Packing Multicast Trees

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Problem

Given

- Directed graph \((V,E)\) with edge capacities \(c(e), e \in E\)
- Multiple multicast sessions \(\{(s_i, T_i)\}\), each with
  - Sender \(s_i\)
  - Receiver set \(T_i\)
  - Transmission rate \(r_i\)

What is the capacity region \(\mathcal{R} = \{\text{achievable } (r_1, \ldots, r_{|S|})\}\)?

What is a transmission scheme that achieves a given transmission rate vector \((r_1, \ldots, r_{|S|}) \in \mathcal{R}\)?
Butterfly Network with Two Unicast Sessions

sender $s_1$
rate $r_1$

sender $s_2$
rate $r_2$

receiver $t_1$

receiver $t_2$

Capacity Region?
Network Coding vs Routing

\[ f_1(y_1, y_2, y_3) \]

\[ f_2(y_1, y_2, y_3) \]
Butterfly Network with Two Unicast Sessions

sender $s_1$
rate $r_1$

sender $s_2$
rate $r_2$

receiver $t_1$

receiver $t_2$

$a \oplus b$

Capacity Region $R$
Characterization of Capacity for Acyclic Graphs

\[ N = \{ Y_s : s \in S \} \cup \{ U_e : e \in E \} \text{ -- } N \text{ random variables} \]

\[ \Gamma_N^* = \{ h \in 2^{N-1} : h \text{ is entropic} \} \]

\[ C_1 = \left\{ h \in 2^{N-1} : h_{Y_s} = \sum_{s \in S} h_{Y_s} \right\} \]

\[ C_2 = \left\{ h \in 2^{N-1} : h_{U_{Out(s)}|Y_s} = 0, \forall s \in S \right\} \]

\[ C_3 = \left\{ h \in 2^{N-1} : h_{U_{Out(v)}|U_{In(v)}} = 0, \forall v \in V \setminus (S \cup T) \right\} \]

\[ C_4 = \left\{ h \in 2^{N-1} : h_{U_e} \leq c(e), \forall e \in E \right\} \]

\[ C_5 = \left\{ h \in 2^{N-1} : h_{Y_{S_t} U_{In(t)}} = 0, \forall t \in T \right\} \]

\[ R = \Lambda \left( \text{proj}_{Y_s} \left( \text{con} \left( \Gamma_N^* \cap C_1 \cap C_2 \cap C_3 \right) \cap C_4 \cap C_5 \right) \right) \]

[Yan, Yeung, Zhang; ISIT 2007]
Single Unicast Session

- **Given**
  - Directed graph \((V,E)\) with edge capacities \(c(e), e \in E\)
  - Single unicast session with
    - Sender \(s\)
    - Receiver \(t\)
    - Transmission rate \(r\)

- \(r \leq \text{MinCut}(s,t)\)

- \(\text{MinCut}(s,t)\) is achievable, i.e., \(\text{MaxFlow}(s,t) = \text{MinCut}(s,t)\), by packing edge-disjoint directed paths

[\text{Menger; 1927}]
Single Broadcast Session

- Given
  - Directed graph \((V,E)\) with edge capacities \(c(e), \; e \in E\)
  - Single broadcast session with
    - Sender \(s\)
    - Receiver set \(T=V\)
    - Transmission rate \(r\)

- \(r \leq \min_{v \in V} \text{MinCut}(s,v)\)
- \(\min_{v \in V} \text{MinCut}(s,v)\) is achievable ("broadcast capacity") by packing edge-disjoint directed spanning trees

[Edmonds; 1972]
Single Broadcast Session

- Given
  - Directed graph \((V,E)\) with edge capacities \(c(e), e \in E\)
  - Single broadcast session with
    - Sender \(s\)
    - Receiver set \(T=V\)
    - Transmission rate \(r\)

- \(r \leq \min_{v \in V} \text{MinCut}(s,v)\)
- \(\min_{v \in V} \text{MinCut}(s,v)\) is achievable (“broadcast capacity”) by packing edge-disjoint directed spanning trees

[Edmonds; 1972]
Single Multicast Session

- \( r \leq \min_{t \in T} \text{MinCut}(s, t) \)
- \( \min_{t \in T} \text{MinCut}(s, t) \) is NOT always achievable by packing edge-disjoint multicast (Steiner) trees

**Given**
- Directed graph \((V, E)\) with edge capacities \(c(e), e \in E\)
- Single multicast session with
  - Sender \(s\)
  - Receiver set \(T \subset V\)
  - Transmission rate \(r\)
Packing Multicast Trees
Insufficient to achieve MinCut

\[ \min_{t \in T} \text{MinCut}(s,t) \text{ is always achievable by network coding} \]

[Alswede, Cai, Li, Yeung; 2000]
Linear Network Coding
Sufficient to achieve MinCut

- Linear network coding is sufficient to achieve $\min_{t \in T} \text{MinCut}(s, t)$

- Polynomial time algorithm for finding coefficients

[Cai, Li, Yeung; 2003]
[Koetter and Médard; 2003]

[Jaggi, Chou, Jain, Effros; Sanders, et al.; 2003]
[Erez, Feder; 2005]
Linear Network Coding
Sufficient to achieve MinCut

- Packing a maximum-rate set of multicast trees is NP hard
  \[\text{[Jain, Mahdian, Salavatipour; 2003]}\]

- Rate gap to \(\min_{t \in T} \text{MinCut}(s, t)\) can be a factor of \(\log n\)
  \[\text{[Jaggi, Chou, Jain, Effros; Sanders, et al.; 2003]}\]
Linear Network Coding NOT Sufficient for Multiple Sessions

- Linear network coding is NOT generally sufficient to achieve capacity for multiple sessions

[Dougherty, Freiling, Zeger; 2005]
P2P Networks

Download bandwidth is bottleneck

Completely connected overlay

\[ V; c_{out}(v), v \in V \]

Upload bandwidth is bottleneck
P2P Network as Special Case

Networks w/ edge capacities

P2P Networks

$c_{out}(v) \rightarrow \infty$
Corollary to Edmonds’ Theorem:

- Given a P2P network with a single broadcast session (i.e., a single sender and all other nodes as receivers), the maximum throughput is achievable by routing over a set of directed spanning trees.
Mutualcast

Cutset analysis: Throughput achieves mincut bound

Throughput:

\[
\left[ c_{\text{out}}(S) - \sum_{n=1}^{N} \frac{c_{\text{out}}(t_n)}{N-1} \right] / N \\
+ \sum_{n=1}^{N} \frac{c_{\text{out}}(t_n)}{N-1}
\]

[Li, Chou, Zhang; 2005]
Mutualcast with Helpers

- Cutset analysis: Throughput achieves mincut bound

Throughput:

\[
\frac{c_{out}(s) - \sum_{n=1}^{N} \frac{c_{out}(t_n)}{N-1} + \sum_{n=1}^{M} \frac{c_{out}(h_n)}{NN-1}}{N} + \sum_{n=1}^{N} \frac{c_{out}(t_n)}{N-1} + \sum_{n=1}^{M} \frac{c_{out}(h_n)}{N}
\]

[Li, Chou, Zhang; 2005]
Given
- P2P network
- Source $s$, receiver set $T$, helper set $H = V - T - s$

Then any rate $r \leq C$ can be achieved by routing over at most $|V|$ depth-1 and depth-2 multicast trees

Capacity is given by

$$C = \min \left\{ c_{out}(s), \left[ c_{out}(s) - \sum_{n=1}^{N} \frac{c_{out}(t_n)}{N-1} - \sum_{n=1}^{M} \frac{c_{out}(h_n)}{N} \right] / N + \sum_{n=1}^{N} \frac{c_{out}(t_n)}{N-1} + \sum_{n=1}^{M} \frac{c_{out}(h_n)}{N} \right\}$$

Remarks:
- Despite existence of helpers, no network coding needed!
- $|V|$ trees instead of $O(|V|^{|V|})$
- Simple, short trees
Multi-session Mutualcast Theorem

- Given
  - P2P network
  - Multiple sources $s_i$ and receiver sets $T_i$, $i=1,...,|S|$, s.t. sets $(s_i \cup T_i)$ are identical or disjoint; no connections between the disjoint receiver sets; $H = V - \cup (s_i \cup T_i)$

- Then any rate vector $(r_1, ..., r_{|S|}) \in \mathbb{R}$ can be achieved by routing over at most $|V| \times |S|$ depth-1 and depth-2 multicast trees

[Sengupta, Chen, Chou, Li; ISIT 2008]
Capacity region is given by set of linear inequalities:

\[ x_{i1} + \cdots + x_{i|V|} = r_i, \quad i = 1, \ldots, |S| \]

\[ \sum_{ij} x_{ij} \deg_{vij} \leq c_{out}(v), \quad v \in V \]

Remark:

- Despite multiple sessions (and helpers), still no inter-session or intra-session network coding is needed!
- Still |V| trees instead of \( O(|V|^{|V|}) \); still simple, short

[Sengupta, Chen, Chou, Li; ISIT 2008]
Application to Video Conferencing

- Maximizing $\sum U_i(r_i)$ over all $(r_1, ..., r_{|S|}) \in \mathbb{R}$ can be found by maximizing $\sum U_i(r_i)$ over all tree rates $x_{ij}$, $i=1, ..., |S|$, $j=1, ..., |V|$, s.t.
  - $x_{i1} + \cdots + x_{i|V|} = r_i$, $i=1, ..., |S|$
  - $\sum_{ij} x_{ij} \deg_{ij} \leq c_{out}(v)$, $v \in V$

- Network Utility Maximization problem
Summary

- For general multi-session multicast over directed graphs with edge capacities
  - Capacity region is stated in terms of $\Gamma^*$
  - Hard to know if a rate vector $r$ is in capacity region
- For restricted case of multi-session multicast with identical or disjoint receiver sets over P2P networks
  - Capacity region is given by $|V|$ inequalities over at most $|V| \times |S|$ variables (rates on each tree)
  - Any point in this region is achievable by routing over at most $|V| \times |S|$ depth-1 and depth-2 trees