Computationally-Feasible Truthful Auctions for Convex Bundles

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Abstract

In many economic settings, like spectrum and real-estate auctions, geometric figures on the plane are for sale. Each bidder bids for his desired figure, and the auctioneer has to choose a set of disjoint figures that maximizes the social welfare. In this work, we design mechanisms that are both incentive compatible and computationally feasible for these environments. Since the underlying algorithmic problem is computationally hard, these mechanisms cannot always achieve the optimal welfare; Nevertheless, they do guarantee a fraction of the optimal solution. We differentiate between two information models - when both the desired figures and their values are unknown to the auctioneer or when only the agents' values are private data. We guarantee different fractions of the optimal welfare for each information model and for different families of figures (e.g., arbitrary convex figures or axis-aligned rectangles). We suggest using a measure on the geometric diversity of the figures for expressing the quality of the approximations that our mechanisms provide.

Key words: Mechanism Design, Auctions, Approximation Algorithms

1 Introduction

In many environments, individuals have preferences over bundles of items. *Combinatorial Auctions* take into account such preferences, and auction bundles instead of auctioning each item separately. In some of these environments,

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the preferences of the bidders are structured in a very particular way: the bidders desire areas on the plane, i.e., the bidders bid for geometric figures and the auctioneer determines a set of disjoint winning figures and the monetary transfers. Prominent examples for such auctions are the FCC spectrum auctions, in which the bidders compete for geographical broadcasting areas. These multi-billion dollar auctions involve thousands of distinct combinations of licenses. Many mechanisms were suggested for these spectrum auctions, and these mechanisms encounter severe economic and computational difficulties (see survey in Cramton et al. (2006)). Other important recent examples are the Adwords auctions run by Google, in which firms should specify the geographical area in which they would like to bid for a particular word (for example, one could bid for having his advertisment appear whenever the word "taxi" is queried from California). Similarly, selling the right to advertise as a banner in popular websites also takes into account the georgarphical location of the viewers. Other examples for settings in which the bidders bid for geometric figures may be auctions for real-estate lots, selling licenses for location-based services (e.g., selling advertising space in a big city) or selling advertisements in newspapers. In these settings, agents may have different preferences about the properties of their desired figures: the size of the figure, its location on the page, whether the figures are rectangular, square, or elliptic etc.

The design of such mechanisms involves both game-theoretic and computational difficulties, and these must by addressed together, when designing dominant-strategy incentive-compatible mechanisms that are also computationally feasible (that is, run in polynomial time). Due to the *revelation principle*, we can limit our search to direct-revelation mechanisms - where bidders are asked to submit their private data to the auctioneer. Thus, in an *incentive-compatible* mechanism, a dominant strategy of each agent is to report her true secret data. A general scheme for achieving welfare-maximizing incentive-compatible mechanisms is the family of Vickrey-Clarke-Groves (VCG) mechanisms (see Vickrey (1961); Clarke (1971); Groves (1973)). However, for implementing such mechanisms the goods must be allocated optimally (otherwise the mechanisms would not necessarily be truthful, see Nisan and Ronen (2001)). As the underlying optimization problem of finding the optimal welfare is computationally hard ("NP-hard" even for the simple case where all the figures are 2×2 squares, Rothkopf et al. (1998)), it is infeasible to build mechanisms that are based on exactly solving the optimization problem.[1]

To overcome the computational hardness, the theory of computer science suggests looking for computationally-feasible algorithms that guarantee some fraction of the optimal solution (an "approximation").[2] In this paper we sug-

[1] Note that the running time of the "brute-force" algorithm for these problems, which enumerates on all possible choices of subsets, clearly grows exponentially with the number of bidders.

[2] Under the standard computational conventions, an algorithm is considered com-

gest exploiting some specific geometric characteristics of the bids, to better handle the challenges in such complex environments. Under reasonable geometric assumptions on the structure and the diversity of the bids (e.g., that the desired figures are convex), we aim to construct dominant-strategy incentive-compatible approximation mechanisms, and to characterize the worst-case fraction of the welfare they achieve. Our main goal in this paper:

Design mechanisms for selling figures on the plane that achieve a "good" approximation for the social welfare, and that are both computationally feasible and incentive compatible.

As we use algorithms that do not solve the optimization problem exactly, our incentive-compatible mechanisms are non-VCG. Almost all the incentive-compatible non-VCG mechanisms currently known are for models where the agents hold one-dimensional private values (*single-parameter* models). A classic paper by Roberts (1979) and a recent work by Lavi et al. (2003), show that in many reasonable settings, essentially no incentive-compatible mechanisms exist for general multi-parameter models, except the family of VCG mechanisms. As mentioned, the VCG mechanisms are computationally infeasible in many important settings. Indeed, we hardly know of non-trivial dominant-strategy implementations for multi-parameter domains (one rare example is in Bartal et al. (2003)). On the other hand, a multitude of non-VCG incentive-compatible mechanisms exist for single-parameter models (e.g., Feigenbaum et al. (2001); Archer and Tardos (2002), see more details below.), The importance of constructing non-VCG mechanisms was observed in Nisan and Ronen (2001), who were the first to consider mechanism-design problems involving a computational perspective.

1.1 Single-Minded Bidders

In light of the impossibilities above, we assume that each agent is interested in a single figure. Lehmann et al. (2002) initiated the study of the *Single-Minded Bidders* model for combinatorial auctions. Each single-minded bidder desires one specific bundle, and has the same positive value for all its supersets. Subsequent papers further studied combinatorial auctions with single-minded bidders (Mua'lem and Nisan (2002)), and extended the model to combinatorial auctions with duplicates (Archer et al. (2004)) and for the auctions in the supply chain (Babaioff and Nisan (2004); Babaioff and Walsh (2005)). We

putationally feasible if it runs for a number of steps that is bounded by some polynomial function of the parameters of the problems (e.g., number of players or number of items). If the worst-case running time of an algorithm grows exponentially with these parameters, then it is considered computationally infeasible. It is commonly believed that every problem that is "NP-hard" cannot be solved by a polynomial-time algorithm. See Cormen et al. (2001) for an introduction for computational analysis of algorithms.

also studied single-minded combinatorial auctions, but for geometric figures on the plane (rather than for discrete set of goods, as in the previous papers). Note that we actually auction an infinite, even uncountable, number of goods.

In our model, the plane ($\mathbb{R}^2$) is for sale. Each agent has a private value for a single compact figure and for any figure that contains it (he has a zero value for any other figure). We study several restrictions on the figures desired by the agents, e.g., convex figures, rectangles, and axis-aligned rectangles. Each agent submits a bid for his desired figure (e.g., an advertiser may want the bottom half of the first page in a newspaper). After receiving all bids, the auctioneer determines the set of winning agents and the payment that each winner should pay. Note that the agents demand figures in fixed locations in the plane, and the figures cannot be rotated of translated.

In this paper, we differentiate between two information models (as was previously done in, e.g., Mua'lem and Nisan (2002); Archer et al. (2003); Babaioff and Walsh (2005)). In the *Known Single-Minded (KSM)* model, the auctioneer knows which figure each agent wants, but does not know how much this agent is willing to pay for this figure. In the *Unknown Single Minded (USM)* model, both the figures and the values are private information. In other words, in the KSM model we should motivate an agent to truthfully declare his true value, where in the USM model an agent might submit untruthful bid both for his desired figure and his value. The USM model is therefore "almost" a single parameter model – the only private information of an agent, except of the value, is the identity of his desired figure. Nevertheless, as we later see, the additional private information seems to make the mechanism design problem harder to solve, and we get different results for the two models. Like previous papers that presented different results for the KSM and the USM models, we have not been able to *prove* that such a difference really exists, and this is still an interesting open question. In some cases, our results for the two models differ significantly - even by an exponential factor.

The characterization of incentive compatibility in the KSM model is well known: a necessary and sufficient condition is that the allocation function will hold the *bid-monotonicity* property, i.e., if a winning agent raised his bid when all other bids are fixed, he would still win his figure. Then, the necessary payments for incentive compatibility are uniquely defined (for normalized models, with losers paying 0) – each agent pays exactly the minimal bid for which he would still win. Achieving incentive compatibility in the USM model, however, is harder, since a bidder may also bid for a *superset* of his desired bundle and it is possible that such bid decreases his payment. Therefore, bid-monotonicity alone is not sufficient in the USM model. Lehmann et al. (2002) presented a characterization of incentive-compatible mechanisms for the USM model. [3]

[3] Babaioff et al. (2005) consider an abstract model with agents that have the same value for all desire outcomes, and the set of desired outcomes is private information. This model generalizes the USM model. The paper presents a characterization of

Basically, for a USM mechanism to be incentive compatible, the mechanism must be monotonic and also make sure that by untruthful bidding for some superset of his desired bundle, an agent either becomes a loser or pay at least what he would have paid for his desired bundle.

1.2 The Geometric Model

From the computational perspective, the welfare-maximization problems in our model belong to a family of hard problems known as "packing" problems - problems in which optimal subsets (with maximal sum of weights) of non-conflicting elements should be chosen. Approximation algorithms for "weighted-packing" problems of geometric shapes on the plane were previously studied, mainly for axis-aligned squares and rectangles. Hochbaum and Maass (1985) and Hochbaum (1997) studied the packing of equi-sized squares, and generalizations for arbitrary squares appear in, e.g., Erlebacj et al. (2001) and Chan (2003). Khanna et al. (1998) used similar methods in a model where axis-aligned rectangles lie in a $n \times n$ grid, and they presented a polynomial time algorithm that achieves an $O(log(n))$-approximation for the optimal welfare [4] and similar problems were also studied in Berman et al. (2001) and Agarwal et al. (1998). We are not aware of any previous approximation schemes for packing arbitrary convex figures (or even rectangles that are not necessarily axis-aligned).

Unlike the previous work, our results allow a large range of figures, without being restricted to axis-aligned squares and rectangles. In our basic model, each agent is interested in some arbitrary *compact convex* figure in $\mathbb{R}^2$. In all the real-life settings presented above, bidders typically bid for sufficiently-"thick" figures - e.g., building a house on a long and winding area is impractical. In many of these settings, a typical figure will even be convex (or without "holes" inside it), but our general results only require that the area of the figures will capture a constant fraction of the area of their convex hull. We call such figures *massive* figures, and in particular, convex figures are massive. In other words, we allow the figures to exhibit non-convexity as long as it is not extreme (e.g., a very narrow banana-shaped figure).

The quality of the approximation ratio that we achieve is closely related to two geometric parameters of the figures demanded by the agents: to the maximal number of disjoint figures that a figure can intersect, and to the ratio between the area of a figure and the total area of the figures that intersect it. For convex (or massive) figures, we are able to bound the parameters above and to use these bounds to attain the approximation results. For these bounds, we

incentive compatibility in such domains, that generalize the USM model described above.

[4] That is, they achieved at least a $\frac{c}{logn}$ fraction of the optimal result, for some constant $c > 0$.

suggest using a measure R on the diversity of the figures. We define the *aspect ratio* R of some family of figures, to be the ratio between the maximal *diameter* of a figure in the family, and the minimal *width* of a figure in the family.[5] In all our mechanisms, as the diversity of the figures' dimensions increases (i.e., R grows), the approximation ratio we guarantee degrades. Given that all the figures have diameters and widths with similar sizes, all our mechanisms always achieve a constant fraction of the optimal welfare (that does not depend, for example, on the number of figures).

1.3 Our Contribution

We study different families of figures, and for each family we construct incentive-compatible mechanisms that guarantee some fraction of the social welfare as a function of the aspect ratio R. We give different results for the KSM model and for the USM model, all attained with dominant-strategy equilibria and admit ex-post individual rationality. Note that unlike Bayesian models, and in the spirit of Wilson's doctrine (Wilson (1987)), we use the strong dominant-strategy equilibrium concept, with no distributional assumptions.

All our results for the USM model are corollaries of a general parameterized theorem presented in Section 4, that holds for any family of compact figures. This general theorem characterizes the approximation ratio attained by a wide range of variants of the "greedy" algorithm (introduced in Lehmann et al. (2002)), according to few geometric parameters of the figures. For each family of figures (e.g., convex figures or rectangles), we find the appropriate geometric parameters and then apply the general result to derive the approximation achieved for the specific family.

We present mechanisms for the following three families of figures. As mentioned, we use a standard computer-science notation and say that a mechanism achieves a c-approximation if it runs in polynomial time and guarantees at least $\frac{1}{c}$ of the optimal social welfare.

- **Massive figures and arbitrary convex figures**: For massive figures, and in particular for convex figures, we present an $O(R^{4/3})$-approximation in a mechanism that is incentive-compatible in the USM model. In the KSM model, we achieve a better $O(R)$-approximation – for which we develop a novel algorithm for "packing" arbitrary convex bundles. As far as we know, this is the first treatment for this computational problem.
- **Arbitrary rectangles**: We present an $O(R)$-approximation for the USM model. (We do not have an improved approximation for the KSM model.)

[5] The diameter of a compact figure is the maximal Euclidean distance between two points in the figure. The width of a figure is the distance between a closest pair of parallel lines bounding the figure.

- **Axis-aligned rectangles**: We observe that a mechanism, based on an algorithm by Khanna et al. (1998) (with few modifications), achieves an $O(\log R)$-approximation for the optimal welfare. Since their algorithm is "bid-monotonic", this mechanism is incentive-compatible in the KSM model. For the USM model, we do not know how to improved upon the mechanism for arbitrary rectangles. Thus, for axis-aligned rectangles, the gap between the approximation ratio that we achieve in the KSM model and in the USM model is exponential.

From an algorithmic perspective, most of our mechanisms use the family of greedy algorithms presented by Lehmann et al. (2002) as building blocks. They presented an incentive-compatible approximation mechanism for combinatorial auction with m items where the bidders are single minded. Their mechanism chooses the figure with the highest *normalized* value to be a winner, removes all the figures that are in conflict with it, and proceeds until no figures are left. For taking into account the size of each bundle, in addition to each value, they normalized each value by $|S|^\alpha$, where $|S|$ is the number of items in the bundle S and α is some real constant. Lehmann et al. (2002) showed that choosing $\alpha = \frac{1}{2}$ guarantees the best approximation ratio that is achievable in polynomial time (a fraction of $\frac{1}{\sqrt{m}}$ of the optimal welfare, where m is the number of goods m) for combinatorial auctions.[6] We use similar algorithms that normalize the values by the *geometric area* of the figures, that is, given $\alpha \in \mathbb{R}$ we assign a normalize value of $\frac{v}{a^\alpha}$ to a bid with a value of v for a figure with a geometric area of a. We show that, somewhat surprisingly, for compact convex figures the optimal value of α is $\frac{1}{3}$.

We also treat a *discrete model*, in which the plane contains predefined atomic building blocks (or tiles), and each agent bids for a set of building blocks. For example, in a spectrum auction, each state or county might be considered as a basic tile that cannot be partitioned. The resolution of the building blocks plays an important role in our analysis: if the ratio between the minimal width of a figure and the dimensions of the tiles (this ratio is denoted by Q) is smaller than the aspect ratio R, we can achieve an approximation of $O(R \cdot Q^{\alpha^*})$, where $\alpha^* = \frac{log(R)}{2log(R)+log(Q)}$. This is an improvement over the $O(R^{4/3})$-approximation achieved for the continuous model. A conclusion from our result in the discrete model is that if the items in specific combinatorial-auction environments can be embedded on the plane (e.g., in spectrum auctions), and if each agent bids for a set of items contained in some convex figure, then our mechanism may improve the approximation ratio achieved by Lehmann et al. (2002).

[6] The algorithm of Lehmann et al. (2002) is optimal if "NP-hard" problems cannot be solved on polynomial time, that is, if $P \neq NP$ (Zuckerman (2005)). The "$P = NP$?" question is probably the most important open question in Computer Science, and it is widely believed that $P \neq NP$.

The paper is organized as follows: Section 2 presents the formal model. Section 3 demonstrates our main techniques by presenting a mechanism for selling convex figures in the USM model. Section 4 describes a general theorem that quantifies the approximation achieved by greedy mechanisms in a general setting. This general theorem is used in Section 5 for proving our approximation results for different families of figures in the USM model. In Section 6 we present improved mechanisms that are incentive-compatible only in the KSM model. We conclude with a list of open questions in Section 7.

2 Model

Let B denote the family (set) of ***bids*** (figure-value pairs) of the set of agents N,[7] that is $B = \{(s_i, v_i)|i \in N\}$. Let F denote the family of agents figures, i.e., $F = \{s_i|i \in N\}$. Given a family of bids B, we aim to maximize the ***social welfare***, i.e., find a collection of non-conflicting bids (bids for disjoint figures) that maximizes the sum of valuations. For a subset $C \subseteq N$ of agents with disjoint figures, denote the value of C by $V(C) = \sum_{i \in C} v_i$. The maximal social welfare, $V(OPT)$, is the welfare achieved by the family OPT, which satisfies

$$V(OPT) = \max_{C \subseteq N| \ \forall i,j \in C \ \ s_i \cap s_j = \emptyset} V(C)$$

A mechanism decides on the set of winners and the payments. A winners receives his reported figure, while a loser receives the empty set.

Definition 1 *A* ***mechanism*** *consists of a pair of functions* (G, P) *where:*

- G *is an* ***allocation scheme*** *which given a profile of bids* B *outputs a set of winners* $G(B) \subseteq N$*, with disjoint figures. That it, for every* $i, j \in G(B)$*, if* $i \neq j$ *then* $s_j \cap s_i = \emptyset$.
- P *is a* ***payment scheme****, where for any profile of bids* B*,* $P(B) \in \mathbb{R}^n_+$*, i.e., agent* i *pays* $P_i(B)$.

All the payment schemes that we consider are *normalized*, that is, a losing agent pays zero. We assume quasi-linear utilities and that the agents have no externalities (the utility for each agent does not depend on the packages received by the other agents), i.e., the utility $u_i(B)$ of agent i is 0 if he is a loser, and $v_i - P_i(B)$ if he is a winner. The agents are rational, so each agent acts to maximize his utility. We use the notation $v_{-i} = (v_1, ..., v_{i-1}, v_{i+1}, ..., v_n)$ to denote the values of all agents but i, and similarly, B_{-i} is the family of all bids except i's bid.

A mechanism is *incentive-compatible (IC)* if declaring their true secret infor-

[7] Since all the mechanisms we consider are incentive-compatible, we use the same notation for the secret information and the declared information (bid), except of the proofs of incentive compatibility.

mation is a dominant strategy for all the agents. Note that we do not assume any prior distributions on the values, but rather use a stronger concept of dominant-strategy incentive compatibility. In the KSM model, it means that for any set of values reported by the other agents, each agent cannot achieve a higher utility by reporting an untruthful value, i.e.,

$$\forall i \quad \forall B_{-i} \quad \forall v_i' \quad u_i((s_i, v_i), B_{-i}) \geq u_i((s_i, v_i'), B_{-i})$$

In the USM model, IC means that the best strategy of each agent is to report *both* his figure and his value truthfully, regardless of the other agents' reports, i.e.,

$$\forall i \quad \forall B_{-i} \quad \forall v_i', s_i' \quad u_i((s_i, v_i), B_{-i}) \geq u_i((s_i', v_i'), B_{-i})$$

Clearly, if a mechanism is truthful in the USM model, it is also truthful in the KSM model. An incentive-compatible mechanism is also *individually rational (IR)* if for any agent i, bidding truthfully ensures him a non-negative utility. That is, $\forall i \quad \forall B_{-i} \quad u_i((s_i, v_i), B_{-i}) \geq 0$. We use the term *truthful mechanism* to describe an incentive-compatible and individually-rational mechanism.

2.1 Geometric Definitions

We state our approximation bounds as functions of a few geometric properties of the figures the agents bid for. We use standard definitions of *diameter* and *width* of compact figures in $\mathbb{R}^2$ (see Figure 1):

Definition 2 *The* ***diameter*** $\mathbf{d_z}$ *of a compact figure z is the maximal distance between any two points in the figure, i.e., $d_z = \max_{p_1, p_2 \in z} ||p_1 - p_2||$ (when $||p_1 - p_2||$ is the Euclidean distance between p_1, p_2). The* ***width*** $\mathbf{w_z}$ *of a compact figure z is the minimal distance between a pair of parallel lines such that the compact figure z lies between them*[8] *.*

Definition 3 *Given a family of compact figures F in $\mathbb{R}^2$, let* L *be the maximal diameter of a figure in F, and let* W *be the minimal width of a figure in F. The* aspect ratio **R** *of F is the ratio between the maximal diameter and the minimal width. That is,*

$$L = \max_{z \in F} d_z, \quad W = \min_{z \in F} w_z, \quad R = \frac{L}{W}$$

The aspect ratio describes how diverse is the family of figures with respect to the figures' diameters and widths. If W is much smaller than L it follows that a "large" number of disjoint figures may intersect a single figure, and this seems to make our problem harder to approximate. Note that L, W and R are properties of F, but we omit the parameter F to simplify the notation.

[8] The width also equals the shortest projection of the figure on any direction.

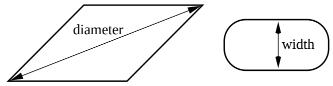

Fig. 1. The width of a figure is the minimal distance between parallel lines bounding the figure, and the diameter of a figure is the maximal distance between two points in this figure. Assume that the left figure has diameter 10 and width 7, while the right figure has diameter 6 and width 2, then the aspect ratio R of this family of two figures is $\frac{10}{2} = 5$.

Denote the *geometric area* of a compact figure z by $q(z)$, and denote its perimeter by p_z (by a slight abuse of notation we use p_z both for the set of points in the perimeter and the perimeter length).

A computational remark: in our analysis, we assume that the geometric data can be easily accessed: we assume that the diameter, width and area of any agent's figure are computable in polynomial time and also that given any two figures, we can decide in polynomial-time if the two figures are disjoint. [9]

2.2 The Greedy Mechanism

The work of Lehmann et al. (2002) have presented a family of *greedy mechanisms* for (discrete) combinatorial auctions. This greedy framework, with small changes to fit the geometric setting, is the framework we use to design our mechanisms for the USM model.

The greedy algorithm:

- **Input:** *A profile of bids for non-empty figures B and some function f on the figures, such that f assigns a positive value to any figure.*
- **Output:** *An allocation (set of winners) ALG.*
- **The greedy allocation algorithm:**
  - · *Sort the bids from high to low according to their values normalized by f (i.e., $\frac{v_1}{f(s_1)} \geq \frac{v_2}{f(s_2)} \geq \ldots \geq \frac{v_n}{f(s_n)}$).*
  - · *While the list is not empty, choose a figure s_i for which the normalized value is the highest, among the bids that are still on the list (with a consistent tie breaking). Add agent i to the allocation ALG and update the list by removing all the bids for figures that intersect s_i.*

The actual algorithm that is used is determined by the choice of the norm function f. Lehmann et al. suggested using the norm $\frac{v_s}{|s|^\alpha}$ for combinatorial auctions, where $|s|$ is the size of the package s and α is some non-negative constant. We generalize this method for compact figures in $\mathbb{R}^2$ and define the **α-greedy algorithm** to use the norm $\frac{v_z}{q(z)^\alpha}$, where $q(z)$ is the area of figure

[9] Note that for polygons the above assumptions hold, and that any compact convex figure can be approximated (as well as one wants) by a polygon.

z.[10] Using the monotonicity properties of this algorithm (see Section 6), we can set up a payment scheme that creates an incentive-compatible mechanism:

Definition 4 *The α-**greedy mechanism** uses the α-greedy algorithm as its allocation scheme, with the following payment scheme: Assume that agent i wins. Let j be the first agent to win, among all the agents whose figures intersect agent i's figure, when the greedy algorithm runs without i. If such j exists, i pays $\frac{q(s_i)^\alpha \cdot v_j}{q(s_j)^\alpha}$, otherwise i pays 0. Losing agents pay 0.*

The properties of the α-greedy mechanisms, proved in Lehmann et al. (2002), also hold in our model:

Theorem 1 *(essentially due to Lehmann et al. (2002)) For every α, the α-greedy mechanism is a polynomial-time truthful mechanism for agents bidding for compact figures in the USM model.*

Throughout the paper, we use standard notations for asymptotic analysis of approximation algorithms. Let $f(R)$ be some non-negative real function of R. A mechanism achieves an $O(f(R))$-approximation, if for some constant $c > 0$ and for any large enough[11] R, the mechanism achieves *at least* a $\frac{c}{f(R)}$-fraction of the optimal welfare for *any* family of bids with an aspect ratio R. For example, if we design a mechanism that for any given family of figures with $R \geq 3$ always achieves a social welfare of at least $\frac{10}{\log R} \cdot V(OPT)$, we say that this algorithm achieves an $O(\log R)$-approximation.

3 Unknown Single-Minded Model for Convex Figures

In Section 4, we describe a parameterized approximation result for the maximal welfare in a general model, and in Section 5 we use this result to construct several approximation mechanisms for different families of figures in the USM model. But first, in this section, we would like to demonstrate the outline of the proof of the general result, by explicitly proving an $O(R^{\frac{4}{3}})$-approximation for unrestricted convex figures. This approximation will be formally proved in Section 5, but the sketch of the proof given in this section should illustrate the intuition for the more abstract arguments presented in Section 4. Later in this section, we also show that no other greedy algorithm can achieve better than this $O(R^{\frac{4}{3}})$-approximation; the proofs shed some light on the nature of the problems study in this paper.

To prove the approximation result, we use a few elementary geometric properties of compact convex figures. First, for any compact convex figure z, the

[10] For example, the 0-greedy algorithm sorts the figures according to their values, and disregards the area. The 1-greedy algorithm sorts the figures according to their value per unit of area, giving priority to small area figures.
[11] That is, there exist some R_0 such that the approximation holds for any family of figures with aspect ratio $R > R_0$.

area of z is of the same order as the product of its diameter and width, $d_z \cdot w_z$. Additionally, the perimeter of z is continuous, and is of the same order as the diameter of z These properties are sufficient for the approximation to hold.

Theorem 2 *Assume that the agents bid for a family of compact convex figures in the USM model. Then, the* $\frac{1}{3}$ - greedy mechanism *is a polynomial-time truthful mechanism that achieves an* $O(R^{\frac{4}{3}})$*-approximation for the social welfare. This is the best asymptotic approximation ratio achievable by an* α*-greedy mechanism (for any* α*).*

Proof sketch: As mentioned, we only present a sketch of the proof to illustrate our main techniques; for the full proof we refer the reader to Section 5.1. By Theorem 1, any α-greedy mechanism is a polynomial-time truthful mechanism. Next, we present the idea behind the proof of the $O(R^{\frac{4}{3}})$-approximation ratio. Lemma 3 below shows that this is better than what any other α-greedy mechanism can achieve.

By the definition of the α-greedy algorithm, any figure is either a winner or intersects a winner. As we will show in the formal proof, it is sufficient to prove the approximation bound for the case where no winner in the optimal solution wins in the α-greedy mechanism (i.e., $OPT \cap ALG = \emptyset$). We map each agent $x \in OPT$ to a winner $z \in ALG$ that intersects x. We then bound the sum of values of any disjoint set of agents' figures that intersects z: we separately bound the total value of the figures that are contained in z and the figures that are not.

Consider a set of disjoint figures $C(z)$ that are contained in z. We first note that due to the definition of the aspect ration R, the size of this set of figures is at most $2R^2$: the diameter of z is at most L, therefore its area is not larger than L^2. The width of any figure in $C(z)$ is at least W, therefore its area is at least [12] $\frac{W^2}{2}$. This implies that we cannot pack more than $L^2/\left(\frac{W^2}{2}\right) = 2R^2$ figures inside z, hence, $|C(z)| \leq 2R^2$.

Since z is a winner, by the properties of the $\frac{1}{3}$-greedy algorithm, for any $x \in C(z)$ it holds that $\frac{v_x}{q(x)^{\frac{1}{3}}} \leq \frac{v_z}{q(z)^{\frac{1}{3}}}$. Summing over all $x \in C(z)$ implies the following leftmost inequality:

$$\sum_{x \in C(z)} v_x \leq \frac{v_z}{q(z)^{\frac{1}{3}}} \cdot \sum_{x \in C(z)} q(x)^{\frac{1}{3}} \leq \frac{v_z}{q(z)^{\frac{1}{3}}} \cdot \left(\sum_{x \in C(z)} q(x) \right)^{\frac{1}{3}} \cdot |C(z)|^{\frac{2}{3}}$$
$$\leq \frac{v_z}{q(z)^{\frac{1}{3}}} \cdot q(z)^{\frac{1}{3}} \cdot 2^{\frac{2}{3}} \cdot R^{\frac{4}{3}} \leq 2R^{\frac{4}{3}} \cdot v_z$$

The second leftmost inequality is a result of Hölder inequality (Theorem 19). The third inequality is due to the observation that $|C(z)| \leq 2R^2$ and the fact that for a family $C(z)$ of disjoint figures contained in z, $\sum_{x \in C(z)} q(x) \leq q(z)$.

[12] A formal statement of this simple fact is given in Proposition 15.4 in the appendix.

Thus we have proved that the sum of values of any set of disjoint figures that are contained in z is bounded by $2R^{\frac{4}{3}}$ times v_z, which derives that the $\frac{1}{3}$-greedy mechanism achieves $2R^{\frac{4}{3}}$-approximation for the total value of any disjoint set of figures that are contained in z.

The proof of the equivalent bound for any family of disjoint figures that intersects z but are not contained in it is more involved. We partition these figures to classes according to their diameters, in exponentially growing intervals. We show that there are at most $O(R)$ figures in each of these classes, then we use Hölder inequality and some algebra to derive the theorem. (The full proof is given in Sections 4 and 5.) $\square$

The obvious question is whether a mechanism that runs in polynomial time and is also incentive-compatible in the USM model can achieve better than $O(R^{4/3})$-approximation. While we could not give a complete answer for this question, we do show that no α-greedy mechanism achieves better than the approximation ratio above. We even prove a stronger result (Lemma 14 in Appendix A.1), showing that an approximation asymptotically better than $R^{\frac{4}{3}}$ cannot be achieved, even if we allow α to be chosen as a function of R (as opposed to being fixed in advance).[13] Here, we prove that any α-greedy mechanism cannot do better than $O(R^{4/3})$-approximation.

For proving this hardness results, we recognize two extreme fatal cases, in which the α-greedy algorithm needs to handle. We give explicit constructions of figures in the plane that form these hard scenarios; one is hard for α-greedy algorithms with small α's and the other is hard for large α's. That is, we show that for any α, the α-greedy algorithm cannot achieve better than $R^{2(1-\alpha)}$ approximation in one case and $R^{1+\alpha}$ approximation in the other case. Since we use a worst-case approximation measure, any α-greedy algorithm will achieve no better than $\max\{R^{2(1-\alpha)}, R^{1+\alpha}\}$ approximation. This expression is minimized when $\alpha = \frac{1}{3}$, implying a $R^{\frac{4}{3}}$ lower bound for the approximation achievable by α-greedy algorithms.

Each of the two lower bounds is achieved by a construction of figures and values that can be built for any sufficiently-large R. The basic idea in both examples is similar: there is one figure z that is chosen by the greedy mechanism, while the socially optimal mechanism choses a family of disjoint figures intersecting z. This is done by assigning a value to z such that its normalized value ($\frac{v_z}{q(z)^\alpha}$) is a bit greater than 1, and the rest of the figures are assigned normalized value of exactly 1. This way, the algorithm achieves the value of the figure z, instead of the sum of values of the intersecting figures, and the approximation lower bound follows.

The first hard instance (Figure 2, left) is composed of one large figure, which

[13] In Section 5.3 we show that for the discrete model, this is not the case: chosing α as a function of the figures diversity improve the asymptotic approximation in the discrete model.

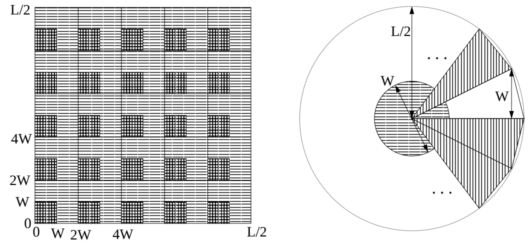

Fig. 2. Approximation bounds for the α-greedy mechanism

contains many (the largest possible number, an order of R^2) small figures. For small α's, the effect of normalizing the value by the geometric areas is small, therefore the algorithm tends to pick large figures. For instance, if $\alpha = 0$ and all the figures have a value of 1, their normalized values will also be 1, and the algorithm achieves a fraction proportional to $\frac{1}{R^2}$ of the optimal welfare. For this construction, we show that the α-greedy algorithm cannot achieve more than a fraction proportional to $\frac{1}{R^{2(1-\alpha)}}$ of the optimal welfare, and we also observe that the approximation improves as α increases.

The second instance (Figure 2, right) is composed of one small figure, intersecting many (the largest possible number, an order of R) large figures. For large α's, the area of the shapes have significant effect on the normalized values of the shapes, and therefore, small figures will be chosen more easily. For this construction, we show that the α-greedy algorithm cannot achieve more than a fraction proportional to $\frac{1}{R^{1+\alpha}}$ of the optimal welfare, and observe that the approximation improves as α decreases.

The two constructions together form the lower bound. Next, we state and formally prove the lower bound.

Lemma 3 *For compact convex figures, no α-greedy mechanism achieves better than an $O(R^{\frac{4}{3}})$-approximation.*

Proof: As explained, this result is a corollary of the two claims below.

Claim 1 *For compact convex figures and for any fixed $\alpha < 1$, no α-greedy mechanism achieves better than $O(R^{2(1-\alpha)})$-approximation.*

Proof: The left part of Figure 2 illustrates the construction we use in our proof. We consider a large square z of side-length $\frac{1}{2}L$ that contains $\frac{R^2}{16}$ small squares of side-length W (assuming that $L \geq 2W$). Note that z's area is $\frac{L^2}{4}$ and the total area of the small squares is $\frac{R^2}{16} \cdot W^2 = \frac{L^2}{16}$. The small squares can be separated by intervals of length W ($\frac{L}{2} \cdot \frac{1}{2W}$ squares in each row or column, see Figure 2). [14]

Assume now that $v_z = (\frac{L}{2})^{2\alpha} + \epsilon$ for some $\epsilon > 0$, and each square $x \neq z$ has

[14] Note that z's diameter is actually $\frac{L}{\sqrt{2}} < L$, but we disregard the constants in this analysis.

a value of $v_x = W^{2\alpha}$. Then, for any α,

$$\forall x \neq z \quad \frac{v_x}{q(x)^\alpha} = \frac{W^{2\alpha}}{(W^2)^\alpha} = 1 < \frac{(\frac{L}{2})^{2\alpha} + \epsilon}{((\frac{L}{2})^2)^\alpha} = \frac{v_z}{q(z)^\alpha}$$

We see that the normalized value of the large square z is greater than the normalized values of the small squares Hence, only z will be picked by the α-greedy allocation algorithm. The ratio between the optimal welfare and the welfare achieved by the greedy algorithm is therefore:

$$\frac{V(OPT)}{V(ALG)} = \frac{\frac{1}{16}R^2 \cdot W^{2\alpha}}{(\frac{L}{2})^{2\alpha} + \epsilon}$$

And this ratio approaches $\frac{4^\alpha}{16}R^{2(1-\alpha)}$ as ϵ goes to zero. We conclude that for any $\alpha < 1$, the α-greedy mechanism cannot achieve an approximation asymptotically better than $\frac{4^\alpha}{16}R^{2(1-\alpha)}$, which is $O(R^{2(1-\alpha)}$. □

Claim 2 *For compact convex figures and any fixed $\alpha > 0$, no α-greedy mechanism achieves better than $O(R^{1+\alpha})$-approximation.*

Proof: The right part of Figure 2 illustrates the construction we use in our proof. In this proof we use polar coordinates, and denote by (β, r) the point at angle β and distance r from $(0, 0)$.

We look at a disk z of radius $\frac{1}{2}W$ around $(0, 0)$. This disk intersects a set of n disjoint isosceles triangles (n will be chosen such that the width of each triangle at least W), each of which is contained and almost covers a wedge of the disk with equal arc length (we shrink the wedges a little to make them disjoint). Let $\epsilon > 0$ be some small real number. For triangle $j \in \{0, \ldots, n-1\}$ we define β_j to be $j \cdot \frac{2\pi}{n}$. Triangle j's vertices lie in the polar coordinates $(\beta_j + \frac{1}{2}\frac{2\pi}{n}, \epsilon)$, $(\beta_j + \frac{\epsilon}{2}, \frac{1}{2}L)$ and $(\beta_j + \frac{2\pi}{n} - \frac{\epsilon}{2}, \frac{1}{2}L)$.

We would like to set the height to the two equal sides of each triangle (that equals the width of the triangle) to be W. For this to happen, we need to choose n such that $\sin\frac{2\pi}{n} = \frac{W}{\frac{1}{2}L} = \frac{2}{R}$. Since $\lim_{x\to 0}\frac{x}{sin(x)} = 1$, it suffices to pick n such that $n \cong \pi \cdot R$.

Disk z's area is $\frac{\pi}{4}W^2$, and we set z's value to be $(\frac{\pi}{4}W^2)^\alpha + \epsilon$. Each triangle's area is $\frac{1}{4}WL$, and we set their values to be $(\frac{1}{4}WL)^\alpha$. Then, for any α, z will have higher normalized value than any triangle, and thus only z will be picked by the α-greedy algorithm. The ratio between the optimal welfare and the result of the α-greedy algorithm is therefore:

$$\frac{V(OPT)}{V(ALG)} = \frac{\pi R \cdot (\frac{1}{4}WL)^\alpha}{(\frac{\pi}{4}W^2)^\alpha + \epsilon}$$

And it approaches $\pi^{1-\alpha}R^{1+\alpha}$ as ϵ goes to zero. We conclude that for any $\alpha > 0$, the α-greedy mechanism cannot achieve an approximation asymptotically better than $\pi^{1-\alpha}R^{1+\alpha}$, which is $O(R^{1+\alpha})$. □

This concludes the proof of the lower bound on the approximation ratio attained by α-greedy algorithms. □

4 Analysis of the α-Greedy Algorithm in a General Model

In this section, we present a general, somewhat sophisticated, parameterized analysis of the performance of greedy algorithms for general families of compact figures in the plane. The theorem presented in this section characterizes the approximation ratio achieved by greedy mechanisms, as a function of several geometric properties of the figures. This theorem is the basis for all of our results for the USM model, described in Section 5. In Section 5 we compute the specific values for these parameters for each family of figures, and use the general theorem, presented in this section, to derive the approximation ratio that the α-greedy achieved for each family of figures.

In the general setting, a figure may be any compact subset of $\mathbb{R}^2$ (not necessarily convex or even connected). The algorithm receives as an input a finite family of compact figures and their weights, $\hat{F} = \{(s_i, v_i)\}_{i=1}^n$, where each v_i is the weight of a compact figure $s_i \subseteq \mathbb{R}^2$ (we slightly abuse notation and also denote the family of figures by $\hat{F}$). In Section 5 we will replace the abstract family $\hat{F}$ with several concrete families, each denoted by F.

The analysis of the α-greedy algorithm is based on the following technique. The optimal welfare is achieved by a family of disjoint figures with a maximal sum of values. In order to bound the result attained by the greedy algorithm with the total value of the optimal solution, we partition the figures that intersect each winning figure into two sets: those who are contained in the convex hull of this winner, and those who are not. We bound the value that can be achieved by each of these sets and, taken together, we derive the final bound for the approximation achieved by the greedy algorithm.

The bound on the first family is based on the fact that the convex hull of a figure cannot contain more than b disjoint figures, if each has at least b fraction of the area of the convex hull. In this case we say that the family of figures is b-balanced. Thus, b is the first geometric parameter in our analysis. The bound on the second family is more involved. The figures that intersect a winner are classified according to their diameters, with exponentially growing intervals. Two parameters are now of interest: First, the maximal number of figures in a class, which we denote by $t(\hat{F})$. Secondly, we will need to bound the ratio between the area of the figure and the total area of the intersecting family of figures. We use the notion of $\gamma(\hat{F})$ to denote the bound (see details below).

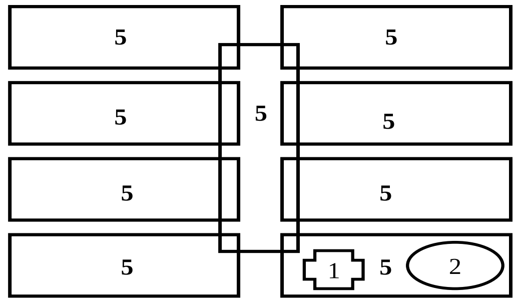

Fig. 3. A family of figures. The area of each figure is written inside it. The family is 5-balanced, since every figure that is contained in another figure x has at least 1/5 of the area of x. The family is also (64,8)-bounded (that is, it is γ-bounded for $\gamma = 64$, and the maximal class size is 8).

4.1 Definitions and Notations

We first present some definitions and notations. As mentioned, we compare the value of a single figure z to the total value of disjoint figures that intersect it. As mentioned, it turns out to be useful to separately analyze such figures that are fully contained in the convex hull of z (we denoted the convex hull of z by $CH(z)$) and between those that are partially outside this convex hull:

Definition 5 *Let $\hat{F}$ be a finite family of compact figures in $\mathbb{R}^2$. For any $z \in \hat{F}$,*

- *$Cont(z)$ denotes the collection of families of disjoint figures in $\hat{F}$, such that each figure intersects z and is contained in the convex hull of z. That is, the family of figures $C(z)$ belongs to the collection $Cont(z)$ if*
  $\forall x_1, x_2 \in C(z) \quad x_1 \cap x_2 = \emptyset$ and

$$C(z) \subseteq \{x \in \hat{F} \mid x \cap z \neq \emptyset \text{ and } x \subseteq CH(z)\}$$

- *$Inter(z)$ denotes the collection of families of disjoint figures in $\hat{F}$, such that each figure intersects z but is not contained in z's convex hull. That is, the family of figures $B(z)$ belongs to the collection $Inter(z)$ if*
  $\forall x_1, x_2 \in B(z) \quad x_1 \cap x_2 = \emptyset$ and

$$B(z) \subseteq \{x \in \hat{F} \mid x \cap z \neq \emptyset \text{ and } x \setminus CH(z) \neq \emptyset\}$$

The first parameter that we define considers figures that are contained in the convex hull of other figures. A set of figures is *b-balanced* if a figure cannot capture more than a $\frac{1}{b}$ fraction of the area of the convex hull of a figure that contains it:

Definition 6 *Let $\hat{F}$ be a finite family of compact figures in $\mathbb{R}^2$. The family of figures $\hat{F}$ is b-**balanced** if*

$$\forall \; x, z \in \hat{F}, \quad x \subseteq CH(z) \;\Rightarrow\; b \cdot q(x) \geq q(CH(z))$$

For example, the family of figures described in Figure 3 is 5-balanced, since every figure that is contained in another figure (e.g., the figures contained in the bottom-right rectangle), have at least 1/5 of the area of the containing figure (the convex hull of every rectangle is the rectangle itself).

We now define parameters for figures that are not contained in the convex hull of the figure that intersect them. For defining these parameters, we should first define a classification of the figures according to their diameters, with exponentially growing intervals.

Definition 7 *Let $\hat{F}$ be a family of compact figures in $\mathbb{R}^2$ with an aspect ratio R. For any $z \in \hat{F}$ and for any family of sets $B(z) \in Inter(z)$, the **partition-by-diameters** of $B(z)$ is a partition of $B(z)$ to $\log(R)+1$ classes, where for each $i \in \{0, \ldots, \log(R)\}$,*

$$B_z^i = \{x \mid x \in B(z) \text{ and } \mathrm{W} \cdot 2^{\mathrm{i}} \leq \mathrm{d_x} < \mathrm{W} \cdot 2^{\mathrm{i+1}}\}$$

For a figure z, the i-th class in the partition (B_z^i) includes a disjoint set of figures, each intersects z but is not contained in z's convex hull, and each has a diameter of at least $W \cdot 2^i$ and at most twice that length. Since for any figure x we have that $W \leq d_x \leq W \cdot R$, this is indeed a partition to disjoint sets.

We are now able to formally define two parameters that help us bounding the total value gained by such classes. The first parameter is the maximal number $t(\hat{F})$ of figures in a class, over all the figures $z \in \hat{F}$ and all the sets $B(z) \in Inter(z)$ (i.e., the sets of disjoint figures that intersect z but are not contained in it).

Definition 8 *The **maximal-class-size** of a family $\hat{F}$ is defined to be*

$$t(\hat{F}) = \max_{z \in \hat{F},\ B(z) \in Inter(z),\ i \in \{0,\ldots,\log(R)\}} |B_z^i|$$

For example, in the family of figures described in Figure 3, the maximal class size is 8, since the 8 rectangles intersecting the central rectangle are disjoint and will appear in the same class in the partition-by-diameters (they all have the same diameter). Other classes will clearly have a smaller size.

The second parameter is a bound on the ratio between the area of a figure and the total area of the figures that it intersects from every class defined above. We use the notion of $\gamma(\hat{F})$ to denote the bound.

Definition 9 *Let γ be a function that assigns a real number to every family of figures. We say that a family $\hat{F}$ of figures is **γ-bounded** if*

$$\forall\ z \in \hat{F} \quad \forall B(z) \in Inter(z) \quad \forall i \in \{0, \ldots, \log(R)\}$$

$$\gamma(\hat{F}) \geq \frac{R}{2^i} \cdot |B_z^i| \cdot \frac{\sum_{x \in B_z^i} q(x)}{q(z)}$$

A family of figures $\hat{F}$ is called (γ,t)-bounded*, if it is γ-bounded and its maximal-class-size $t(\hat{F})$ is at most t.*

For example, the family of figures described in Figure 3 is 64-bounded. We can check this on the set of eight disjoint rectangles intersecting the central rectangle. Their total area is $8 \cdot 5$ and the area of the central rectangle is 5, thus clearly $64 \geq 1 \cdot 8 \cdot \frac{8 \cdot 5}{5}$ (since all these rectangles have the maximal diameter, we clearly have $\frac{R}{2^i} = 1$).

4.2 The Main General Result

We are now ready to prove the approximation achieved by the α-greedy algorithm. This result is later used to approximate the optimal social welfare in different families of figures (like convex figures and rectangles), by providing the best possible bounds on the characteristic parameters of the family (i.e., providing γ, t and b for the family of figures). The remainder of this section presents our general result and its proof.

Theorem 4 *Let $\hat{F}$ be a finite family of compact figures in $\mathbb{R}^2$. Assume that $\hat{F}$ is (γ, t)-bounded and b-balanced. Then for any $\alpha \in (0, \frac{1}{2}]$, the α-greedy algorithm achieves a*

$$\frac{2^\alpha}{2^\alpha - 1} \cdot \gamma(\hat{F})^\alpha \cdot t(\hat{F})^{1-2\alpha} + b^{1-\alpha}$$

approximation to the maximal total value of any family of disjoint figures.

Proof: Let OPT be a family of disjoint figures with the maximal sum of values. Let ALG be the family of disjoint figures picked by the α-greedy algorithm. We first show that it is sufficient to prove the approximation when we disregard the figures that are in both allocations (OPT and ALG). Let $I = OPT \cap ALG$, $opt = OPT \setminus I$ and $alg = ALG \setminus I$. Denote the value of a set $S \in \hat{F}$ by $V(S) = \sum_{x \in S} v_x$. Then,

$$\frac{V(OPT)}{V(ALG)} = \frac{V(OPT \setminus I) + V(I)}{V(ALG \setminus I) + V(I)} \leq \frac{V(OPT \setminus I)}{V(ALG \setminus I)} = \frac{V(opt)}{V(alg)}$$

The inequality holds since all the figures' values are non-negative and $V(OPT) \geq V(ALG)$. We next turn to bound $\frac{V(opt)}{V(alg)}$ from above.

By the definition of the α-greedy algorithm, any figure $x \in \hat{F}$ is either a winner or intersects a winner. Since $opt \cap alg = \emptyset$, this implies that there are families of figures $\{C(z)\}_{z \in alg}$ and $\{B(z)\}_{z \in alg}$ (as defined in Definition 5) such that $opt \subseteq \bigcup_{z \in alg} (C(z) \cup B(z))$. Therefore, for these families of figures,

$$V(opt) = \sum_{z \in opt} v_z \leq \sum_{z \in alg} (\sum_{x \in C(z)} v_x + \sum_{x \in B(z)} v_x)$$

We use the two lemmas below to bound each summand of $\sum_{x \in C(z)} v_x + \sum_{x \in B(z)} v_x$ for every figure $z \in alg$. The theorem will follow from summing up these bounds for all $z \in alg$. We use the assumptions that family $\hat{F}$ is (γ, t)-bounded and b-balanced.

Lemma 6 below shows that for a (γ, t)-bounded family $\hat{F}$, for any $z \in alg$ and for any $\alpha \in (0, \frac{1}{2}]$,

$$\forall B(z) \in Inter(z) \quad \sum_{x \in B(z)} v_x \leq \frac{2^\alpha}{2^\alpha - 1} \cdot v_z \cdot \gamma(\hat{F})^\alpha \cdot t(\hat{F})^{1-2\alpha}$$

Lemma 5 below shows that for a b-balanced family $\hat{F}$, for any $z \in alg$ and for any $\alpha \in [0, 1]$,

$$\forall C(z) \in Cont(z) \quad \sum_{x \in C(z)} v_x \leq v_z \cdot b^{1-\alpha}$$

Summing over all $z \in alg$, we have that for every $B(z) \in Inter(z)$ and every $C(z) \in Cont(z)$ we have that:

$$\sum_{z \in alg} \Big(\sum_{x \in B(z)} v_x + \sum_{x \in C(z)} v_x \Big) \leq \sum_{z \in alg} \left(\frac{2^\alpha}{2^\alpha - 1} \cdot v_z \cdot \gamma(\hat{F})^\alpha \cdot t(\hat{F})^{1-2\alpha} + v_z \cdot b^{1-\alpha} \right)$$

$$= \left(\frac{2^\alpha}{2^\alpha - 1} \cdot \gamma(\hat{F})^\alpha \cdot t(\hat{F})^{1-2\alpha} + b^{1-\alpha} \right) \cdot V(alg)$$

The above holds, in particular, for the families of figures $\{C(z)\}_{z \in alg}$ and $\{B(z)\}_{z \in alg}$ such that $opt \subseteq \bigcup_{z \in alg} (C(z) \cup B(z))$. Thus,

$$V(opt) \leq \left(\frac{2^\alpha}{2^\alpha - 1} \cdot \gamma(\hat{F})^\alpha \cdot t(\hat{F})^{1-2\alpha} + b^{1-\alpha} \right) \cdot V(alg)$$

□

Note that this general theorem only holds for positive α which is not greater than $\frac{1}{2}$, but this suffices for all our mechanisms (according to our lower bound, other α's are asymptotically worse). In the remainder of this section we prove the two claims used in the proof of the above theorem.

Lemma 5 *Let $\hat{F}$ be a finite b-balanced family of compact figures in $\mathbb{R}^2$. For any $\alpha \in [0, 1]$, if the α-greedy algorithm picks $z \in \hat{F}$ as a winner, then for any $C(z) \in Cont(z)$ we have $\sum_{x \in C(z)} v_x \leq v_z \cdot b^{1-\alpha}$.*

Proof: We first show that for any $C(z) \in Cont(z)$, $|C(z)| \leq b$. For any $C(z) \in Cont(z)$, let $\bar{x} \in C(z)$ be a figure with minimal area, that is $q(\bar{x}) = \min_{x \in C(z)} q(x)$.

$$b \geq \frac{q(CH(z))}{q(\bar{x})} \geq \sum_{x \in C(z)} \frac{q(x)}{q(\bar{x})} \geq \sum_{x \in C(z)} 1 = |C(z)|$$

The leftmost inequality holds since $\hat{F}$ is b-balanced, thus, for $\bar{x} \in C(z)$ it holds that $\bar{x} \subseteq CH(z)$ and therefore $b \cdot q(\bar{x}) \geq q(CH(z))$. The next inequality holds since $C(z)$ is a set of disjoint figures, each contained in $CH(z)$, thus $q(CH(z)) \geq \sum_{x \in C(z)} q(x)$. The last inequality is due to the definition of $\bar{x}$ as the figure with minimal area in $C(z)$.

Since z is a winner, by the properties of the α-greedy algorithm, for any $x \in C(z)$ it holds that $\frac{v_x}{q(x)^\alpha} \leq \frac{v_z}{q(z)^\alpha}$. Summing over all $x \in C(z)$ implies the left inequality:

$$\begin{aligned}\sum_{x \in C(z)} v_x \leq \frac{v_z}{q(z)^\alpha} \cdot \sum_{x \in C(z)} q(x)^\alpha \quad &\leq \quad \frac{v_z}{q(z)^\alpha} \cdot \left(\sum_{x \in C(z)} q(x) \right)^\alpha \cdot |C(z)|^{1-\alpha} \\ \leq \frac{v_z}{q(z)^\alpha} \cdot q(z)^\alpha \cdot b^{1-\alpha} \quad &= \quad v_z \cdot b^{1-\alpha}\end{aligned}$$

where the second inequality is a result of Hölder inequality (see Theorem 19 in the Appendix). The third inequality is due to the observation that $|C(z)| \leq b$, combined with the assumption that $\alpha \in [0, 1]$. $\square$

Lemma 6 *Let $\hat{F}$ be a finite $(\gamma,\ t)$-bounded family of compact figures in $\mathbb{R}^2$. For any $\alpha \in (0, \frac{1}{2}]$, if an α-greedy algorithm picks $z \in \hat{F}$ as a winner, then*

$$\forall B(z) \in Inter(z) \qquad \sum_{x \in B(z)} v_x \leq \frac{2^\alpha}{2^\alpha - 1} \cdot v_z \cdot \gamma(\hat{F})^\alpha \cdot t(\hat{F})^{1-2\alpha}$$

Proof: For any $B(z) \in Inter(z)$, since z is a winner, by the properties of the α-greedy algorithm, for any $x \in B(z)$ it holds that $\frac{v_x}{q(x)^\alpha} \leq \frac{v_z}{q(z)^\alpha}$. Summing over all the figures in $B(z)$, implies that

$$\sum_{x \in B(z)} v_x \leq \frac{v_z}{q(z)^\alpha} \cdot \sum_{x \in B(z)} q(x)^\alpha \tag{1}$$

We use the fact that $\hat{F}$ is (γ, t)-bounded to bound the expression $\sum_{x \in B(z)} q(x)^\alpha$.

$$\begin{aligned}\sum_{x \in B(z)} q(x)^\alpha = \sum_{i=0}^{\log(R)} (\sum_{x \in B_z^i} q(x)^\alpha) \quad &= \quad \sum_{i=0}^{\log(R)} \left(\sum_{x \in B_z^i} q(x)^\alpha \cdot 1^{1-\alpha} \right) \\ \leq \sum_{i=0}^{\log(R)} \left(\left(\sum_{x \in B_z^i} q(x) \right)^\alpha \left(\sum_{x \in B_z^i} 1 \right)^{1-\alpha} \right) \quad &= \quad \sum_{i=0}^{\log(R)} \left(|B_z^i|^{1-\alpha} \left(\sum_{x \in B_z^i} q(x) \right)^\alpha \right)\end{aligned}$$

Again, the inequality is implied by Hölder inequality, and by the fact that $\alpha \in (0, \frac{1}{2}]$. Since $\hat{F}$ is $(\gamma,\ t)$-bounded, for $i \in \{0, \ldots, \log(R)\}$, we have that

$$\sum_{x \in B_z^i} q(x) \le \frac{2^i}{R \cdot |B_z^i|} q(z) \cdot \gamma(\hat{F}) \tag{2}$$

Therefore, bounding $\sum_{x \in B_z^i} q(x)$ in Inequality 2 yields (when $\alpha > 0$):

$$\sum_{x \in B(z)} q(x)^\alpha \le \sum_{i=0}^{\log(R)} \left(|B_z^i|^{1-\alpha} \cdot \left(\frac{2^i}{R \cdot |B_z^i|} q(z) \cdot \gamma(\hat{F}) \right)^\alpha \right) \tag{3}$$

$$\le q(z)^\alpha \cdot \gamma(\hat{F})^\alpha \cdot t(\hat{F})^{1-2\alpha} \cdot \sum_{i=0}^{\log(R)} \left(\frac{2^i}{R} \right)^\alpha \tag{4}$$

where the last inequality is derived from $1 - 2\alpha \ge 0$ (as $\alpha \in (0, \frac{1}{2}]$), and the definition of $t(\hat{F})$ which ensures that for all $i \in \{0, \ldots, \log(R)\}$, $|B_z^i| \le t(\hat{F})$. Since $2^\alpha > 1$, we have:

$$\sum_{i=0}^{\log(R)} \left(\frac{2^i}{R} \right)^\alpha = R^{-\alpha} \sum_{i=0}^{\log(R)} (2^\alpha)^i = R^{-\alpha} \frac{(2^\alpha)^{\log(R)+1} - 1}{2^\alpha - 1} \le \frac{2^\alpha}{2^\alpha - 1} \tag{5}$$

We now combine Equations 1, 3 and 5, and conclude the proof of this lemma:

$$\sum_{x \in B(z)} v_x \le \frac{v_z}{q(z)^\alpha} \cdot q(z)^\alpha \cdot \gamma(\hat{F})^\alpha \cdot t(\hat{F})^{1-2\alpha} \cdot \sum_{i=0}^{\log(R)} \left(\frac{2^i}{R} \right)^\alpha$$

$$\le \; v_z \cdot t(\hat{F})^{1-2\alpha} \cdot \gamma(\hat{F})^\alpha \cdot \sum_{i=0}^{\log(R)} \left(\frac{2^i}{R} \right)^\alpha \; \le \; \frac{2^\alpha}{2^\alpha - 1} \cdot v_z \cdot t(\hat{F})^{1-2\alpha} \cdot \gamma(\hat{F})^\alpha$$

□

5 The Unknown Single-Minded model: Massive Figures, Rectangles and Discrete Tiles

In this section we use the general result presents in the previous section, to derive our results for three specific families of figures. We present a different result for each family. For massive figures, and in particular for convex figures, we prove an $O(R^{\frac{4}{3}})$-approximation; for rectangles, not necessarily axis parallel, we prove an $O(R)$-approximation; and finally, for convex figures in the discrete model (where the agents bid for sets of atomic figures), we improve the $O(R^{\frac{4}{3}})$-approximation by a factor that depends on the resolution of the atomic figures.

5.1 Massive Figures

A family of compact figures will be called β-"massive", if the area of each compact figure z in the family is at least a β fraction of the area of its bounding

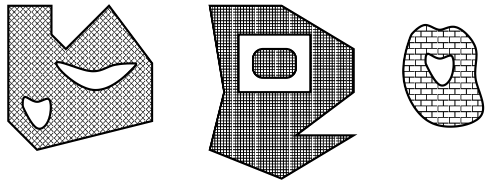

Fig. 4. Our mechanism handles "massive" figures although they are not convex. A "massive" figure captures a constant fraction of the area of its convex hull.

box, that is the product of its width w_z and its diameter d_z (as we later observe, this means that the figure captures a constant fraction of the area of its convex hull). In particular, we will show that any family of compact convex figures in the plane is $\frac{1}{2}$-massive. Figure 4 illustrates massive figures that are not convex.

Definition 10 *A family F of compact figures in $\mathbb{R}^2$ is β-**massive** (for some $\beta > 0$), if for every figure z in F, $q(z) \geq \beta \cdot d_z \cdot w_z$*

We use Theorem 4, which was proved to general compact figures (not necessarily massive nor convex), to determine the approximation achieved by the $\frac{1}{3}$-greedy mechanism for massive figures (note that a skecth of the proof of this theorem, in the special case of convex figures, was given in Section 3). We show that the family of β-massive figures is $\frac{R^2}{\beta}$-balanced and (γ, t)-bounded with some γ and t, which are sufficient to derive the approximation result.

Theorem 7 *For a fixed $\beta > 0$, assume that a set of agents are bidding for a β-massive family of figures in the USM model. Then, the $\frac{1}{3}$ -* greedy mechanism *is a polynomial-time truthful mechanism that achieves an $O(R^{\frac{4}{3}})$-approximation for the social welfare.*

Proof: By Theorem 1, the mechanism is truthful and it runs is polynomial time. We are left to prove the welfare approximation. We first state, in the following claim, few properties of any β-massive family of compact figures in $\mathbb{R}^2$. The proofs need some geometric calculations and can be found in Appendix A.2.

Claim 3 *Let F be a β-massive family of compact figures in $\mathbb{R}^2$ that has an aspect ratio R. Then:*

- *F is $\frac{R^2}{\beta}$-balanced.*
- *For the maximal-class-size of F it holds that $t(F) \leq 8\pi(\pi+1)\frac{1}{\beta} \cdot R$.*
- *F is γ-bounded with $\gamma(F) = 8\pi(\pi+1)\frac{1}{\beta} \cdot t(F) \cdot R^2$.*

By Theorem 4, for any $\alpha \in (0, \frac{1}{2}]$, the α-greedy mechanism achieves a $\{\frac{2^\alpha}{2^\alpha-1} \cdot \gamma(F)^\alpha \cdot t(F)^{1-2\alpha} + b^{1-\alpha}\}$-approximation to the social welfare. We apply the theorem with the properties stated in Claim 3. For $\alpha = \frac{1}{3}$ (the value of α that make the two terms to be of the same order), we get that the approximation is at least as good as what we wanted to prove:

$$\frac{2^{\frac{1}{3}}}{2^{\frac{1}{3}}-1} \cdot \left(8\pi(\pi+1)\frac{1}{\beta} \cdot \left(8\pi(\pi+1)\frac{1}{\beta} \cdot R\right) \cdot R^2\right)^{\frac{1}{3}} \cdot \left(8\pi(\pi+1)\frac{1}{\beta} \cdot R\right)^{1-\frac{2}{3}} +$$
$$\left(\frac{1}{\beta}R^2\right)^{1-\frac{1}{3}} = \left(8\left(\frac{2^{\frac{1}{3}}}{2^{\frac{1}{3}}-1}\right)\pi(\pi+1)\frac{1}{\beta} + \beta^{-\frac{2}{3}}\right) \cdot R^{\frac{4}{3}} \qquad \square$$

A family F of compact convex figures in $\mathbb{R}^2$ is $\frac{1}{2}$-massive (Proposition 4 in Appendix A.2). Thus, we derive the following corollary. As shown in Section 3, the $O(R^{\frac{4}{3}})$-approximation is the best asymptotic approximation achieved by any α-greedy algorithm.

Corollary 8 *Assume that a set of agents are bidding for a family of compact convex figures in the USM model. Then, the* $\frac{1}{3}$ - greedy mechanism *is a polynomial-time truthful mechanism that achieves an $O(R^{\frac{4}{3}})$-approximation for the social welfare.*

5.2 An Improved Mechanism for Rectangles

If agents are only interested in rectangles (not necessarily axis parallel), then we can derive a stronger result of an $O(R)$-approximation for the social welfare. The difference from arbitrary convex figures can be illustrated as follows: While the construction of a big rectangle containing $\frac{1}{16}R^2$ small rectangles (as presented in Lemma 3) is still possible, the second construction is not. That is, it is impossible for a small rectangle to intersect many disjoint rectangles that are not contained in it. For a rectangle to intersect k disjoint rectangle that are not contained in it, its area must be sufficiently large (at least an order of kW^2) and we use this property to improve the bound. Technically, the improved approximation (over general compact convex figures) is a result of a "better" γ function.

Theorem 9 *Assume that the agents bid for rectangles in the plane. Then the* $\frac{1}{2}$ - greedy mechanism *is a polynomial-time truthful mechanism, in the USM model, that achieves an $O(R)$-approximation for the social welfare.*

Proof: By Theorem 1, the mechanism is truthful and it runs is polynomial time. We are left to prove the welfare approximation. Again, the approximation result is derived from Theorem 4. Lemma 20 in the Appendix presents an improved bound on γ. It shows that there exists a constant $C_1 > 0$, such that for any family F of rectangles in $\mathbb{R}^2$, family F is γ-bounded with $\gamma(F) = C_1 \cdot R^2$. We also use the properties from Claim 3 (in the proof of Theorem 7) for convex figures which are $\frac{1}{2}$-massive. Thus, the approximation is at least as good as (for some $C_2 > 0$): $\frac{\sqrt{2}}{\sqrt{2}-1} \cdot (C_1 R^2)^{\frac{1}{2}} \cdot (16\pi(\pi+1) \cdot R)^{1-2\cdot\frac{1}{2}} + (2R^2)^{1-\frac{1}{2}} = C_2 R$.
□

Next, we show that the $\frac{1}{2}$-greedy mechanism achieves the best approximation over all the α-greedy mechanisms. Actually, we prove a somewhat stronger proposition. This proposition shows that an $O(R)$-approximation is the best over *all* the greedy allocation algorithms that sort the bids according to a function of the value and the area of the rectangles (not only by the specific function $\frac{v_z}{q(z)^\alpha}$). Note that this hardness result holds even for axis-parallel rectangles (the $O(\log R)$ algorithm in Section 6.3 is indeed more sophisticated).

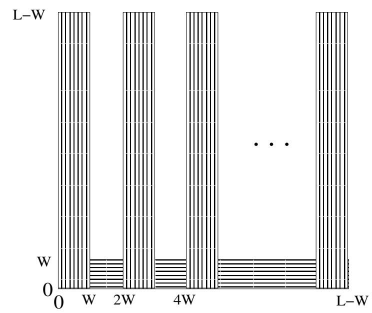

Fig. 5. For this family of figures, any allocation scheme that is based only on the areas and the values (e.g., greedy algorithms) achieves at most R-approximation.

Proposition 10 *Let ALG be an allocation algorithm for rectangles, which chooses rectangles greedily according to some function of the reported values and the rectangles area. Assume that the function is strictly monotonic in the reported value. Then, ALG achieves no better than an $O(R)$-approximation for the social welfare.*

Proof: For any big enough R, we present a construction for which no greedy mechanism with the specified properties achieves a better approximation. Figure 5 illustrates the construction we use to prove this proposition. We look at $\frac{1}{2}R+1$ rectangles of dimensions $W \times (L-W)$ (each has a diameter smaller than L): $\frac{1}{2}R$ disjoint rectangles are perpendicular to the rectangle z, where the area of each intersection is exactly $W \times W$.

Assume that all the $\frac{1}{2}R$ rectangles have the same value v, and z has slightly higher value, that is $v_z = v+\epsilon$ for some $\epsilon > 0$. Any greedy allocation algorithm based only on the area and the value (e.g., α-greedy mechanisms), will pick z and not the other $\frac{1}{2}R$ rectangles (all the rectangles have the same area). For $R \geq 4$, the socially efficient mechanism would have picked the other $\frac{1}{2}R$ rectangles. Hence, the ratio between the efficient allocation and the algorithm's result is $\frac{(\frac{1}{2}R)\cdot v}{v+\epsilon}$, and it approaches $\frac{R}{2}$ as ϵ tends to zero. □

5.3 Convex Figures in the Discrete Model

In the discrete model, there is a set of atomic building blocks (we call *tiles*) embedded in the plane. All the tiles are assumed to have similar dimensions, or specifically, we assume that each tile contains a disk of a diameter W_0 and its diameter is at most $2W_0$.[15] Each agent bids for a set of tiles that are fully contained in some convex figure. A winning agent receives all the tiles that are contained in his reported figure, and only them. The allocation in the discrete model allocates tiles to the agents, and it allocates each tile only

[15] Since the width of a convex figure is at most 3 times its inradius, see Scott and Awyong (2000), any tile that is a convex figure with a width of at least $\frac{3}{2}W_0$ and a diameter of at most $2W_0$ has these desired properties.

once. Therefore, two figures in this model are disjoint, if there is no tile which is *fully* contained in both of them.

The resolution of the tiles is an important factor in our analysis of the discrete model, and we give the approximation guarantee as a function of R and Q: for a given family of bids, we define the ***width-ratio*** $\mathbf{Q} = \frac{W}{W_0}$. The width-ratio gives an upper bound on the width of any figure, with respect to the size of the tiles. Clearly, we can assume that $Q \geq 1$.

We present a mechanism that achieves an improved approximation over our previous results in the "continuous" model. The improved approximation is due to the fact that now, for a figure to intersect k disjoint figures that are not contained in it, its area must be at least of an order of kW_0W. This was not the case for "continuous" convex figures, for which a figure with area of $O(W^2)$ could similarly intersect R disjoint figures (see Claim 2). Technically, the improved approximation is a result of a "better" γ function.

Our mechanism for the discrete model may choose the values of α as a function of R and Q.[16] This comes as opposed to the the continuous model, where we showed that allowing the mechanism to determine α^* as a function of R does not improve the asymptotic approximation ratio (see Lemma 14 in Appendix A.1).

The Discrete-Model Greedy Mechanism:
Given bids for compact convex figures in the discrete model, do the following: If $Q \geq R$ then it runs the $\frac{1}{3}$-greedy mechanism, *and if $Q \leq R$ then it runs the* α^* - greedy mechanism *for* $\alpha^* = \frac{log(R)}{2log(R)+log(Q)}$.

We are now ready to present the properties of the mechanism.

Theorem 11 *Consider that the agents bid for compact convex figures in $\mathbb{R}^2$ in the discrete model, with an aspect-ratio R and a width-ratio Q. Then, the* Discrete Model Greedy Mechanism *achieves an $O(R^{\frac{4}{3}})$-approximation for the social welfare. Moreover, when $Q \leq R$ it achieves a better approximation of $O(R \cdot Q^{\alpha^*})$. The mechanism is asymptotically better than the α-greedy mechanism for any α. Additionally, the mechanism is truthful for the USM model, and it runs in polynomial time.*

Proof: By Theorem 1, the mechanism is a polynomial-time truthful mechanism. We are left to prove the welfare approximation.

If $Q \geq R$: Clearly, the $O(R^{\frac{4}{3}})$-approximation that the $\frac{1}{3}$-greedy mechanism achieves for general compact convex figures (Theorem 8), holds for the discrete model as well.

[16] Note that an agent might try to manipulate the values of α by affecting R and Q. Therefore, for truthfulness to hold, we need to assume that the mechanism knows the true values of R and Q, but the mechanism is not required to know the specific figures demanded by each agent.

If $Q \leq R$: Once again, the result is derived from our main parameterized approximation result (Theorem 4). Lemma 21 in Appendix A.4 presents an improved bound on γ. It shows that any family F of compact convex figures in $\mathbb{R}^2$ in the discrete model, is γ-bounded with $\gamma(F) = C_3 \cdot R^2 \cdot Q$ (for some constant C_3). We use the improved bound on γ, together with the properties presented in Claim 3 (in the proof of Theorem 7) and show that the approximation we get is at least as good as:

$$\begin{aligned}
& \frac{2^{\alpha^*}}{2^{\alpha^*}-1} \cdot \left(C_3 \cdot R^2 \cdot Q\right)^{\alpha^*} \cdot t(F)^{1-2\alpha^*} + (2R^2)^{1-\alpha^*} \\
\leq \;& C_4 \cdot \left(R^{2\alpha^*} \cdot Q^{\alpha^*} \cdot (16\pi(\pi+1) \cdot R)^{1-2\alpha^*} + R^{2(1-\alpha^*)}\right) \\
\leq \;& C_5 \cdot \left(R \cdot Q^{\alpha^*} + R^{2(1-\alpha^*)}\right) = 2C_5 \cdot RQ^{\alpha^*}
\end{aligned}$$

This holds for some constants $C_4 > 0$ and $C_5 > 0$, and for any $\alpha \in [\frac{1}{3}, \frac{1}{2}]$. Note that C_4 and C_5 are independent of the value of α. The rightmost inequality holds since for the given α^* it is easy to see that $R \cdot Q^{\alpha^*} = R^{2(1-\alpha^*)}$ (this is exactly the value of α that makes the two terms to be of the same order). Note also that since $Q \geq 1$, for any R and for any Q, such that $R > Q$, it holds that $\alpha^* \in (\frac{1}{3}, \frac{1}{2}]$ – thus the approximation ratio in such cases is better than $O(R^{4/3})$.

Next, we show that in the discrete model, the given mechanism achieves asymptotically better approximation than any other α-greedy mechanism. Claim 1 (in the proof of Lemma 3) shows that for any $\alpha < \alpha^*$, the approximation of the α-greedy mechanism is asymptotically worse than the α^*-greedy mechanism. This claim constructed a set of figures and values for which the α-greedy mechanism achieves no better than $O(R^{2(1-\alpha)})$-approximation.[17] The bound presented in Claim 2 (also in the proof of Lemma 3) uses a construction with a small disk intersecting an order of R triangles. This construction is possible only if $Q \geq R$, otherwise the intersection between each triangle and the disk is too small to contain a tile with W_0 diameter disk inside it. Thus, we should use a different construction for cases where $Q \leq R$. The basic idea behind this construction (see Figure 6): The intersection between a narrow rectangle z (with dimensions of order of $W \times RW_0$) and each of R narrow trapezoids (of dimensions of order $W \times L$) contains a disk of diameter W_0. Thus, once the rectangle is chosen by the greedy algorithm, no trapezoid can be picked. The normalized value of the rectangle z can be set to be slightly more than 1, where all the trapezoids have normalized values of 1. Therefore, the approximation is asymptotically at least as bad as $\frac{R \cdot (WL)^{\alpha}}{(WRW_0)^{\alpha}} = R \cdot Q^{\alpha}$. It is not hard to build the construction according to the outline above. Due to space limitations, we omit the details of this construction.

[17] The construction in that lemma also holds for the discrete model. Since $W \geq W_0$ we can embed a W_0 diameter disk in each of the small squares, and these disks are the construction's tiles

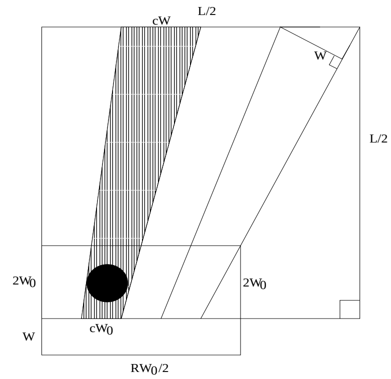

Fig. 6. For this family of figures, any α-greedy mechanism achieves no better than $O(RQ^{\alpha})$-approximation in the discrete model.

From both bounds, we conclude that the approximation is asymptotically at least as bad as $max\{R^{2(1-\alpha)}, RQ^{\alpha}\}$ for any α. Thus, the best worst-case asymptotic approximation ratio will be achieved when these terms are equal. This yields the value α^*. □

6 The Known Single-Minded Model

This section studies the KSM model in which the auctioneer knows the desired figure of each agent, but does not know the value each agent is willing to pay for her figure. We start by presenting an auction for general compact convex figures; We achieve an $O(R)$-approximation for the social welfare, which improves the $O(R^{\frac{4}{3}})$-approximation that we proved for the USM model. For the case of axis-aligned rectangles, we present an exponential improvement, from $O(R)$ in the USM model, to $O(log(R))$ in the KSM model.

6.1 The Characterization of Incentive-Compatible Mechanisms

For environments with one-dimensional types for the players, the characterization of dominant-strategy incentive compatibility is well known (for a proof see, for example, Archer and Tardos (2001); Babaioff and Nisan (2004)). Specifically, a necessary and sufficient condition for the implementability in dominant strategy of an allocation algorithm is *bid monotonicity*, i.e., the property that no winning agent becomes a loser by increasing his bid:

Definition 11 *An allocation scheme S is* ***bid monotonic*** *if for any profile of reported values $v = (v_i, v_{-i})$ and any agent i, if i wins according to S when he reports v_i, then he will also win when he reports $\tilde{v}_i > v_i$ (when v_{-i} is fixed).*

An equivalent property to bid monotonicity is the existence of a ***critical value*** C_i for each agent such that if i bids more than C_i he wins, and if he bids less than C_i he loses (when v_{-i} is fixed). Therefore, in our proof, we are able to prove the monotonicity of the allocation rule by presenting critical values for the players. It is well known that these critical values are exactly the payments that guarantee incentive compatibility: a (normalized) mechanism is incentive compatible *if and only if* its allocation rule is bid monotonic and each trading agent i pays his critical value C_i. Note that the payments above are identical to the VCG payments only when the mechanism is socially efficient; mechanisms that approximate the welfare, like all the mechanisms in this paper, are clearly not VCG mechanisms.

6.2 A Mechanism for Convex Figures

Consider the mechanism called the *"Classes-by-Area Greedy Mechanism"* (**CBAG mechanism**) presented in Figure 7. This mechanism divides the bids of the agents to classes according to the figures' geometric area (with exponentially growing intervals), runs a 0-greedy algorithm in each class,[18] and allocates the figures to agents in the class that achieved the highest result. We show in the proof that this allocation scheme is bid-monotonic, and set the payments to be the "critical values". This implies that this mechanism is incentive compatible in the KSM model.[19] From an algorithmic aspect, our algorithm is the first algorithm for "packing" weighted convex figures that we know of. It achieves an $O(R)$-approximation, and whether there exists a better approximation scheme is an open question.

We first observe that the CBAG mechanism is not incentive-compatible in the USM model. The basic idea is that an agent in a losing class may bid for a larger figure in order to join a winning class. For example, consider three disjoint rectangles: one with an area of 1 and a value of 3, and two rectangles with an area of 5 and a value of 2. When the agents bid truthfully, the two large rectangles will be in the same class and win, while the small rectangle loses. However, if the small rectangle untruthfully declared a larger rectangle of area 5 that includes his original rectangle (this rectangle may intersect the other rectangles), he would clearly become a winner.

The intuition behind the computation of the payments in the CBAG mechanism is as follows: to win the auction, each agent i should be both a winner in his class and his class should beat all other classes; Bidding above the value z_i in the mechanism's description (in Figure 7), guarantees that agent i wins in his class. However, if agent i bids below $V^2 - V^1_{-i}$ and still wins in his class, his class will definitely lose. Therefore, the critical value for agent i should be the maximum over these two values.

Theorem 12 *When the agents bid for compact convex figures in* $\mathbb{R}^2$ *with an*

[18] Actually any α-greedy allocation algorithm can be run instead.

[19] In fact, the auctioneer only needs to know the area of each figure and not the exact figure.

The Classes-by-Area Greedy (CBAG) mechanism:

Allocation:

Step 1: Divide the given input to $m = 2log(R)$ classes according to their area. A figure s belongs to class c if $q(s) \in [W^2 \cdot 2^c, W^2 \cdot 2^{c+1})$ (for $c \in \{0, \ldots, m-1\}$).

Step 2: Perform the 0-greedy algorithm per each class. Denote the welfare achieved by class c by V^c.

Step 3: Output the allocation in the class c for which the 0-greedy algorithm achieved the highest welfare, i.e., $c \in argmax_{\tilde{c}\in\{0,\ldots,m-1\}} V^{\tilde{c}}$.

Payments:

Denote the winning class as class 1, and the class with the second-highest welfare as class 2. Let $V^1_{-i} = V^1 - v_i$, and let j be the figure that intersects figure i and wins, when we run the greedy algorithm where agent i is removed. Let z_i be v_j if such j exists, and 0 otherwise. A winning agent i pays: $\mathbf{P(i)} = \mathbf{max}\{\mathbf{V^2 - V^1_{-i}}, \mathbf{z_i}\}$, and any losing agent pays 0.

Fig. 7. A mechanism for arbitrary convex figures that is incentive compatible in the KSM model and achieves an $O(R)$-approximation for the social welfare.

aspect ratio R, the CBAG *mechanism achieves an $O(R)$-approximation. This mechanism is truthful for the KSM model, and runs in polynomial time.*

Proof: Denote the sets of figures assigned to each of the m classes by $S_0, ..., S_{m-1}$, and let R_c denote the aspect ratio of the set of figures S_c (i.e. R_c is the ratio between the maximal diameter L_c and the minimal width W_c of the figures in class S_c). We start by proving the approximation ratio.

The area of each figure in class S_c is in the range $[W^2 2^c, W^2 2^{c+1})$, and due to Proposition 4 in Appendix A.2 if a figure s is compact and convex then its geometric area $q(s)$ is approximately the product of its width and its diameter, i.e. $q(s) \in [\frac{d_s w_s}{2}, d_s w_s]$. We use these properties to derive the following bounds on the diameter and the width of each figure.

Since the widths of all figures is at least W, the diameter of any figure s in S_c is bounded as follows: $d_s \leq \frac{2q(s)}{w_s} \leq \frac{2W^2 2^{c+1}}{W} = W2^{c+2}$. Since the diameter of each figure is not greater than L, the following holds for the width of every figure $s \in S_c$: $w_s \geq \frac{q(s)}{d_s} \geq \frac{W^2 2^c}{L}$. Using the inequalities above, we bound each R_c by the following two upper bounds:

$$R_c = \frac{\max_{s\in S_c} d_s}{\min_{s\in S_c} w_s} \leq \frac{W2^{c+2}}{W} = 2^{c+2}, \quad R_c = \frac{\max_{s\in S_c} d_s}{\min_{s\in S_c} w_s} \leq \frac{L}{\left(\frac{W^2 2^c}{L}\right)} = R^2 2^{-c}$$

We now show that each figure $z \in S_c$ intersects $O(R_c)$ disjoint figures from S_c. The area of the figure with the diameter of L_c is[20] at least $\frac{L_c W_c}{2}$, thus,

[20] Since L_c is the maximal diameter of a figure in S_c, it must be the diameter of

by the definition of the classes, the area of any other figure in S_c is at least $\frac{L_c W_c}{4}$. Clearly, all the figures in S_c that intersect z are contained in the shape created by extending z with the union of all the disks with radius L_c that their center lies on the perimeter of z (the L_c-extension of z - see Proposition 15.2 in Appendix A.2). Since the area of this extension, $q(E(z, L))$, is $O(L_c^2)$ (see Proposition 3 in Appendix A.2), then there are $O(\frac{L_c^2}{L_c W_c}) = O(R_c)$ disjoint figures that intersect z.

Next, we show that the welfare alg_c achieved by the 0-greedy algorithm in class c is an $O(R_c)$-approximation for the optimal social welfare opt_c in this class. That is, for some constant $b > 0$:

$$\sum_{y \in opt_c} v_y \leq \sum_{x \in alg_c} \sum_{y \in opt_c,\ y \cap x \neq \emptyset} v_y \leq \sum_{x \in alg_c} \sum_{y \in opt_c,\ y \cap x \neq \emptyset} v_x \leq bR_c \sum_{x \in alg_c} v_x$$

The leftmost inequality holds since every figure in opt_c intersects some figure in alg_c (otherwise it would be chosen by alg_c). The second leftmost inequality holds since if a figure x is chosen by the algorithm, and figure y intersects it, then $v_x \geq v_y$. The rightmost inequality holds since we proved that each figure chosen by the greedy algorithm intersects $O(R_c)$ disjoint figures in its class (i.e., less than $b \cdot R_c$ figures for some constant b), and the figures in opt_c are clearly disjoint. We conclude that the 0-greedy algorithm achieves an $O(R_c)$-approximation for the welfare in each class c. An easy observation is that the approximation ratio achieved by choosing the best class over a set of classes, each with approximation ratio $O(R_c)$, is $O(\sum_{c=0}^{m-1} R_c)$.[21] Thus, by applying one of the above upper bounds on R_c for each half of the classes, the $O(R)$ approximation ratio follows:

$$\sum_{c=0}^{m-1} R_c \leq \sum_{c=0}^{log(R)-1} 2^{c+2} + \sum_{c=log(R)}^{2log(R)-1} R^2 2^{-c} \leq 4R + 2R = 6R$$

Finally, we show that the given mechanism is incentive compatible. As mentioned, it suffices to show that the allocation scheme is bid monotonic and the payments are by critical values (the mechanism is normalized by definition). Assume that agent i's declaration is $\widetilde{v_i}$. We prove that the payments $P(i) = \max\{V^2 - (V^1 - v_i), z_i\}$ are indeed the critical values for the agents: if agent i bids $\widetilde{v_i} < P(i)$ he loses, and if $\widetilde{v_i} > P(i)$ he wins (fixing the declarations of the other agents). Due to Theorem 1, the value z_i is the critical value for agent i in the 0-greedy mechanism in his class. Thus, if $\widetilde{v_i} < z_i$ agent i loses in his class. if $\widetilde{v_i} > z_i$, agent i wins in his class, but his class might lose to

some figure in this class, and this figure's width is at least W_c.

[21] To see this, let OPT_c be the value that each class c contributes to the optimal welfare. For each class the greedy algorithm achieves $ALG_c \geq \frac{OPT_c}{R_c}$. Multiplying by R_c, and summing over all classes, we get that $\sum_c ALG_c R_c \geq \sum_c OPT_c$. The algorithm's result ALG is clearly no worse than any ALG_c, and therefore $ALG \sum_c R_c \geq \sum_c OPT_c = OPT$ and the claim follows.

another class. When $\widetilde{v_i} < V^2 - (V^1 - v_i)$, then if agent i still wins, the other winning figures in the greedy allocation will clearly stay the same. Thus, the social welfare in class 1 will be $(V^1 - v_i) + \widetilde{v_i}$ which is smaller than V^2, thus class 2 will be chosen and agent i will lose. If $\widetilde{v_i} > V^2 - (V^1 - v_i)$, and agent i wins in his class, similar considerations derive that class 1 will be chosen, thus agent i wins. It follows that $\max\{V^2 - (V^1 - v_i), z_i\}$ is indeed the critical value for agent i.

Since critical values exist, the mechanism is bid monotonic and it follows that the given mechanism is incentive compatible. The greedy mechanism runs in polynomial time (Theorem 1). We run this mechanism once for each class, and one more time for each winning agent for calculating the payments. Thus, the CBAG algorithm runs in polynomial-time. □

6.3 An Improved Mechanism for Axis-Aligned Rectangles

When we restrict our discussion to axis-aligned rectangles, then a significantly better approximation ratio of $O(\log R)$ can be achieved in the KSM model. This mechanism is based on an algorithm by Khanna et al. (1998) with some minor changes. Khanna et al. (1998) studied a model where axis-aligned rectangles lie in an $n \times n$ array, and they presented an $O(log(n))$-approximation algorithm for the weighted-packing problem. This approximation ratio is the best polynomial-time approximation currently known for weighted packing of axis-aligned rectangles, and it is unknown whether a better polynomial-time approximation is possible. Their algorithm divides the figures to classes according to their heights (that is, their projections on the y axis); For each class, an allocation is chosen that achieves a constant-factor approximation for the optimal result in this class, using a special dynamic programming procedure, and again, the class that achieves the maximal values is chosen. In our model, where figures can lie in arbitrary locations on the plane, a similar algorithm can be easily shown to achieve an $O(log(R))$-approximation.

Theorem 13 *(Essentially due to Khanna et al. (1998).) When the agents bid axis-aligned rectangles with an aspect ratio R, there exists a polynomial-time algorithm that achieves an $O(log(R))$-approximation for the social welfare.*

The detailed description of this algorithm is tedious, and thus it is left out of the scope of this paper. Our main contribution here is the observation that this complex allocation algorithm is bid-monotonic, and hence it is incentive compatible. The basic idea is to identify the "critical values" for the players in this algorithm; the existence of these values implies the monotonicity of the algorithm, and these values are exactly the payments of the winners that ensures incentive compatibility. The characterization of these critical values is not technically involved, but it closely depends on the subtle details of the algorithm, and thus it remains out of the scope of this paper. This results in a mechanism that is incentive compatible in the KSM model and that attains an $O(log(R))$-approximation for the social welfare.

7 Conclusion and Further Research

In this paper, we studied auctions in which the agents bid for convex figures on the plane. We presented incentive-compatible mechanisms that are also computationally feasible. We suggested using the aspect ratio R, which measures the diversity of the dimensions of the figures, for analyzing the economic efficiency of the mechanisms.

We managed to construct incentive-compatible mechanisms (in the KSM model) that achieve the best approximation results currently known to be achievable in polynomial time. Whether the best polynomial-time approximation can always be implemented with dominant strategies is an important open problem.

Our results indicate that a gap may exist between the approximation achievable in the KSM model and in the USM model; For general convex figures, the approximation we achieve in the KSM and the USM models are $O(R)$ and $O(R^{\frac{4}{3}})$, respectively; For axis-parallel rectangles, the gap in our results is even exponential. This question is open for other settings as well.

Open Problem: *Is there indeed a gap between the best approximation achievable in the KSM and in the USM models?*

The main algorithmic open question is how well can the the social welfare be approximated by polynomial-time algorithms - even without incentive considerations. In our paper, we present novel algorithms for packing convex figures and rectangles. We have not been able to show that these results are tight. This comes in addition to the existing open problem of whether a better than $O(log(R))$-approximation exists for packing of axis-parallel rectangles.

Open Problem: *Can one achieve better than $O(R)$-approximation for packing general convex figures, or even rectangles (not necessarily axis-parallel).*

Finally, we note that the general theorem presented in Section 4 should be useful in deriving approximation results for the problem of packing weighted convex bodies in dimensions higher than two [22]. In addition, there are results for rectangle packing in higher dimensions (e.g., Berman et al. (2001)) that could be used to build new mechanisms for these problems.

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[22] The economic interpretation of such auctions is not always clear. (Examples in 3 dimensions: storage space in boats, selling figures in two dimensions over time.)

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A The Unknown Single-Minded Model

A.1 A Lower Bound for Compact Convex Figures

Lemma 14 *Let $\epsilon > 0$. For any function $\alpha : [1, \infty) \rightarrow \Re$ that chooses α as a function of R ($\alpha = \alpha(R)$), the asymptotic approximation of the $\alpha(R)$-greedy mechanism is not better than $R^{\frac{4}{3}-\epsilon}$. That is, if the approximation achieved by the $\alpha(R)$-greedy mechanism is $f(R)$, then for a large enough R, $f(R) > R^{\frac{4}{3}-\epsilon}$.*

Proof: Assume that we combine both constructions of Claim 1 and Claim 2 to one construction, then the approximation ratio is as bad as :
$\frac{4^\alpha}{16} R^{2(1-\alpha)} + \pi^{1-\alpha} R^{1+\alpha} = \frac{R^2}{16} \left(4R^{-2}\right)^\alpha + \pi R (\pi^{-1} R)^\alpha$.

By the first order condition, this function is minimized the following $\alpha(R)$:

$$\alpha(R) = \frac{\ln(R) - \ln(16\pi) + \ln\left(\frac{2\ln(R) - \ln(4)}{\ln(R) - \ln(\pi)}\right)}{3\ln(R) - \ln(4\pi)}$$

$\alpha(R)$ converges to $\frac{1}{3}$ as R grows to infinity. Thus, for any fixed $\epsilon > 0$ and large enough R, a better approximation than $R^{\frac{4}{3}-\epsilon}$ is impossible. $\square$

A.2 β-massive Figures

Proof of Claim 3 from Section 5.1:

Proof: We prove the claim using the results proved below. Proposition 16 shows that F is $\frac{R^2}{\beta}$-balanced and Lemma 18 shows that $t(F) \leq \left(8\pi(\pi+1)\frac{1}{\beta}\right) R$. Lemma 17 shows that F is γ-bounded with $\gamma(F) = 8\pi(\pi+1)\frac{1}{\beta} \cdot t(F) \cdot R^2$. $\square$

Definition 12 *For any compact figure $z \subseteq \mathbb{R}^2$ the r-**extension** of z is the union of disks of the **extension radius** $r \geq 0$ around all points of the perimeter of z, that is $E(z, r) = \bigcup_{c \in p_z} b(c, r)$ (where $b(c, r)$ denotes the disk of radius $r \geq 0$ around a center point c). We denote the area of the r-extension of z by $q(E(z, r))$.*

Proposition 15 ***(Geometric facts)*** *For any compact convex figure z:*

(1) $q(z) \leq d_z \cdot w_z$.
(2) For any compact figure x, if $x \cap p_z \neq \emptyset$ and $d_x \leq r$ then $x \subseteq E(z, r)$.

(3) *If $d_z \le D$ for some $D \ge 0$, then for any $r \le D$, $q(E(z,r)) \le 4\pi(\pi+1)\cdot r \cdot D$.*

(4) *$\frac{1}{2}d_z \cdot w_z \le q(z) \le d_z \cdot w_z$ and $2d_z \le p_z \le \pi d_z$. In particular, any family F of compact convex figures in $\mathbb{R}^2$ is $\frac{1}{2}$-massive.*

Proof: (1). We look at two parallel lines of distance w_z bounding z. There are two parallel lines perpendicular to these lines that bound z, and they are at most d_z apart. So z is contained in the rectangle of area $d_z \cdot w_z$, therefore $q(z) \le d_z \cdot w_z$.

(2). Let P be a point in $x \cap p_z$. Since x's diameter is not greater than r, x is contained is the disk of radius r around P, which is contained in the r-extension of z.

(3). Since the perimeter of z is continuous, we can cover the r-extension of z with $\lceil \frac{p_z}{r} \rceil$ disks of radius $2r$. For compact convex figure it holds that $\pi p_z \ge d_z$ (Scott and Awyong (2000)), thus:

$$q(E(z,r)) \le \lceil \frac{p_z}{r} \rceil (\pi(2r)^2) \le \frac{p_z + r}{r}(4\pi\cdot r^2) \le 4\pi\cdot(\pi\cdot d_z + D)\cdot r \le 4\pi(\pi+1)\cdot r \cdot D$$

(4). For any compact convex figure z in $\mathbb{R}^2$, $2d_z \le p_z \le \pi d_z$ (Scott and Awyong (2000)). By Observation 15.1, $q(z) \le d_z \cdot w_z$ for any compact figure in $\mathbb{R}^2$. Next we show that $q(z) \ge \frac{1}{2}d_z \cdot w_z$. Let P_1 and P_2 be two points in z of distance d_z. We look at the two parallel lines that are parallel to $[P_1, P_2]$ (the line passing through P_1 and P_2). The distance between these two lines is at least w_z, and assume that they touch z at Q_1 and Q_2. The two triangles $Q_1P_1P_2$ and $Q_2P_1P_2$ are contained in z, have a base of length d_z and sum of their heights is at least w_z, so $q(z) \ge \frac{1}{2}d_z \cdot w_z$. □

Proposition 16 *Let F be a family of compact figures in $\mathbb{R}^2$ that is β-massive and has an aspect ratio R. Then F is $\frac{R^2}{\beta}$-balanced. with respect to the geometric area function q.*

Proof: By Observation 15.1, for any compact figures z, $d_z \cdot w_z \ge q(z)$. Since F is β-massive, for all $x \in F$, $q(x) \ge \beta \cdot d_x \cdot w_x$. Therefore:

$$\forall z, x \in F \quad \frac{q(z)}{q(x)} \le \frac{d_z \cdot w_z}{\beta \cdot d_x \cdot w_x} \le \frac{d_z \cdot d_z}{\beta \cdot w_x \cdot w_x} \le \frac{1}{\beta} \cdot \frac{L^2}{W^2} = \frac{R^2}{\beta}$$

□

Lemma 17 *Let F be a family of compact figures in $\mathbb{R}^2$ that is β-massive and has an aspect ratio R. Then F is γ-bounded, with $\gamma(F) = 8\pi(\pi+1)\frac{1}{\beta}\cdot t(F)\cdot R^2$.*

Proof: We prove that for any $z \in F$, any $B(z) \in Inter(z)$ and any $i \in \{0, \ldots, \log(R)\}$ we prove that

$$\frac{R}{2^i} \cdot |B_z^i| \cdot \frac{\sum_{x \in B_z^i} q(x)}{q(z)} \le \gamma(F)$$

We first prove the following claim.

Claim 4 *For any* $z \in F$,$B(z) \in Inter(z)$ *and* $i \in \{0, \ldots, \log(R)\}$ *it holds that* $\sum_{x \in B_z^i} q(x) \leq 4\pi(\pi+1) \cdot (W \cdot 2^{i+1}) \cdot L$

Proof: By Proposition 15.2, for any $x \in B_z^i$, $x \subseteq E(CH(z), W \cdot 2^{i+1})$ ($CH(z)$ is the convex hull of z, and any $x \in B_z^i$ intersects the perimeter of $CH(z)$ and satisfies $d_x \leq W \cdot 2^{i+1}$). Since B_z^i is a family of disjoint sets, by the above claim for each $x \in B_z^i$ it holds that $x \subseteq E(CH(z), W \cdot 2^{i+1})$, and q is an area function, thus $\sum_{x \in B_z^i} q(x) \leq q(E(CH(z), W \cdot 2^{i+1}))$.

The diameter of z and the diameter of the convex hull of z are equal, that is $d_z = d_{CH(z)}$, it holds that $d_{CH(z)} \leq L$. By Proposition 3 on $CH(z)$ for $D = L \geq d_{CH(z)}$ and $r = W \cdot 2^{i+1} \leq L = D$ ($r = L$ for $i = \log(R)$),

$$q(E(CH(z), W \cdot 2^{i+1})) \leq 4\pi(\pi+1) \cdot (W \cdot 2^{i+1}) \cdot L$$

□

Additionally, since F is β-massive, we have $q(z) \geq \beta \cdot d_z \cdot w_z \geq \beta \cdot W^2$. We conclude that $\forall z \in F \;\; \forall B(z) \in Inter(z) \;\; \forall i \in \{0, \ldots, \log(R)\}$,

$$\frac{R}{2^i} \cdot |B_z^i| \frac{\sum_{x \in B_z^i} q(x)}{q(z)} \leq \frac{R}{2^i} t(F) \frac{4\pi(\pi+1)(W 2^{i+1})L}{\beta W^2} = 8\pi(\pi+1)\frac{1}{\beta} t(F) \cdot R^2$$

□

Lemma 18 *Let* F *be a family of compact figures in* $\mathbb{R}^2$ *that is* β*-massive and has an aspect ratio* R*. Then, for the maximal-class-size* $t(F)$ *it holds that* $t(F) \leq 8\pi(\pi+1)\frac{1}{\beta} \cdot R$.

Proof: For any $z \in F$, any $B(z) \in Inter(z)$ and any $i \in \{0, \ldots, \log(R)\}$ we prove that $|B_z^i| \leq 8\pi(\pi+1)\frac{1}{\beta} \cdot R$.

By Claim 4, for any $z \in F$,$B(z) \in Inter(z)$ and $i \in \{0, \ldots, \log(R)\}$ it holds that $\sum_{x \in B_z^i} q(x) \leq 4\pi(\pi+1) \cdot (W \cdot 2^{i+1}) \cdot L$.

Additionally, F is β-massive, so for any $x \in B_z^i$, $q(x) \geq \beta \cdot d_x \cdot w_x \geq \beta \cdot 2^i \cdot W^2$, and since the figures are disjoint we conclude that $\forall z \in F \;\; \forall B(z) \in Inter(z) \;\; \forall i \in \{0, \ldots, \log(R)\}$, the size of each class is as required:

$$|B_z^i| \leq \frac{4\pi(\pi+1) \cdot (W \cdot 2^{i+1}) \cdot L}{\beta \cdot 2^i \cdot W^2} \leq 8\pi(\pi+1)\frac{1}{\beta} \cdot R$$

□

Theorem 19 ***(Hölder Inequality - a weak version)*** *If* $p, q \geq 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*, then for any two vectors of non-negative* n *real numbers* $\overrightarrow{X}, \overrightarrow{Y}$*,* $\sum_{k=1}^{n} X_k \cdot Y_k \leq \left(\sum_{k=1}^{n} X_k^p\right)^{1/p} \left(\sum_{k=1}^{n} Y_k^q\right)^{1/q}$

Lemma 20 *There exists a constant $C_1 > 0$, such that for any family F of rectangles in $\mathbb{R}^2$ with aspect ratio R, family F is γ-bounded with $\gamma(F) = C_1 \cdot R^2$.*

Proof: We use the same arguments of Lemma 17 with a tighter bounds on the area of z. We show that for any $B(z) \in Inter(z)$, $q(z) \geq \frac{1}{80\pi} \cdot |B(z)| \cdot W^2$.

We look at the W-extension of z. By Proposition 4, $\frac{1}{2}p_z \geq d_z \geq w_z \geq W$, we can use Proposition 3 for $D = \frac{1}{2}p_z$ and $r = W$ to conclude that $q(E(z, W)) \leq 20\pi \cdot W \cdot \frac{1}{2}p_z$. If x intersects z but is not contained in it, it must intersect the perimeter of z. This implies that the W-extension of z and x both contain at least a $\frac{1}{4}$ of the disk with a radius of W around the intersection point.

The total area of $|B(z)|$ disjoint rectangles that intersect z but are not contained in it is at least $|B(z)| \cdot \frac{1}{4}\pi \cdot W^2$, so

$$|B(z)| \cdot \frac{1}{4}\pi \cdot W^2 \leq q(E(z, W)) \leq 20\pi \cdot W \cdot \tfrac{1}{2}p_z$$

which means that $p_z \geq |B(z)| \cdot \frac{1}{40}W$. Since by Proposition 4, $q(z) \geq \frac{1}{2}d_z w_z \geq \frac{1}{2}(\frac{p_z}{\pi})w_z$, we conclude that

$$q(z) \geq \tfrac{1}{2}\left(\frac{1}{\pi}p_z\right) w_z \geq \frac{1}{2\pi}\left(|B(z)| \cdot \frac{1}{40}W\right) w_z \geq \frac{1}{80\pi} \cdot |B(z)| \cdot W^2$$

We now use this tighter bound on the area of z (on $B(z) = B_z^i$) , with the arguments presented in Lemma 17 to get that, $\forall z \in F \;\; \forall B(z) \in Inter(z) \;\; \forall i \in \{0, \ldots, \log(R)\}$

$$\frac{R}{2^i} \cdot |B_z^i| \cdot \frac{\sum_{x \in B_z^i} q(x)}{q(z)} \leq \frac{R}{2^i} \cdot |B_z^i| \cdot \frac{20\pi \cdot (W \cdot 2^{i+1}) \cdot L}{\frac{1}{80\pi} \cdot |B_z^i| \cdot W^2} = 3200\pi^2 \cdot R^2 = C_1 \cdot R^2$$

□

A.4 The Discrete Model

Lemma 21 *There exists a constant $C_3 > 0$, such that for any family F of compact convex figures in $\mathbb{R}^2$ in the discrete model, family F is γ-bounded with $\gamma(F) = C_3 \cdot R^2 \cdot Q$.*

Proof: We use the same arguments of Lemma 17 with the tighter bounds on the area of z. In Lemma 22, we show that in the discrete model, for any $z \in F$ and any $B(z) \in Inter(z)$, it holds that $q(z) \geq \frac{1}{300\pi^2} \cdot |B(z)| \cdot W_0 \cdot W$. By the same arguments of Lemma 17 we conclude that for $C_3 = 12000\pi^3 > 0$, and

$\forall z \in B \;\; \forall B(z) \in Inter(z) \;\; \forall i \in \{0, \dots, \log(R)\}$

$$\frac{R}{2^i} \cdot |B_z^i| \cdot \frac{\sum_{x \in B_z^i} q(x)}{q(z)} \leq \frac{R}{2^i} \cdot |B_z^i| \cdot \frac{20\pi \cdot (W \cdot 2^{i+1}) \cdot L}{\frac{1}{300\pi^2} \cdot |B_z^i| \cdot W_0 \cdot W} = C_3 \cdot R^2 \cdot Q$$

□

Lemma 22 *If F is a family of compact convex figures in $\mathbb{R}^2$ in the discrete model, then for any $z \in F$ and any $B(z) \in Inter(z)$, it holds that $q(z) \geq \frac{1}{300\pi^2} \cdot |B(z)| \cdot W_0 \cdot W$.*

Proof: For any z and any $B(z) \in Inter(z)$, we first present a lower bound on the perimeter of z. We show that $p_z \geq \frac{1}{150\pi} W_0 \cdot |B(z)|$. Later on, we use this to derive the a lower bound for the area.

Let P_x be a point such that $P_x \in x \cap p_z$. Let $T(x,z) \subseteq x \cap z$ be a tile with $P_x \in T(x,z)$. Let c_x be the center of the disk of diameter W_0 in $T(x,z)$

We look at the $\frac{1}{2}W$-radius disk around c_x. Since $c_x \in x$ and $w_x \geq W$, there exist a point Q_x at the intersection of x and the disk circumference. Thus, x contains a triangle with an area of at least $\frac{1}{2}(\frac{1}{2}W)W_0$ (the diameter of the W_0-radius disk is the base of the triangle, and $D(Q_x, c_x)$ is its height, where $D(p,q)$ denotes the Euclidean distance between the points p and q.). Since $P_x, c_x \in T(x,z)$ and $d_{T(x,z)} \leq 2W_0 \leq 2W$, then $D(P_x, c_x) \leq 2W$. Since the triangle is contained in a disk of radius $\frac{1}{2}W$ around c_x, the triangle is contained in the $\frac{5}{2}W$-extension of z.

$B(z)$ is a set of disjoint figures, therefore the triangles (described above) of the different figures are disjoint, and we conclude that $q(E(z, \frac{5}{2}W)) \geq |B(z)|(\frac{1}{4}W_0 W)$. Since by Proposition 4 $\frac{1}{2}p_z \geq d_z \geq w_z \geq W$, we can use Proposition 3 for $D = \frac{5}{4}p_z$ and $r = \frac{5}{2}W$ to see that $q(E(z, \frac{5}{2}W)) \leq \frac{125}{2}\pi W p_z$. We conclude that $\frac{125}{2}\pi W p_z \geq |B(z)|(\frac{1}{4}W_0 W)$, and it follows that $p_z \geq \frac{1}{150\pi}|B(z)| W_0$.

We use the bound on the perimeter of z to bound the area of z. By Proposition 4, $q(z) \geq \frac{1}{2} d_z w_z \geq \frac{1}{2}(\frac{p_z}{\pi}) w_z$ and $w_z \geq W$ and our lemma follows:

$$q(z) \geq \tfrac{1}{2}\left(\frac{1}{\pi}p_z\right) w_z \geq \tfrac{1}{2}\frac{1}{\pi}\left(\frac{1}{150\pi}|B(z)| \cdot W_0\right) w_z \geq \frac{1}{300\pi^2} \cdot |B(z)| \cdot W_0 \cdot W$$

□

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