Operatory Theory Approach to Discrete Time-Frequency Distributions

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Introduction

The field of quadratic time-frequency distributions (TFDs) assumed importance when Cohen [1] developed a systematic way of obtaining bilinear, joint bivariate densities of time and frequency using operator theory. In a more recent paper [2], he presented a generalized approach for obtaining joint representations for arbitrary quantities (not just time and frequency) using the characteristic function operator methods of Moyal [3] and Ville [4].

Even though the continuous-time versions of the quadratic TFDs are backed by rich theory, they have to undergo the process of discretization (resulting in discrete-time/discrete-frequency mass functions) before they can be implemented in a computer and be of any use. A considerable amount of work has gone into the study of the ill-effects of sampling the continuous-time TFDs and techniques for eliminating them (e.g. [5]).

We propose to provide a direct theoretical link between the discrete-time sequence $x[n]$ that resides in the computer and its discrete time-frequency distributions. Our approach to obtaining discrete time-frequency distributions is at a more fundamental and general level than that was adopted by Richman et. al. in [6]. For example, our approach can provide a variety of distributions other than a Wigner type.

Existing theory links only their continuous-time counterparts. It is not clear whether the rich theory that binds $x(t)$ and $X(t, f)$ automatically translates into the corresponding binding between $x[n]$ and $X[n, k]$.

Our intent in this paper is to build a theoretical framework that would link $x[n]$ and $X[n, k]$. We begin by assuming that all we have is a finite-length, discrete sequence $x[n]$ of length N (which is assumed to be odd throughout this paper), whose N point discrete Fourier transform (DFT) is given by $X[k]$. We then work towards obtaining $X[n, k]$ by developing discrete (matrix) operators corresponding to the discrete variable time and the discrete variable frequency. From here, we can calculate our distribution $X[n, k]$. Our results indicate that important differences exist between the set of sampled continuous-time quadratic

TFDs and the set of discrete TFDs obtained through discrete operator theory. Through operators, we are able to obtain additional insights into existing TFDs as well as establish a framework for formulating new distributions.

1 Derivation and Properties of the Discrete Fourier Frequency Operators

We shall first derive the discrete frequency operator in the discrete-time domain. In the continuous case, it is well known that the frequency operator in the time domain is $-j\frac{d}{dt}$. This can be found by solving for the operator A in the expected value equation [2]

$$\int x^*(t)Ax(t)dt = \int X^*(\omega)\omega X(\omega)d\omega. \tag{1}$$

Using k, $X[k]$, $x[n]$ and summation instead of integration, we can solve the same problem in the discrete domain. We obtain

$$A\{x[n]\} = o[n] \otimes x[n], \tag{2}$$

where $\otimes$ denotes circular convolution and $o[n]$ is the IDFT of the sequence $\{-(N-1)/2, -(N-1)/2+1, \ldots, -1, 0, 1, \ldots, (N-1)/2-1, (N-1)/2\}$. (To avoid the use of the scaling term $1/N$ in the IDFT, a symmetric definition of the DFT is used in this paper.)

$$o[n] = \begin{cases} \frac{(j/2)(-1)^n}{\sin(\pi n/N)} & n \neq 0 \\ 0 & n = 0 \end{cases} \tag{3}$$

Note that the operator A expressed in the form of $o[n]$ operates convolutionally on a signal $x[n]$. A can also be written in matrix notation as a circulant matrix K. Similarly, the discrete time operator can be expressed in the time domain as the diagonal matrix L formed using the numbers $\{-(N-1)/2, -(N-1)/2+1, \ldots, -1, 0, 1, \ldots, (N-1)/2-1, (N-1)/2\}$. A very similar set of operators can be derived to calculate the expected value of $x[n]$, but we ignore those here for clarity.

The K operator, like its continuous-time counterpart, is purely imaginary and Hermitian and exhibits a shift property:

$$\exp(j2\pi K/N)x[n] = x[n+m]$$

Unlike the continuous operators however, the commutator of time and frequency C is not a constant:

$$C\{x[n]\} = (LK - KL)x[n] = (no[n]) \otimes x[n]$$

2 Calculation of energy density in the discrete Fourier representation

The energy density function $P[k]$ can be expressed as the DFT of the characteristic function, $\langle \exp(j2\pi mk/N) \rangle$, where $\langle \rangle$ indicates the expectation operation in k. The characteristic function is

$$\begin{aligned} \langle \exp(j2\pi mk/N) \rangle &= \sum_k \exp(j2\pi mk/N) P[k] \\ &= \sum_n x^*[n] \exp(j2\pi mK/N) x[n] \\ &= \sum_n x^*[n] x[n+m], \end{aligned} \tag{4}$$

and taking the DFT of the last term yields $|X[k]|^2$. This result validates the use of the K operator for calculating energy density functions.

3 Calculation of discrete time-frequency distributions

For a discrete joint time-frequency distribution $P[n,k]$, we can define the characteristic function as

$$M(\theta, m) = \langle \exp(j2\pi\theta L/N + j2\pi mK/N) \rangle \tag{5}$$

Since L and K are non-commuting operators, the exponent can be written in several ways, each leading to a distinct characteristic function. For instance,

$$\langle \exp(j2\pi\theta L/N + j2\pi mK/N) \rangle,$$
$$\langle \exp(j2\pi\theta L/N) \exp(j2\pi mK/N) \rangle, \text{ and}$$
$$\langle \exp(j\pi\theta L/N) \exp(j2\pi mK/N) \exp(j\pi\theta L/N) \rangle$$

are three different characteristic functions yielding three different TFDs. We shall now use two of these characteristic functions and attempt to derive two well-known discrete TFDs.

3.1 TFD 1

Using the characteristic function $\langle \exp(j2\pi\theta L/N) \exp(j2\pi mK/N) \rangle$, we have

$$\begin{aligned} M(\theta, m) &= \langle \exp(j2\pi\theta L/N) \exp(j2\pi mK/N) \rangle \\ &= \sum_n x^*[n] \exp(j2\pi\theta L/N) \exp(j2\pi mK/N) x[n] \\ &= \sum_n x^*[n] \exp(j2\pi\theta L/N) x[n+m]. \end{aligned} \tag{6}$$

Thus, $P[n,k]$ is given by

$$\begin{aligned} P[n,k] &= \sum_m \sum_\theta M(\theta,m) \exp(-j2\pi mk/N) \exp(j2\pi\theta n/N) \\ &= x^*[n] \exp(j2\pi nk/N) X[k] \end{aligned} \tag{7}$$

which is the well-known discrete Rihaczek TFD [7]. This result is analogous to the continuous-time result.

3.2 TFD 2

Using the continuous-time equivalent of the characteristic function

$$\langle \exp(j2\pi\theta L/N + j2\pi mK/N) \rangle$$

results in the Wigner distribution. This is because the continuous-time time and frequency operators commute with the commutator, resulting in a simplification of the characteristic function. In the discrete-time case, no such simplification is possible, since the operators do not commute with the commutator. Thus, the characteristic function $\langle \exp(j2\pi\theta L/N + j2\pi mK/N) \rangle$, instead of yielding the sampled continuous-time Wigner distribution, would result in a TFD that is more complicated and difficult to compute.

4 Advantages of the Discrete Approach

The discrete operator theory approach offers new analysis tools that enable us to not only analyze the properties of existing distributions, but also to search for new distributions.

For example, several attempts have been made in the past to "non-negativize" the continuous Wigner distribution by convolving it with a smoothing function [8],[9]. We shall show here that, even in the discrete domain, all such attempts yield a standard spectrogram. We first formulate a non-negativity condition for the joint distribution of any two variables. When we apply this condition to the time-frequency case, the spectrogram falls out in a very natural way.

Rewriting Eqn.(5) as a summation, we have

$$M(\theta,m) = \sum_n x^*[n] \phi[\theta,m] \exp(j2\pi\theta L/N) \exp(j2\pi mK/N) x[n] \tag{8}$$

Here, we make use of the correspondence rule [2], which maps permutations of the terms in the exponent to a kernel function $\phi[\theta,m]$, which is a square matrix of order N. Expressing the entire Eqn. 8 in matrix notation (and using H to indicate the conjugate transpose), we obtain

$$M(\theta,m) = x^H \phi \Lambda_1 G \Lambda_2 G^H x \tag{9}$$

where Λ_1 is the diagonal matrix $\exp(j2\pi\theta L/N)$, G is the unitary eigenvector matrix corresponding to the operator K, ($G = KLK^H$), and Λ_2 is the diagonal matrix $\exp(j2\pi mL/N)$. Taking the two-dimensional DFT of Eqn. (9), we obtain the distribution

$$P[n,k] = x^H A x = x^h (\Phi .* G) G^H x, \tag{10}$$

where the operator (.*) stands for a element by element product, and Φ is the two dimensional DFT of ϕ whose rows and columns have been have been cyclically shifted by n and k, respectively.

For a distribution to be nonnegative, matrix A needs to be nonnegative definite for any n and k, i.e., we must be able to express A as WW^H, where W is a non-singular matrix.

For the Fourier case, G = F, the DFT matrix. Thus, for $n = 0$ and $k = 0$

$$(\Phi .* F)F^H = WW^H \rightarrow \Phi = (WW^H F)./F, \tag{11}$$

where the operator (./) stands for a element by element product. Shifting the rows of Φ by n units and columns by k units, and substituting the result in Eqn. (10), we obtain

$$W = w[n-m] \exp(jw\pi mk/N). \tag{12}$$

$$P[n,k] = x^H W W^H x = |x^H W|^2 = |\sum_{m=0}^{N-1} x[m] w[n-m] \exp(j2\pi mk/N)|^2 \tag{13}$$

It can now be observed that the application of the positivity constraint to the general discrete-time quadratic time-frequency representation results in the squared magnitude of the short-time Fourier Transform (spectrogram)! Hence, any non-negative discrete quadratic time-frequency distributions must be a spectrogram. The positivity constraint precludes satisfaction of the marginals, as was shown by Wigner.

We might well ask if proper quadratic distributions (distributions which are nonnegative and also satisfy the marginals for all signals) can exist at all. Though Wigner showed that no such time-frequency distribution exists, this is an inherent limitation of the Fourier transform and does not mean that such a proper distribution cannot exist when frequency is replaced by another variable. For example, it can be shown that the Pauli spin operator

$$G = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix} \tag{14}$$

generates a proper quadratic distribution. This result suggests that there could exist physically meaningful proper quadratic distributions that are built around a non-Fourier transform.

5 Conclusions and Contributions

1. We have developed a theory for constructing discrete quadratic time-frequency distributions using operator theory methods.

2. The discrete methods give us additional insight into the discrete case in particular, rather than forcing us to infer from the continuous case.
3. We have proven that, even in the discrete case, spectrograms are the only positive quadratic time-frequency distributions.
4. We have argued that proper distributions may be developed using non-Fourier transforms.

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