

Approximating Hereditary Discrepancy via Small Width Ellipsoids

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Abstract

The *Discrepancy* of a hypergraph is the minimum attainable value, over two-colorings of its vertices, of the maximum absolute imbalance of any hyperedge. The *Hereditary Discrepancy* of a hypergraph, defined as the maximum discrepancy of a restriction of the hypergraph to a subset of its vertices, is a measure of its complexity. Lovász, Spencer and Vesztergombi (1986) related the natural extension of this quantity to matrices to rounding algorithms for linear programs, and gave a determinant based lower bound on the hereditary discrepancy. Matoušek (2011) showed that this bound is tight up to a polylogarithmic factor, leaving open the question of actually computing this bound. Recent work by Nikolov, Talwar and Zhang (2013) showed a polynomial time $\tilde{O}(\log^3 n)$ -approximation to hereditary discrepancy, as a by-product of their work in differential privacy. In this paper, we give a direct simple $O(\log^{3/2} n)$ -approximation algorithm for this problem. We show that up to this approximation factor, the hereditary discrepancy of a matrix A is characterized by the optimal value of simple geometric convex program that seeks to minimize the largest ℓ_∞ norm of any point in an ellipsoid containing the columns of A . This characterization promises to be a useful tool in discrepancy theory.

1 Introduction

Discrepancy theory, in the broadest sense, studies the fundamental limits to approximating a “complex measure” (i.e. continuous, or with large support) with a “simple measure” (i.e. counting measure, or a measure with small support) with respect to a class of “distinguishers”. A prototypical “continuous discrepancy” question is how uniform can a set P of n points in the unit square $[0, 1]^2$ be, where uniformity is measured with respect to a class of geometric shapes, e.g. axis-aligned rectangles [Sch72]. A prototypical “discrete discrepancy” question asks whether we can color the n vertices of a hypergraph of $O(n)$ edges with two colors, red and blue, so that each edge has approximately the same number of red vertices as blue vertices [Spe85]. The two kinds of questions are deeply related, and transference theorems between different discrepancy measures are known [BS95].

Questions related to discrepancy theory are raised throughout mathematics, e.g. number theory, Diophantine approximation, numerical integration. Unsurprisingly, they also naturally appear in computer science – questions about the (im)possibility of approximating continuous, average objects with discrete ones are central to pseudorandomness, learning theory, communication complexity, approximation algorithms, among others. For a beautiful survey of applications of discrepancy theory to computer science, we refer the reader to Chazelle’s *The Discrepancy Method* [Cha00].

Despite discrepancy theory’s many applications in computer science, we have only recently began to understand the computational complexity of measures of discrepancy themselves. In this work, we address the problem of approximately computing hereditary discrepancy, one of the fundamental discrepancy measures. Hereditary discrepancy is a robust version of combinatorial discrepancy, which is the hypergraph coloring problem mentioned above. More precisely, the combinatorial discrepancy $\text{disc}(\mathcal{H})$ of a hypergraph $\mathcal{H} = (H_1, \dots, H_m)$ on the vertices $[n] = \{1, \dots, n\}$ is the minimum over colorings $\chi : [n] \rightarrow \{-1, 1\}$ of the maximum “imbalance” over hyperedges $\max_{i=1}^m |\sum_{j \in H_i} \chi(j)|$. While relatively simple, $\text{disc}(\mathcal{H})$ is a brittle quantity, which can make it intractable to estimate. We may wish to say that $\text{disc}(\mathcal{H})$ measures the complexity of \mathcal{H} , but it can be 0 for intuitively complex \mathcal{H} for trivial reasons. For example let (V, E) be a complex hypergraph all of whose sets have equal size, such as $\binom{[n]}{n/2}$. Consider the hypergraph formed by taking two identical copies of (V, E) , say (V_1, E_1) and (V_2, E_2) and defining the new hypergraph as $(V_1 \cup V_2, E' \triangleq \{e_1 \cup e_2 : e_1 \in E_1, e_2 \in E_2\})$. By coloring V_1 as $+1$ and V_2 as -1 , we get discrepancy zero for each edge in E' , despite the intuitive complexity of E . For this reason it is often more convenient to work with the more robust *hereditary discrepancy*. Hereditary discrepancy is the maximum discrepancy over restricted hypergraphs, i.e. $\text{herdisc}(\mathcal{H}) = \max_{W \subseteq [n]} \text{disc}(\mathcal{H}|_W)$, where $\mathcal{H}|_W = (H_1 \cap W, \dots, H_m \cap W)$. Notice, for example, that the hereditary discrepancy of the above example is in fact $\Omega(n)$ – a more fitting measure of the complexity of the hypergraph.

Discrepancy and hereditary discrepancy have natural generalizations to matrices. The discrepancy of a matrix A is equal to $\text{disc}(A) = \min_{x \in \{-1, 1\}^n} \|Ax\|_\infty$, and hereditary discrepancy is equal to $\max_{S \subseteq [n]} \text{disc}(A|_S)$, where $A|_S$ is the submatrix of A consisting of columns indexed by elements of S . These quantities coincide with hypergraph discrepancy when evaluated on the incidence matrix of the hypergraph, and are also natural themselves. For example, a classical result of Lovász, Spencer, and Vesztergombi [LSV86] states that for any matrix A , any vector $c \in [-1, 1]^n$ can be rounded to $x \in \{-1, 1\}^n$ so that $\|Ax - Ac\|_\infty \leq 2 \text{herdisc}(A)$. In the context of a linear program, this means that we can round fractional solutions to integral ones while still approximately satisfying the linear

constraints defined by A . This fact was recently used by Rothvoß to design an improved approximation algorithm for bin packing [Rot13].

The robustness of hereditary discrepancy in comparison with discrepancy is evident in the hardness of approximating each of the two measures. By an important result of Spencer [Spe85], whenever the number of edges in \mathcal{H} is $m = O(n)$, the discrepancy is at most $O(\sqrt{n})$. It turns out that it is **NP**-hard to distinguish between this worst-case upper bound and discrepancy zero [CNN11]. By contrast, recently Nikolov, Talwar, and Zhang gave a polylogarithmic approximation to hereditary discrepancy [NTZ13]. At first glance, this is surprising, because hereditary discrepancy is a maximum over exponentially many **NP**-hard minimization problems. Thus, hereditary discrepancy is not even obviously in **NP** (but is **NP**-hard to approximate within a factor of 2, see [AGH13]). However, the structure and robustness of hereditary discrepancy explain its more tractable nature. As one classical illustration to this, we note that the hypergraphs with hereditary discrepancy 1 are exactly the hypergraphs with totally unimodular incidence matrices [GH62], and are recognizable by a polynomial time algorithm [Sey80].

Our Results and Techniques. When approximating a function f , we need to provide (nearly matching) upper bounds and lower bounds on f . When we approximate **NP**-optimization problems, usually proving either the upper (for maximization problems) or the lower bound (for minimization problems) is relatively straightforward: it is given by a combinatorial lower (or upper) bound or a convex relaxation. The challenge is to design a bound which is nearly tight. On the other hand, in a max-min problem like hereditary discrepancy, both upper and lower bounds are challenging to prove. Nevertheless, a convex relaxation of *discrepancy* still turns out to be very useful. The relaxation, vector discrepancy, is derived by relaxing the condition on the coloring $\chi : [n] \rightarrow \{-1, 1\}$ to the weaker $\chi : [n] \rightarrow \mathbb{S}^{n-1}$, where \mathbb{S}^{n-1} is the unit sphere in \mathbb{R}^n . Then the vector discrepancy $\text{vecdisc}(\mathcal{H})$ is the minimum over such χ of $\max_{i=1}^m \|\sum_{j \in H_i} \chi(j)\|_2$. Similarly, the extension $\text{vecdisc}(A)$ to matrices A is the minimum over $\chi : [n] \rightarrow \mathbb{S}^{n-1}$ of $\max_{i=1}^m \|\sum_{j=1}^n A_{ij} \chi(j)\|_2$. These quantities can be efficiently approximated to within any prescribed accuracy by solving a semidefinite program. The *hereditary* vector discrepancy $\text{hvecdisc}(A)$ is defined analogously to hereditary discrepancy as the maximum vector discrepancy over submatrices. Clearly, $\text{vecdisc}(A) \leq \text{disc}(A)$ for any matrix A . There exist matrices A with $m = O(n)$ for which $\text{vecdisc}(A) = 0$ and $\text{disc}(A) = \Omega(\sqrt{n})^1$. Nevertheless, in a recent breakthrough, Bansal showed that an upper bound on hereditary vector discrepancy is useful in efficient discrepancy minimization.

Theorem 1 ([Ban10]). *Let A be an m by n matrix with $\text{hvecdisc}(A) \leq \lambda$. Then there exists a randomized polynomial time algorithm that computes $x \in \{-1, 1\}^n$ with discrepancy at most $\|Ax\|_\infty \leq O(\log m) \cdot \lambda$.*

Theorem 1 implies that the gap between hvecdisc and herdisc is at most logarithmic.

Corollary 2. *For any $m \times n$ matrix A*

$$\text{hvecdisc}(A) \leq \text{herdisc}(A) \leq O(\log m) \text{hvecdisc}(A).$$

While $\text{vecdisc}(A)$ can be approximated to within any degree of accuracy in polynomial time, it is not clear if $\text{hvecdisc}(A)$ can be computed efficiently: notice that hereditary vector discrepancy is the maximum of the objective functions of an exponential number of convex

¹This is the case, for example, for the matrix which contains three copies of each column of a Hadamard matrix.

minimization problems. Nevertheless, by using vector discrepancy we remove one of the two quantifiers over exponentially large sets.

In this paper we prove the following approximation result for hvecdisc .

Theorem 3. *There exists a polynomial time algorithm that approximates $\text{hvecdisc}(A)$ within a factor of $O(\log m)$ for any $m \times n$ matrix A . Moreover, the algorithm finds a submatrix $A|_S$ of A , such that $\text{hvecdisc}(A) = O(\log m) \text{vecdisc}(A|_S)$.*

Theorem 3 follows from a geometric characterization of hereditary vector discrepancy. We show that, up to a factor of $O(\log m)$, $\text{hvecdisc}(A)$ is equal to the smallest value of $\|E\|_\infty$ over all ellipsoids that contain the columns of A . Here, $\|E\|_\infty$ is just the maximum ℓ_∞^m norm of all points in E , or, equivalently, the maximum width of E in the directions of the standard basis vectors e_1, \dots, e_m . *A priori*, it is not clear how to relate this quantity in either direction to the $\text{hvecdisc}(A)$, as it is not a fractional “relaxation” in the traditional sense. It is in fact non-trivial to prove either of the two inequalities relating the geometric quantity to $\text{hvecdisc}(A)$.

Proving that this quantity is an upper bound on hereditary discrepancy relies on a recent result of Nikolov that upper bounds the vector discrepancy of matrices with columns bounded in Euclidean norm by 1 [Nik13]. We need a slight generalization of Nikolov’s result that shows that the vector discrepancy of such matrices can be bounded by 1 *in any direction*. We then transform linearly the containing ellipsoid E to a unit ball, so that Nikolov’s result applies; because of the transformation, we need to make sure that in the transformed space the vector discrepancy is low in a set of directions different from the standard basis. While, on the face of things, this argument only upper bounds the vector discrepancy of A , it in fact also upper bounds the vector discrepancy of *any submatrix* as well, because if E contains all columns of A , it also contains all the columns of any submatrix of A . This simple observation is crucial to the success of our arguments.

To show that the smallest value of $\|E\|_\infty$ over all containing ellipsoids also gives a lower bound on hereditary vector discrepancy, we analyze the *convex dual* of the problem of finding containing ellipsoids of small width and show that we can transform dual certificates for this problem to dual certificates for vector discrepancy of some submatrix of A . The dual of the problem of minimizing $\|E\|_\infty$ for a matrix A is a problem of maximizing the *nuclear norm* (i.e. the sum of singular values) over re-weightings of the columns and rows of A . To get dual certificates for vector discrepancy for some submatrix, we need to be able to extract a submatrix with a large least singular value from a matrix of large nuclear norm. We accomplish this using the *restricted invertibility principle* of Bourgain and Tzafriri [BT87]: a powerful theorem from functional analysis which states, roughly, that any submatrix with many approximately equal singular values contains a large well-conditioned submatrix. Using a constructive proof of the theorem by Spielman and Srivastava [SS10], we can also find the well-conditioned submatrix in deterministic polynomial time; this gives us a submatrix of A on which hereditary vector discrepancy is approximately maximized.

Theorem 3 immediately implies a $O(\log^2 m)$ approximation of herdisc via Bansal’s theorem. However, we can improve this bound to an $O(\log^{3/2} m)$ approximation.

Theorem 4. *There exists a polynomial time algorithm that approximates $\text{herdisc}(A)$ within a factor of $O(\log^{3/2} m)$ for any $m \times n$ matrix A . Moreover, the algorithm finds a submatrix $A|_S$ of A , such that $\text{herdisc}(A) \leq O(\log^{3/2} m) \text{vecdisc}(A|_S)$.*

To prove Theorem 4, we lower bound hereditary vector discrepancy as before, in order to lower bound hereditary discrepancy. However, for the upper bound, rather than upper

bounding vector discrepancy in terms of $\|E\|_\infty$ for a containing ellipsoid, and then upper bounding discrepancy in terms of vector discrepancy, we directly upper bound discrepancy in terms of $\|E\|_\infty$. For this purpose, we use another discrepancy bound – this time a theorem due to Banaszczyk [Ban98] that shows that for any convex body K of large Gaussian volume, and a matrix A with columns of at most unit Euclidean norm, there exists a $x \in \{-1, 1\}^n$ such that $Ax \in CK$ for a constant C . We use this theorem analogously to the way we used Nikolov’s theorem: we linearly transform E to the unit ball, and we specify a body K such that if some ± 1 combination of the columns of A is in K after the transformation, then in the preimage the combination is in an infinity ball scaled by $O(\sqrt{\log m})$.

After a preliminary version of this paper was made available, Matoušek [Mat14] has shown that our analysis of both the upper and the lower bound on $\text{herdisc}(A)$ in terms of the minimum of $\|E\|_\infty$ over containing ellipsoids E is tight. He also used our characterization of $\text{herdisc}(A)$ in terms of the minimum of $\|E\|_\infty$ and our analysis of the dual to give new proofs of classical and new results in discrepancy theory.

Comparison with Related Works. Lovász, Spencer and Vesztergombi [LSV86] defined a determinant based lower bound on the hereditary discrepancy of a matrix. Matoušek [Mat13] showed that this lower bound is tight up to $O(\log^{3/2} m)$. These results did not immediately yield an approximation algorithm for hereditary discrepancy, as the determinant lower bound is a maximum over exponentially many quantities and not known to be efficiently computable.

Nikolov, Talwar and Zhang [NTZ13] recently studied hereditary discrepancy as a tool for designing near optimal differentially private mechanisms for linear queries, and as a by-product, derived an $\tilde{O}(\log^3 n)$ -approximation algorithm for hereditary discrepancy, where the \tilde{O} notation hides sub-logarithmic factors. Small width containing ellipsoids were implicit in their work. The current paper is the first that explicitly considers this natural geometric object in the context of discrepancy. While the proof of the upper bound on discrepancy in [NTZ13] was via a connection between discrepancy and differential privacy due to Nikolov and Muthukrishnan [MN12], here we give a *tight* and *direct* argument using results of Nikolov [Nik13] and Banaszczyk [Ban98] on the Komlós problem. The arguments via differential privacy cannot give the tight relationship between the minimum width of a containing ellipsoid and hereditary discrepancy: they necessarily lose a logarithmic factor, because the relationship between discrepancy and privacy is itself not tight. Moreover, our arguments are simpler and more transparent. The proof of our lower bound is also more natural: we relate the duals of the two convex optimization problems under consideration, i.e. the problem of minimizing vector discrepancy, and the problem of minimizing the width of a containing ellipsoid. Via this approach we arrive at a new discrepancy lower bound, which is at least as strong as the determinant lower bound (up to a logarithmic factor), is tight with respect to hereditary discrepancy up to the same asymptotic factor of $O(\log^{3/2} m)$, and is efficiently computable. We believe our lower bound will have future applications in discrepancy theory.

A separate but related line of recent works [Ban10, LM12, Rot14] gives constructive versions of existence proofs in discrepancy theory.

2 Preliminaries

We start by introducing some basic notation.

For a $m \times n$ matrix A and a set $S \subseteq [n]$, we denote by $A|_S$ the submatrix of A consisting of those columns of A indexed by elements of S . \mathcal{P}_k is the set of orthogonal projection matrices

onto k -dimensional subspaces of \mathbb{R}^m . We use $\text{range}(A)$ for the range, i.e. the span of the columns, of A . By $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$ we denote, respectively, the smallest and largest singular value of A . I.e., $\sigma_{\min}(A) = \min_{x: \|x\|_2=1} \|Ax\|_2$ and $\sigma_{\max}(A) = \max_{x: \|x\|_2=1} \|Ax\|_2$. In general, we use σ_i for the i -th largest singular value of A .

By $X \succeq 0$ ($X \succ 0$) we denote that X is a positive semidefinite (resp. positive definite) matrix, and by $X \preceq Y$ that $Y - X \succeq 0$.

Recall that for a block matrix

$$X = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

the *Schur complement* of an invertible block C in X is $A - B^T C^{-1} B$. When $C \succ 0$, $X \succeq 0$ if and only if $A - B^T C^{-1} B \succeq 0$.

For a positive semidefinite (PSD) matrix $X \succeq 0$, we denote by $X^{1/2}$ the principal square root of X , i.e. the matrix $Y \succeq 0$ such that $Y^2 = X$.

2.1 Matrix Norms and Restricted Invertibility

The Schatten 1-norm of a matrix A , also known as the trace norm or the nuclear norm, is equal to $\|A\|_{S_1} = \sum_i \sigma_i(A) = \text{tr}((AA^T)^{1/2})$.

For a matrix A , we denote by $\|A\|_2 = \sigma_{\max}(A)$ the spectral norm of A and $\|A\|_{HS} = \sqrt{\sum_i \sigma_i^2(A)} = \sqrt{\sum_{i,j} a_{i,j}^2}$ the Hilbert-Schmidt (or Frobenius) norm of A . We use $\|A\|_{1 \rightarrow 2}$ for the maximum Euclidean length of the columns of the matrix $A = (a_i)_{i=1}^n$, i.e. $\|A\|_{1 \rightarrow 2} = \max_{x: \|x\|_1=1} \|Ax\|_2 = \max_{i \in [n]} \|A_i\|_2$.

A matrix A trivially contains an invertible submatrix of k columns as long as $k \leq \text{rank}(A)$. An important result of Bourgain and Tzafriri [BT87] (later strengthened by Vershynin [Ver01], and Spielman and Srivastava [SS10]) shows that when k is strictly less than the robust rank $\|A\|_{HS}^2 / \|A\|_2^2$ of A , we can find k columns of A that form a *well-invertible* submatrix. This result is usually called the *restricted invertibility principle*. Next we state a weighted version of it, which can be proved by slightly modifying the proof of Spielman and Srivastava [SS10]. Rather than describe the modification, we give a reduction of the weighted version to the standard statement in Appendix A.

Theorem 5. *Let $\epsilon > 0$, let A be an m by n real matrix, and let Q be a diagonal matrix such that $Q \succeq 0$ and $\text{tr}(Q) = 1$. For any integer k such that $k \leq \epsilon^2 \frac{\|AQ^{1/2}\|_{HS}^2}{\|AQ^{1/2}\|_2^2}$ there exists a subset $S \subseteq [n]$ of size $|S| = k$ such that $\sigma_{\min}(A|_S)^2 \geq (1 - \epsilon)^2 \|AQ^{1/2}\|_{HS}^2$. Moreover, S can be computed in deterministic polynomial time.*

2.2 Geometry

Let $\text{conv}\{a_1, \dots, a_n\}$ be the convex hull of the vectors a_1, \dots, a_n .

A *convex body* is a convex compact subset of \mathbb{R}^m . For a convex body $K \subseteq \mathbb{R}^m$, the *polar body* K° is defined by $K^\circ = \{y : \langle y, x \rangle \leq 1 \ \forall x \in K\}$. A basic fact about polar bodies is that for any two convex bodies K and L , $K \subseteq L \Leftrightarrow L^\circ \subseteq K^\circ$. Moreover, a symmetric convex body K and its polar body are dual to each other, in the sense that $(K^\circ)^\circ = K$.

A convex body K is (*centrally*) *symmetric* if $-K = K$. The *Minkowski norm* $\|x\|_K$ induced by a symmetric convex body K is defined as $\|x\|_K \triangleq \min\{r \in \mathbb{R} : x \in rK\}$. The Minkowski norm induced by the polar body K° of K is the *dual norm* of $\|x\|_K$ and also has the form $\|y\|_{K^\circ} = \max_{x \in K} \langle x, y \rangle$. It follows that we can also write $\|x\|_K$ as $\|x\|_K = \max_{y \in K^\circ} \langle x, y \rangle$. For a vector y of unit Euclidean length, $\|y\|_{K^\circ}$ is the *width* of K in

the direction of y , i.e. half the Euclidean distance between the two supporting hyperplanes of K orthogonal to y . For symmetric body K , we denote by $\|K\| = \max_{x \in K} \|x\|$ the diameter of K under the norm $\|\cdot\|$.

Of special interest are the ℓ_p^m norms, defined for any $p \geq 1$ and any $x \in \mathbb{R}^m$ by $\|x\|_p = (\sum_{i=1}^m |x_i|^p)^{1/p}$. The ℓ_∞^m norm is defined for as $\|x\|_\infty = \max_{i=1}^m |x_i|$. The norms ℓ_p^m and ℓ_q^m are dual if and only if $\frac{1}{p} + \frac{1}{q} = 1$, and ℓ_1^m is dual to ℓ_∞^m . We denote the unit ball of the ℓ_p^m norm by $B_p^m = \{x : \|x\|_p \leq 1\}$. As with the unit ball of any norm, B_p^m is convex and centrally symmetric for $p \in [1, \infty]$.

An *ellipsoid* in \mathbb{R}^m is the image of the ball B_2^m under an affine map. All ellipsoids we consider are symmetric, and therefore, are equal to an image FB_2^m of the ball B_2^m under a linear map F . A full dimensional ellipsoid $E = FB_2^d$ can be equivalently defined as $E = \{x : x^T (FF^T)^{-1} x \leq 1\}$. The polar body of a symmetric ellipsoid $E = FB_2^d$ is the ellipsoid $E^\circ = \{x : x^T FF^T x \leq 1\}$. It follows that for $E = FB_2^m$ and for any x , $\|x\|_E = \sqrt{x^T (FF^T)^{-1} x}$ and for any y , $\|y\|_{E^\circ} = \sqrt{y^T (FF^T) y}$.

2.3 Convex Duality

Assume we are given the following optimization problem:

$$\text{Minimize } f_0(x) \tag{1}$$

s.t.

$$\forall 1 \leq i \leq m : f_i(x) \leq 0. \tag{2}$$

The Lagrange dual function associated with (1)–(2) is defined as $g(y) = \inf_x f_0(x) + \sum_{i=1}^m y_i f_i(x)$, where the infimum is over the intersection of the domains of f_1, \dots, f_m , and $y \in \mathbb{R}^m$, $y \geq 0$. Since $g(y)$ is the infimum of affine functions, it is a concave function. Moreover, g is upper semi-continuous, and therefore continuous over the convex set $\{y : g(y) > -\infty\}$.

For any x which is feasible for (1)–(2), and any $y \geq 0$, $g(y) \leq f_0(x)$. This fact is known as *weak duality*. The *Lagrange dual problem* is defined as

$$\text{Maximize } g(y) \text{ s.t. } y \geq 0. \tag{3}$$

Strong duality holds when the optimal value of (3) equals the optimal value of (1)–(2). Slater's condition is a commonly used sufficient condition for strong duality. We state it next.

Theorem 6 (Slater's Condition). *Assume f_0, \dots, f_m in the problem (1)–(2) are convex functions over their respective domains, and for some $k \geq 0$, f_1, \dots, f_k are affine functions. Let there be a point x in the relative interior of the domains of f_0, \dots, f_m , so that $f_i(x) \leq 0$ for $1 \leq i \leq k$ and $f_j(x) < 0$ for $k+1 \leq j \leq m$. Then the minimum of (1)–(2) equals the maximum of (3), and the maximum of (3) is achieved if it is finite.*

For more information on convex programming and duality, we refer the reader to the book by Boyd and Vandenberghe.

2.4 Hereditary Discrepancy and Relaxations

For a $m \times n$ matrix A , discrepancy and hereditary discrepancy are defined as

$$\text{disc}(A) \triangleq \min_{x \in \{\pm 1\}^n} \|Ax\|_\infty \qquad \text{herdisc}(A) \triangleq \max_{S \subseteq [n]} \text{disc}(A|_S).$$

Vector discrepancy is a convex relaxation of discrepancy in which one can assign arbitrary unit vectors rather than ± 1 to the columns of A . Formally, let \mathbb{S}^{n-1} be the unit sphere in \mathbb{R}^n , and define

$$\text{vecdisc}(A) \triangleq \min_{u_1, \dots, u_n \in \mathbb{S}^{n-1}} \max_{i=1}^m \left\| \sum_{j=1}^n A_{ij} u_j \right\|_2$$

From a computational complexity perspective, the important property of vector discrepancy $\text{vecdisc}(A)$ is that it can be approximated to within an arbitrarily small additive constant ϵ in time polynomial in $\log \frac{1}{\epsilon}$ by (approximately) solving a semidefinite program. The following lower bound on vector discrepancy follows by weak convex duality for semidefinite programming (see [Mat13]).

Lemma 7. *For any $m \times n$ matrix A , and any $m \times m$ diagonal matrix $P \succeq 0$ with $\text{tr}(P) = 1$, we have $\text{vecdisc}(A) \geq \sqrt{n} \sigma_{\min}(P^{1/2} A)$.*

We define a *spectral lower bound* based on Lemma 7.

$$\text{specLB}(A) \triangleq \max_{k=1}^n \max_{S \subseteq [n]: |S|=k} \max_P \sqrt{k} \sigma_{\min}(P^{1/2} A|_S),$$

where P ranges over positive (i.e. $P \succeq 0$) $m \times m$ diagonal matrices satisfying $\text{tr}(P) = 1$. Lemma 7 implies immediately that $\text{hvecdisc}(A) \geq \text{specLB}(A)$.

Notice that it is not clear whether $\text{specLB}(A)$ can be computed efficiently. One of our main contributions is to develop a lower bound on hereditary (vector) discrepancy which is tractable and can be related to $\text{specLB}(A)$.

2.5 Vector Balancing and Banaszczyk's Theorem

A well-known conjecture by Komlós states that $\text{disc}(A) \leq C \|A\|_{1 \rightarrow 2}$, for an absolute constant C . The conjecture remains open, and the best known result towards resolving it is a discrepancy bound of $O(\sqrt{\log n} \|A\|_{1 \rightarrow 2})$ due to Banaszczyk [Ban98]. Banaszczyk's result in fact concerns the more general vector balancing problem of determining sufficient conditions under which there exists an assignment of signs $x \in \{\pm 1\}^n$ such that $Ax \in K$ for given $m \times n$ matrix A and convex body K . This general version of Banaszczyk's result is crucial to our argument.

Theorem 8 ([Ban98]). *There exists a universal constant C such that the following holds. Let A be an m by n real matrix such that $\|A\|_{1 \rightarrow 2} \leq 1$, and let K be a convex body in \mathbb{R}^m such that $\Pr[g \in K] \geq 1/2$ where $g \in \mathbb{R}^m$ is a standard m -dimensional Gaussian random vector, and the probability is taken over the choice of g . Then there exists $x \in \{-1, 1\}^n$ such that $Ax \in CK$.*

Another partial result towards the Komlós conjecture is a recent bound by Nikolov [Nik13] on the vector discrepancy of matrices A satisfying the condition $\|A\|_{1 \rightarrow 2} \leq 1$. Here we state a version of the bound that is stronger than the one stated in [Nik13]. However, a minor variation of the same proof shows this stronger bound; we give the argument in Appendix B.

Theorem 9 ([Nik13]). *For any $m \times n$ matrix A satisfying $\|A\|_{1 \rightarrow 2} \leq 1$, there exists a $n \times n$ matrix $X \succeq 0$ such that $\forall i \in [n]: X_{jj} = 1$ and $\|AXA^T\|_2 \leq 1$.*

Note that the above theorem implies $\text{vecdisc}(A) \leq 1$ because $\|AXA^T\|_2 \leq 1$ implies $e_i^T (AXA^T) e_i \leq 1$ for all standard basis vectors e_i . However, the spectral norm bound is formally stronger than the vector discrepancy upper bound, and this will be essential in our proofs.

3 Ellipsoid Upper Bounds on Discrepancy

In this section we show that small-width ellipsoids provide upper bounds on both hereditary vector discrepancy and hereditary discrepancy. Giving such an upper bound is in general challenging because it must hold for all submatrices simultaneously. The proofs use Theorems 8 and 9. We start with the two main technical lemmas.

Lemma 10. *Let $A = (a_j)_{j=1}^n \in \mathbb{R}^{m \times n}$, and let $F \in \mathbb{R}^{m \times m}$ be a rank m matrix such that $\forall j \in [n] : a_j \in E = FB_2^m$. Then there exists a matrix $X \succeq 0$ such that $\forall j \in [n] : X_{jj} = 1$ and $AXA^T \preceq FF^T$.*

Proof. Observe that, $a_j \in E \Leftrightarrow F^{-1}a_j \in B_2^m$. This implies $\|F^{-1}A\|_{1 \rightarrow 2} \leq 1$, and, by Theorem 9, there exists an X with $X_{jj} = 1$ for all j such that $(F^{-1}A)X(F^{-1}A)^T \preceq I$. Multiplying on the left by F and on the right by F^T , we have $AXA^T \preceq FF^T$, and this completes the proof. \square

Lemma 10 is our main tool for approximating hereditary vector discrepancy. By the relationship between vector discrepancy and discrepancy established by Bansal (Corollary 2), this is sufficient for a poly-logarithmic approximation to hereditary discrepancy. However, to get tight upper bounds on discrepancy from small width ellipsoids (and improved approximation ratio), we give a direct argument using Banaszczyk's theorem.

Lemma 11. *Let $A = (a_j)_{j=1}^n \in \mathbb{R}^{m \times n}$, and let $F \in \mathbb{R}^{m \times m}$ be a rank m matrix such that $\forall j \in [n] : a_j \in E = FB_2^m$. Then, for any set of vectors $v_1, \dots, v_k \in \mathbb{R}^m$, there exists $x \in \{\pm 1\}^n$ such that $\forall i \in [k] : |\langle Ax, v_i \rangle| \leq C\sqrt{(v_i^T FF^T v_i) \log k}$ for a universal constant C .*

Proof. Let $P = \{y : |\langle y, v_i \rangle| \leq \sqrt{v_i^T FF^T v_i} \forall i \in [k]\}$. We need to prove that there exists an $x \in \{-1, 1\}^n$ such that $Ax \in (C\sqrt{\log k})P$ for a suitable constant C . Notice that the polar body of P is

$$P^\circ = \text{conv}\{(v_i^T FF^T v_i)^{-1/2} v_i\}_{i=1}^k.$$

Set $K = F^{-1}P$. To show that there exists an x such that $Ax \in (C\sqrt{\log k})P$, we will show that there exists an $x \in \{-1, 1\}^n$ such that $F^{-1}Ax \in (C\sqrt{\log k})K$. For this, we will use Theorem 8. As in the proof of Lemma 10, $\|F^{-1}A\|_{1 \rightarrow 2} \leq 1$. To use Theorem 8, we also need to argue that for a standard Gaussian g , $\Pr[g \in (C\sqrt{\log k})K] \geq \frac{1}{2}$. To this end, we compute the polar body of K as

$$\begin{aligned} K^\circ &= \{y : \langle y, F^{-1}x \rangle \leq 1 \forall x \in P\} = \{y : \langle (F^T)^{-1}y, x \rangle \leq 1 \forall x \in P\} \\ &= \{F^T z : \langle z, x \rangle \leq 1 \forall x \in P\} = F^T P^\circ \end{aligned}$$

By the definition of Minkowski norm, for any $t \in \mathbb{R}$, $y \in tK$, if and only if

$$t \geq \|y\|_K = \sup_{z \in K^\circ} \langle y, z \rangle = \sup_{z \in P^\circ} \langle y, F^T z \rangle = \max_{i=1}^k \frac{1}{\sqrt{v_i^T FF^T v_i}} |\langle y, F^T v_i \rangle|,$$

where the first equality is by the duality of $\|\cdot\|_K$ and $\|\cdot\|_{K^\circ}$, and the final equality holds because the linear functional $\langle y, F^T z \rangle$ is maximized at a vertex of P° . We have then that $y \in tK$ if and only if $\forall i \in [k] : |\langle y, F^T v_i \rangle|^2 \leq t^2 (v_i^T FF^T v_i)$. Let g be a standard m -dimensional Gaussian vector. Then $\mathbb{E}_g |\langle g, F^T v_i \rangle|^2 = v_i^T FF^T v_i$; by standard concentration bounds, $\Pr[|\langle g, F^T v_i \rangle|^2 > t^2 (v_i^T FF^T v_i)] < \exp(-t^2/2)$. Setting $t = \sqrt{2 \ln 2k}$ and taking a union bound over all $i \in [k]$ gives us that $\Pr[g \notin \sqrt{2 \ln 2k} K] < 1/2$. By Theorem 8, this implies that there exists an $x \in \{-1, 1\}^n$ such that $F^{-1}Ax \in \sqrt{2 \ln 2k} K$, and, by multiplying on both sides by F , it follows that $Ax \in \sqrt{2 \ln 2k} P$. \square

Notice that the quantity $\sqrt{v_i^T F F^T v_i}$ is just the width of E in the direction of v_i . The property that all columns of a matrix A are contained in E is *hereditary*: if it is satisfied for A , then it is satisfied for any submatrix of A . This elementary fact lends the power of Lemmas 10 and 11: the bound given by ellipsoids is *universal* in the sense that the discrepancy bound for any direction v_i holds for all submatrices $A|_S$ of A simultaneously. This fact makes it possible to upper bound hereditary discrepancy in arbitrary norms, and in the sequel we do this for ℓ_∞^m , which is the norm of interest for standard definitions of discrepancy. We consider ellipsoids E that contain the columns of A and minimize the quantity $\|E\|_\infty$: the largest ℓ_∞ norm of the points of E . Note that $\|E\|_\infty$, for an ellipsoid $E = FB_2^m$, can be written as

$$\|E\|_\infty = \max_{x \in E, y: \|y\|_1=1} \langle x, y \rangle = \max_{y: \|y\|_1=1} \|y\|_{E^\circ} = \max_{i \in [n]} \sqrt{e_i^T F F^T e_i}, \quad (4)$$

where the first identity follows since ℓ_1 is the dual norm to ℓ_∞ , and the final identity follows from the formula for $\|\cdot\|_{E^\circ}$ and the fact that a convex function over the ℓ_1 ball is always maximized at a vertex, i.e. a standard basis vector. The next theorem gives our main upper bound on hereditary (vector) discrepancy, which is in terms of $\|E\|_\infty$.

Theorem 12. *Let $A = (a_i)_{i=1}^n \in \mathbb{R}^{m \times n}$, and let F be a rank m matrix such that $\forall i \in [n]: a_i \in E = FB_2^m$. Then $\text{hvecdisc}(A) \leq \|E\|_\infty$, and $\text{herdisc}(A) = O(\sqrt{\log m})\|E\|_\infty$.*

Proof. Let $A|_S$ be an arbitrary submatrix of A ($S \subseteq [n]$). Since all columns of A are contained in E , this holds for all columns of $A|_S$ as well, and by Lemma 10, we have that there exists $X \succeq 0$ with $X_{jj} = 1$ for all $j \in S$, and $(A|_S)X(A|_S)^T \preceq FF^T$. Therefore, for all $i \in [m]$, $e_i^T (A|_S)X(A|_S)^T e_i \leq e_i^T FF^T e_i \leq \|E\|_\infty^2$, by (4). Since S was arbitrary, this implies the bound on $\text{hvecdisc}(A)$. For bounding $\text{herdisc}(A)$, in Lemma 11, set $k = m$ and $v_i = e_i$ for $i \in [m]$ and e_i the i -th standard basis vector. \square

4 Lower Bounds on Discrepancy

In Section 3 we showed that the hereditary (vector) discrepancy of a matrix A can be *upper bounded* in terms of the $\|E\|_\infty$ for any E containing the columns of A . In this section we analyze the properties of the minimal such ellipsoid and show that it provides lower bounds for discrepancy as well. We use convex duality and the restricted invertibility theorem for this purpose. The lower bound we derive is new in discrepancy theory and of independent interest.

4.1 The Ellipsoid Minimization Problem and Its Dual

To formulate the problem of minimizing $\|E\|_\infty = \max_{x \in E} \|x\|_\infty$ as a convex optimization problem we need the following well-known lemma, which shows that the matrix inverse is convex in the PSD sense.

Lemma 13. *For any two $m \times m$ matrices $X \succ 0$ and $Y \succ 0$, $(\frac{1}{2}X + \frac{1}{2}Y)^{-1} \preceq \frac{1}{2}X^{-1} + \frac{1}{2}Y^{-1}$.*

Proof. Define the matrices

$$U = \begin{pmatrix} X^{-1} & I \\ I & X \end{pmatrix} \quad V = \begin{pmatrix} Y^{-1} & I \\ I & Y \end{pmatrix}.$$

The Schur complement of X in U is 0, and therefore $U \succeq 0$, and analogously $V \succeq 0$. Therefore $U + V \succeq 0$, and the Schur complement of $X + Y$ in $U + V$ is also positive

semidefinite, i.e. $X^{-1} + Y^{-1} - 4(X + Y)^{-1} \succeq 0$. This completes the proof, after re-arranging terms. \square

Consider a matrix $A = (a_j)_{j=1}^n \in \mathbb{R}^{m \times n}$ of rank m . Let us formulate $\min\{\|E\|_\infty : \forall j \in [n] : a_j \in E\}$ as a convex minimization problem. The problem is defined as follows

$$\text{Minimize } t \text{ s.t.} \quad (5)$$

$$X \succ 0 \quad (6)$$

$$\forall i \in [m] : e_i^T X^{-1} e_i \leq t \quad (7)$$

$$\forall j \in [n] : a_j^T X a_j \leq 1. \quad (8)$$

Lemma 14. *For a rank m matrix $A = (a_j)_{j=1}^n \in \mathbb{R}^{m \times n}$, the optimal value of the optimization problem (5)–(8) is equal to $\min\{\|E\|_\infty^2 : \forall j \in [n] : a_j \in E\}$. Moreover, the objective function (5) and constraints (7)–(8) are convex over $t \in \mathbb{R}$ and $X \succ 0$.*

Proof. Let λ be the optimal value of (5)–(8) and $\mu = \min\{\|E\|_\infty : \forall j \in [n] : a_j \in E\}$. Given a feasible X for (5)–(8), set $E = X^{-1/2} B_2^m$ (this is well-defined since $X \succ 0$). Then for any $j \in [n]$, $\|a_j\|_E = a_j^T X a_j \leq 1$ by (8), and, therefore, $a_j \in E$. Also, by (4), $\|E\|_\infty^2 = \max_{i=1}^m e_i^T X e_i \leq t$. This shows that $\mu \leq \lambda$. In the reverse direction, let $E = F B_2^m$ be such that $\forall j \in [n] : a_j \in E$. Then, because A is full rank, F is also full rank and invertible, and we can define $X = (F F^T)^{-1}$ and $t = \|E\|_\infty^2$. Analogously to the calculations above, we can show that X and t are feasible, and therefore $\lambda \leq \mu$.

The objective function and the constraints (8) are affine, and therefore convex. To show (7) are also convex, let X_1, t_1 and X_2, t_2 be two feasible solutions. Then, Lemma 13 implies that for any i , $e_i^T (\frac{1}{2} X_1 + \frac{1}{2} X_2)^{-1} e_i \leq \frac{1}{2} X_1^{-1} + \frac{1}{2} X_2^{-1} \leq \frac{1}{2} t_1 + \frac{1}{2} t_2$, and constraints (7) are convex as well. \square

Theorem 15. *Let $A = (a_j)_{j=1}^n \in \mathbb{R}^{m \times n}$ be a rank m matrix, and let $\mu = \min\{\|E\|_\infty : \forall j \in [n] : a_j \in E\}$. Then,*

$$\mu^2 = \max \|P^{1/2} A Q^{1/2}\|_{S_1}^2 \text{ s.t.} \quad (9)$$

$$\text{tr}(P) = \text{tr}(Q) = 1 \quad (10)$$

$$P, Q \succeq 0; P, Q \text{ diagonal.} \quad (11)$$

Proof. We shall prove the theorem by showing that the convex optimization problem (5)–(8) satisfies Slater's condition, and its Lagrange dual is equivalent to (9)–(11). Let us first verify Slater's condition. We define the domain for constraints (7) as the open cone $\{X : X \succ 0\}$, which makes the constraint $X \succ 0$ implicit. Let $r = \|A\|_{1 \rightarrow 2}$, $X = \frac{1}{r} I$, and $t = r + \varepsilon$ for some $\varepsilon > 0$. Then the affine constraints (8) are satisfied exactly, and the constraints (7) are satisfied with slack since $\varepsilon > 0$. Moreover, by Lemma 14, all the constraints and the objective function are convex. Therefore, (5)–(8) satisfies Slater's condition, and consequently strong duality holds.

The Lagrange dual function for (5)–(8) is

$$g(p, q) = \inf_{t, X \succ 0} t + \sum_{i=1}^m p_i (e_i^T X^{-1} e_i - t) + \sum_{j=1}^n q_j (a_j^T X a_j - 1),$$

with dual variables $p \in \mathbb{R}^m$ and $q \in \mathbb{R}^n$, $p, q \geq 0$. Equivalently, writing p as a diagonal matrix $P \in \mathbb{R}^{m \times m}$, $P \succeq 0$, q as a diagonal matrix $R \in \mathbb{R}^{n \times n}$, $R \succeq 0$, we have $g(P, R) =$

$\inf_{t, X \succ 0} t + \text{tr}(PX^{-1}) - \text{tr}(tP) + \text{tr}(ARA^T X) - \text{tr}(R)$. If $\text{tr}(P) \neq 1$, then $g(P, R) = -\infty$, since we can take t to $-\infty$ while keeping X fixed. On the other hand, for $\text{tr}(P) = 1$, the dual function simplifies to

$$g(P, R) = \inf_{X \succ 0} \text{tr}(PX^{-1}) + \text{tr}(ARA^T X) - \text{tr}(R). \quad (12)$$

Since $X \succ 0$ implies $X^{-1} \succ 0$, $g(P, R) \geq -\text{tr}(R) > -\infty$ whenever $\text{tr}(P) = 1$. Therefore, $g(P, R)$ is continuous over the set of diagonal positive semidefinite P, R such that $\text{tr}(P) = 1$. For the rest of the proof we assume that P and ARA^T are rank m . This is without loss of generality by the continuity of g and because both assumptions can be satisfied by adding arbitrarily small perturbations to P and R . (Here we use the fact that A is rank m .)

After differentiating the right hand side of (12) with respect to X , we get the first-order optimality condition

$$X^{-1}PX^{-1} = ARA^T. \quad (13)$$

Multiplying by $P^{1/2}$ on the left and the right and taking square roots gives the equivalent condition $P^{1/2}X^{-1}P^{1/2} = (P^{1/2}ARA^TP^{1/2})^{1/2}$. This equation has a unique solution, since P and ARA^T were both assumed to be invertible. Since $\text{tr}(PX^{-1}) = \text{tr}(P^{1/2}X^{-1}P^{1/2})$ and also, by (13), $\text{tr}(ARA^T X) = \text{tr}(X^{-1}P) = \text{tr}(PX^{-1})$, we simplify $g(P, R)$ to

$$g(P, R) = 2\text{tr}((P^{1/2}ARA^TP^{1/2})^{1/2}) - \text{tr}(R) = 2\|P^{1/2}AR^{1/2}\|_{S_1} - \text{tr}(R). \quad (14)$$

We showed that (5)–(8) satisfies Slater's condition and therefore strong duality holds, so by Theorem 6 and Lemma 14, $\mu^2 = \max\{g(P, R) : \text{tr}(P) = 1, P, R \succeq 0, \text{ diagonal}\}$. Let us define new variables Q and c , where $c = \text{tr}(R)$ and $Q = R/c$. Then we can re-write $g(P, R)$ as

$$g(P, R) = g(P, Q, c) = 2\|P^{1/2}A(cQ)^{1/2}\|_{S_1} - \text{tr}(cQ) = 2\sqrt{c}\|P^{1/2}AQ^{1/2}\|_{S_1} - c.$$

From the first-order optimality condition $\frac{dg}{dc} = 0$, we see that maximum of $g(P, Q, c)$ is achieved when $c = \|P^{1/2}AQ^{1/2}\|_{S_1}^2$ and is equal to $\|P^{1/2}AQ^{1/2}\|_{S_1}^2$. Therefore, maximizing $g(P, R)$ over diagonal positive semidefinite P and R such that $\text{tr}(P) = 1$ is equivalent to the optimization problem (9)–(11). This completes the proof. \square

4.2 Spectral Lower Bounds via Restricted Invertibility

In this subsection we relate the dual formulations of the min-ellipsoid problem from Section 4.1 to the dual of vector discrepancy, and specLB in particular. The connection is via the restricted invertibility principle and gives our main lower bounds on hereditary (vector) discrepancy.

Lemma 16. *Let A be an m by n real matrix, and let $Q \succeq 0$ be a diagonal matrix such that $\text{tr}(Q) = 1$. Then there exists a submatrix $A|_S$ of A such that $|S|\sigma_{\min}(A|_S)^2 \geq \frac{c^2\|AQ^{1/2}\|_{S_1}^2}{(\log m)^2}$, for a universal constant $c > 0$. Moreover, given A as input, S can be computed in deterministic polynomial time.*

Proof. By homogeneity of the nuclear norm and the smallest singular value, it suffices to show that if $\|AQ^{1/2}\|_{S_1}^2 = 1$, then $|S|\sigma_{\min}(A|_S)^2 \geq \frac{c^2}{(\log m)^2}$ for a set $S \subseteq [n]$.

Let $T_k = \{i \in [m] : 2^{-k-1} \leq \sigma_i(AQ^{1/2}) \leq 2^{-k}\}$ for an integer $0 \leq k \leq \log_2 m$, and $R = \{i \in [m] : \sigma_i(AQ^{1/2}) \leq \frac{1}{2m}\}$. Then

$$\sum_{k=0}^{\log_2 m} \sum_{i \in T_k} \sigma_i(AQ^{1/2}) = 1 - \sum_{i \in R} \sigma_i(AQ^{1/2}) \geq 1/2,$$

since $|R| \leq m$. Therefore, by averaging, there exists a k^* such that $\sum_{i \in T_{k^*}} \sigma_i(AQ^{1/2}) \geq \frac{1}{2^{\log_2 m}}$. Let Π be the projection operator onto the span of the left singular vectors of $AQ^{1/2}$ corresponding to the singular values $\sigma_i(AQ^{1/2})$ for $i \in T_{k^*}$. Setting $\tau = \frac{1}{2^{\log_2 m}}$ and $r = |T_{k^*}| = \text{rank}(\Pi AQ^{1/2})$, we have $\|\Pi AQ^{1/2}\|_{S_1} \geq \tau$ by the choice of k^* , and $\|\Pi AQ^{1/2}\|_2 \leq 2\tau/r$ because all values of $\Pi AQ^{1/2}$ are within a factor of 2 from each other. Finally, applying Cauchy-Schwarz to the singular values of $\Pi AQ^{1/2}$, we have that $\|\Pi AQ^{1/2}\|_{HS} \geq \tau/r^{1/2}$. By Theorem 5 applied with $\epsilon = \frac{1}{2}$, there exists a set S of size $|S| \geq r/16$ such that $\sigma_{\min}(\Pi A|_S)^2 \geq \tau^2/4r$, implying that

$$|S| \sigma_{\min}(A|_S)^2 \geq |S| \sigma_{\min}(\Pi A|_S)^2 \geq \frac{1}{64} \tau^2.$$

Moreover, S can be computed in deterministic polynomial time. \square

Theorem 17. Let $\mu = \min\{\|E\|_\infty : \forall j \in [n] : a_j \in E\}$ for a rank m matrix $A = (a_j)_{j=1}^n$. Then

$$\mu = O(\log m) \text{ hvecdisc}(A).$$

Moreover, we can compute in deterministic polynomial time a set $S \subseteq [n]$ such that $\mu = O(\log m) \text{ vecdisc}(A|_S)$.

Proof. Let P and Q be optimal solutions for (9)-(11). By Theorem 15, $\mu = \|P^{1/2}AQ^{1/2}\|_{S_1}$. Then, by Lemma 16, applied to the matrices $P^{1/2}A$ and Q , there exists a set $S \subseteq [n]$, computable in deterministic polynomial time, such that

$$\text{specLB}(A) \geq \sqrt{|S|} \sigma_{\min}(P^{1/2}A|_S) \geq \frac{c\|P^{1/2}AQ^{1/2}\|_{S_1}}{\log m} = \frac{c\mu}{\log m}. \quad (15)$$

\square

The determinant lower bound of Lovasz, Spencer, and Vesztergombi [LSV86] is equal to the maximum of $|\det A_{S,T}|$ over all submatrices $A_{S,T}$ of A . We note that up to the log factor, the lower bound (15) is at least as strong. In particular, assume the determinant lower bound is maximized by a $k \times k$ submatrix $A_{S,T}$ induced by a subset S of the rows and a subset T of the columns. Then we can set $P = \frac{1}{k} \text{diag}(\mathbf{1}_S)$ and $Q = \frac{1}{k} \text{diag}(\mathbf{1}_T)$, and $\|P^{1/2}AQ^{1/2}\|_{S_1} = \frac{1}{k} \|A_{S,T}\|_{S_1}$ is at least as large as $|\det A_{S,T}|$ by the geometric mean-arithmetic mean inequality applied to the singular values of $A_{S,T}$.

5 The Approximation Algorithm

We are now ready to give our approximation algorithm for hereditary vector discrepancy and hereditary discrepancy. In fact, the algorithm is a straightforward consequence of the upper and lower bounds we proved in the prior sections.

Theorem 18 (Theorems 3,4, restated). *There exists a polynomial time algorithm that, on input an $m \times n$ real matrix $A = (a_j)_{j=1}^n$, computes a value α such that the following inequalities hold*

$$\begin{aligned}\alpha &\leq \text{hvecdisc}(A) \leq O(\log m) \cdot \alpha \\ \alpha &\leq \text{herdisc}(A) \leq O(\log^{3/2} m) \cdot \alpha\end{aligned}$$

Moreover, the algorithm finds a submatrix $A|_S$ of A , such that $\alpha \leq \text{vecdisc}(A|_S)$.

Proof. We first ensure that the matrix A is of rank m by adding a tiny full rank perturbation to it, and adding extra columns if necessary². By making the perturbation small enough, we can ensure that it affects $\text{herdisc}(A)$ and $\text{hvecdisc}(A)$ negligibly. Let $\mu = \min\{\|E\|_\infty : \forall j \in [n] : a_j \in E\}$. The value α we output is $\alpha = \mu/(C \log m)$, where C is a sufficiently large constant so that the asymptotic expression in Theorem 17 holds. The approximation guarantees follow from Theorems 12 and 17, and S is computed as in Theorem 17.

To compute α in polynomial time, we solve (5)–(8). By Lemma 14, this is a convex minimization problem, and as such can be solved using the ellipsoid method up to an ϵ -approximation in time polynomial in the input size and in $\log \epsilon^{-1}$. The optimal value is equal to μ by Lemma 14, and, therefore, we can compute an arbitrarily good approximation to α in polynomial time. \square

6 A Geometric Consequence

In this section we derive a geometric consequence of Lemma 16. Specifically, we prove that any convex body K is contained in an ellipsoid whose Gaussian width is bounded in terms of the Kolmogorov widths of K . While not necessary for our approximation algorithm, this result may be of independent interest.

Let us first define the Kolmogorov widths for a convex body K .

Definition 1. *The Kolmogorov width $d_k(K)$ of a symmetric convex body $K \subseteq \mathbb{R}^n$ is equal to $d_k(K) \triangleq \min_{\Pi \in \mathcal{P}_{n-k+1}} \|\Pi K\|_2$, i.e. the minimum radius (in ℓ_2) of any projection of K of co-dimension $k - 1$.*

We note that Kolmogorov width is more generally defined for linear operators between Banach spaces, and the definition above is the special case of the Kolmogorov width of the identity operator $I : X \rightarrow \ell_2$, where X is a finite-dimensional Banach space with unit ball K .

Lemma 16 implies the following result.

Theorem 19. *Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body. There exists an ellipsoid $E = FB_2^n$ containing K such that*

$$\|F\|_{HS} \leq (C \log n) \max_{k=1}^n \sqrt{k} d_k(K),$$

for a universal constant C .

²There are other, more numerically stable ways to reduce to the full rank case, e.g. by projecting A onto its range and modifying the norms we consider accordingly. We choose the perturbation approach for simplicity.

The proof of the result relies on Lemma 16 and an optimization problem over ellipsoids containing K that is closely related to the problem of minimizing width in coordinate directions, discussed in prior sections. Let v_1, \dots, v_N be points in \mathbb{R}^n , and consider the problem of minimizing $\|F\|_{HS}^2$ subject to $v_1, \dots, v_N \in E = FB_2^n$. The problem can be formulated as

$$\text{Minimize } \text{tr}(X^{-1}) \text{ s.t.} \quad (16)$$

$$X \succ 0 \quad (17)$$

$$\forall i \in [N] : v_i^T X v_i \leq 1. \quad (18)$$

The constraints (18) are affine, and the convexity of the objective (16) follows from Lemma 13. An argument analogous to the one in the proof of Theorem 15 shows that the Lagrange dual function for (16)–(18) is

$$g(R) = \|VR^{1/2}\|_{S_1} - \text{tr}(R),$$

where V is the matrix whose columns are v_1, \dots, v_N , and R is a non-negative $N \times N$ diagonal matrix. Again analogously to Theorem 15, the Lagrange dual problem is to maximize $g(R)$ over all non-negative diagonal R , and by strong duality we have the equality

$$\begin{aligned} \min\{\|F\|_{HS}^2 : \forall i \in [N] \ v_i \in E = FB_2^n\} &= \max\{g(R) : R \succeq 0, \text{diagonal}\} \\ &= \max\{\|VQ^{1/2}\|_{S_1}^2 : Q \succeq 0, \text{diagonal}, \text{tr}(Q) = 1\}. \end{aligned} \quad (19)$$

Proof of Theorem 19: Let v_1, \dots, v_N be chosen to form a sufficiently dense net on the boundary of K and let $E = FB_2^n$ be the ellipsoid containing v_1, \dots, v_N that minimizes $\|F\|_{HS}^2$; by taking N sufficiently large, we can ensure that $K \subseteq (1 + \epsilon)E$ for an arbitrary small ϵ .

Let V be the matrix whose columns are the points v_1, \dots, v_N . By (19) and Lemma 16 (with V used in the role of A), there exists a set $S \subseteq [N]$ and an absolute constant c such that

$$|S| \sigma_{\min}(V|_S)^2 \geq \frac{c}{(\log n)^2} \|F\|_{HS}^2. \quad (20)$$

We claim that for $s = |S|$, $k = \lceil s/2 \rceil$, and any $\Pi \in \mathcal{P}_{n-k+1}$ there exists an $i \in S$ such that $\|\Pi v_i\|_2^2 \geq \sigma_{\min}(V|_S)^2/2$. This suffices to prove the theorem, since, together with (20), it implies that $kd_k(K)^2 \geq \frac{c/2}{(\log n)^2} \|F\|_{HS}^2$.

Define $M \triangleq (V|_S)(V|_S)^T$ and fix some $\Pi \in \mathcal{P}_{n-k+1}$. By averaging, it suffices to show that

$$\frac{1}{s} \sum_{i \in S} \|\Pi v_i\|_2^2 = \frac{1}{s} \text{tr}((V|_S)^T \Pi (V|_S)) \geq \sigma_{\min}(V|_S)^2/2.$$

Let $\Pi = UU^T$, where U is a matrix with $n - k + 1$ mutually orthogonal unit columns. Then, by the Cauchy interlace theorem (see Lemma 22 in Appendix B),

$$\lambda_k(U^T M U) \geq \lambda_{2k-1}(M) \geq \lambda_s(M) = \sigma_{\min}(V|_S)^2.$$

Therefore, we have

$$\begin{aligned} \frac{1}{s} \text{tr}((V|_S)^T \Pi (V|_S)) &= \frac{1}{s} \text{tr}((U^T V|_S)^T (U^T V|_S)) = \frac{1}{s} \text{tr}((U^T V|_S)(U^T V|_S)^T) \\ &= \frac{1}{s} \text{tr}(U^T M U) \geq \frac{k}{s} \lambda_k(U^T M U) \geq \frac{1}{2} \sigma_{\min}(V|_S)^2. \end{aligned}$$

As remarked above, this completes the proof of the theorem together with (20). \square

The Hilbert-Schmidt norm $\|F\|_{HS}$ has several natural geometric interpretations in terms of the ellipsoid $E = FB_2^n$. On one hand, $\|F\|_{HS}^2$ is equal to the sum of squared lengths (in ℓ_2) of the major axes of E . By an easy calculation, $\|F\|_{HS}$ is also equivalent up to constants to the norm $\ell^*(K) \triangleq \mathbb{E}\|g\|_{K^\circ} = \mathbb{E}\max_{x \in K} |\langle x, g \rangle|$, where g is a standard m -dimensional Gaussian random variable (see, e.g. [Tal05]). This quantity is also known as the Gaussian width of K . Phrased in these terms, Theorem 19 shows that for any n -dimensional convex symmetric K there exists an ellipsoid E containing K such that

$$\ell^*(E) \leq (C \log n) \max_{k=1}^n \sqrt{k} d_k(K). \quad (21)$$

A qualitatively weaker bound follows from a theorem of Carl and Dudley's chaining bound. Carl [Car81] showed that for an absolute constant C_1 ,

$$\max_{k=1}^n \sqrt{k} e_k(K) \leq C_1 \max_{k=1}^n \sqrt{k} d_k,$$

where $e_k(K)$ is the k -th entropy number of K , i.e. the least r such that K can be covered by at most 2^{k-1} copies of rB_2^n . Dudley's chaining argument [Dud67, Tal05] implies that there exists a constant C_2 such that $\ell^*(K) \leq (C_2 \log n) \max_{k=1}^n \sqrt{k} e_k(K)$; combining the two bounds we have that

$$\ell^*(K) \leq (C_3 \log n) \max_{k=1}^n \sqrt{k} d_k(K), \quad (22)$$

where $C_3 = C_1 C_2$. This is readily implied by (21) (up to the value of the constant), because $K \subseteq E$ implies $\ell^*(K) \leq \ell^*(E)$. However, there are examples where (21) is near-tight while (22) is loose. For example, for the ℓ_1^n -ball B_1^n , $\max_{k=1}^n \sqrt{k} d_k(B_1^n) = \Omega(\sqrt{n})$ and $\ell^*(B_1^n) = \Theta(\sqrt{\log n})$, while for any ellipsoid E containing B_1 we have $\ell^*(E) = \Omega(\sqrt{n})$, as can be seen from (19).

7 Conclusion

We gave an $O(\log^{3/2} n)$ -approximation algorithm for the hereditary discrepancy of a matrix A , by approximately characterizing the hereditary vector discrepancy of a matrix in terms of a simple convex program: that of minimizing $\|E\|_\infty$ over all E that contain the columns of A .

Our lower bound is “constructive”: we can construct in polynomial time a submatrix $A|_S$ demonstrating the lower bound on $\text{hvecdisc}(A)$ and hence on $\text{herdisc}(A)$. Our upper bound for $\text{hvecdisc}(A)$ is also “constructive” in that the ellipsoid E^* gives a recipe to construct a vector solution to the $\text{vecdisc}(A|_S)$ given any S . Our $O(\log^{3/2})$ upper bound for $\text{herdisc}(A)$ is however non-constructive as it uses the result of Banaszczyk [Ban98], whose proof does not yield an efficient algorithm for computing the sign vector x . We can however use the result of Bansal to algorithmically get a coloring for any given S , at the cost of losing a factor of $\sqrt{\log n}$ in the approximation.

We leave open several questions of interest. One natural question is whether our approximation ratios can be improved. The best known hardness of approximating hereditary discrepancy is 2, but we conjecture that the hardness is superconstant. Another interesting question is whether the guarantee for Bansal's algorithm (Theorem 1) can be improved by a factor of $O(\sqrt{\log m})$, which would make it tight. This question was previously posed

by Matoušek [Mat13]. Such an improvement would also imply a constructive proof of Banaszczyk’s theorem. A further question concerns the complexity of computing $\text{herdisc}(A)$ exactly. Deciding whether $\text{herdisc}(A) \leq t$ for any t is naturally in Π_2^P , but not known to be in NP. Is this problem complete for Π_2^P ?

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References

- [AGH13] Per Austrin, Venkatesan Guruswami, and Johan Håstad. $(2 + \epsilon)$ -SAT is NP-hard. In *ECCC*, 2013.
- [Ban98] Wojciech Banaszczyk. Balancing vectors and gaussian measures of n-dimensional convex bodies. *Random Structures & Algorithms*, 12(4):351–360, 1998.
- [Ban10] N. Bansal. Constructive algorithms for discrepancy minimization. In *Foundations of Computer Science (FOCS), 2010 51st Annual IEEE Symposium on*, pages 3–10. IEEE, 2010.
- [BS95] József Beck and Vera T. Sós. Discrepancy theory. In R. L. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics (vol. 2)*, pages 1405–1446. MIT Press, Cambridge, MA, USA, 1995.
- [BT87] J. Bourgain and L. Tzafriri. Invertibility of large submatrices with applications to the geometry of banach spaces and harmonic analysis. *Israel journal of mathematics*, 57(2):137–224, 1987.
- [Car81] Berndt Carl. Entropy numbers, s -numbers, and eigenvalue problems. *Journal of Functional Analysis*, 41(3):290–306, 1981.
- [Cha00] Bernard Chazelle. *The Discrepancy Method: Randomness and Complexity*. Cambridge University Press, 2000.
- [CNN11] M. Charikar, A. Newman, and A. Nikolov. Tight hardness results for minimizing discrepancy. In *Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1607–1614. SIAM, 2011.
- [Dud67] Richard M Dudley. The sizes of compact subsets of hilbert space and continuity of gaussian processes. *Journal of Functional Analysis*, 1(3):290–330, 1967.
- [GH62] Alain Ghouila-Houri. Caractérisation des matrices totalement unimodulaires. *CR Acad. Sci. Paris*, 254:1192–1194, 1962.
- [LM12] Shachar Lovett and Raghu Meka. Constructive discrepancy minimization by walking on the edges. In *Foundations of Computer Science (FOCS), 2012 IEEE 53rd Annual Symposium on*, pages 61–67. IEEE, 2012.
- [LSV86] L. Lovász, J. Spencer, and K. Vesztegombi. Discrepancy of set-systems and matrices. *European Journal of Combinatorics*, 7(2):151–160, 1986.

- [Mat13] Jiří Matoušek. The determinant bound for discrepancy is almost tight. *Proceedings of the American Mathematical Society*, 141(2):451–460, 2013.
- [Mat14] Jiri Matoušek. Personal communication, 2014.
- [MN12] S. Muthukrishnan and Aleksandar Nikolov. Optimal private halfspace counting via discrepancy. In *STOC '12: Proceedings of the 44th symposium on Theory of Computing*, pages 1285–1292, New York, NY, USA, 2012. ACM.
- [Nik13] Aleksandar Nikolov. The Kórmlos conjecture holds for vector colorings. *Under submission.*, 2013.
- [NTZ13] Aleksandar Nikolov, Kunal Talwar, and Li Zhang. The geometry of differential privacy: the sparse and approximate cases. In *Proceedings of the 45th annual ACM symposium on Symposium on theory of computing*, STOC '13, pages 351–360, New York, NY, USA, 2013. ACM.
- [Rot13] Thomas Rothvoß. Approximating bin packing within $O(\log \text{OPT} * \log \log \text{OPT})$ bins. In *FOCS*, pages 20–29, 2013.
- [Rot14] Thomas Rothvoß. Constructive discrepancy minimization for convex sets. *CoRR*, abs/1404.0339, 2014.
- [Sch72] Wolfgang Schmidt. Irregularities of distribution, vii. *Acta Arithmetica*, 21(1):45–50, 1972.
- [Sey80] Paul D Seymour. Decomposition of regular matroids. *Journal of combinatorial theory, Series B*, 28(3):305–359, 1980.
- [Spe85] Joel Spencer. Six standard deviations suffice. *Transactions of the American Mathematical Society*, 289(2):679–706, 1985.
- [SS10] D.A. Spielman and N. Srivastava. An elementary proof of the restricted invertibility theorem. *Israel Journal of Mathematics*, pages 1–9, 2010.
- [Tal05] Michel Talagrand. *The Generic Chaining: Upper and Lower Bounds for Stochastic Processes*. Springer, 2005.
- [Ver01] R. Vershynin. John’s decompositions: Selecting a large part. *Israel Journal of Mathematics*, 122(1):253–277, 2001.

A Weighted Restricted Invertibility

In this section we show that the weighted restricted invertibility principle reduces to the standard version of the principle. Let us first give a statement of the principle, in a version proved by Spielman and Srivastava.

Theorem 20 ([SS10]). *Let $\epsilon > 0$, and let A be an m by n real matrix. For any integer k such that $k \leq \epsilon^2 \frac{\|A\|_{HS}^2}{\|A\|_2^2}$ there exists a subset $S \subseteq [n]$ of size $|S| = k$ such that $\sigma_{\min}(A|_S)^2 \geq (1 - \epsilon)^2 \frac{\|A\|_{HS}^2}{n}$. Moreover, S can be computed in deterministic polynomial time.*

The reduction of Theorem 5 to Theorem 20 is based on the following simple technical lemma.

Lemma 21. *Let $Q \succeq 0$ be a diagonal matrix with rational entries, such that $\text{tr}(Q) = 1$. Then for any m by n matrix $A = (a_j)_{j=1}^n$, there exists a $m \times n_Q$ matrix A_Q such that $A_Q A_Q^T = n_Q A Q A^T$. Moreover, all columns of A_Q are columns of A .*

Proof. Let n_Q be the least common denominator of all diagonal entries of Q , that is $n_Q Q = R$ for an integral diagonal matrix R . Let then A_Q be a matrix with R_{jj} copies of each a_j . Clearly,

$$A_Q A_Q^T = \sum_{j=1}^n R_{jj} a_j a_j^T = A R A^T = n_Q A Q A^T.$$

Observe, finally, that the number of columns of A_Q is equal to $\sum_{j=1}^n R_{jj} = n_Q \sum_{j=1}^n Q_{jj} = n_Q$. \square

Proof of Theorem 5: By introducing a tiny perturbation to Q , we can make it rational while changing $\|A Q^{1/2}\|_{HS}$ and $\|A Q^{1/2}\|_2$ by an arbitrarily small amount. Therefore, we assume that Q is rational. Then, by Lemma 21, there exists a matrix A_Q with n_Q columns all of which are columns of A , such that $A_Q A_Q^T = n_Q A Q A^T$. Let k be arbitrary integer such that

$$k \leq \epsilon^2 \frac{\|A Q^{1/2}\|_{HS}^2}{\|A Q^{1/2}\|_2^2} = \epsilon^2 \frac{\text{tr}(A Q A^T)}{\lambda_{\max}(A Q A^T)} = \epsilon^2 \frac{n_Q \text{tr}(A_Q A_Q^T)}{n_Q \lambda_{\max}(A_Q A_Q^T)} = \epsilon^2 \frac{\|A_Q\|_{HS}^2}{\|A_Q\|_2^2},$$

where $\lambda_{\max}(M)$ is used to denote the largest eigenvalue of a matrix M . By Theorem 20, there exists a set S_Q of size k , such that

$$\sigma_{\min}(A_Q|_{S_Q})^2 \geq (1 - \epsilon)^2 \frac{\|A_Q\|_{HS}^2}{n_Q} = (1 - \epsilon)^2 \frac{\text{tr}(A_Q A_Q^T)}{n_Q} = (1 - \epsilon)^2 \text{tr}(A Q A^T) = (1 - \epsilon)^2 \|A Q^{1/2}\|_{HS}^2.$$

But since all columns of A_Q are also columns of A , and no column in $A_Q|_{S_Q}$ can be repeated or otherwise $\sigma_{\min}(A_Q|_{S_Q}) = 0$, there exists a set $S \subseteq [n]$ such that $\sigma_{\min}(A|_S)^2 \geq (1 - \epsilon)^2 \|A Q^{1/2}\|_{HS}^2$. \square

B Vector Discrepancy for the Komlós Problem

Theorem 9 follows from the arguments in [Nik13] with few modifications. Here we sketch the argument.

We need a well-known lemma, also known as the Cauchy Interlace Theorem. It follows easily from the variational characterization of eigenvalues.

Lemma 22. *Let M be a symmetric real matrix with eigenvalues $\lambda_1 \geq \dots \geq \lambda_n$. Let also $U \in \mathbb{R}^{n \times k}$ be a matrix with mutually orthogonal unit columns. Then for $1 \leq i \leq k$,*

$$\lambda_{n-k+i}(M) \leq \lambda_i(U^T M U) \leq \lambda_i(M).$$

The following is an immediate consequence of Lemma 22.

Lemma 23. *Let $M \in \mathbb{R}^{n \times n} : M \succeq 0$ be a symmetric real matrix with eigenvalues $\sigma_1 \geq \dots \geq \sigma_n \geq 0$. Let also $U \in \mathbb{R}^{n \times k}$ be a matrix with mutually orthogonal unit columns. Then $\det(U^T M U) \leq \sigma_1 \dots \sigma_k$.*

The next lemma follows from strong duality for semidefinite programming (see [Mat13]).

Lemma 24. *For any real $m \times n$ matrix A , the minimum of $\|AXA^T\|_2^2$ over $X \succeq 0$ such that $\forall j \in [n] : X_{jj} = 1$ is equal to D^2 if and only if there exists a matrix $P \succeq 0$, $\text{tr}(P) = 1$, and a diagonal matrix $Q \geq 0$, such that*

$$\text{tr}(Q) \geq D^2 \tag{23}$$

and for all $z \in \mathbb{R}^n$

$$z^T A^T P A z \geq z^T Q z. \tag{24}$$

The final lemma we need was proved in [Nik13]. It can be proved using the fact that the function $\sum_{i=1}^n e^{z_i}$ is symmetric and convex in $z = (z_i)_{i=1}^n$, and therefore is Schur-convex.

Lemma 25. *Let $x_1 \geq \dots \geq x_n > 0$ and $y_1 \geq \dots \geq y_n > 0$ such that*

$$\forall k \leq n : x_1 \dots x_k \geq y_1 \dots y_k$$

Then,

$$\forall k \leq n : x_1 + \dots + x_k \geq y_1 + \dots + y_k.$$

Theorem 26 (Theorem 9, restated). *For any $m \times n$ matrix $A = (a_i)_{i=1}^n$ satisfying $\forall i \in [n] : \|a_i\|_2 \leq 1$ there exists a $n \times n$ matrix $X \succeq 0$ such that $\forall i \in [n] : X_{jj} = 1$ and $\|AXA^T\|_2 \leq 1$.*

Proof. We will use Lemma 24 with $D = \sqrt{1+\epsilon}$ for an arbitrary $\epsilon > 0$. Assume for contradiction that there exist P and Q such that (23) and (24) are satisfied. Let us define $q_i = Q_{ii}$, and also $p_i = \sigma_i(P)$. Let, without loss of generality, $q_1 \geq \dots \geq q_n > 0$. Denote by $A_{[k]}$ the matrix (a_1, \dots, a_k) and by Q_k the diagonal matrix with q_1, \dots, q_k on the diagonal. We first show that

$$\forall k \leq n : \det(A_{[k]}^T P A_{[k]}) \leq p_1 \dots p_k. \tag{25}$$

Let u_1, \dots, u_k be an orthonormal basis for the range of $A_{[k]}$ and let U_k be the matrix (u_1, \dots, u_k) . Then $A_{[k]} = U_k U_k^T A_{[k]}$. Each column of the square matrix $U_k^T A_{[k]}$ has norm at most 1, and, by Hadamard's inequality,

$$\det(A_{[k]}^T U_k) = \det(U_k^T A_{[k]}) \leq 1.$$

Therefore,

$$\forall k \leq n : \det(A_{[k]}^T P A_{[k]}) \leq \det(U_k^T P U_k).$$

By Lemma 23, we have that $\det(U_k^T P U_k) \leq p_1 \dots p_k$, which proves (25).

By (24) we know that for all k and for all $u \in \mathbb{R}^k$, $u^T A_{[k]}^T P A_{[k]} u \geq u^T Q_k u$, since we can freely choose z such that $z_i = 0$ for all $i > k$. Then, we have that

$$\forall k \leq n : \det(A_{[k]}^T P A_{[k]}) \geq \det(Q_k) = q_1 \dots q_k \quad (26)$$

Combining (25) and (26), we have that

$$\forall k \leq n : p_1 \dots p_k \geq q_1 \dots q_k \quad (27)$$

By Lemma 25, (27) implies that $1 = \sum_{j=1}^m p_j \geq \sum_{j=1}^n p_j \geq \sum_{i=1}^n q_i \geq 1 + \epsilon$, a contradiction. \square