^*(\Gamma)$, and it is negative for $b_i > b_i^*(\Gamma)$. The results then follow by the following augments:

**(1)** If **(a)** $b_i^*(\Gamma) \in (b^-, b^+)$ then it follows from the derivative $\frac{d\Pi_i(b_i)}{db_i}$ that $b = b_i^*(\Gamma)$ uniquely maximizes $\Pi_i(b)$ for bids $b$ within the interval $(b^-, b^+)$. If **(b)** $b_i^*(\Gamma) = b^+$ then $d\Pi_i(b_i)/db_i > 0$ for $(b^-, b^+)$. Therefore $\Pi_i(b_i)$ is strictly higher at $b^+$ than at any $b_i \in (b^-, b^+)$ because $\Pi_i(b_i)$ is either continuous at $b^+$ or increases discretely at $b^+$ (depending on whether or not $j$ has an atom at $b^+$.). In either case (a) or case (b), bid $b = b_i^*(\Gamma)$ is also strictly better than bidding $b^-$, either by continuity at $b^-$ if $j$ does not have an atom at $b^-$ or by Lemma 40 if $j$ does have an atom at $b^-$. Therefore part (1) holds.

**(2)** **(a)** Suppose $b_i^*(\Gamma) \leq b^-$: In this case, $d\Pi_i(b_i)/db_i < 0$ for $(b^-, b^+)$ and there is no optimal bid in $(b^-, b^+)$. If $j$ bids an atom at $b^+$ then $b^+$ is not an optimal bid for $i$ by Lemma 40. If $j$ does not bid an atom at $b^+$, then $\Pi_i(b_i)$ is continuous from the left at $b^+$. Therefore $\Pi_i(b_i)$ is strictly lower at $b^+$ than at any other $b_i \in (b^-, b^+)$. In either case there is no optimal bid within $(b^-, b^+]$. **(b)** Suppose $b_i^*(\Gamma) > b^+$: Inspection of equation (13) shows that $v_j^{win}(b)$ is non-decreasing in $b$. This fact and Lemma 41 imply that any optimal bid $b_i > b^-$ must be at least $b_i^*(\Gamma)$ because $v_i^{win}(b_i) = b_i^*(\Gamma)$ for all $b_i \in (b^-, b^+)$. Therefore there is no optimal bid within $(b^-, b^+]$. Thus part (2) holds.

**(3)** From parts (1) and (2) it follows that if $b \in (b^-, b^+]$ is an optimal bid, either $b = b_i^*(\Gamma)$ or $b = b^+$. It therefore remains to show that if $b = b^+$ is an optimal bid that $b^+ = b_i^*(\Gamma)$. Proof: Suppose that $b^+$ is an optimal bid. Then $j$ does not have an atom at $b^+$ (Lemma 40). By part (2), $b_i^*(\Gamma) \in (b^-, b^+]$. Suppose that $b_i^*(\Gamma) \in (b^-, b^+)$. Then $d\Pi_i(b_i)/db_i < 0$ for all $b \in (b, b^+)$. By continuity of $\Pi_i(b)$ at $b^+$ (as $j$ has not atom at $b^+$), bidding $b_i^*(\Gamma)$ therefore strictly dominates bidding $b^+$, a contradiction. Thus $b_i^*(\Gamma) = b^+$. This completes the proof of part(3).

$\square$

**Corollary 43.** *At $\mu^\epsilon$ the following must hold. If bidder $i \in \{1, 2\}$ bids an atom at $b \in [0, 1]$, then $b = b_i^*(G_j(b))$.*

*Proof.* Lemma 40 implies that $j$ does not bid in the interval $(b - \delta, b]$ for some $\delta > 0$. We can apply Lemma 42 for $b^+ = b$ and $b^- = b - (\delta/2)$, and the result is now implied by part (3) of the lemma. $\square$

Define $\underline{b}_i$ to be the infimum bid by $i \in \{1, 2\}$, $\underline{b}_i = \inf\{b : G_i(b) > 0\}$. Let $\underline{b} = min\,\{\underline{b}_1, \underline{b}_2\}$ be the infimum of all bids of any bidder with a high signal. Let $b_{min} = \max\{\underline{b}_1, \underline{b}_2\}$.

**Corollary 44.** *At $\mu^\epsilon$ the following must hold. Suppose that $j \in \{1, 2\}$ has an optimal bid $b$ at or below $\underline{b}_i$. Then $b = v_j$.*

*Proof.* Note that $G_i(b) = 0$ because Lemma 40 implies that $i$ does not have an atom at $\underline{b}_i$ if $b = \underline{b}_i$. Thus, as $i$ does not bid strictly below $\underline{b}_i$ but $j$ has an optimal bid $b$ weakly below $\underline{b}_i$, Lemma 42 part (3) implies $b = b_j^*(0) = v_j$.

$\square$

**Lemma 45.** *At $\mu^\epsilon$ the following must hold. Assume that both bids $b^- \geq 0$ and $b^+ > b^-$ are optimal bids for bidder $i \in \{1, 2\}$. Then for bidder $j \neq i$ it holds that $G_j(b^+) > G_j(b^-)$.*

*Proof.* Proof is by contradiction. Suppose that $G_j(b^+) = G_j(b^-)$. By Lemma 42 part (3), $b^+ = b_i^*(G_j(b^+))$. By Lemma 42 part (1) the bid $b^+$ strictly dominates $b^-$, a contradiction. $\square$

**Lemma 46.** *At $\mu^\epsilon$ the following must hold.*

1. *Suppose both bidders have the same infimum bid: $\underline{b}_i = \underline{b}_j = \underline{b} = b_{min}$. Then $\underline{b} = \max\{v_i, v_j\}$. If $v_i = v_j$, then neither bidder bids an atom at $\underline{b}$ (that is, $G_j(\underline{b}) = G_i(\underline{b}) = 0$). However, if $v_i < v_j$ then $j$ bids an atom at $\underline{b} = v_j$ and $i$ does not bid at $\underline{b}$.*
2. *Suppose bidder $i$ has a strictly higher infimum bid: $\underline{b}_i > \underline{b}_j$. Then $\underline{b} = \underline{b}_j = v_j$ and $j$ bids an atom with some positive weight $\Gamma > 0$ at $v_j$ but nowhere else at or below $\underline{b}_i$:*

$$G_j(b) = \begin{cases} 0 & b < v_j \\ \Gamma & b \in [v_j, \underline{b}_i] \end{cases}$$

*Moreover, $b_{min} = \underline{b}_i > v_i$.*

*Proof.* (1) It cannot be the case that both bidders have an atom at $\underline{b}$. Suppose $i$ does not have an atom at $\underline{b}$. Then $\Pi_j(b)$ is continuous at $\underline{b}$ and therefore $\underline{b}$ is an optimal bid for $j$. ($\underline{b}_j = \underline{b}$

implies that $j$ bids with positive probability at $\underline{b}$ or in every neighborhood above $\underline{b}$.) Because $j$ has an optimal bid at $\underline{b}_i$, Corollary 44 implies that $\underline{b}_i = v_j$. Moreover, $\underline{b}_i \geq v_i$ by Lemma 39. Therefore $v_i \leq v_j$ and $\underline{b} = \max\{v_i, v_j\}$.

Suppose that $v_i < v_j$ and $j$ does not bid an atom at $\underline{b}$. Then $\Pi_i(b)$ is continuous at $\underline{b}$ and hence $\underline{b}$ is an optimal bid for $i$ and Corollary 44 implies $\underline{b} = v_i$, which is a contradiction. Thus $v_i < v_j$ implies $j$ has an atom at $\underline{b}$. (Hence by Lemma 40 $i$ does not bid at $\underline{b}$.)

Suppose that $v_i = v_j$ and $j$ has an atom of weight $\Gamma > 0$ at $\underline{b}$. Then by Lemma 41, bidder $i$ must bid at least $v_i^{win}(\underline{b}) > v_i$, which is a contradiction. Thus $v_i = v_j$ implies neither bidder has an atom at $\underline{b}$.

(2) The assumption $\underline{b}_j < \underline{b}_i$ implies that $j$ bids with some positive probability $\Gamma > 0$ below $\underline{b}_i$. By Corollary 44, $j$ can only bid below $\underline{b}_i$ at $v_j$. Therefore $j$ bids with atom $\Gamma$ at $\underline{b}_j = v_j$ and nowhere else below $\underline{b}_i$. Moreover, Lemma 41 implies that for all bids $b \geq \underline{b}_i$, bidder $i$ must bid at least $v_i^{win}(\underline{b}_i) = b_i^*(\Gamma) > v_i$.

□

**Lemma 47.** *If for all $\delta > 0$, bidder $i$ has an optimal bid in the interval $(b-\delta, b]$ then $b$ is an optimal bid for $i$.*

*Proof.* By Lemma 40, $j$ does not have an atom at $b$ and hence $\Pi_i(b_i)$ is continuous from the left at $b_i = b$. Since $i$ has an optimal bid at $b$ or arbitrarily close to $b$, continuity implies that $b$ must be an optimal bid. □

Suppose that bidder $j$ has an atom at $b > 0$. By Lemma 40, bidder $i$ does not bid in $(b-\delta, b]$ for some $\delta > 0$. Define $x_i(b)$ to be the supremum point below $b$ at which bidder $i$ does place a bid

$$x_i(b) = \sup\left\{x : G_i(x) < G_i^-(b)\right\} = \inf\left\{x : G_i(x) = G_i^-(b)\right\}. \tag{17}$$

Lemma 40 implies $x_i(b) < b$. Similarly, let

$$x_j(b) = \inf\left\{x : G_j(x) = G_j^-(b)\right\}.$$

Note that if $i$ does not bid below $b$ ($\underline{b}_i \geq b$) then $x_i(b) = -\infty$.

Our goal is to prove that if $j$ has an atom at $b$ then $b$ is $j$'s infimum bid. We first prove some helpful claims.

**Lemma 48.** *If $j$ has an atom at $b > 0$ and $b$ is not $j$'s infimum bid ($0 \leq \underline{b}_j < b$) then:*

1. *It holds that $v_j \leq x_j(b) < x_i(b) < b$.*
2. *In the interval $(x_j(b), b]$, $i$ bids an atom at $x_i(b) = b_i^*(G_j(x_i(b)))$ but nowhere else.*
3. *$j$ bids with an atom at $x_j(b) = b_j^*(G_i(x_j(b)))$.*

*4.* $b = b_j^*\left(G_i(b)\right)$.

*Proof.* We prove the claims:

1. We prove that $v_j \leq x_j(b) < x_i(b) < b$:
   - $x_j(b) \geq v_j$: By assumption ($\underline{b}_j < b$) bidder $j$ bids with positive probability below $b$. Such bids must be at least $v_j$.
   - $x_i(b) < b$: follows from Lemma 40.
   - $x_j(b) < x_i(b)$: suppose not and $x_i(b) \leq x_j(b) < b$. There are two cases. (i) First, if $x_j(b) > x_i(b)$, then there exists some bid $b^- \in [x_i(b), b)$ where $j$ bids. Then by Lemma 45, $G_i(b^-) < G_i(b)$ which contradicts $G_i\left(x_i(b)\right) = G_i(b)$ and $x_i(b) < b^- < b$. (ii) Second, if $x_j(b) = x_i(b)$ then by Lemma 47 $x_i\left(b\right)$ is an optimal bid for $j$. Then by Lemma 45, $G_i(x_i) < G_i(b)$ which contradicts $G_i\left(x_i\right) = G_i(b)$.
2. By part (1) and definition of $x_i(b)$, $j$ does not bid with positive probability in the interval $(x_j(b), b)$ but $i$ does. As a result, part (3) of Lemma 42 implies part (2).
3. There are two cases, either $\underline{b}_i = x_i(b)$ or $\underline{b}_i < x_i(b)$. (i) By part (1), $j$ bids with positive probability below $x_i(b)$. Therefore, if bidder $i$'s infimum bid is at $\underline{b}_i = x_i(b)$, Lemma 46 implies that $j$ bids with an atom at $x_j(b) = v_j$. (ii) bidder $i$ bids with positive probability below $x_i(b)$ and $\underline{b}_i < x_i(b)$. Parts (1) and (2) of the Lemma can be applied to the atom at $x_i(b)$ and these imply that $j$ bids with an atom at $x_j(b) = b_j^*(G_i(x_j(b))) > v_i$.
4. Part (4) follows from Corollary 43.

□

**Lemma 49.** *If $j \in \{1,2\}$ has an atom at $b$ then $b$ is $j$'s infimum bid: $b = \underline{b}_j$.*

*Proof.* Suppose not and $j$ bids with positive probability in a neighborhood of $b^- < b$. Then by Lemma 48, $j$ bids with an atom at $x_j(b) = b_j^*\left(G_i\left(x_j(b)\right)\right)$, $i$ bids with an atom at $x_i(b) \in (x_j(b), b)$, $b = b_j^*\left(G_i(b)\right)$, and there are no other bids in the interval $\left(x_j\left(b\right), b\right)$. We will show a contradiction by showing that $\Pi_j(b) > \Pi_j\left(x_j(b)\right)$. Let $\Gamma_1 = G_i\left(x_j(b)\right)$ and $\Gamma_2 = G_i\left(x_i(b)\right) = G_i(b)$.

Let $\Pi_j^-$ and $\Pi_j^+$ be the left and right hand limits of $\Pi_j$ respectively. I will write down the difference in profit between bidding at $x_j(b)$ and $b$ for bidder $j$ in three parts corresponding to $\Pi_j^-\left(x_i(b)\right) - \Pi_j\left(x_j(b)\right)$, $\Pi_j^+\left(x_i(b)\right) - \Pi_j^-\left(x_i(b)\right)$, and $\Pi_j(b) - \Pi_j^+\left(x_i(b)\right)$:

$$\begin{aligned}
\Pi_j(b) - \Pi_j\left(x_j(b)\right) &= \left(G_i\left(x_j(b)\right)\Pr\left[H_i|H_j\right] + \Pr\left[L_i|H_j\right]\right)\int_{x_j(b)}^{x_i(b)}\left(b_j^*\left(\Gamma_1\right) - t\right)\hat{r}\left(t\right)dt \\
&\quad + \Pr\left[H_i|H_j\right]\left(G_i\left(x_i(b)\right) - G_i\left(x_j(b)\right)\right)\hat{R}\left(x_i(b)\right)\left(1 - x_i(b)\right) \\
&\quad + \left(G_i\left(x_i(b)\right)\Pr\left[H_i|H_j\right] + \Pr\left[L_i|H_j\right]\right)\int_{x_i(b)}^{b}\left(b_j^*\left(\Gamma_2\right) - t\right)\hat{r}\left(t\right)dt
\end{aligned}$$

The third term $\Pi_j(b) - \Pi_j(x_i(b))$ is positive since $b = b_j^*(\Gamma_2)$ implies the following integral is positive:

$$\int_{x_i(b)}^{b} \left(b_j^*(\Gamma_2) - t\right) \hat{r}(t)\, dt = \int_{x_i(b)}^{b} (b - t)\, \hat{r}(t)\, dt > 0. \tag{18}$$

The fact that $b_j^*(\Gamma_1) = x_j(b)$ provides a lower bound to the integral in the first term:

$$\int_{x_j(b)}^{x_i(b)} \left(b_j^*(\Gamma_1) - t\right) \hat{r}(t)\, dt \geq -\left(\hat{R}(x_i(b)) - \hat{R}(x_j(b))\right)(x_i(b) - x_j(b)) \geq -\hat{R}(x_i(b))(x_i(b) - x_j(b)). \tag{19}$$

The inequalities in equations (18) and (19) imply that

$$\Pi_j(b) - \Pi_j(x_j(b)) \quad > \quad -\left(G_i(x_j(b)) \Pr[H_i|H_j] + \Pr[L_i|H_j]\right) \hat{R}(x_i(b))(x_i(b) - x_j(b)) \tag{20}$$

$$+ \Pr[H_i|H_j]\left(G_i(x_i(b)) - G_i(x_j(b))\right) \hat{R}(x_i(b))(1 - x_i(b)) \tag{21}$$

Substituting $x_j(b) = b_j^*(G_i(x_j(b))) = \frac{G_i(x_j(b))\Pr(H_i|H_j)+\Pr(L_i|H_j)v_j}{G_i(x_j(b))\Pr(H_i|H_j)+\Pr(L_i|H_j)}$ into the right-hand side of equation (21) and canceling and regrouping terms gives

$$\hat{R}(x_i(b))\left(G_i(x_i(b))\Pr(H_i|H_j) + \Pr(L_i|H_j)\right)\left(\frac{G_i(x_i(b))\Pr(H_i|H_j) + \Pr(L_i|H_j)\, v_j}{\left(G_i(x_i(b))\Pr(H_i|H_j) + \Pr(L_i|H_j)\right)} - x_i(b)\right).$$

Finally, since $G_i(x_i(b)) = G_i(b)$ and $b = b_j^*(G_i(b))$ we can substitute in $b = \frac{G_i(x_i(b))\Pr(H_i|H_j)+\Pr(L_i|H_j)v_j}{G_i(x_i(b))\Pr(H_i|H_j)+\Pr(L_i|H_j)}$ yielding

$$\hat{R}(x_i(b))\left(G_i(x_i(b))\Pr(H_i|H_j) + \Pr(L_i|H_j)\right)(b - x_i(b)),$$

which is positive since $b > x_i(b)$. Thus $\Pi_j(b) - \Pi_j(x_j(b)) > 0$. $\square$

Recall the definition $b_{min} = \max\{\underline{b}_1, \underline{b}_2\}$. In addition, define $\bar{b}_i = \inf\{x : G_i(x) = 1\}$ and $\bar{b}_j = \inf\{x : G_j(x) = 1\}$. Finally, define $b_{max} = \max\{\bar{b}_1, \bar{b}_2\}$. Notice that $b_{max} \geq b_{min}$.

**Lemma 50.** *At $\mu^\epsilon$ the following must hold.*

1. *If $b_{max} > b_{min}$ then both bidders have the same supremum bid: $\bar{b}_i = \bar{b}_j = b_{max}$.*
2. *Both $G_1$ and $G_2$ are continuous for all $b > b_{min}$. Moreover, both $G_1$ and $G_2$ are strictly increasing over the interval $(b_{min}, b_{max})$.*
3. *Suppose that $\underline{b}_i > \underline{b}_j$ so that $b_{min} = \underline{b}_i > \underline{b} = \underline{b}_j$. Then $j$ bids an atom at $\underline{b} = \underline{b}_j = v_j$ and $i$ bids an atom at $b_{min} = \underline{b}_i = b_i^*(G_j(v_j))$ with weight $\Gamma_i$. Moreover the size of $i$'s atom at $b_{min}$ is $1$ if $b_{max} = b_{min}$ and otherwise is:*

$$\Gamma_i = \frac{\Pr[L_i|H_j]}{\Pr[H_i|H_j]} \frac{\int_{v_j}^{b_{min}} (x - v_j)\, \hat{r}(x)\, dx}{\hat{R}(b_{min})(1 - b_{min})}$$

*Proof.* (1) Suppose not and $b_{max} = \bar{b}_i > \bar{b}_j$. Then $j$ does not bid over $(\bar{b}_j, \bar{b}_i)$ but $i$ bids with positive probability in $(\bar{b}_j, \bar{b}_i]$. By part (3) of Lemma 42 and the definition of $\bar{b}_i$, this positive probability must be concentrated at a single atom at $\bar{b}_i$. By Lemma 46, $\bar{b}_i$ is $i$'s infimum bid, that is $\bar{b}_i = \underline{b}_i$, thus $\bar{b}_i = \underline{b}_i \leq b_{min} \leq b_{max} = \bar{b}_i$, so $b_{min} = b_{max}$, a contradiction.

(2) By Lemma 49 $G_i$ and $G_j$ are continuous for all $b > b_{min}$. To show that they must also be strictly increasing over $(b_{max}, b_{min})$ we consider and rule out two types of flat spots. Throughout, we assume $b_{max} > b_{min}$ (the claim is trivially satisfied for $b_{max} = b_{min}$).

First, suppose that at least one bidder, say $i$, does not bid in an interval $(b_{min}, b^+)$ so that $G_i(b_{min}) = G_i^-(b^+) = \Gamma$ for some $b^+ > b_{min}$. Note that $b_{max} > b_{min}$ and part (1) imply $\Gamma < 1$ and no atoms above $b_{min}$ implies $G_i^-(b^+) = G_i(b^+)$. Moreover, let $b^+$ be the upper bound of the flat spot: $b^+ = sup\{b : G_i(b) = G_i(b_{min})\}$. By part (3) of Lemma 42, $j$ can place at most one bid over $(b_{min}, b^+]$. By definition, $b_{min}$ must be the infimum bid of one or both bidders. As neither bidder bids in $(b_{min}, b^+)$, this implies one (but not both by Lemma 40) bidder has an atom at $b_{min}$. By the definition of $b^+$ and the fact that $j$ does not bid an atom at $b^+$, $b^+$ must be an optimal bid for $i$.

Suppose (i) $i$ has the atom at $b_{min}$. Then $i$ has optimal bids at $b_{min}$ and $b^+$ but $G_j(b_{min}) = G_j(b^+)$, contradicting Lemma 45.

Suppose instead (ii) that $j$ has the atom at $b_{min}$. By Lemma 45, $b^+$ is not an optimal bid for $j$ because $b_{min}$ is optimal but $G_i(b_{min}) = G_i(b^+)$. Because $i$ does not bid an atom at $b^+$, $\Pi_j(b)$ is continuous at $b^+$ and $j$ does not have an optimal bid in a neighborhood $(b^+ - \beta, b^+ + \beta)$ for $\beta > 0$ sufficiently small. However $i$ must bid with positive probability in this interval by definition of $b^+$. By part (3) of Lemma 42, this probability must be concentrated at an atom, which contradicts no atoms above $b_{min}$.

Second, suppose that at least one bidder, say $i$, does not bid in an interval $(b^-, b^+)$ such that $G_i(b^-) = G_i^-(b^+) = \Gamma$ where

$$b_{min} < b^- = inf\{b : G_i(b) = \Gamma\} < b^+ = sup\{b : G_i(b) = \Gamma\} < b_{max}.$$

Note that $b^- > b_{min}$ implies $\Gamma > 0$ and $b_{max} > b^+$ implies $\Gamma < 1$. Because there are no atoms above $b_{min}$, both agent's utility functions are continuous at $b^-$ and $b^+$. Thus the definitions of $b^-$ and $b^+$ (and $\Gamma \in (0, 1)$) therefore imply that $b^-$ and $b^+$ are both optimal bids for $i$. By part (3) of Lemma 42, $j$ can place at most one bid over $(b^-, b^+]$, and because $j$ cannot have an atom, this implies $G_j(b^-) = G_j(b^+)$. By Lemma 45, this contradicts optimality of $b^-$ and $b^+$ for $i$.

(3) Lemma 46 and $\underline{b}_j < \underline{b}_i$ imply that $j$ bids an atom at $\underline{b} = v_j$ but nowhere else below $b_{min}$. The final step in the proof is to show that $i$ bids an atom at $b_{min}$. Then Corollary 43 implies $b_{min} = b_i^*(G_j(b_{min}))$. Finally $G_j(b_{min}) = G_j(v_j)$ because $j$ does not bid in $(v_j, b_{min}]$ (Lemmas 40 and 46).

To show that $i$ bids an atom at $b_{min}$, there are two cases. (1) $b_{max} = b_{min}$: This implies that $j$'s atom at $v_j$ has mass 1 and that $i$ bids $b_{min}$ with probability 1. (2) $b_{max} > b_{min}$: Then by part (1) of this Lemma, for any $\delta > 0$ bidder $j$ has optimal bid within the interval $(b_{min}, b_{min} + \delta)$. This means that bidder $i$ must have an atom at $b_{min} = \underline{b}_i$. Suppose not and $G_i(b_{min}) = G_i(0)$. Then $b_{min}$ will be an optimal bid for $j$ by continuity but $\underline{b}$ is also an optimal bid for $j$. This contradicts Lemma 45 given $G_i(b_{min}) = G_i(0)$.

To compute $\Gamma_i$ we observe that the utility of $j$ is the same across all bids in the support, in particular at his atom at $\underline{b}_j = v_j$ and at any optimal bid $b_j > b_{min}$ that is arbitrarily close to $b_{min}$ (such bid exists for any $\delta > 0$ in the interval $(b_{min}, b_{min} + \delta)$ since $b_{max} > b_{min}$). Thus the change in utility from increasing the bid from $v_j$ to such $b_j$ is zero. The next equation presents this utility change in the limit when $b_j$ tends to $b_{min}$ from above.

$$\hat{R}(b_{min}) \Pr[H_i|H_j]\Gamma_i(1 - b_{min}) - \Pr[L_i|H_j] \int_{v_j}^{b_{min}} (x - v_j)\, \hat{r}\,(x)\, dx = 0$$

Or equivalently,

$$\Gamma_i = \frac{\Pr\,[L_i|H_j]}{\Pr\,[H_i|H_j]} \frac{\int_{v_j}^{b_{min}} (x - v_j)\, \hat{r}\,(x)\, dx}{\hat{R}\,(b_{min})\,(1 - b_{min})}$$

□

Recall that $\underline{b}_i = \inf\{b : G_i(b) > 0\}$ and $\bar{b}_i = \inf\{x : G_i(x) = 1\}$ for agent $i \in \{1, 2\}$. Note that when agent never submit dominated bids by definition it holds that $1 \geq b_{max} = \max\{\bar{b}_i, \bar{b}_j\} \geq b_{min} = \max\{\underline{b}_1, \underline{b}_2\} \geq \underline{b} = \min\{\underline{b}_1, \underline{b}_2\} \geq 0$.

We are now ready to prove Lemma 24.

*Proof.* (of Lemma 24) (1) Follows from Lemma 46. (2) Follows from the definition of $b_{max}$ and Lemma 50 parts 1 and 2. (3) $G_i(b) = 0$ for $b < \underline{\mathrm{b}}$ follows from the definition of $\underline{\mathrm{b}}$. $G_i(b) = G_i(\underline{\mathrm{b}})$ for $b \in [\underline{\mathrm{b}}, b_{min})$ follows from Lemma 46 part 2. (4) Follows from Lemma 46 part 1. (5) Follows almost entirely from Lemma 50 part 3. The fact that $\max\{v_i, v_j\} < b_{min} \leq v(H_i)$ and $b_{min} = v(H_i)$ if and only if $G_j(v_j) = 1$ follows from the definition of $b_{min}$, inspection of equation (6), and the fact that $v(H_i) = \Pr[H_j|H_i] + v_i \Pr[L_j|H_i]$. □

## F.2 Proof of Lemma 25 (Necessary Conditions Part II)

In this section we prove Lemma 25.

Recall that to simplify the notation we denote $v_1 = V_{HL}$ and $v_2 = V_{LH}$. As the domain is non-degenerate, $Pr[H_1, H_2] > 0$ and for any bidder $i \in \{1, 2\}$ it holds that $1 > Pr[H_i] > 0$. The assumption that $0 < Pr[H_1, L_2](1 - v_1) \leq Pr[L_1, H_2](1 - v_2) < 1$ implies $\max\{v_1, v_2\} < 1$ and that $\min\{Pr[H_1, L_2], Pr[L_1, H_2]\} > 0$ and combining with the above implies that $\min\{Pr[L_2|H_1], Pr[L_2|H_1]\} > 0$. Additionally, as the domain is non-degenerate, $\min\{Pr[H_1|H_2], Pr[H_2|H_1]\} > 0$.

### F.2.1 Characterizations of the CDFs of $G_1$ and $G_2$ for $b > b_{min}$

**Lemma 51.** *At $\mu^\epsilon$, for each bidder $i \in 1, 2$ and $j \neq i$, the following must hold. For every bid $b \in (b_{min}, b_{max})$, if $G_j(b)$ is differentiable at $b$ then it holds that*

$$\frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{b - v_i}{1 - b} \cdot \frac{\hat{r}(b)}{\hat{R}(b)} = g_j(b) + \frac{\hat{r}(b)}{\hat{R}(b)} \cdot G_j(b) \tag{22}$$

*Proof.* By Lemma 50 part (2), all bids $b \in (b_{min}, b_{max})$ are optimal for bidder $i$. $\Pi_i(b)$ is differentiable at $b$ because $G_j(b)$ is differentiable at $b$ by assumption. Therefore the first-order condition $d\Pi_i(b)/db = 0$ is necessary for optimality of $i$'s bid $b$. Equation (22) follows from setting equation (15) to zero and rearranging terms. □

Our next lemma follows by applying the following well known differential-equation result to the first-order condition derived in Lemma 51.

**Theorem 52.** *Assume that $q(x) = u'(x) + p(x) \cdot u(x)$ holds for every $x \in (b_{min}, b)$ but a set of measure zero, and $p(x)$ and $q(x)$ are continuous on the interval. Define $z(x) = e^{\int_{b_{min}}^{x} p(y)dy}$. Then every function $u(x)$ that satisfies the assumption is of the form*

$$u(b) - \frac{u(b_{min})}{z(b)} = \frac{1}{z(b)} \int_{b_{min}}^{b} z(x)q(x)dx + C \tag{23}$$

*for some $C$.*

**Lemma 53.** *At $\mu^\epsilon$, for each bidder $i \in 1, 2$ and $j \neq i$, the following must hold. For every bid $b \in [b_{min}, b_{max}]$, it must hold that*

$$G_j(b) = \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{\epsilon}{\hat{R}(b)} \cdot \int_{b_{min}}^{b} \frac{x - v_i}{1 - x} r(x)dx + G_j(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)} \tag{24}$$

*Proof.* At any bid $b \in (b_{min}, b_{max})$ for which $G_j$ is differentiable, equation (22) holds by Lemma 51. $G_j$ is differentiable almost everywhere because it is nondecreasing.[22]

This is a first-order ODE. We apply Theorem 52 with $u(b) = G_j(b)$, $u'(b) = g_j(b)$, $q(b) = \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{b-v_i}{1-b} \cdot \frac{\hat{r}(b)}{\hat{R}(b)}$ and $p(b) = \frac{\hat{r}(b)}{\hat{R}(b)}$. We observe that $z(x) = \int_{b_{min}}^{x} p(y)dy = \int_{b_{min}}^{x} \frac{\hat{r}(y)}{\hat{R}(y)}dy =$

[22] See, for example, Theorem 31.2 in (Billingsley 1995).

$log(\hat{R}(x)) - log(b_{min})$ and thus $z(x) = e^{\int_{b_{min}}^{x} p(y)dy} = \hat{R}(x)/\hat{R}(b_{min})$. Therefore

$$G_j(b) - G_j(b_{min})\frac{\hat{R}(b_{min})}{\hat{R}(b)} = \frac{\hat{R}(b_{min})}{\hat{R}(b)}\left(\int_{b_{min}}^{b} \frac{\hat{R}(x)}{\hat{R}(b_{min})}\frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{x - v_i}{1 - x} \cdot \frac{\hat{r}(x)}{\hat{R}(x)}dx + C\right) = \tag{25}$$

$$\frac{1}{\hat{R}(b)}\left(\int_{b_{min}}^{b} \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{x - v_i}{1 - x} \cdot \hat{r}(x)dx + C\right) \tag{26}$$

Evaluating the above at $b = b_{min}$ shows that $C$ must equal 0. Equation (24) then follows for all $b \in (b_{min}, b_{max})$. The equation holds also at $b = b_{min}$ where it reduces to $G_j(b_{min}) = G_j(b_{min})$. It also holds at $b = b_{max}$, by continuity (Lemma 50 part (2)). □

### F.2.2 Preliminary small $\epsilon$ results

We next show that for sufficiently small $\epsilon$ it holds that $b_{max} > b_{min}$ (ruling out the case $b_{max} = b_{min}$ allowed for in Lemma 24).

**Lemma 54.** *At $\mu^\epsilon$ the following must hold. If $\epsilon > 0$ is small enough then $b_{max} > b_{min}$.*

*Proof.* Assume that $b_{max} = b_{min}$. Clearly it cannot be the case that $b_{min} = \underline{b}$ as it means that both agents are bidding an atom (of size 1) at $\underline{b}$. If $b_{min} < 1$, this contradicts Lemma 40. If $b_{min} \geq 1$, bidder $i \in \{1, 2\}$ could earn strictly more by deviating to bid $v_i$. Reducing the bid to $v_i$ means that bidder $i$ loses every time bidder $j \neq i$ has a high signal. In these cases the value is 1, but the payment would have been 1, so bidder $i$ is indifferent to losing rather than tying. In addition, reducing the bid to $v_i$ means that bidder $i$ now loses every time that bidder $j$ has a low signal and the random bidder bids between $v_i$ and $i$. Thus the bid reduction avoids overpayment with positive probability. This contradicts optimality of bidder $i$ bidding 1. We conclude that $b_{min} > \underline{b}$.

Given $b_{max} = b_{min} > \underline{b}$, Lemma 24 implies that one agent, say $j$, is bidding an atom of size 1 at $v_j$, while the other agent $i$ is bidding an atom of size 1 at $b_{min} = b_i^*(1)$. We note that Equation (6) shows that for $v_i < 1$ there exists $\zeta < 1$ which is independent of $\epsilon$ such that $b_{min} < \zeta$. When $\epsilon$ is small enough agent $j$ can deviate and get strictly higher utility by bidding $b^+ \in (b_{min}, 1)$. This deviation has two effects. First it means that $j$ has additional wins when $i$ has a low signal and the random bidder bids between $v_j$ and $b^+$ causing $j$ to pay more than the value $v_j$. This costs bidder $j$

$$\epsilon \Pr[L_i|H_j] \int_{v_j}^{b^+} (x - v_j)\, r(x)\, dx < \epsilon$$

which is proportional to $\varepsilon$. In addition, the deviation means that $j$ has additional wins when $i$ has a high signal and the random bidder bids below $b^+$. All of these incremental wins are valued at 1 but cost no more than $b^+$ so increase $j$'s payoff. Considering just those incremental wins for which the random bidder bids below $b_{min}$, this benefit is bounded below by $\Pr[H_i|H_j](1-b_{min}) > \Pr[H_i|H_j](1-\zeta)$. Thus $\epsilon < \Pr[H_i|H_j](1-\zeta)$ is a sufficient condition for the deviation to be strictly profitable. This contradiction shows $b_{max} > b_{min}$. □

We further show that $b_{max}$ tends to 1 as $\epsilon$ goes to 0.

**Lemma 55.** *Fix a small $\delta > 0$. At $\mu^\epsilon$ the following must hold. If $\epsilon > 0$ is small enough then it holds that $1 > b_{max} > 1-\delta$ ($b_{max}$ tends to 1 as $\epsilon$ goes to 0).*

*Proof.* By Lemma 53, for each bidder $i \in \{1,2\}$ and $j \neq i$, $b_{max}$ must satisfy:

$$1 = \frac{Pr[L_i|H_j]}{Pr[H_i|H_j]} \cdot \frac{\epsilon}{\hat{R}(b_{max})} \int_{b_{min}}^{b_{max}} \frac{x-v_i}{1-x} \cdot r(x)dx + G_i(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b_{max})} \tag{27}$$

The integral $\int_{b_{min}}^{b_{max}} \frac{x-v_i}{1-x} \cdot r(x)dx$ is finite for any $b_{max} < 1$ but it can be shown that it approaches infinity in the limit as $b_{max}$ goes to 1. (This relies on the fact that $r(x)$ is bounded away from zero by $\underline{r} = \inf_{x\in[0,1]} r(x)$.) For any fixed $\epsilon > 0$, for the right-hand side of equation (27) to be finite it must hold that $b_{max} < 1$. As $\epsilon$ approaches zero, Lemma 54, Lemma 24, and equation (7) imply that for some bidder $i$ either $G_i(b_{min}) = 0$ or $G_i(b_{min})$ approaches zero. Let us fix this bidder to be named $i$ and consider equation (27). For the first term to approach 1 as $\epsilon$ approaches zero requires the integral to approach infinity. Thus $b_{max}$ tends to 1 as $\epsilon$ goes to 0. □

Next, let $T(b) = \frac{\int_{b_{min}}^{b} \frac{1}{1-x} \cdot r(x)dx}{\int_{b_{min}}^{b} \frac{x}{1-x} \cdot r(x)dx}$. We characterize the behavior of $T(b)$ as $b$ approaches 1, which turns out to be useful for understanding implications of the fact that $b_{max}$ approaches 1 as $\epsilon$ goes to zero.

**Lemma 56.** *Fix any $0 \leq b_{min} < 1$ and a standard distribution $R$ with density $r$. The function*

$$T(b) = \frac{\int_{b_{min}}^{b} \frac{1}{1-x} \cdot r(x)dx}{\int_{b_{min}}^{b} \frac{x}{1-x} \cdot r(x)dx} \tag{28}$$

*monotonically decreases to 1 as $b$ increases from $b_{min}$ to 1.*

*Additionally,*

$$\frac{\int_{b_{min}}^{b} \frac{x-v_1}{1-x} r(x)dx}{\int_{b_{min}}^{b} \frac{x-v_2}{1-x} r(x)dx} = \frac{1-v_1 \cdot T(b)}{1-v_2 \cdot T(b)} \tag{29}$$

*tends to $\frac{1-v_1}{1-v_2}$ as $b$ tends to 1. If $v_1 > v_2$ it is monotonically increasing to its limit, and if $v_1 < v_2$ it is monotonically decreasing to its limit.*

*Proof.* Let $c \geq b_{min}$ be some number such that $0 < c < 1$ (say, $c = b_{min}$ unless $b_{min} = 0$, in this case $c = 1/2$). Assume $b \geq c$. Since $r$ is continuous on a compact set its infimum is obtained. Since $r$ is positive for every $x$, $\exists \underline{r} > 0$ such that $r(x) \geq \underline{r}$ for every $x$. Then

$$\int_{b_{min}}^{b} \frac{1}{1-x} r(x)dx \geq \int_{b_{min}}^{b} \frac{x}{1-x} r(x)dx \geq \underline{r} \cdot \int_{c}^{b} \frac{x}{1-x} dx \geq c \cdot \underline{r} \cdot \int_{c}^{b} \frac{1}{1-x} dx$$

Now we observe that both the numerator and the denominator of $T(b)$ tend to infinity when $b$ tends to 1 as

$$\lim_{b\to 1} \int_{c}^{b} \frac{1}{1-x} dx = \lim_{b\to 1} \left(\ln(1-c) - \ln(1-b)\right) = \infty$$

Thus by L'Hôpital's rule,

$$\lim_{b\to 1} \frac{\int_{b_{min}}^{b} \frac{1}{1-x} r(x)dx}{\int_{b_{min}}^{b} \frac{x}{1-x} r(x)dx} = \lim_{b\to 1} \frac{\frac{d}{db}\int_{b_{min}}^{b} \frac{1}{1-x} r(x)dx}{\frac{d}{db}\int_{b_{min}}^{b} \frac{x}{1-x} r(x)dx} = \lim_{b\to 1} \frac{\frac{1}{1-b} r(b)}{\frac{b}{1-b} r(b)} = \lim_{b\to 1} \frac{1}{b} = 1.$$

Next we show that $T(b)$ monotonically decreases to 1 as $b$ increases to 1. For any $b < 1$ all terms are finite, so we can compute the derivative:

$$\frac{d}{db} \frac{\int_{b_{min}}^{b} \frac{1}{1-x} r(x)dx}{\int_{b_{min}}^{b} \frac{x}{1-x} r(x)dx} = \frac{1}{1-b} r(b) \frac{\int_{b_{min}}^{b} \frac{x-b}{1-x} r(x)dx}{\left(\int_{b_{min}}^{b} \frac{x}{1-x} r(x)\, dx\right)^2} < 0.$$

For $b < 1$, $\frac{1}{1-b} > 0$. For $0 \leq b_{min} < b < 1$ and $x \in [b_{min}, b]$, $\frac{x-b}{1-x} < 1$. Therefore $T(b)$ is monotonically decreasing to 1 as $b$ increases to 1.

Observe that

$$\frac{\int_{b_{min}}^{b} \frac{x-v_1}{1-x} r(x)dx}{\int_{b_{min}}^{b} \frac{x-v_2}{1-x} r(x)dx} = \frac{1 - v_1 \cdot T(b)}{1 - v_2 \cdot T(b)} = 1 - \frac{v_1 - v_2}{1/T(b) - v_2}. \tag{30}$$

When $v_1 > v_2$ it is monotonically increasing to $\frac{1-v_1}{1-v_2}$ as $b$ increases to 1, since $T(b)$ decreases to 1 and $v_1 - v_2 > 0$. Similar argument shows that when $v_1 < v_2$ it is monotonically decreasing to its limit. □

Finally, we use Lemmas 25, 55, and 56 to find the limits of $G_1(b_{min})$ and $G_2(v_2)$ as $\epsilon$ goes to zero; they are useful when proving existence of Nash equilibrium in the tremble and proving convergence to the TRE.

**Lemma 57.** *Fix a standard distribution $R$, a sequence of $\epsilon$ converging to zero, and an associated sequence of NE $\{\mu^\epsilon\}$ in the trembles $\lambda(\epsilon, R)$. Then it holds that $\lim_{\epsilon\to 0} G_1(b_{min}) = 0$ and*

$$\lim_{\epsilon\to 0} G_2(v_2) = 1 - \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \cdot \frac{1-v_1}{1-v_2}. \tag{31}$$

*Proof.* By Lemma 25, $G_1(b_{min})$ must satisfy equation (9) and $G_2(v_2)$ must satisfy equation (10) for all $\mu^\epsilon$. By inspection, it is clear that $\lim_{\epsilon\to 0} G_1(b_{min}) = 0$. Turning to $G_2(v_2)$, we note that: (1) $\frac{\hat{R}(b_{max})}{\hat{R}(b_{min})} = \frac{1-\epsilon+\epsilon R(b_{max})}{1-\epsilon+\epsilon R(b_{min})}$ approaches 1. (2) Lemmas 55 and 56 jointly imply that $\frac{\int_{b_{min}}^{b_{max}} \frac{x-v_1}{1-x} r(x)dx}{\int_{b_{min}}^{b_{max}} \frac{x-v_2}{1-x} r(x)dx}$ approaches $\frac{1-v_1}{1-v_2}$. Therefore equation (31) holds. $\square$

### F.2.3 Proof of Lemma 25

Recall that we assume that $0 < Pr[H_1, L_2](1 - v_1) \leq Pr[L_1, H_2](1 - v_2)$, and additionally, we assume that if $Pr[H_1, L_2](1 - v_1) = Pr[L_1, H_2](1 - v_2)$ then $v_1 \geq v_2$.

We next prove Lemma 25.

*Proof.* Following Lemma 54, we assume that $b_{max} > b_{min}$ throughout the proof. For brevity, we define $\alpha_1 = \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]}$ and $\alpha_2 = \frac{Pr[L_2|H_1]}{Pr[H_2|H_1]}$. Observe that

$$\frac{\alpha_2}{\alpha_1} = \frac{Pr[L_2|H_1]}{Pr[L_1|H_2]} \cdot \frac{Pr[H_1|H_2]}{Pr[H_2|H_1]} = \frac{Pr[L_2|H_1]}{Pr[L_1|H_2]} \cdot \frac{Pr[H_1, H_2]}{Pr[H_2]} \cdot \frac{Pr[H_1]}{Pr[H_1, H_2]} = \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \tag{32}$$

**Claim 1.** *Assume $b_{max} > b_{min}$. Let $\beta(b) = \hat{R}(b_{min})/\hat{R}(b)$. It holds that*

$$\frac{1 - G_2(b_{min}) \cdot \beta(b_{max})}{1 - G_1(b_{min}) \cdot \beta(b_{max})} = \frac{\alpha_2}{\alpha_1} \cdot \frac{1 - v_1 T(b_{max})}{1 - v_2 T(b_{max})} \tag{33}$$

*Proof.* Recall that by Lemma 24 for $b_{max}$ it holds that $G_1(b_{max}) = G_2(b_{max}) = 1$. By Lemma 53, for every bid $b \in [b_{min}, b_{max}]$ equation (24) holds. Therefore:

$$1 - G_2(b_{min}) \cdot \beta(b_{max}) = \alpha_2 \cdot \frac{\epsilon}{\hat{R}(b_{max})} \int_{b_{min}}^{b_{max}} \frac{x - v_1}{1 - x} \cdot r(x)dx,$$

$$1 - G_1(b_{min}) \cdot \beta(b_{max}) = \alpha_1 \cdot \frac{\epsilon}{\hat{R}(b_{max})} \int_{b_{min}}^{b_{max}} \frac{x - v_2}{1 - x} \cdot r(x)dx.$$

The claim follows from dividing the two equations (since for $b_{max} > b_{min}$ both sides of the two equations are not 0, thus such a division is well defined). $\square$

**Claim 2.** *Assume $b_{max} > b_{min}$. There are no atoms ($G_1(b_{min}) = G_2(b_{min}) = 0$) if and only if both bidders are symmetric: $v_1 = v_2$ and $Pr[H_1, L_2] = Pr[L_1, H_2]$.*

*Proof.* By Lemma 24 if $G_1(b_{min}) = G_2(b_{min}) = 0$ then $\underline{b} = b_{min} = v_1 = v_2$. In such a case equation (33) reduces to $\alpha_2 = \alpha_1$. Now, recall that $\frac{\alpha_2}{\alpha_1} = \frac{Pr[H_1,L_2]}{Pr[L_1,H_2]}$, thus if there are no atoms in both $G_1$ and $G_2$ then $\underline{b} = b_{min} = v_1 = v_2$ and $Pr[H_1, L_2] = Pr[L_1, H_2]$, that is, the two agents are completely symmetric.

Now, assume that both bidders are symmetric, that is, $v = v_1 = v_2$ and $Pr[H_1, L_2] = Pr[L_1, H_2]$, we want to show that no bidder has an atom. We next show that it cannot be the case that $b_{min} > \underline{b}$. This is sufficient as, by Lemma 24, $b_{min} = \underline{b}$ and $v_1 = v_2$ imply that no bidder has an atom, that is $G_2(b_{min}) = G_1(b_{min}) = 0$.

We next show that symmetry and $b_{min} > \underline{b}$ implies a contradiction. For symmetric bidders Equation (33) implies that $G_1(b_{min}) = G_2(b_{min})$. Using Lemma 24 we observe the following. One bidder, w.l.o.g. bidder 2, bids an atom at $\underline{b} = v_1 = v_2 = v$ and the other bidder (bidder 1) bids an atom at $b_{min} > \underline{b} = v$. Denote $\Gamma = G_1(b_{min}) = G_2(\underline{b})$. By Equation (6),

$$b_{min} = b_1^*(\Gamma) = \frac{\Pr[H_2|H_1]\,\Gamma + v_1 \Pr[L_2|H_1]}{\Pr[H_2|H_1]\,\Gamma + \Pr[L_2|H_1]},$$

or equivalently,

$$\Gamma = \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]} \cdot \frac{b_{min} - v_1}{1 - b_{min}}.$$

By Equation (7),

$$\Gamma = \frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \cdot \frac{\int_{v_2}^{b_{min}} (x - v_2)\,\hat{r}(x)dx}{\hat{R}(b_{min})\,(1 - b_{min})}.$$

Thus,

$$\frac{\Pr[L_1|H_2]}{\Pr[H_1|H_2]} \cdot \frac{\int_{v_2}^{b_{min}} (x - v_2)\,\hat{r}(x)dx}{\hat{R}(b_{min})\,(1 - b_{min})} = \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]} \cdot \frac{b_{min} - v_1}{1 - b_{min}},$$

or due to symmetry in conditional probabilities ($\alpha_1 = \alpha_2$) and values ($v_1 = v_2 = v$),

$$\int_v^{b_{min}} (x - v)\,\hat{r}(x)dx = \hat{R}(b_{min})\,(b_{min} - v)\,.$$

Integration by parts implies that

$$\int_v^{b_{min}} (x - v)\,\hat{r}(x)dx \quad = \quad (b_{min} - v)\,\hat{R}(b_{min}) - \int_v^{b_{min}} \hat{R}(x)dx,$$

and this can only equal $\hat{R}(b_{min})\,(b_{min} - v)$ when $b_{min} = v$, a contradiction. □

We next consider the case that $Pr[H_1, L_2](1 - v_1) = Pr[L_1, H_2](1 - v_2)$ but the bidders are not symmetric ($v_1 > v_2$ and $Pr[H_1, L_2] < Pr[L_1, H_2]$ ).

**Claim 3.** *Assume $b_{max} > b_{min}$ and that $\epsilon$ is small enough. Assume that $Pr[H_1, L_2](1 - v_1) = Pr[L_1, H_2](1 - v_2)$ but the bidders are not symmetric, and it holds that $v_1 > v_2$ and $Pr[H_1, L_2] < Pr[L_1, H_2]$. Then bidder 1 has an atom at $b_{min} = \underline{b}_1 > v_1$ and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$.*

*Proof.* By Claim 2 as bidders are not symmetric it cannot be the case that both bidders have no atom.

We next show that it cannot be the case that only one bidder has an atom. By Lemma 24 if only one bidder has an atom and $v_1 > v_2$ it must be the case that $\underline{b} = b_{min} = v_1 > v_2$ and bidder 1 has the atom at $v_1$. But in this case, as $G_2(b_{min}) = 0$, the LHS of Equation (33) equals to $\frac{1}{1 - G_1(b_{min}) \cdot \beta(b_{max})} > 1$ (as $0 < \beta(b_{max}) \leq 1$ and $G_1(b_{min}) > 0$), while the RHS of

Equation (33) is at most 1 since by Lemma 56 it is monotonically increasing to its limit 1, a contradiction.

We conclude that both bidders have an atom, each at his infimum bid. We next figure out which bidder has an atom at $\underline{b}$ and which has an atom at $b_{min}$. We first show that it must be the case that both $G_1(b_{min})$ and $G_2(b_{min})$ tend to 0 as $\epsilon$ goes to 0. By Equation (7) for one bidder $i$ it holds that $G_i(b_{min})$ must tend to 0 as $\epsilon$ goes to 0 (as $b_{min}$ does not tend to 1 the denominator does not tend to 0, while the numerator tends to 0). Now, as the RHS of Equation (33) tends to 1 as $\epsilon$ goes to 0, $G_1(b_{min}) - G_2(b_{min})$ must tend to 0. Now, as both $G_1(b_{min})$ and $G_2(b_{min})$ tend to 0 as $\epsilon$ goes to 0, by Equation (6) the bid of bidder $i$ that is bidding at $\underline{b}_i = b_{min}$ must tend to $v_i$, that is $b_{min} - v_i$ tends to 0. Now recall that in that case it holds that $b_{min} > \underline{b}_j = v_j$. Thus, if $v_i < v_j$ we get a contradiction as $b_{min} - v_i > v_j - v_i$ and $v_j - v_i$ is some positive constant (bounded away from 0). We conclude that $b_{min} = \underline{b}_1 > \underline{b} = \underline{b}_2 = v_2$, that is, bidder 1 has an atom at $b_{min} = \underline{b}_1 > v_1$ and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$, as we need to show. □

**Claim 4.** *Assume $b_{max} > b_{min}$ and that $\epsilon$ is small enough. Assume that $Pr[H_1, L_2](1 - v_1) < Pr[L_1, H_2](1 - v_2)$. Then either bidder 1 has no atom and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} = b_{min}$. Or, bidder 1 has an atom at $b_{min} = \underline{b}_1 > v_1$ and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$.*

*Proof.* By Claim 2 as bidders are not symmetric it cannot be the case that both bidders have no atom. We next consider the case that at least one bidder has an atom. By Lemma 55 $b_{max}$ tends to 1 as $\epsilon$ goes to 0. Additionally, $T(b)$ tends to 1 as $b$ tends to 1 (by Lemma 56). Thus, the RHS of Equation (33) tends to $\chi = \frac{Pr[H_1,L_2](1-v_1)}{Pr[L_1,H_2](1-v_2)} < 1$ as $\epsilon$ goes to 0. Equation (33) combined with $\chi < 1$ implies that $G_1(b_{min}) < G_2(b_{min})$.

Now, if only one bidder has an atom it must be bidder 2, since $G_2(b_{min}) = 0$ implies $G_1(b_{min}) < 0$, a contradiction. If on the other hand both bidders have an atom, we claim that bidder 1 has an atom at $b_{min} = \underline{b}_1$ and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$. Observe also that $\beta(b_{max}) = \frac{\hat{R}(b_{min})}{\hat{R}(b_{max})}$ tends to 1 as $\epsilon$ goes to 0. Now, if bidder 2 is the bidder with the atom at $b_{min}$, by Equation (7) $G_2(b_{min})$ must tend to 0 as $\epsilon$ goes to 0 (as $b_{min}$ does not tend to 1 the denominator does not tend to 0, while the numerator tends to 0). Combining with $G_1(b_{min}) < G_2(b_{min})$ this will imply that $G_1(b_{min})$ must also tend to 0 as $\epsilon$ goes to zero. But then the LHS of Equation (33) tends to 1 while the RHS tends to $\chi < 1$, a contradiction. We conclude that bidder 1 has an atom at $b_{min} = \underline{b}_1$ and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$. □

By Equation (33) $G_2(b_{min})$ must satisfy

$$G_2(b_{min}) = \frac{1}{\beta(b_{max})} - \left(\frac{1}{\beta(b_{max})} - G_1(b_{min})\right) \cdot \frac{Pr[H_1, L_2]}{Pr[L_1, H_2]} \cdot \frac{1 - v_1 T(b_{max})}{1 - v_2 T(b_{max})} \tag{34}$$

Now Equation (10) follows from the definition of $\beta(b_{max})$ and $T(b_{max})$. The other claims in the lemma for the case that bidder 1 has an atom at $b_{min} = \underline{b}_1$ and bidder 2 has an atom at $v_2 = \underline{b}_2 = \underline{b} < b_{min}$ directly follow from Lemmas 24 and 53.

□

## F.3 Proof of Lemma 26 (Existence of NE in $\lambda(\epsilon, R)$)

We next show that for any standard distribution $R$, if $\epsilon$ is small enough then there exists a mixed NE in the game $\lambda(\epsilon, R)$. We prove existence of one of three types of equilibria depending on parameter values. For symmetric bidders, we show the existence of an equilibrium with no atoms (case 1). For asymmetric bidders we show the existence of either a one-atom (case 2) or a two-atom (case 3) equilibrium depending on whether or not equation (38) in the proof is satisfied. The following observation indicates why equation (38) determines whether asymmetric equilibria involve one or two atoms.

**Observation 58.** *If $\epsilon$ is small enough and $G_1(b_{min}) > 0$ (bidder 1 has an atom, which implies that bidder 2 also has an atom) then it must hold that*

$$\alpha_2 \cdot \frac{v_2 - v_1}{1 - v_2} \leq 1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1 - v_1}{1 - v_2} \tag{35}$$

*Proof.* If $G_1(b_{min}) > 0$ then Equation (6) holds. In particular it must hold that

$$\frac{G_2(v_2) + v_1\alpha_2}{G_2(v_2) + \alpha_2} = 1 - \frac{\alpha_2(1 - v_1)}{G_2(v_2) + \alpha_2} > v_2 \tag{36}$$

Lemma 57 states that $G_2(v_2)$ tends to $1 - \frac{Pr[H_1,L_2](1-v_1)}{Pr[L_1,H_2](1-v_2)} = 1 - \frac{\alpha_2(1-v_1)}{\alpha_1(1-v_2)}$ as $\epsilon$ goes to zero. Thus it must hold that

$$1 - \frac{\alpha_2(1 - v_1)}{\left(1 - \frac{\alpha_2(1-v_1)}{\alpha_1(1-v_2)}\right) + \alpha_2} \geq v_2 \tag{37}$$

and the claim follows from reorganizing the last equation. □

### F.3.1 Proof of Lemma 26

*Proof.* Let $\bar{v} = \max\{v_1, v_2\}$. Throughout the proof we index bidders 1 and 2 such that either 1) $\alpha_1(1 - v_2) = \alpha_2(1 - v_1)$ and $v_1 > v_2$, or 2) $\alpha_1(1 - v_2) > \alpha_2(1 - v_1)$. Moreover, we often distinguish between three cases:

1. *No atom case.* Bidders are symmetric: $v = v_1 = v_2$ and $Pr[H_1, L_2] = Pr[L_1, H_2]$. In this case we show there exists an equilibrium in which $b_{min} = v$ and neither bidder has an atom: $G_1(b_{min}) = G_2(v_2) = 0$.

2. *One atom case.* Bidders are asymmetric ($v_1 \neq v_2$ or $Pr[H_1, L_2] \neq Pr[L_1, H_2]$) and equation (38) holds:

$$\alpha_2 \cdot \frac{v_2 - v_1}{1 - v_2} \geq 1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1 - v_1}{1 - v_2}. \tag{38}$$

   Note that asymmetry and equation (38) imply that $\alpha_1(1 - v_2) > \alpha_2(1 - v_1)$ and $v_2 > v_1$. This is so as by assumption the RHS of equation (38) is non-negative, this implies that $v_2 \geq v_1$. If $v_2 = v_1$ Then the equation implies that $\alpha_1 = \alpha_2$ which means the bidders are symmetric, a contradiction. Therefore $v_2 > v_1$ and thus $\alpha_1(1 - v_2) > \alpha_2(1 - v_1)$ (since in the case that $\alpha_1(1 - v_2) = \alpha_2(1 - v_1)$ we assume that $v_1 > v_2$).

   In this case we show that there exists an equilibrium in which $b_{min} = v_2$ and only bidder 2 has an atom: $G_2(v_2) > 0$ and $G_1(b_{min}) = 0$.

3. *Two atom case.* Bidders are asymmetric ($v_1 \neq v_2$ or $Pr[H_1, L_2] \neq Pr[L_1, H_2]$) and equation (38) is violated. Note that either 1) $\alpha_1(1 - v_2) = \alpha_2(1 - v_1)$ and $v_1 > v_2$, or 2) $\alpha_1(1 - v_2) > \alpha_2(1 - v_1)$ are both feasible. In this case we show that there exists an equilibrium in which $b_{min} > \max\{v_1, v_2\}$ and both bidders have atoms: $G_2(v_2) > 0$ and $G_1(b_{min}) > 0$.

In all cases, bidder $i$ with signal $L_i$ is bidding $V_{LL} = 0$. We construct distributions $G_1$ and $G_2$ using the necessary conditions in Lemma 25 and show that they form a NE. Equations (11) and (12) define $G_1$ and $G_2$ as a function of the four parameters $b_{min}$, $b_{max}$, $G_1(b_{min})$, and $G_2(v_2)$. There are three main steps to the proof. First we show existence of parameters $b_{min}$, $b_{max}$, $G_1(b_{min})$, and $G_2(v_2)$ that satisfy the necessary conditions in Lemma 25. Second, we show that, for the chosen parameters, $G_1$ and $G_2$ are well defined distributions (non-decreasing, and satisfying $G_1(0) = G_2(0) = 0$ and $G_1(1) = G_2(1) = 1$). Third we show that the constructed bid distributions are best responses. By construction, bidder $i \in \{1, 2\}$ is indifferent to all bids in the support of his bid distribution and we show that every bid outside the support gives weakly lower utility.

**Step 1.** Existence of parameters $b_{min}$, $b_{max}$, $G_1(b_{min})$, and $G_2(v_2)$:

**Case 1 (no atoms):** First consider the case that the bidders are symmetric. We define $b_{min} = v$ and $G_1(b_{min}) = G_2(v_2) = 0$. By the necessary conditions at $b_{max}$ it must hold that

$$1 = G_1(b_{max}) = \frac{Pr[L_1|H_2]}{Pr[H_1|H_2]} \cdot \frac{\epsilon}{\hat{R}(b_{max})} \cdot \int_v^{b_{max}} \frac{x - v}{1 - x} r(x)dx \tag{39}$$

The RHS increases from zero to infinity as $b_{max}$ increases from $v$ to 1 (Claim 6), so there exists a unique value of $b_{max} \in (v, 1)$ that solves this equation. It is clear that $b_{max}$ must tend to 1 as $\epsilon$ goes to 0. Note that all the necessary conditions presented in Lemma 25 for the symmetric case are now satisfied.

**Case 2 (one atom):** Next consider the case that bidders are asymmetric and equation (38) holds (implying $\alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2)$ and $v_1 < v_2$). We define $b_{min} = v_2$ and

$G_1(b_{min}) = 0$. As $G_1(b_{min}) = 0$, $b_{max} \in (v, 1)$ can be determined exactly as in the symmetric case. Finally, we set $G_2(v_2)$ using Equation (10). Observe that $G_2(v_2)$ as defined tends to $1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1-v_1}{1-v_2} \in (0, 1)$ as $\epsilon$ tends to 0, thus for sufficiently small $\epsilon$ it is positive.

**Case 3 (two atoms):** Finally, consider the case that bidders are asymmetric and equation (38) is violated. We define $G_1(b_{min})$ as a function of $b_{min}$ and $b_{max}$ by equation (7). We define $G_2(v_2)$ as a function of $b_{min}$ by equation (6), or equivalently by:

$$G_2(v_2) = \frac{\Pr[L_2|H_1]}{\Pr[H_2|H_1]} \frac{b_{min} - v_1}{1 - b_{min}}. \tag{40}$$

The arguments below show that $b_{min} > \max\{v_1, v_2\}$, which ensures that $G_1(b_{min}) > 0$ and $G_2(v_2) > 0$. By substituting $G_1(b_{min})$ and $G_2(v_2)$ into equations (11) and (12), which determine $G_1(b)$ and $G_2(b)$, and evaluating these equations at $b_{max}$, for which it must hold that $G_1(b_{max}) = G_2(b_{max}) = 1$, we derive that we need to find $b_{min}$ and $b_{max}$ that satisfy the following pair of equations:

$$1 = \alpha_1 \cdot \frac{1}{\hat{R}(b_{max})} \int_{b_{min}}^{b_{max}} \frac{x - v_2}{1 - x} \cdot \hat{r}(x)dx + \alpha_1 \cdot \frac{1}{\hat{R}(b_{max})} \int_{v_2}^{b_{min}} \frac{x - v_2}{1 - b_{min}} \cdot \hat{r}(x)dx \tag{41}$$

$$1 = \alpha_2 \cdot \frac{1}{\hat{R}(b_{max})} \int_{b_{min}}^{b_{max}} \frac{x - v_1}{1 - x} \cdot \hat{r}(x)dx + \frac{b_{min} - v_1}{1 - b_{min}} \cdot \alpha_2 \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b_{max})} \tag{42}$$

We first show that when $\epsilon$ is small enough, for any $b_{min} \in [\bar{v}, v(H_1)]$ we can find a unique $b_{max} \in (b_{min}, 1)$ that solves equation (41). We denote such a solution by $b_{max}(b_{min})$. When $b_{max} = b_{min}$, the RHS of equation (41) equals $\epsilon \cdot h(b_{min})$ for $h(b_{min}) = \frac{\alpha_1}{\hat{R}(b_{min})} \int_{v_2}^{b_{min}} \frac{x-v_2}{1-b_{min}} \cdot r(x)dx$. As $h$ is a continuous function on a compact set it is bounded, thus $\epsilon \cdot h(b_{min}) < 1$ for any $b_{min} \in [\bar{v}, v(H_1)]$ as long as $\epsilon$ is small enough. Now, for every fixed $b_{min} \in [\bar{v}, v(H_1)]$, the RHS of equation (41) is continuously increasing in $b_{max}$ (by Claim 6 below) and goes to infinity when $b_{max}$ tends to 1. Therefore there exists a unique $b_{max} \in (b_{min}, 1)$ that solves the equation. Note that $b_{max}(b_{min})$ is a continuous function of $b_{min}$ and, for any fixed $b_{min}$, $b_{max}(b_{min})$ tends to 1 as $\epsilon$ tends to 0.

Now we substitute $b_{max}(b_{min})$ into equation (42) and get the following equation in $b_{min}$

$$1 = \alpha_2 \cdot \frac{1}{\hat{R}(b_{max}(b_{min}))} \int_{b_{min}}^{b_{max}(b_{min})} \frac{x - v_1}{1 - x} \cdot \hat{r}(x)dx + \frac{b_{min} - v_1}{1 - b_{min}} \cdot \alpha_2 \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b_{max}(b_{min}))} \tag{43}$$

To complete the proof we need to show that there exists $b_{min} \in [\bar{v}, v(H_1)]$ that satisfies equation (43). The RHS of this equation is a continuous function of $b_{min}$ on the compact set $[\bar{v}, v(H_1)]$. It will therefore be sufficient to show that for $b_{min} = v(H_1)$ the RHS is strictly larger than 1, while for $b_{min} = \bar{v}$ the RHS is strictly smaller than 1. Once this is shown (below) we conclude that there exists $b_{min} > \bar{v}$ such that the RHS is exactly 1. This $b_{min}$ together with $b_{max} = b_{max}(b_{min})$ solve both equations (41) and (42) and satisfy $1 > b_{max} > b_{min} > \bar{v}$.

To prove the remaining two inequalities, define:

$$z(b_{min}) = \alpha_1 \cdot \frac{1}{\hat{R}(b_{max}(b_{min}))} \int_{b_{min}}^{b_{max}(b_{min})} \frac{x - v_2}{1 - x} \cdot \hat{r}(x)dx.$$

Now, the RHS of equation (43) can be written as

$$z(b_{min}) \cdot \frac{\alpha_2 \cdot \int_{b_{min}}^{b_{max}(b_{min})} \frac{x-v_1}{1-x} \cdot \hat{r}(x)dx}{\alpha_1 \cdot \int_{b_{min}}^{b_{max}(b_{min})} \frac{x-v_2}{1-x} \cdot \hat{r}(x)dx} + \frac{b_{min} - v_1}{1 - b_{min}} \cdot \alpha_2 \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b_{max}(b_{min}))} \tag{44}$$

Fix $b_{min}$. Note that equation (41) implies that $z(b_{min}) \leq 1$ and $z(b_{min})$ tends to 1 as $\epsilon$ goes to 0, as the second term of the RHS of equation (41) is positive and tends to 0. By Lemma 56, $\frac{\alpha_2 \cdot \int_{b_{min}}^{b_{max}(b_{min})} \frac{x-v_1}{1-x} \cdot \hat{r}(x)dx}{\alpha_1 \cdot \int_{b_{min}}^{b_{max}(b_{min})} \frac{x-v_2}{1-x} \cdot \hat{r}(x)dx}$ tends to $\frac{\alpha_2(1-v_1)}{\alpha_1(1-v_2)}$, and thus, as $\epsilon$ tends to 0, the RHS of equation (43) tends to

$$\frac{\alpha_2(1 - v_1)}{\alpha_1(1 - v_2)} + \frac{b_{min} - v_1}{1 - b_{min}} \cdot \alpha_2 \tag{45}$$

For $b_{min} = v(H_1)$, equation (45) strictly exceeds 1 since by equation (6) it holds that $b_{min} = v(H_1)$ if and only if $G_2(v_2) = \frac{b_{min}-v_1}{1-b_{min}} \cdot \alpha_2 = 1$, and the first term is strictly positive by assumption. Thus, for sufficiently small $\epsilon$, the RHS of equation (43) also strictly exceeds 1 for $b_{min} = v(H_1)$.

If $b_{min} = \bar{v}$ we show that the RHS of equation (43) is strictly less than 1 for sufficiently small $\epsilon$. We consider two cases separately. First, if $b_{min} = \bar{v} = v_2 \geq v_1$, equation (45) is strictly less than 1 as equation (38) is violated. Thus, for sufficiently small $\epsilon$, the RHS of equation (43) is also strictly less than 1. Second, if $b_{min} = \bar{v} = v_1 > v_2$, equation (45) is weakly (but not necessarily strictly) less than 1. However, we show that equation (44) (and hence the RHS of equation (43)) is strictly less than equation (45) for all $\epsilon > 0$. This follows because $b_{min} > v_2$ implies that the second term on the RHS of equation (41) is strictly positive so that $z(b_{min}) < 1$ and $v_1 > v_2$ implies (by Lemma 56) that $\frac{\alpha_2 \cdot \int_{b_{min}}^{b_{max}} \frac{x-v_1}{1-x} \cdot \hat{r}(x)dx}{\alpha_1 \cdot \int_{b_{min}}^{b_{max}} \frac{x-v_2}{1-x} \cdot \hat{r}(x)dx}$ is increasing to its limit (which is at most 1).

**Step 2. $G_1$ and $G_2$ are well defined:** We next argue that $G_1$ and $G_2$, as defined above by Step 1 and equations (11) and (12), are well defined distributions. The way we have chosen the parameters in Step 1 ensures that $\max\{v_1, v_2\} \leq b_{min} < b_{max} \leq 1$, $G_1(b_{min}), G_2(v_2) \geq 0$, and $G_1(b_{max}) = G_2(b_{max}) = 1$. The two distributions are continuous from the right at $b_{min}$, and by Claim 6 and Claim 5 are strictly increasing on $(b_{min}, b_{max})$. Thus both are monotonically non-decreasing on $[0, \infty)$ with $G_1(0) = G_2(0) = 0$ and $G_1(b_{max}) = G_2(b_{max}) = 1$.

**Step 3. Constructed bid distributions are best responses:** To see that $\mu^\epsilon$ is indeed a mixed NE we show that each bidder is best responding to the other. Observe that, by construction, $G_1$ and $G_2$ ensure that each bidder is indifferent between all the bids in the support her bid distribution. It only remains to show that all other bids earn weakly lower payoffs.

First consider bids above $b_{max}$. As $0 < Pr[H_1, L_2](1 - V_{HL}) \leq Pr[L_1, H_2](1 - V_{LH})$ it holds that $\max\{v(H_1), v(H_2)\} < 1$. Therefore, as $b_{max}$ tends to 1 when $\epsilon$ tends to 0, for small enough $\epsilon$ it holds that $b_{max} > \max\{v(H_1), v(H_2)\}$. Noticing that $b_i^*(1) = v(H_i)$, this means that $b_{max}$ exceeds both $b_1^*(1)$ and $b_2^*(1)$. Therefore, for small enough $\epsilon$, part (2) of Lemma 42 implies that $b_{max}$ strictly dominates any higher bid $b > b_{max}$.

Second note that for bidder $i$, bidding $v_i$ weakly dominates any lower bid $b < v_i$.

Third, we consider bids $b \in [v_i, b_{min}]$ by bidder $i \in \{1, 2\}$ outside the support of bidder $i$'s bid distribution for each of the three cases.

Consider case 1 (no atoms) in which $b_{min} = v_1 = v_2 = v$. In this case, the utility from bidding $b_{min} = v$ equals the utility of any bid in $[v, b_{max}]$ by continuity.

Consider case 2 (one atom) in which $b_{min} = v_2$, $\alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2)$, and $v_2 > v_1$. Bidder 2 bids an atom at $v_2$ so there are no other bids to check. For bidder 1, Lemma 40 implies that any bid in $(v_2, b_{max})$ strictly dominates bidding $v_2$. By Lemma 42 part (1), the bid with the highest payoff strictly below $v_2$ is $v_1$. By bidding $v_1$, bidder 1 never wins when bidder 2 gets the high signal $H_2$. Since $1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1-v_1}{1-v_2} > 0$ the size of the atom of bidder 2 does not tend to 0 as $\epsilon$ tends to 0, and clearly the gain by bidding above the atom of bidder 2 at $v_2$ instead of bidding $v_1$ is positive if $\epsilon$ is small enough.

Consider case 3 (two atoms) in which $b_{min} > \max\{v_1, v_2\}$. Bidder 2 bids an atom at $v_2$, which by Lemma 42 part (1) dominates any bid $b < b_{min}$. Moreover, for bidder 2, bidding $b_{min}$ is dominated by bids in the support by Lemma 40. Now turn to bidder 1. Lemma 42 part (1) and Lemma 40 imply that $i$'s atom at $b_{min}$ dominates any bid in $[v_2, b_{min})$ because $b_{min}$ is defined by equation (6). For $v_1 \geq v_2$, $[v_2, b_{min})$ includes all bids $[v_1, b_{min})$ and we are done. For $v_1 < v_2$, we must also consider bids $[v_1, v_2)$, of which $v_1$ gives the highest payoff to bidder 1 by Lemma 42 part (1). As $v_1 < v_2$ implies $\alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2)$, $b_{min}$ must dominate $v_1$ for sufficiently small $\epsilon$ by the same argument applied above in the one-atom case.

**Claim 5.** *In all three cases (no atoms, one atom, two atoms) $G_2(b)$ as defined above is increasing in $b$ for every $b \in (b_{min}, b_{max})$.*

*Proof.* We need to show that in all three cases $G_2(b)$ is increasing in $b$ for every $b \in (b_{min}, b_{max})$. For any such $b$, $G_2(b)$ satisfies Equation (24), and its derivative with respect to $b$ is

$$g_2(b) = \frac{\hat{r}(b)}{\hat{R}(b)} \left( \alpha_2 \cdot \frac{b - v_1}{1 - b} - G_2(b) \right).$$

To prove the claim it is sufficient to show that for every $b \in (b_{min}, b_{max})$:

$$g_2(b) \cdot \frac{\hat{R}(b)}{\hat{r}(b)} = \alpha_2 \cdot \frac{b - v_1}{1 - b} - G_2(b) > 0. \tag{46}$$

If $G_2(b) \leq 0$ the claim follows from $1 \geq b_{max} > b > b_{min} \geq \max\{v_1, v_2\}$. Next assume that $G_2(b) \geq 0$. We observe that for small enough $\epsilon$ this is an increasing function in $b$ for $b \in (b_{min}, b_{max})$:

$$\begin{aligned}
\frac{d}{db}\left(\frac{\hat{R}(b)}{\hat{r}(b)}g_2(b)\right) &= \alpha_2\frac{1-v_1}{(1-b)^2} - g_2(b) = \alpha_2\frac{1-v_1}{(1-b)^2} - \frac{\hat{r}(b)}{\hat{R}(b)}\left(\alpha_2\frac{b-v_1}{1-b} - G_2(b)\right) \\
&\geq \alpha_2\frac{1}{(1-b)^2}\left((1-v_1) - \frac{\hat{r}(b)}{\hat{R}(b)}(b-v_1)(1-b)\right) \\
&\geq \alpha_2\frac{1}{(1-b_{min})^2}\left(1-v_1-\varepsilon\frac{r(b)}{1-\epsilon}\right).
\end{aligned}$$

As $1 > v_1$ and $r(b)$ is bounded from above ($r$ is continuous on a compact interval), for small enough $\epsilon$ this is positive.

Thus, as the function $\frac{\hat{R}(b)}{\hat{r}(b)}g_2(b)$ is increasing, to prove that it is positive for any $b > b_{min}$ it would be sufficient to show that it is at least 0 at $b_{min}$, or equivalently, that the following holds:

$$\alpha_2 \cdot \frac{b_{min} - v_1}{1 - b_{min}} \geq G_2(b_{min}). \tag{47}$$

We show that equation (47) is satisfied for each of the three cases.

In the first case (no atoms), $G_2(v_2) = 0$, and equation (47) clearly holds because $b_{min} \geq v_1$. In the third case (two atoms), $G_2(v_2)$ satisfies equation (6), which is exactly equivalent to equation (47) holding with equality.

Finally we consider the second case (one atom) in which $\alpha_2 \cdot (1 - v_1) < \alpha_1 \cdot (1 - v_2)$, equation (38) holds and $G_2(b_{min}) = G_2(v_2) > 0$ satisfies equation (10) with $G_1(b_{min}) = 0$, and additionally, $b_{min} = v_2 > v_1$ (this corresponds to the case that only bidder 2 has an atom). These conditions imply that

$$G_2(v_2) = \frac{\hat{R}(b_{max})}{\hat{R}(v_2)}\left(1 - \frac{\alpha_2 \int_{v_2}^{b_{max}} \frac{x-v_1}{1-x} r(x)dx}{\alpha_1 \int_{v_2}^{b_{max}} \frac{x-v_2}{1-x} r(x)dx}\right).$$

Which means that we need to show that

$$\alpha_2\frac{v_2 - v_1}{1 - v_2} \geq \frac{\hat{R}(b_{max})}{\hat{R}(v_2)}\left(1 - \frac{\alpha_2 \int_{v_2}^{b_{max}} \frac{x-v_1}{1-x} r(x)dx}{\alpha_1 \int_{v_2}^{b_{max}} \frac{x-v_2}{1-x} r(x)dx}\right) = G_2(v_2)$$

Equation (39) determines $b_{max}$ and implies that $\hat{R}(b_{max}) = \alpha_1 \int_{v_2}^{b_{\max}} \frac{x-v_2}{1-x} \hat{r}(x)dx$, thus:

$$\frac{\hat{R}(b_{max})}{\hat{R}(v_2)} = \frac{\hat{R}(b_{max})}{\hat{R}(b_{max}) - \int_{v_2}^{b_{\max}} \hat{r}(x)dx} = \frac{\alpha_1 \int_{v_2}^{b_{\max}} \frac{x-v_2}{1-x} r(x)dx}{\int_{v_2}^{b_{\max}} \left(\alpha_1 \frac{x-v_2}{1-x} - 1\right) r(x)dx}$$

We can now express $G_2(v_2)$ as a function of $b_{max}$ as follows:

$$
\begin{aligned}
G_2(v_2) &= \frac{\alpha_1 \int_{v_2}^{b_{\max}} \frac{x-v_2}{1-x} r(x)dx}{\int_{v_2}^{b_{\max}} \left(\alpha_1 \frac{x-v_2}{1-x} - 1\right) r(x)dx} \left(1 - \frac{\alpha_2 \int_{v_2}^{b_{\max}} \frac{x-v_1}{1-x} r(x)dx}{\alpha_1 \int_{v_2}^{b_{\max}} \frac{x-v_2}{1-x} r(x)dx}\right) \\
&= \frac{\int_{v_2}^{b_{\max}} \left(\alpha_1 (x-v_2) - \alpha_2 (x-v_1)\right) \frac{r(x)}{1-x} dx}{\int_{v_2}^{b_{\max}} \left(\alpha_1 \frac{x-v_2}{1-x} - 1\right) r(x)dx}
\end{aligned}
$$

$b_{max}$ tends to 1 as $\epsilon$ goes to 0 (Lemma 55) and $G_2(v_2)$ tends to $1 - \frac{\alpha_2}{\alpha_1} \cdot \frac{1-v_1}{1-v_2}$ as $b_{max}$ tends to 1 (implied by Lemma 57). By Equation (38) it is thus sufficient to prove that $G_2(v_2)$ is nondecreasing in $b_{\max}$: $\frac{d}{db_{\max}} G_2(v_2) \geq 0$.

$$
\begin{aligned}
\frac{dG_2(v_2)}{db_{\max}} &= \frac{1}{\left(\int_{v_2}^{b_{\max}} \left(\alpha_1 \frac{x-v_2}{1-x} - 1\right) r(x)dx\right)^2} \cdot \frac{r(b_{max})}{1-b_{\max}} \\
&\quad \cdot \left( \begin{array}{c} (\alpha_1 (b_{\max} - v_2) - \alpha_2 (b_{\max} - v_1)) \int_{v_2}^{b_{\max}} (\alpha_1 (x - v_2) - (1-x)) \frac{r(x)}{1-x} dx \\ - (\alpha_1 (b_{\max} - v_2) - (1 - b_{\max})) \int_{v_2}^{b_{\max}} (\alpha_1 (x-v_2) - \alpha_2 (x - v_1)) \frac{r(x)}{1-x} dx \end{array} \right) \\
&= \frac{1}{\left(\int_{v_2}^{b_{\max}} \left(\alpha_1 \frac{x-v_2}{1-x} - 1\right) r(x)dx\right)^2} \cdot \frac{r(b_{max})}{1-b_{\max}} \\
&\quad \cdot \int_{v_2}^{b_{\max}} \frac{b_{\max} - x}{1-x} \left(\alpha_1 \alpha_2 (v_2 - v_1) - \alpha_1 (1-v_2) + \alpha_2 (1-v_2)\right) r(x)dx \\
&= \alpha_1 (1-v_2) \left(\alpha_2 \frac{v_2 - v_1}{1-v_2} - \left(1 - \frac{\alpha_2 (1-v_2)}{\alpha_1 (1-v_2)}\right)\right) \frac{\frac{r(b_{max})}{1-b_{\max}} \int_{v_2}^{b_{\max}} \frac{b_{\max}-x}{1-x} r(x)dx}{\left(\int_{v_2}^{b_{\max}} \left(\alpha_1 \frac{x-v_2}{1-x} - 1\right) r(x)dx\right)^2}
\end{aligned}
$$

By Equation (38), $\alpha_2 \frac{v_2-v_1}{1-v_2} \geq \left(1 - \frac{\alpha_2(1-v_2)}{\alpha_1(1-v_2)}\right)$, thus $\frac{dG_2(v_2)}{db_{\max}} \geq 0$ holds. (Moreover, when $\alpha_2 \frac{v_2-v_1}{1-v_2} = \left(1 - \frac{\alpha_2(1-v_2)}{\alpha_1(1-v_2)}\right)$, $\frac{dG_2(v_2)}{db_{\max}} = 0$ and $G_2(v_2)$ attains its limit for any $b_{\max} < 1$). □

**Claim 6.** *In all three cases (no atoms, one atom, two atoms) $G_1(b)$ as defined above is increasing in $b$ for every $b \in (b_{min}, b_{max})$.*

*Proof.* The same arguments as the ones presented in the proof of Claim 5 show that it is sufficient to prove that

$$\alpha_1 \cdot \frac{b_{min} - v_2}{1 - b_{min}} \geq G_1(b_{min}). \tag{48}$$

When bidder 1 does not have an atom (when no bidder has an atom, or only bidder 2 has an atom), this trivially holds since $b_{min} \geq v_2$. We are left to prove the claim when both bidders have an atom and $G_1(b_{min}) > 0$ satisfies Equation (7). We need to show that

$$\alpha_1 \cdot \frac{b_{min} - v_2}{1 - b_{min}} \geq \alpha_1 \cdot \frac{\int_{v_2}^{b_{min}} (x - v_2) \, \hat{r}(x) \, dx}{\hat{R}(b_{min})(1 - b_{min})}, \tag{49}$$

which trivially holds since $\hat{R}(b_{min}) \geq \int_{v_2}^{b_{min}} \hat{r}(x) \, dx = \hat{R}(b_{min}) - \hat{R}(v_2)$.

□

□

## F.4 Proof of Lemma 27 (Convergence) and Theorem 19

We first we provide a bound on $G_j$ in Lemma 59. Then we apply this bound with necessary conditions in Lemma 25 to prove the convergence result in Lemma 27. Finally we note that the Theorem follows from Lemmas 26-27.

**Lemma 59.** *If $\epsilon$ is small enough then the following holds. For every bidder $i \in \{1, 2\}$ and $j \neq i$ and every $b \in (b_{min}, b_{max})$ it holds that:*

$$G_j(b) - G_j(b_{min}) \leq \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{\epsilon}{1 - \epsilon} \cdot r_{max} \cdot (-b - log(1 - b)) \tag{50}$$

*where $r_{max} = sup_{x \in [0,1]} r(x)$ is finite.*

*Proof.* By Lemma 53

$$G_j(b) = \frac{Pr[L_j|H_i]}{Pr[H_j|H_i]} \cdot \frac{\epsilon}{\hat{R}(b)} \cdot \int_{b_{min}}^{b} \frac{x - v_i}{1 - x} r(x) dx + G_j(b_{min}) \cdot \frac{\hat{R}(b_{min})}{\hat{R}(b)} \tag{51}$$

For a standard distribution $R$, $r_{max}$ is a finite upper bound for $r(x)$. As $v_i \geq 0$ and $r(b) \leq r_{max}$ for all $b$,

$$\int_{b_{min}}^{b} \frac{x - v_i}{1 - x} r(x) dx \leq \int_{b_{min}}^{b} \frac{x}{1 - x} r(x) dx \leq r_{max} \int_0^b \frac{x}{1 - x} dx = r_{max}(-b - log(1 - b)) \tag{52}$$

As $\hat{R}(b) \geq \hat{R}(b_{min}) \geq 1 - \epsilon$, equation (50) follows. □

**Corollary 60.** *Let $\bar{b}_{min} = max\{v_2, v(H_1)\} < 1$. Fix any $b \in [\bar{b}_{min}, 1)$ and any $\delta > 0$. For small enough $\epsilon > 0$, for every bidder $j \in \{1, 2\}$ it holds that $b \geq b_{min}$ and $G_j(b) - G_j(b_{min}) < \delta$.*

*Proof.* First, by Lemma 25, for any small enough $\epsilon > 0$ it holds that either $b_{min} = v_2 < 1$ or $b_{min}$ is determined by equation (8) and in this case $v_2 < b_{min} = b_1^*(G_2(v_2)) \leq b_1^*(1) = \frac{\Pr[H_2|H_1] + v_1 \Pr[L_2|H_1]}{\Pr[H_2|H_1] + \Pr[L_2|H_1]} = v(H_1) < 1$ (the inequality is true since $b_1^*(G_2(v_2))$ is an increasing function of $G_2(v_2)$). Thus we get that in either case $\bar{b}_{min} \geq b_{min}$, and therefore for $b \geq \bar{b}_{min}$ it holds that $b \geq b_{min}$. Second, for $b \in [\bar{b}_{min}, 1)$ and sufficiently small $\epsilon$, equation (50) holds by Lemma 59. As for any fixed positive $b < 1$ the RHS tends to 0 when $\epsilon$ tends to 0, the claim follows. $\square$

### F.4.1 Proof of Lemma 27

*Proof.* Fix a standard distribution $R$. We make three claims: (1) First, $G_2(b) = 0$ for $b \in [0, v_2)$ and $G_2(b_{min}) = G_2(v_2)$ for all $\epsilon$ sufficiently small. (2) Second, $\lim_{\epsilon \to 0} G_1(b_{min}) = 0$ and $\lim_{\epsilon \to 0} G_2(v_2) = 1 - \frac{Pr[H_1,L_2](1-v_1)}{Pr[L_1,H_2](1-v_2)}$. (3) Third, for any $b \in [\bar{b}_{min}, 1)$ where $\bar{b}_{min}$ is as defined in corollary 60, $\lim_{\epsilon \to 0}(G_i(b) - G_i(b_{min})) = 0$ for both $i \in 1, 2$. It then follows that in the limit as $\epsilon$ approaches zero, bidder 1 bids 1 with probability 1 while bidder 2 bids $v_2$ with probability $1 - \frac{Pr[H_1,L_2](1-v_1)}{Pr[L_1,H_2](1-v_2)}$ and 1 with complementary probability. Claims (1) and (2) follow from Lemmas 25 and 57. Claim (3) follows from Corollary 60. $\square$

### F.4.2 Proof of Theorem 19

Lemma 26 implies that for any standard distribution $R$, there exists a sequence of $\epsilon$ converging to zero and an associated sequence of NE $\{\mu^\epsilon\}$ corresponding to the trembles $\lambda(\epsilon, R)$. Lemma 27 shows that the limit of any such sequence $\{\mu^\epsilon\}$ must converge to $\mu$ as $\epsilon$ goes to zero. It then follows that $\mu$ is the *unique* TRE.

## F.5 Notation Summary

Throughout Section F, we use $i$ to denote a bidder, either bidder 1 or 2. When we want to refer to the other bidder we use $j$ to denote that bidder, and assume that $j \neq i$. To simplify the notation we denote $v_1 = V_{HL}$ and $v_2 = V_{LH}$ and (without loss of generality) normalize $V_{LL} = 0$ and $V_{HH} = 1$. We assume that $0 < Pr[H_1, L_2](1 - v_1) \leq Pr[L_1, H_2](1 - v_2) < 1$, and that in case of equality $v_1 \geq v_2$. Note that this implies that $\min\{Pr[H_1, L_2], Pr[L_1, H_2]\} > 0$. In Table 1, we summarize additional notation used in the proof.

Table 1: Notation Summary for Section F

| Notation | Definition | Reference |
|---|---|---|
| $v(H_i)$ | $v(H_i) = E[v\|H_i] = \Pr[H_j\|H_i] + \Pr[L_j\|H_i]v_i$ | |
| $\mu^\epsilon$ | NE of tremble $\lambda(\varepsilon, R)$ | |
| $\Pi_i(b_i)$ | $i$'s E[profit] when bidding $b_i$ with signal $H_i$ | Equation (14), page 49. |
| $G_i$ | CDF of $i$'s bids conditional on $H_i$ ($G_i = \mu^\epsilon(H_i)$). | |
| $G^-(b)$ | $\sup_{x<b} G(x)$ (left-hand limit of $G$ evaluated at $b$) | |
| $\underline{b}_i$ | $\inf\{b : G_i(b) > 0\}$ (infimum bid by $i \in \{1, 2\}$ with signal $H_i$) | |
| $\underline{b}$ | $\max\{\underline{b}_1, \underline{b}_2\}$ (infimum bid of any bidder with a high signal) | |
| $b_{min}$ | $\max\{\underline{b}_1, \underline{b}_2\}$ | |
| $\bar{b}_i$ | $\inf\{x : G_i(x) = 1\}$ | |
| $\bar{b}_j$ | $\inf\{x : G_j(x) = 1\}$ | |
| $b_{max}$ | $\max\{\bar{b}_1, \bar{b}_2\} \geq b_{min}$ | |
| $x_i(b)$ | $i$'s supremum bid below $b$ | Equation (17), page 50 |
| $b_i^*(\Gamma)$ | | Equation (16), page 50. |
| $v_j^{win}(b)$ | expected value given $j$ wins with bid $b$ | Equation (13), page 48. |
| $\alpha_1, \alpha_2$ | $\alpha_1 = \frac{Pr[L_1\|H_2]}{Pr[H_1\|H_2]}$, $\alpha_2 = \frac{Pr[L_2\|H_1]}{Pr[H_2\|H_1]}$ | |
| $T(b)$ | $\frac{\int_{b_{min}}^{b} \frac{1}{1-x} \cdot r(x)dx}{\int_{b_{min}}^{b} \frac{x}{1-x} \cdot r(x)dx}$ | |
| $\beta(b)$ | $\hat{R}(b_{min})/\hat{R}(b)$ | |