Performance Analysis of Contention Based Medium Access Control Protocols

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***Abstract*—This paper studies the performance of contention based medium access control (MAC) protocols. In particular, a simple and accurate technique for estimating the throughput of the IEEE 802.11 DCF protocol is developed. The technique is based on a rigorous analysis of the Markov chain that corresponds to the time evolution of the back-off processes at the contending nodes. An extension of the technique is presented to handle the case where service differentiation is provided with the use of heterogeneous protocol parameters, as, for example, in IEEE 802.11e EDCA protocol. Our results provide new insights into the operation of such protocols. The techniques developed in the paper are applicable to a wide variety of contention based MAC protocols.**

***Index Terms*—IEEE 802.11, IEEE 802.11e, CSMA/CA, Performance evaluation, Fixed point analysis, Fluid limit, Diffusion Approximation, Wireless LANs, Performance of the MAC protocols.**

I. INTRODUCTION

Wireless local area networks (WLANs) based on the IEEE 802.11 standard are one of the fastest growing wireless access technologies in the world today. They provide an effective means of achieving wireless data connectivity in homes, public places and offices. The low-cost and high-speed WLANs can be integrated within the cellular coverage to provide hotspot coverage for high-speed data services, thus becoming an integral part of next generation wireless communication networks.

The fundamental access mechanism of IEEE 802.11 MAC is the Distributed Coordination Function (DCF). The DCF is a carrier sense multiple access protocol with collision avoidance (CSMA/CA). In addition to DCF, the IEEE 802.11 standard also defines an optional Point Coordination Function (PCF), which uses a central coordinator for assigning the transmission right to stations, thus guaranteeing a collision free access to the shared wireless medium. While DCF has gained enormous popularity and been widely deployed, the use of PCF has been rather limited.

Whereas the IEEE 802.11 standard was targeted at best-effort service for data transfer, it is expected that in the future WLANs will need to support a mix of QoS-sensitive, multimedia and interactive traffic, in addition to data traffic which is only sensitive to the throughput. Future WLANs must therefore provide service differentiation in order to better support the diverse QoS requirements of the applications running on them. A new standard, namely IEEE 802.11e, has been proposed for this purpose; it defines two new access mechanisms: EDCA (an enhancement to DCF), and HCF (an enhancement to PCF). Of the two, EDCA appears to be gaining more early acceptance.

In this paper we study the performance of contention based MAC protocols, with a specific emphasis on DCF and a simplified version of EDCA. There have been several previous works on the performance of DCF; these include simulation studies [1], [2] as well as analytical studies based on simplified models of DCF [3], [4], [5], [6], [7], [8]. Most of the analytical work is based on a decoupling approximation, first proposed by Bianchi in [3]; we henceforth refer to the simplified model with this decoupling assumption as Bianchi's model.

More recently, several studies [9], [10], [11], [12], [13] have evaluated the performance of EDCF, an earlier version of EDCA (see [14]). With the exception of [13], where the authors propose an extension of the Bianchi's model for analyzing EDCF, all these studies are simulation based.

The main contribution of this paper is a novel technique for estimating the throughput and other parameters of interest for the contention based MAC protocols. Our technique is based on a rigorous analysis of the drift of the Markovian model of the system, and does not require the decoupling assumption of Bianchi. In fact, through the insights it yields into the system dynamics, it provides an intuitive justification of Bianchi's simplifying assumptions. The technique is easy to apply, and we use it to analyze DCF as well as a simplified version of EDCA. We now briefly sketch the key ideas behind our approach.

A common feature of all the contention based MAC protocols is the concept of *back-off stage* for a station. The stations can be in different back-off stages; the back-off stage for a station depends on the number of collisions that it has encountered since its last successful transmission (and, possibly, other information) and can be thought of as its estimate of the current level of contention at all stations. The stations use this estimate to control their access probabilities. The key observation we make in this paper is that, when the number of stations is large, the Markov chain associated with the back-off process stays close to what we call a *typical* state, which can be obtained as the unique *equilibrium* point of the drift equations associated with the back-off process. We can obtain quite accurate estimates of the throughput and other parameters of interest by assuming the system to be in this typical state at all times.

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We find that the accuracy of the throughput estimates obtained using our technique is about the same as those obtained using Bianchi's analysis. But, in addition, we are able to provide some key insights about the system dynamics; in fact, our results provide a justification for Bianchi's approximation, which may be of separate interest.

The rest of the paper is organized as follows. We provide a brief description of DCF and EDCA, and discuss some related work, in the next section. Our technique for performance evaluation is discussed in the context of DCF in section III. An extension of our technique in the context of EDCA is discussed in Section V. Some concluding remarks are presented in Section V. Due to space constraints, all technical details and proofs are deferred to Appendix A and B.

II. DCF, EDCA, AND RELATED WORK

In this section, we provide a brief description of DCF and EDCA, and discuss some related work in the literature. We start with a description of DCF.

A. IEEE 802.11 DCF

The DCF is a Carrier Sense Multiple Access with Collision Avoidance (CSMA/CA) MAC protocol. The collision avoidance scheme of DCF is based on the binary exponential back-off (BEB) scheme [15], [16]. The DCF defines two access mechanisms for packet transmissions: *basic access mechanism*, and *RTS/CTS access mechanism*.

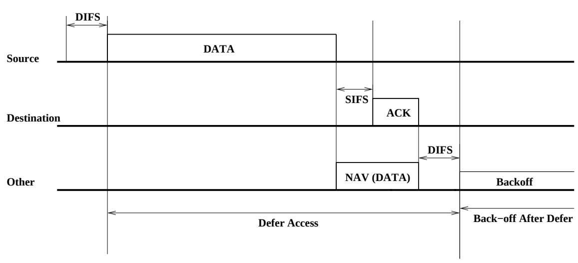

Fig. 1. Basic Access Method.

In the basic access mechanism (see Figure 1), any station, before transmitting a DATA frame, senses the channel for a duration of time equal to the Distributed Interframe Space (DIFS) to check if it is idle. If the channel is determined to be idle, the station starts the transmission of a DATA frame. All stations which hear the transmission of the DATA frame set their Network Allocation Vector (NAV) to the expected length of the transmission, as indicated in the Duration/ID field of the DATA frame. This is called the *virtual carrier sensing* mechanism. The channel is considered to be busy if either the physical carrier sensing or the virtual carrier sensing indicates so, and in that case, the station enters into a wait period determined by the back-off procedure to be explained later. Upon successful reception of the DATA frame, the destination station waits for a SIFS interval following the DATA frame, and then sends an ACK frame back to the source station, indicating successful reception of the DATA frame.

The RTS/CTS access mechanism uses a four-way handshake in order to reduce bandwidth loss due to the hidden terminal problem (see, for example, [17]). A station that wishes to send a DATA frame first senses the channel for a DIFS duration. If the channel is determined to be idle, then a RTS frame is sent to the destination. Otherwise, the back-off algorithm is triggered after the end of the current transmission and a further DIFS interval. Upon successful transmission of the RTS frame, the destination waits for a SIFS interval, and then sends a CTS frame back to the source. The source can start sending the DATA frame a SIFS interval after the reception of the CTS frame. As in the basic access mechanism, upon successful reception of the DATA frame, the destination waits for a SIFS interval, and then sends an ACK frame back to the source. A station that hears either the RTS, CTS, or DATA frame updates its NAV based on the Duration/ID field of the corresponding frame (see Figure 2). The four way handshake prevents any DATA-DATA collisions that might occur due to the hidden terminal problem. Since the RTS and CTS frames are very small in size, the RTS/CTS access scheme significantly reduces bandwidth loss due to collisions.

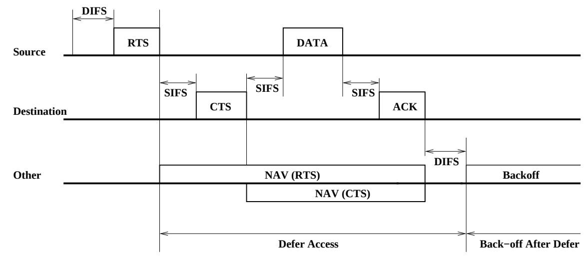

Fig. 2. RTS/CTS Access Method.

The back-off procedure is implemented by means of the *back-off counter* and *back-off stages*. Initially, upon receiving a new frame to be transmitted, the station starts in back-off stage 0, window (CW) size set to CW_{min}. Following an unsuccessful transmission attempt (collision), the back-off stage is incremented by 1 and the contention window size is doubled until the maximum size of the contention window, CW_{max}, is reached, after which the back-off stage and the contention window size remain unchanged on subsequent collisions. The back-off window size as well as the back-off stage are set back to their initial values of CW_{min} and 0 after a successful transmission attempt or if the *retry count* limit for the frame is reached. At the start of each back-off stage, the back-off counter is set to an integer chosen uniformly at random between zero and the value $CW-1$ of the contention window for the current back-off stage. The back-off counter is decremented by 1 in every subsequent slot, as long as the channel is sensed idle in that slot. (Here, a slot is an interval of fixed duration specified by the protocol.) If a transmission by some other station is detected, then the station freezes its back-off counter, and resumes its count from where it left off after the end of the transmission plus an additional DIFS interval.

When the back-off counter reaches 0, the station transmits*.

The scheme described above treats all the stations equally. We now briefly describe the enhanced distributed channel access (EDCA) mechanism, which is an extension of the DCF mechanism, and aims at providing service differentiation.

B. IEEE 802.11e EDCA

The EDCA has been designed from the perspective of providing QoS in WLANs. The EDCA defines four different ACs, each maintaining its own channel access function (an enhanced variant of the DCF). The main differences between the EDCA and DCF are:

1) The minimum specified idle duration time, called the *arbitration inter frame space* (AIFS), is not a constant value unlike the DIFS in the case of DCF.
2) The contention window limits, $CW_{min}{}^\dagger$ and CW_{max}, are different for different ACs.

In section IV we consider a heterogeneous setting similar to the one as under the above EDCA mechanism.

C. Related Work

One of the earliest analyses of the throughput of DCF was carried out in [4] using a greatly simplified back-off model, namely that the back-off counter value is geometrically distributed with *constant* parameter p, irrespective of the current back-off stage of the station. A more realistic model was proposed in the seminal paper of Bianchi [3]. Here, the evolution of the back-off stage at each node is described by a Markov process; the Markov chains at different nodes evolve independently, but in an *environment* specified by the collision probability p for any transmission attempt. The parameter p is a constant derived from the *mean* transmission probability in the associated Markov chains. This formulation leads to a fixed point equation for p. Note that the model is analogous to mean-field models in statistical physics; the only interaction between the Markov processes at different nodes is through the parameter p, which represents a mean value of the environment. It is not a goal in [3] to provide a rigorous justification of the mean-field assumption. The assumption is justified through simulations, which show that the model predictions are quite accurate.

Several subsequent studies have built on the work of Bianchi. In [7], the authors obtain similar fixed point equations using the same decoupling assumption but without the Markovian assumptions of Bianchi; extensions of this fixed point formulation are studied in [8].‡ In [6], the authors present an approximate delay analysis based on Bianchi's model, and also extend the model to account for channel errors.

Recently, Proutiere et al. [**?**] have shown that the mean field analysis of Bianchi is asymptotically (in the infinite station limit) accurate. In particular, they have used ideas from the theory of propagation of chaos to show that Bianchi type decoupling holds aysmptotically as number of stations is allowed to increase to infinity.

Several works have evaluated the performance of EDCF, an earlier version of EDCA (see [14]). Most of these have employed simulation [9], [10], [11], [12]. An exception is [13], where the authors use an extension of Bianchi's model to analyze the performance of IEEE 802.11e MAC protocol. More recently, the performance of EDCA has been analyzed in [20], [21], using theoretical models based on Bianchi type assumptions.

Our approach differs fundamentally from the work described above in that we do not make the decoupling assumption introduced by Bianchi, and common to all of them except [**?**]. Instead, starting from a Markov chain description that explicitly takes into account the interactions between stations, we show that in a large system, namely one with a large number of stations, the Markov chain converges to a *typical* state. Thus, one can approximate the collision probability seen by any single station by that seen in the typical state. Our analysis therefore provides a rigorous justification for Bianchi's model, which has been the basis of much subsequent work. In addition, it provides an alternative approach to performance analysis of MAC protocols; performance measures of interest can be derived directly from analysis of the typical state. We validate this approach by showing that the performance predictions thus obtained are close to those seen in simulations.

Finally, we point out that we focus on DCF and EDCA protocols in this paper because they are likely to be the two most widely deployed wireless MAC protocols in the near future; however, we do not specifically advocate their use. Several works (see, for example, [22], [23], [24], and the references therein) have identified the limitations of these protocols, and proposed alternative MAC protcols that can provide better performance. The techniques developed in this paper are very general, and can be applied to evaluate the performance of these alternative MAC protocols as well.

III. Performance Evaluation of IEEE 802.11 DCF

In this section, we present a performance analysis of DCF. We start with a description of our model.

A. The Model

We consider a wireless LAN with n stations employing the IEEE 802.11 DCF. Every station can hear every other station in the network, i.e., there are no hidden stations. Our discussion covers both *ad hoc networks*, where there is no central access point (AP) through which all the traffic must pass, as well as *intrastructure based networks*, where an AP connects the wireless network with the wired infrastructure. In order to simplify the analysis, we assume, in common with most

*As in [3] and majority of the related literature, in our analysis, we ignore the facts that (i) the back-off procedure is not invoked immediately after a successful transmission or during the transmission of the first data packet, and (ii) the back-off counter is not decremented if the channel is sensed to be busy. For a more accurate model of the back-off procedure, we refer the reader to [18].

†The parameters aCW_{min} and aCW_{max} depend on the physical layer.

‡In order to avoid confusion arising from the superficially similar terminology, we emphasize that the *fixed points* we talk of in this work are different from the fixed points in [3], [7], [8]. Their fixed points are for the 1-dimensional coupling parameter p; our fixed points are for the n-dimensional state descriptor in a joint Markovian representation of the back-off stages at all n stations. The details are provided in the next section.

related work, that all stations always have a packet to send. The throughput obtained under such saturation conditions is commonly referred to as the *saturation throughput*. In some cases (see, for example, [25]), it can be shown that the queues at all the nodes are stable if the arrival rate at each node is less than the saturation throughput. s We make the following additional assumptions:

- ($\mathbf{A1}$) The back-off durations are geometrically distributed, i.e., when a station is in back-off stage i, it makes a transmission attempt in the next slot with a probability p_i. In order to maintain the same average waiting time as in the IEEE 802.11 DCF, we set $p_i = \frac{2}{W_i+1}$, where W_i is the contention window size in back-off stage i.
- ($\mathbf{A2}$) The back-off stage is reset to 0 only after a successful transmission, i.e., the retry count limit, as defined in Section II-A, is infinite. This assumption is not necessary for our analysis, but simplifies the exposition considerably.

All stations use the same back-off parameters. There are $M+1$ back-off stages, labeled 0 to M. We adopt a discrete time model indexed by the slot number t. To avoid confusion, note that the term "slot" in our usage refers to a different quantity from the slot in the IEEE 802.11 protocol description. We use the term to denote the time period at the end of which stations may modify their back-off counters. In particular, the duration of a slot is not a fixed physical layer parameter, but varies depending on whether it represents an idle slot, a successful transmission or a collision.

The state of the system at time t can be represented by a vector $X_n(t) = (X_{n0}(t), ..., X_{nM}(t))$ denoting the number of stations in each of the back-off stages 0 through M. It is easy to see that $X_n(t)$, $t = 0, 1, \dots$ forms an irreducible and aperiodic Markov chain on the state space

$$S_n \triangleq \left\{ x \in \mathbb{Z}^{M+1} : \sum_{i=0}^{M} x_i = n; x_i \geq 0 \text{ for all } i \right\}.$$

In principle, one could solve for the stationary distribution of $X_n(t)$ and thereby obtain parameters of interest about the system. However, the number of states, n^{M+1}, is too large to make this feasible for systems of practical interest. The key insight we provide in this paper is that, when n is large (and exact computation expensive), the Markov chain $X_n(t)$ stays close to what we call a *typical state*. Moreover, accurate estimates of various parameters such as throughput can be obtained by assuming that $X_n(t)$ is in this typical state at all times.

We remark for purposes of comparison that Bianchi [3] models the system as a Markov chain with (typically) an even larger state space of size $(M+1)^n$ by considering the back-off stage at each station. The analysis is simplified by replacing this n-dimensional Markov chain by n 1-dimensional Markov chains (with M+1 states each) which are *assumed* to be conditionally independent, conditional on the collision probability p. We do not make any such independence assumptions.

We now proceed with the analysis of the Markov chain $X_n(t)$. Let us look at the expected change in $X_n(t)$ over one time slot. For $x^{(n)} \in S_n$, let

$$\begin{aligned} f^{(n)}(x^{(n)}) &\triangleq \mathbb{E}\{X_n(t+1) - X_n(t) | X_n(t) = x^{(n)}\} \\ &= \sum_{l: x^{(n)}+l \in S_n} l P_l^{(n)}(x^{(n)}), \end{aligned}$$

where $P_l^{(n)}(x^{(n)})$ is the probability of making a transition from $x^{(n)}$ to $x^{(n)} + l$ over one time slot. We now compute $f_i^{(n)}(x^{(n)})$ for $i \in \{0, 1, ..., M\}$.

First consider $i = 0$. Let $I(x^{(n)}) \triangleq \prod_{i=0}^{M} (1-p_i)^{x_i^{(n)}}$, where p_i denotes the transmission probability for a station in back-off stage i. Note that $I(x^{(n)})$ is the probability of an idle slot when the system is in state $x^{(n)}$. The following events can result in a change in the number of stations in back-off stage 0:

- A successful transmission by a station in back-off stage i, $i \in \{1, 2, ..., M\}$, resulting in an increase in the number of stations in back-off stage 0.
- An unsuccessful transmission attempt by one or more stations in back-off stage 0, resulting in a decrease in the number of stations in back-off stage 0.

For the former event to occur, the station itself must transmit and no other station in the network should transmit; this has probability $\frac{p_i}{1-p_i} I(x^{(n)})$. Noting that there are $x_i^{(n)}$ stations in the back-off stage i to choose from, and summing over i, we obtain

$$\sum_{i=1}^{M} x_i^{(n)} p_i \frac{I(x^{(n)})}{1-p_i} \tag{1}$$

to be the expected increase in the number of stations in back-off stage 0 due to successful transmissions by stations in other back-off stages. Likewise, a node in the back-off stage 0 transmits unsuccessfully with probability $p_0 \left(1 - \frac{I(x^{(n)})}{1-p_0}\right)$ and moves to back-off stage 1. Therefore,

$$x_0^{(n)} p_0 \left(1 - \frac{I(x^{(n)})}{1-p_0}\right) \tag{2}$$

is the expected decrease in the number of stations in the back-off stage 0 due to unsuccessful transmission attempts by stations in the back-off stage 0. Combining Eqs.(1) and (2), we obtain

$$f_0^{(n)}(x^{(n)}) = \sum_{i=0}^{M} x_i^{(n)} p_i \frac{I(x^{(n)})}{1-p_i} - x_0^{(n)} p_0. \tag{3}$$

Next, let $i \in \{1, 2, ..., M-1\}$. We now need to consider the following events:

- A transmission attempt by a station in back-off stage i.
- An unsuccessful attempt by a station in back-off stage $i-1$.

A station in back-off stage i attempts to transmit with probability p_i, following which, it either moves to back-off stage 0 (successful transmission) or to back-off stage $i+1$ (collision). Thus, the expected decrease in the number of stations in back-off stage i at time t is

$$x_i^{(n)} p_i. \tag{4}$$

A station in back-off stage $i-1$ transmits with probability p_{i-1} and moves to back-off stage i if it suffers a collision, i.e., if one or more others station in the network also transmit, which happens with probability $\left(1 - \frac{I(x^{(n)})}{1-p_{i-1}}\right)$. Thus,

$$x_{i-1}^{(n)} p_{i-1} \left(1 - \frac{I(x^{(n)})}{1-p_{i-1}}\right). \tag{5}$$

is the expected increase in the number of stations in back-off stage i due to unsuccessful transmission attempts by stations in back-off stage $i-1$. Combining Eqs.(4) and (5), we obtain

$$f_i^{(n)}(x^{(n)}) = x_{i-1}^{(n)} p_{i-1} \left(1 - \frac{I(x^{(n)})}{1-p_{i-1}}\right) - x_i^{(n)} p_i \tag{6}$$

for $i \in \{1, 2, ..., M-1\}$.

Finally, let $i = M$. In this case, we need to consider the following events:

- A successful transmission attempt by a station in back-off stage M.
- An unsuccessful transmission attempt by a station in back-off stage $M-1$.

A station in back-off stage M transmits with probability p_M and, if no other station in the network transmits, an event of probability $\frac{I(x^{(n)})}{1-p_M}$, then the station moves to back-off stage 0; otherwise it stays in the back-off stage M. The expected decrease in the number of stations in back-off stage M at time t due to a successful transmission is thus:

$$x_M^{(n)} p_M \frac{I(x^{(n)})}{1-p_M}. \tag{7}$$

A station in back-off stage $M-1$ transmits with a probability p_{M-1} and, if at least one other station in the network also transmits, an event of probability $\left(1 - \frac{I(x^{(n)})}{1-p_{M-1}}\right)$, then the station enters into back-off stage M. The expected increase in the number of stations in back-off stage M at time t due to collisions is thus

$$x_{M-1}^{(n)} p_{M-1} \left(1 - \frac{I(x^{(n)})}{1-p_{M-1}}\right). \tag{8}$$

Combining Eqs.(7) and (8), we obtain

$$f_M^{(n)}(x^{(n)}) = x_{M-1}^{(n)} p_{M-1} \left(1 - \frac{I(x^{(n)})}{1-p_{M-1}}\right) - x_M^{(n)} p_M \frac{I(x^{(n)})}{1-p_m}. \tag{9}$$

Collecting Eqs.(3), (6), and (9), at one place, we have

$$f_0^{(n)}(x^{(n)}) = \sum_{i=0}^{M} x_i^{(n)} p_i \frac{I(x^{(n)})}{1-p_i} - x_0^{(n)} p_0, \tag{10}$$

$$f_i^{(n)}(x^{(n)}) = x_{i-1}^{(n)} p_{i-1} \left(1 - \frac{I(x^{(n)})}{1-p_{i-1}}\right) - x_i^{(n)} p_i, 0 < i < M, \tag{11}$$

$$f_M^{(n)}(x^{(n)}) = x_{M-1}^{(n)} p_{M-1} \left(1 - \frac{I(x^{(n)})}{1-p_{M-1}}\right) - x_M^{(n)} p_M \frac{I(x^{(n)})}{1-p_M}. \tag{12}$$

Let

$$B_n \triangleq \{x \in \mathbb{R}^{M+1} : \sum_{i=0}^{M} x_i = n; x_i \geq 0\},$$

and $E \triangleq B_n/n$. Let $f^{(n)} : \mathbb{R}^{M+1} \to \mathbb{R}^{M+1}$ be the function with components $f_i^{(n)}$ specified by Eqs.(3), (6), and (9). It is essentially the one-step drift of the Markov chain $X_n(t)$. We have so far defined the function $f^{(n)}(x)$ for $x \in S_n$ only; we now extend the definition of $f^{(n)}$ to $x \in B_n$ by using the same equations on the extended domain.

In Appendix A, we analyze an appropriately scaled version, $Y_n(t) = X_n(\lfloor nt \rfloor)/n$, of the process $X_n(t)$ for $n = 1, 2, ...$, and show that for all $t \geq 0$, it satisfies:

$$\lim_{n\to\infty} \sup_{0 \leq s \leq t} \|Y_n(s) - Y(s)\| = 0 \ a.s.,$$

where $Y(t)$ is a deterministic process given by the unique solution of the differential equation

$$\frac{dY(t)}{dt} = F(Y(t)) \text{ for } t \geq 0,$$

with initial condition $y^0 = \lim_{n\to\infty} Y_n(0) = X(0)/n$, where $F(x) = \lim_{n\to\infty} f^{(n)}(nx)$ for $x \in E$. In words, we prove a functional 'law of large numbers' limit theorem for the process $Y_n(\cdot)$. We also show that the error involved in approximating $X_n(t)$ with $nY(t/n)$ is (almost surely) $O(n^\beta)$ for all $\beta > 1/2$.

In Appendix B, we show that the equation $F(x) = 0$ has a unique solution If $M = 1$, we can further show that $Y(t)$ converges to x from all possible initial states. We conjecture that such a result holds for all M (as observed in our simulations).

In view of the results in Appendix A and B, we can expect that, for large t, the process $X_n(t)$ remains close to the unique point $x^{(n)} \in B_n$ satisfying $f^{(n)}(x^{(n)}) = 0$, which will henceforth be referred to as the equilibrium point of the system.

B. *Throughput Calculation*

We now estimate the throughput of IEEE 802.11 DCF, assuming that the system stays close to its equilibirium point $x^{(n)}$ at all times. Let

- $T \triangleq$ The normalized throughput of the system.
- $P_c \triangleq$ The conditional collision probability.
- $I \triangleq$ The probability of an idle slot in state $x^{(n)}$.
- $P \triangleq$ The payload duration§.
- $T_c \triangleq$ The average time the channel is sensed busy during a collision.
- $T_s \triangleq$ The average time the channel is sensed busy because of a successful transmission.
- $\sigma \triangleq$ The duration of an idle slot.

Note that some of the above defined quantities may vary with n. For the sake of brevity, we do not make explicit this dependence.

To calculate the throughput, observe that a station in back-off stage i, transmits with a probability p_i, and the transmission is successful if no other station in the network transmits, an event of probability

$$\frac{I}{1-p_i}.$$

Since there are $x_i^{(n)}$ stations in back-off stage i, the probability that a station in back-off stage i transmits successfully is

$$x_i^{(n)} p_i \frac{I}{1-p_i}.$$

§In this paper, we consider the payload duration to be fixed. Variable payload duration can also be analyzed as in [3].

TABLE I
IEEE 802.11 DSSS PHY PARAMETER SET [26] AND OTHER PARAMETERS USED TO OBTAIN NUMERICAL RESULTS

PARAMETER	VALUE
Basic Bit Rate (BBR)	1Mb/s
Bit Rate (BR)	11Mb/s
PHY Header (PH)	192 bits
MAC Header (MH)	272 bits
H	PH/BBR + MH/BR
ACK	112/BR + PH/BBR
RTS	160/BR + PH/BBR
CTS	112/BR + PH/BBR
Propagation Delay (δ)	1μs
SIFS	10μs
Slot Time (σ)	20μs
DIFS	50μs

Summing over all possible back-off stages, we obtain the probability of a successful transmission to be

$$\sum_{i=0}^{M} x_i^{(n)} p_i \frac{I}{1-p_i}.$$

Since the probability that at least one station transmits in a given slot is $1-I$, we have

$$P_c = 1 - \frac{\sum_{i=0}^{M} x_i^{(n)} p_i \frac{I}{1-p_i}}{1-I} \tag{13}$$

The normalized throughput of the system can be expressed as

$$T = \frac{\text{Expected Payload duration per slot}}{\text{Slot duration}}. \tag{14}$$

The expected payload duration per slot is $(1-I)(1-P_c)P$. The expected duration of a slot is readily obtained considering that, with a probability I a slot is idle; with a probability of $(1-I)(1-P_c)$ it contains a successful transmission, and with a probability of $(1-I)P_c$ it contains a collision. And plugging this is Eq.(5),we obtain

$$T = \frac{(1-I)(1-P_c)P}{(1-I)(1-P)T_s + (1-I)P_cT_c + I\sigma} \tag{15}$$

The values of T_c and T_s depend on the access mechanism being used. Let δ be the propagation delay, then one can readily obtain (for details, see [3])

$$\begin{aligned} T_s^{rts} &= RTS + CTS + H + P + ACK + 3SIFS + 4\delta + DIFS \\ T_c^{rts} &= RTS + DIFS + \delta \\ T_s^{bas} &= H + P + ACK + SIFS + 2\delta + DIFS \\ T_c^{bas} &= H + P + DIFS + \delta \end{aligned} \tag{16}$$

where T_c^{rts} (correspondingly, T_c^{bas}) and T_s^{rts} (correspondingly, T_s^{bas}) represent the T_c and T_s values for the RTS/CTS based access (correspondingly, basic access) mechanism, respectively; the parameters RTS, CTS, H, ACK, $DIFS$, and $SIFS$ are all physical layer dependent. We will use the values of these parameters as defined in the DSSS PHY (see Table II).

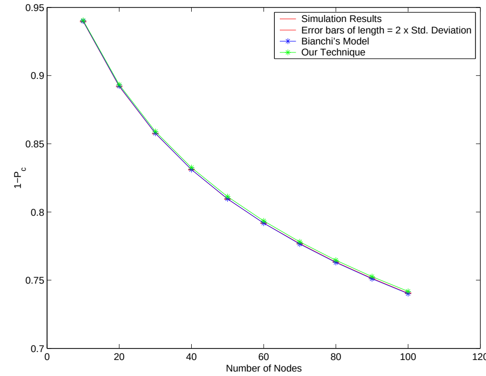

Fig. 3. Success probability $(1-P_c)$ for $M=5$ and $W_0=128$.

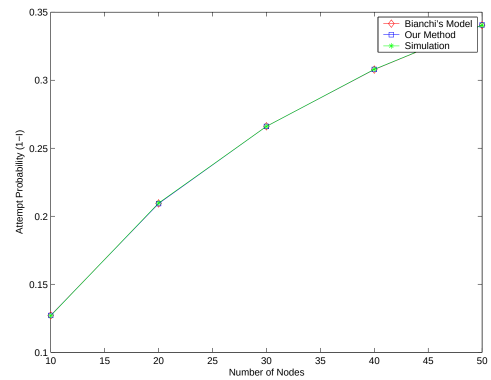

Fig. 4. Attempt probability $(1-I)$ for $M=5$ and $W_0=128$.

C. Performance Comparison

We have performed extensive simulations with different values of M and W_0. The simulation results match extremely well with the numerical results obtained using our technique and Bianchi's model. The results for the RTS/CTS access mechanism with $M=5$ and $W_0=128$ are shown in Figures 3-5. As is evident in these figures (error bars are barely visible), the variation of results across various simulation runs is quite small, thereby showing the high confidence level of the simulation results. An interesting thing to note is that although our technique and Bianchi's model are fundamentally different, they both result in (roughly) the same fixed point (in terms of P_c and I), and correspondingly, the estimates of throughput obtained using the two techniques are very close. Similar results have been obtained for the basic access mechanism as well. An interesting special case ($M=1$) under which it is possible to calculate the exact throughput of DCF is discussed in Appendix C.

IV. EXTENSION TO A HETEROGENEOUS SETTING

The analysis presented in the previous section can easily be extended to a heterogeneous setting, where different nodes can run with different values of the protocol parameters. Indeed, in Appendix D we consider a setting where multiple access categories, each using its own unique set of protocol parameters, are running simultaneously at each node as specified in the EDCA mechanism. However, unlike the EDCA

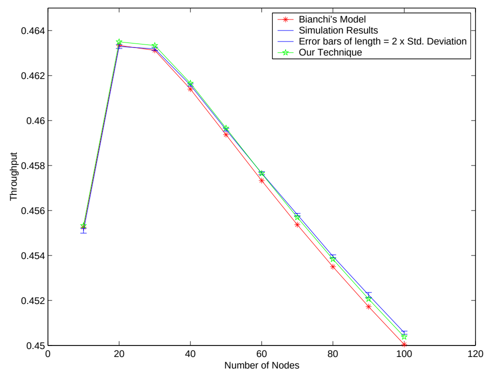

Fig. 5. Throughput (T) for $M = 5$ and $W_0 = 128$.

mechanism, our analysis does not allow for the variable inter-frame spacing. We plan to address this issue in our future work.

V. Concluding remarks

We studied the performance of contention based medium access control (MAC) protocols. We developed a novel technique for estimating the throughput, and other parameters of interest, of such protocols. Our technique is based on a rigorous analysis of a Markovian framework developed in the paper. The analysis shows that in a limiting regime of large system sizes, the stochastic evolution of the back-off stages at different stations converges to a deterministic evolution; moreover, this deterministic process has a unique fixed point. Thus, our analysis provides insight into the dynamics of the MAC protocols, showing that they guide the system to a *typical operating point*. This then allows us to obtain the saturation throughput and other performance measures of interest without having to calculate the stationary distribution of the Markov chain, which would be infeasible for systems of realistic size.

To the best of our knowledge, our technique for performance analysis of MAC protocols is the first one of its kind with a quantifiable accuracy. Our results provide a justification for the decoupling approximation of Bianchi [3]. Finally, although we focused on two representative MAC protcols (IEEE 802.11 DCF and IEEE 802.11e EDCA), the techniques developed in the paper are quite general and are applicable to a wide variety of MAC protocols.

Our performance analysis is based on the assumption that the system remains at its equilibrium point at all times. A natural refinement is to consider fluctuations around this point, which will typically be small. A mathematical framework for studying such fluctuations is provided by the diffusion approximation (a functional central limit theorem for the Markov process). This is a topic for future research. Secondly, our current framework cannot fully handle protocols like EDCA that allow the use of different inter-frame spacing as a means of achieving service differentiation. It remains to extend our analysis techniques to deal with this, and with other forms of heterogeneity. Finally, we have assumed throughout that all nodes can hear each other; accounting for the hidden node problem remains an important research challenge.

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References

[1] B. P. Crow, "Performance evaluation of the IEEE 802.11 Wireless Local Area Network Protocol," M. S. Thesis, Dept. of ECE, Univ. of Arizona, Tuscon, AZ, 1996.

[2] J. Weinmiller, M. Schlager, A. Festag, and A. Wolisz, "Performance study of access control in wireless LANs - IEEE 802.11 DFWMAC and ETSI RES 10 HIPERLAN," *Mobile Networks and Applications*, vol. 2, no. 3, pp. 55–67, 1997.

[3] G. Bianchi, "Performance Analysis of the IEEE 802.11 Distributed Coordination Function," *IEEE Journal on Selected Areas in Communications*, vol. 18, no. 3, pp. 535–547, Mar 2000.

[4] F. Cali, M. Conti, and E. Gregori, "IEEE 802.11 wireless LAN: Capacity analysis and protocol enhancement," in *IEEE INFOCOM*, 1998.

[5] H.S. Chaya and S. Gupta, "Performance modeling of asynchronous data transfer methods of IEEE 802.11 MAC protocol," *Wireless Networks*, vol. 3, no. 3, pp. 217–234, Mar 1997.

[6] N. Gupta and P. R. Kumar, "A Performance Analysis of the IEEE 802.11 Wireless LAN Medium Access Control," in *CISS*, 2004.

[7] A. Kumar, E. Altman, D. Miorandi, and M. Goyal, "New Insights from a Fixed Point Analysis of Single Cell IEEE 802.11 WLANs," in *IEEE INFOCOM*, 2005.

[8] V. Ramaiyan, A. Kumar, and E. Altman, "Fixed Point Analysis of Single Cell IEEE 802.11e WLANs: Uniqueness, Multistability and Throughput Differentiation," in *ACM SIGMETRICS*, 2005.

[9] X. Yang, "IEEE 802.11e wireless LAN for quality of service," in *IEEE Wireless Comm. and Net. Conf.*, Mar 2003, pp. 1291–1296.

[10] A. Lindgren, A. Almquist, and O. Schelen, "Evaluation of quality of service schemes for IEEE 802.11 wireless LANs," in *IEEE Conf. Local Comp. Net.*, Nov 2001, pp. 348–351.

[11] S. Mangold et al., "IEEE 802.11e wireless LAN for quality of service," in *European Wireless*, Feb 2002, pp. 32–39.

[12] A. Grilo and M. Nunes, "Performance evaluation of IEEE 802.11e," in *Proc. Conf. IEEE Personal, Indoor, Mobile Radio Communications*, Sep 2002, pp. 511–517.

[13] B. Li and R. Battiti, "Performance Analysis of an Enhanced IEEE 802.11 Distributed Coordination Function Supporting Service Differentiation," in *Workshop on Quality of Future Internet Services*, 2003, pp. 152–161.

[14] "Draft Supplement to Part 11, Wireless Medium Access Control (MAC) and Physical Layer (PHY) specifications: Medium Access Control (MAC) Quality of Service (QoS) Enhancements," IEEE Std. 802.11e/D4.0, Nov 2002.

[15] J. Goodman, A.G. Greenberg, N. Madras, and P. March, "Stability of Binary Exponential Backoff," in *Proc. ACM Symp. on Thoery of Comp.*, 1985.

[16] J. Hastad, T. Leighton, and B. Rogoff, "Analysis of Backoff Protocols for Multiple Access Channels," *SIAM Jour. Computing*, vol. 25, no. 4, pp. 770–774, Aug 1996.

[17] L. Kleinrock and F. Tobagi, "Packet Switching in Radio Channels, Part II - The Hidden Terminal Problem in Carrier Sense Multiple Access and the Busy Tone Solution," *IEEE Trans. Comm.*, vol. 23, no. 12, pp. 1417–1433, Dec 1975.

[18] E. Ziouva and T. Antonakopoulos, "CSMA/CA performance under high traffic conditions: throughput and delay analysis," *Computer Communications*, vol. 25, pp. 313–321, 2002.

[19] "Draft Supplement to Part 11, Wireless Medium Access Control (MAC) and Physical Layer (PHY) Specifications: Medium Access Control (MAC) Quality of Service (QoS) Enhancements," IEEE Std. 802.11e/D7.0, Jan 2004.

[20] J. Hui and M. Devetsikotis, "A Unified Model for the Performance Analysis of IEEE 802.11e EDCA," *IEEE Trans. on Communications*, vol. 53, no. 9, pp. 1498–1510, Sep 2005.

[21] Z-N. Kong, D.H.K. Tsang, B. Bansaou, and D. Gao, "Performance Analysis of IEEE 802.11e Contention-Based Channel Access," *IEEE JSAC*, vol. 22, no. 10, pp. 2095–2105, Dec 2004.

[22] P. Gupta, Y. Sankarasubramaniam, and A.L. Stolyar, "Random-Access Scheduling with Service Differentiation in Wireless Networks," in *IEEE INFOCOM*, Mar 2005.

[23] J. Deng and Z. J. Haas, “Dual Busy Tone Multiple Access (DBTMA): A New Medium Access Control for Packet Radio Networks,” in *IEEE ICUPC*, Oct 1998.

[24] F. Cali, M. Conti, and E. Gregori, “Dynamic tuning of the IEEE 802.11 Protocol to achieve a theoretical throughput limit,” *IEEE/ACM Trans. on Networking*, vol. 8, no. 6, pp. 785–799, 2000.

[25] A. Kumar and D. Patil, “Stability and throughput analysis of unslotted CDMA-ALOHA with finite number of users and code sharing,” *Telecommunication Systems*, vol. 8, pp. 257–275, 1997.

[26] “Wireless LAN Medium Access Control (MAC) and Physical Layer (PHY) Specifications,” IEEE Std. 802.11, Aug 1999.

[27] S. N. Ethier and T. G. Kurtz, *Markov Processes: Characterization and Convergence*, John Wiley and Sons, 1985.

[28] T. G. Kurtz, *Approximation of Population Processes*, Society for Industrial and Applied Mathematics, 1981.

[29] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968.

APPENDIX A

Recall the setting of Section III-A. For each n, let $E_n \triangleq \{k/n : k \in S_n\}$, and consider the family of stochastic process $\{Y_n(t)\}$ defined as follows:

$$Y_n(t) \triangleq \frac{X_n(\lfloor nt \rfloor)}{n}. \tag{17}$$

Observe that each for each n, $Y_n(t)$ is just a scaled version (where the scaling is both in time as well as magnitude) of $X_n(t)$. We also define another family of stochastic process $\{Z_n(t)\}$ as follows:

$$Z_n(t) \triangleq \frac{X_n(N(nt))}{n}, \tag{18}$$

where $N(t)$ is a Poisson process with unit intensity, independent of the sequence of Markov chains $\{X_n(t)\}$. Observe that $\{Z_n(t)\}$ is a sequence of jump Markov process on E_n, with transition rates (intensities) $q^{(n)}_{k,k+l/n} = nP^{(n)}_l(nk)$, for $k \in E_n$.

We will need the following assumption:

Assumption 1: There exist positive constants $c_0, c_1, ..., c_M$ such that $p^{(n)}_i = c_i/n$ for $i \in 0, 1, ..., M$ and all n.

Remark 1: We note that $p^{(n)}_i$ and M are kept fixed in the IEEE 802.11 DCF, independent of the number of nodes in the network. We allow for the $p^{(n)}_i$ to scale with n to avoid trivialites; for example, if M and $p^{(n)}_i$ were kept fixed for all n, then as $n \to \infty$ the throughput would drop to zero and all the nodes would eventually be in the back-off stage M with probability 1. The above choice of $p^{(n)}_0$ precludes this possibility. Note that the way transmission probabilities are chosen in the IEEE 802.11 DCF, Assumption 1 would imply that $c_0 = 2c_1 = \cdots = 2^M c_M$.

We need some preparation before we can state our main result. Henceforth, we use $\|x\|$ to denote the L^2 norm of x. For $x^{(n)} \in E_n$, let

$$I(x^{(n)}) \triangleq \prod_{i=0}^{M} (1 - p^{(n)}_i)^{x^{(n)}}.$$

Strictly speaking, the function I is not really the same for different n; for the sake of brevity, we will continue to follow the above notation. We start with the following simple result:

Lemma 1: Consider $x^{(n)} \in E_n$ and $x^{(m)} \in E_m$. Suppose Assumption 1 holds. Then,

$$\Big| I(nx^{(n)}) - I(mx^{(m)}) \Big| \le 2c_0 M \|x^{(n)} - x^{(m)}\|,$$

whenever n and m are large enough.

Proof: Without loss of generality, suppose $m \ge n$. Using the definition of $I(x^{(n)})$, and observing that for $M > 1$ $e^{\frac{-2Mc_0}{M+1}} \le (1 - p^{(n)}_i)^n \le (1 - p^{(m)}_i)^m$, we have for n large enough:

$$\begin{aligned} \frac{I(nx^{(n)})}{I(mx^{(m)})} &= \prod_{i=0}^{M} \frac{(1 - p^{(n)}_i)^{nx^{(n)}_i}}{(1 - p^{(m)}_i)^{mx^{(m)}_i}} \\ &\le \prod_{i=0}^{M} (1 - p^{(n)}_i)^{nx^{(n)}_i - nx^{(m)}_i} \\ &\le \prod_{i=0}^{M} (1 - p^{(n)}_i)^{-n\|x^{(n)} - x^{(m)}\|} \\ &\le (1 - p^{(n)}_0)^{-Mn\|x^{(n)} - x^{(m)}\|} \\ &\le e^{2c_0 M\|x^{(n)} - x^{(m)}\|}. \end{aligned}$$

Now we have

$$\begin{aligned} \Big| I(nx^{(n)}) - I(mx^{(m)}) \Big| &= \Big| I(nx^{(n)}) \Big| \left| 1 - \frac{I(mx^{(m)})}{I(nx^{(n)})} \right| \\ &\le \left| 1 - \frac{I(mx^{(m)})}{I(nx^{(n)})} \right| \\ &\le \Big| 1 - e^{-2c_0 M\|x^{(n)} - x^{(m)}\|} \Big| \\ &\le 2c_0 M \|x^{(n)} - x^{(m)}\|, \end{aligned}$$

proving the claim. ■

The following corollary is an easy consequence of the proof of Lemma 1.

Corollary 1: Consider $x^{(n)} \in E_n$ and $x^{(m)} \in E_m$. Suppose Assumption 1 holds. Then,

$$\left| \frac{I(nx^{(n)})}{1 - p^{(n)}_i} - \frac{I(mx^{(m)})}{1 - p^{(m)}_i} \right| \le 2c_0 M \|x^{(n)} - x^{(m)}\|,$$

$i \in \{0, 1, ..., M\}$, whenever n and m are large enough.

For each n, define a function $F^{(n)}(x^{(n)})$ on E_n by setting $F^{(n)}(x^{(n)}) = f^{(n)}(nx^{(n)})$. We have the following result:

Lemma 2: Suppose Assumption 1 holds. Then the sequence $\{F^{(n)}\}$ is uniformly bounded, i.e., there exists a constant $C < \infty$ such that $\|F^{(n)}(x^{(n)})\| \le C$ for all $x^{(n)} \in E_n$ and for all n. Moreover, for $x^{(m)} \in E_m$, $x^{(n)} \in E_n$, and m, n large enough, we have

$$\|F^{(n)}(x^{(n)}) - F^{(m)}(x^{(m)})\| \le \eta(c_0, M)\|x^{(n)} - x^{(m)}\|,$$

whenever

$$\|x^{(n)} - x^{(m)}\| < 1/c_0 M,$$

where $\eta(c_0, M)$ is a constant that depends only on c_0 and M.

Proof: To prove that $\{F^{(n)}\}$ is uniformly bounded, observe that in view of Assumption 1, we have $|F^{(n)}_i(x^{(n)})| \le c_0$ for $i \in \{1, ..., M - 1\}$ and for all n. Thus, we have $\|F^{(n)}(x^{(n)})\| \le \sum_{i=0}^{M} |F^{(n)}_i(x^{(n)})| \le c_0(M + 1)$ for all

$x^{(n)} \in E_n$ and for all n.

Now observe that $|nx_i^{(n)}p_i^{(n)} - mx_i^{(m)}p_i^{(m)}| \leq c_0\|x^{(n)} - x^{(m)}\|$. Using Eqs. (10), (11), and (12), along with Corollary 1, for $1 \leq i \leq M-1$ we have

$$
\begin{aligned}
&|F_i^{(n)}(x^{(n)}) - F_i^{(m)}(x^{(m)})| \leq |nx_{i-1}^{(n)}p_{i-1}^{(n)} - mx_{i-1}^{(m)}p_{i-1}^{(m)}| \\
&+ |nx_i^{(n)}p_i^{(n)} - mx_i^{(m)}p_i^{(m)}| \\
&+ \left| \frac{nx_{i-1}^{(n)}p_{i-1}^{(n)}I^{(n)}(x^{(n)})}{1-p_{i-1}^{(n)}} - \frac{mx_{i-1}^{(m)}p_{i-1}^{(m)}I^{(m)}(x^{(m)})}{1-p_{i-1}^{(m)}} \right| \\
&\leq 2c_0\|x^{(n)} - x^{(m)}\| + \frac{I^{(n)}(x^{(n)})}{1-p_{i-1}^{(n)}}|nx_{i-1}^{(n)}p_{i-1}^{(n)} - mx_{i-1}^{(m)}p_{i-1}^{(m)}| \\
&+ mx_{i-1}^{(m)}p_{i-1}^{(m)} \left| \frac{I^{(n)}(x^{(n)})}{1-p_{i-1}^{(n)}} - \frac{I^{(m)}(x^{(m)})}{1-p_{i-1}^{(m)}} \right| \\
&= (3c_0 + 2c_0^2M)\|x^{(n)} - x^{(m)}\|
\end{aligned}
$$

Similarly, it can be shown that $|F_0^{(n)}(x^{(n)}) - F_0^{(m)}(x^{(m)})| \leq (2c_0 + c_0M + 2c_0^2M + 2c_0^2M^2)\|x^{(n)} - x^{(m)}\|$ and $|F_M^{(n)}(x^{(n)}) - F_M^{(m)}(x^{(m)})| \leq (3c_0 + 4c_0^2M)\|x^{(n)} - x^{(m)}\|$. Now since

$$\|F^{(n)}(x^{(n)}) - F^{(m)}(x^{(m)})\| \leq \sum_{i=0}^{M} |F_i^{(n)}(x^{(n)}) - F_i^{(m)}(x^{(m)})|,$$

the result follows by taking $\eta(c_0, M) = 2c_0 + 4c_0M + 4c_0^2M + 4c_0^2M^2$. ■

Remark 2: The above lemma implies that for a Cauchy sequence $\{x^{(n)}\}$ in E, the sequence $\{F_n(x^{(n)})\}$ is Cauchy in $\mathbb{R}^{M+1}$.

Define a function $F(x)$ on E as follows:

$$F(x) = \lim_{n\to\infty} F^{(n)}(x^{(n)}), \tag{19}$$

where $\{x^{(n)}\}$ is any sequence in E satisfying $x^{(n)} \in E_n$ and $x^{(n)} \to x$. The existence of the limit in Eq. (19) follows from Remark 2. To prove the uniqueness, let $\{x^{(n)}\}$ and $\{y^{(n)}\}$ be two sequences in E, satisfying:

$$x^{(n)}, y^{(n)} \in E_n, \text{ and } \lim_{n\to\infty} x^{(n)} = \lim_{n\to\infty} y^{(n)} = x.$$

Then for n large enough, we would have $\|y^{(n)} - x^{(n)}\| \leq \epsilon < 1/c_0M$, which, in view of Lemma 2, implies that $\|F^{(n)}(x^{(n)}) - F^{(n)}(y^{(n)})\| \leq \eta(c_0, M)\epsilon$, showing that $\lim_{n\to\infty} F^{(n)}(x^{(n)}) = \lim_{n\to\infty} F^{(n)}(y^{(n)})$.

Remark 3: The definition of $F(x)$ and Lemma 2 imply that $F(x) \leq C$ for all $x \in E$.

Remark 4: An alternative, but equivalent, way of defining the function F could be to first define for all $n \geq 1$ a function $\hat{F}^{(n)}$ on E by setting $\hat{F}^{(n)}(x) = f^{(n)}(nx)$, for $x \in E$, and then take F as the pointwise limit of the sequence of functions $\hat{F}^{(n)}(x)$.

The following result is a direct consequence of Lemma 2, the definition of $F(x)$, and the boundedness of $F(x)$ (see Remark 3).

Lemma 3: Suppose Assumption 1 holds. Then the fucntion $F(x)$ is Lipschitz continuous, i.e., there exists a constant $K < \infty$ such that for all $x, y \in E$, we have

$$\|F(x) - F(y)\| \leq K\|x - y\|.$$

Next, we obtain a closed form expression for the function $F(x)$:

Lemma 4: Suppose Assumption 1 holds. Then the function $F(x) = (F_0(x), ..., F_M(x))$, defined by Eq.(19), satisfies:

$$F_0(x) = \sum_{i=0}^{M} c_i x_i L(x) - x_0 c_0, \tag{20}$$

$$F_i(x) = x_{i-1}c_{i-1}(1 - L(x)) - x_i c_i, \textit{ for } i = 1, ..., M-1, \tag{21}$$

$$F_M(x) = x_{M-1}c_{M-1}(1 - L(x)) - x_M c_M L(x), \tag{22}$$

where $c_i = lim_{n\to\infty} np_i^{(n)}$ for $i \in \{0, ..., M\}$, and $L(x) = \prod_{i=0}^{M} e^{-c_i x_i}$.

Proof: Consider $x \in E$, with rational co-ordinates, i.e., $x_i = p_i/q_i$ for $i \in \{0, 1, ..., M\}$, where p_i, q_i are nonnegative integers. Let $q = LCM(q_0, ..., q_M)$, where LCM denotes the least commom multiple. Observe that $x \in E_n$ for $n = qk$, where $k \geq 1$ is an integer. In view of the definition of $F(x)$, we have that

$$F(x) = \lim_{k\to\infty} F^{(qk)}(x) = \lim_{k\to\infty} f^{(qk)}(qkx),$$

which, in view of Eqs.(10)-(12) satisfies Eqs.(20)-(22). For an irrational $x \in E$, the result now follows by appealing to the Lipschitz continuity of $F(x)$ (see Lemma 3). ■

The following result (which is similar to the notion of uniform convergence) is now an easy consequence of the definition of $F^{(n)}$, Eqs.(10)-(12), and Lemma 4.

Lemma 5: Suppose Assumption 1 holds. Then there exists a sequence $\{\delta_n\}$ of numbers satisfying:

$$\sup_{x^{(n)} \in E_n} \|F^{(n)}(x^{(n)}) - F(x^{(n)})\| \leq \delta_n \text{ and } \lim_{n\to\infty} \delta_n = 0.$$

For $l \in \mathbb{Z}^{M+1}$, let $\beta_l^{(n)} \triangleq \sup_{x \in S_n} P_l^{(n)}(x)$. We have the following result:

Lemma 6: For $n \geq 1$, let $g_n = \sum_l \|l\|\beta_l^{(n)}$ and $h_n = \sum_l \|l\|^2\beta_l^{(n)}$. Then the sequences $\{g_n\}$ and $\{h_n\}$ are uniformly bounded, i.e., there exists a constant $G, H < \infty$ such that $g_n \leq G$ and $h_n \leq H$.

Proof: Consider the set of states $S_0 = \{l : \|l\| \leq M\}$. Observe that for all $l \in S_0$, we have $|l_i| \leq \|l\| \leq M$ for $i \in \{0, 1, ..., M\}$. Thus, l_i can take at most $2M + 1$ values, and therefore the total number of states in S_0 is no more than $(2M+1)^{M+1}$. Hence, we have

$$\sum_{l \in S_0} \|l\|\beta_l^{(n)} \leq (2M+1)^{M+1}.$$

Now consider the set of states $S_k = \{l : kM \leq \|l\| \leq (k+1)M\}$ for $k \geq 1$. A similar argument as above shows that the number of states in S_k can be no more than $(2kM+1)^{M+1}$. Also, note that for a jump of magnitude $\|l\| \geq kM$ to occur, more than $\lfloor kM/2 \rfloor$ nodes must transmit during the current slot; the probability of which is smaller than

$$\binom{n}{\lfloor kM/2\rfloor}\left(\frac{c_0}{n}\right)^{\lfloor kM/2\rfloor} \leq \frac{(2c_0)^{\lfloor kM/2\rfloor}}{\lfloor kM/2\rfloor!}, \tag{23}$$

for $\lfloor kM/2\rfloor \leq n$. Thus, we have

$$\sum_{l\in S_k} \|l\|\beta_l^{(n)} \leq (2kM+1)^{M+1}\frac{(2c_0)^{\lfloor kM/2\rfloor}}{\lfloor kM/2\rfloor!}.$$

Let

$$D_0 \triangleq (2M+1)^{M+1},$$

and

$$D_k \triangleq (2kM+1)^{M+1}\frac{(2c_0)^{\lfloor kM/2\rfloor}}{\lfloor kM/2\rfloor!}$$

for $k \geq 1$. Now observing that $\beta_l(n) = 0$ for all l such that $\|l\| > 2n$, we obtain

$$g_n = \sum_{l:\|l\|\leq 2n} \|l\|\beta_l^{(n)} \leq \sum_{k=1}^{\infty} D_k \triangleq G < \infty,$$

proving the claim regarding $\{g_n\}$. The claim regarding $\{h_n\}$ can be proved in a similar fashion. ■

We are now ready to prove the almost sure convergence of the sequence $\{Z_n(t)\}$ to a deterministic process.

Theorem 1: Suppose Assumption 1 holds, $\lim_{n\to\infty} Z_n(0) = z^0$, and $Z(t)$ satisfies:

$$Z(t) = z^0 + \int_0^t F(Z(s))ds \text{ for } t \geq 0.$$

Then for every $t \geq 0$, we have

$$\lim_{n\to\infty} \sup_{0\leq s\leq t} \|Z_n(s) - Z(s)\| = 0 \ a.s.$$

Proof: From Theorem 4.1 in [27, Chapter 6, pp. 327], we have that the jump Markov process $Z_n(t)$ with intensities $q_{k,k+l/n}^{(n)} = nP_l^{(n)}(nk)$ satisfies for t less than the first infinity of jumps:

$$Z_n(t) = Z_n(0) + \sum_l ln^{-1}Y_l\left(n\int_0^t P_l^{(n)}(nZ_n(s))ds\right), \tag{24}$$

where $Y_l(u)$ are independent standard Poisson processes. Now for each $l \in E_n$, let $\hat{Y}_l(u) \triangleq Y_l(u) - u$, then $\hat{Y}_l(u)$ is a Poisson process centered at its mean. It is well known that $\hat{Y}_l(u)$ satisfies:

$$\lim_{n\to\infty} \sup_{0\leq u\leq v} n^{-1}\hat{Y}_l(nu) = 0 \ a.s., \text{ for all } v \geq 0. \tag{25}$$

Now observe that for $x^{(n)} \in E_n$, we have

$$F^{(n)}(x^{(n)}) = f^{(n)}(nx^{(n)}) = \sum_{l:nx^{(n)}+l\in S_n} lP_l^{(n)}(nx^{(n)}),$$

and therefore,

$$\begin{aligned} Z_n(t) = Z_n(0) &+ \sum_l ln^{-1}\hat{Y}_l\left(n\int_0^t P_l^{(n)}(nZ_n(s))ds\right) \\ &+ \int_0^t F^{(n)}(Z_n(s))ds. \end{aligned} \tag{26}$$

Let

$$\epsilon_n(t) \triangleq \sup_{0\leq s\leq t}\left\| Z_n(s) - Z_n(0) - \int_0^s F^{(n)}(Z_n(u))du\right\|,$$

then using Lemma 6, we obtain

$$\begin{aligned} \epsilon_n(t) &= \sup_{0\leq s\leq t}\left\|\sum_l ln^{-1}\hat{Y}_l\left(n\int_0^s P_l^{(n)}(nZ_n(u))du\right)\right\| \\ &\leq \sum_l \|l\|n^{-1}\sup_{0\leq s\leq t}\left\|\hat{Y}_l\left(n\beta_l^{(n)}s\right)\right\| \\ &\leq \sum_l \|l\|n^{-1}(Y_l(n\beta_l^{(n)}t) + n\beta_l^{(n)}t). \end{aligned} \tag{27}$$

The strong law of large numbers (applied to the independent increment process $Y_l(.)$), the uniform boundedness of the sequence $\{g_n\} = \{\sum_l \|l\|\beta_l^{(n)}\}$ (see Lemma 6), and the dominated convergence theorem, together imply that

$$\lim_{n\to\infty}\epsilon_n(t) \leq \sum_l \lim_{n\to\infty}\|l\|n^{-1}\sup_{0\leq s\leq t}\left\|\hat{Y}_l\left(n\int_0^t \beta_l^{(n)}ds\right)\right\| = 0 \text{ a.s.}$$

Using the Lipschitz continuity of F and Lemma 5, we have for $t \geq 0$ that

$$\begin{aligned} &\|Z_n(t) - Z(t)\| \leq \|Z_n(0) - z^0\| + \epsilon_n(t) \\ &+ \int_0^t \|F^{(n)}(Z_n(s)) - F(Z(s))\|ds \\ &\leq \|Z_n(0) - z^0\| + \epsilon_n(t) \\ &+ \int_0^t (\|F^{(n)}(Z_n(s)) - F(Z_n(s))\| + \|F(Z_n(s)) - F(Z(s))\|)ds \\ &\leq \|Z_n(0) - z^0\| + \epsilon_n(t) + \delta_n t + \int_0^t K\|Z_n(s) - Z(s)\|ds. \end{aligned}$$

Appealing to Gronwall's Inequality (see, for example, [27, Appendix 5, pp. 498]), it follows that

$$\|Z_n(t) - Z(t)\| \leq (\|Z_n(0) - z^0\| + \epsilon_n(t) + \delta_n t)e^{Kt}.$$

The result now follows by noting that

$$\lim_{n\to\infty}(\|Z_n(0) - z^0\| + \epsilon_n(t) + \delta_n t) = 0.$$

■

Our goal is to prove a result similar to Theorem 1 for the sequence of stochastic processes $\{Y_n(t)\}$. We will do this by comparing $\{Y_n(t)\}$ with $\{Z_n(t)\}$ as follows:

Theorem 2: Suppose Assumption 1 holds. Then the sequences $\{Y_n(t)\}$ and $\{Z_n(t)\}$, defined by Eqs.(17) and (18), respectively, satisfy:

$$\lim_{n\to\infty}\sup_{0\leq s\leq t}\|Y_n(s) - Z_n(s)\| = 0 \text{ a.s., for all } t \geq 0.$$

Proof: Let $\gamma_n(t) \triangleq \sup_{0\leq s\leq t}\|Y_n(s) - Z_n(s)\|$. Note that

$$\gamma_n(t) = \sup_{0\leq s\leq t}\|X_n(N(ns)) - X_n([ns])\|.$$

We need to prove that $\lim_{n\to\infty}\gamma_n(t) = 0 \ a.s.$ for all $t \geq 0$.

Let $\beta \triangleq 1/16$ and $\alpha \triangleq 7/8$. We have

$$\begin{aligned}\mathbb{P}\left(\gamma_n(t) > n^{-\beta}\right) &\leq \mathbb{P}\left(\sup_{0\leq s\leq t}|N(ns)-[ns]| > n^{\alpha}\right)\\ &+\mathbb{P}\left(\gamma_n(t) > n^{-\beta}|\sup_{0\leq s\leq t}|N(ns)-[ns]|\leq n^{\alpha}\right)\\ &\leq \mathbb{P}\left(\sup_{0\leq s\leq t}|N(ns)-ns| > n^{\alpha}-1\right)\\ &+\mathbb{P}\left(\gamma_n(t) > n^{-\beta}|\sup_{0\leq s\leq t}|N(ns)-[ns]|\leq n^{\alpha}\right) \quad (28)\end{aligned}$$

Now observing that $N(ns)-ns$ for $s\geq 0$ is a martingale, it follows that $|N(ns)-ns|$ for $s\geq 0$ is a submartingale. Using the L^p maximum inequality for $p=4$, we obtain

$$\mathbb{E}\left\{\left(\sup_{0\leq s\leq t}|N(ns)-ns|\right)^4\right\} \leq \left(\frac{4}{3}\right)^4 \mathbb{E}\{(N(nt)-nt)^4\}.$$

Using the Markov Inequality, and observing that $\mathbb{E}\{(N(nt)-nt)^4\} = nt+2n^2t^2$, we obtain

$$\begin{aligned}pr\left(\sup_{0\leq s\leq t}|N(ns)-ns| > n^{\alpha}-1\right) &\leq \frac{\mathbb{E}\{(N(nt)-nt)^4\}}{(n^{\alpha}-1)^4}\\ &\leq \left(\frac{4}{3}\right)^4\frac{nt+2n^2t^2}{(n^{\alpha}-1)^4} \leq 12n^{-3/2}t^2, \quad (29)\end{aligned}$$

for large enough n. Now we claim that for all $x\in S_n$ and all $p\geq 0$, the Markov chain $X_n(t)$ satisties:

$$\mathbb{P}\left(\sup_{p\leq q\leq p+n^{\alpha}}\|X_n(q)-x\| > n^{1-\beta}|X_n(p)=x\right) \leq 2n^{\alpha}e^{-n^{1/8}/16} \quad (30)$$

To prove the above claim observe that for the event

$$\left\{\sup_{p\leq q\leq p+n^{\alpha}}\|X_n(q)-x\| > n^{1-\beta}\right\}$$

to occur, there must be at least one time slot, out of the n^{α} time slots following the p^{th} time slot, in which $n^{1-\alpha-\beta}/2 = n^{1/16}/2$ or more nodes transmit. Since the probability of a node transmitting is no bigger than $p_0^{(n)} = c_0/n$, we have that the random variable $N = Bernoulli(n, c_0/n)$ stochastically dominates the random variable corresponding to the number of nodes that transmit during a time slot. A standard application of Chernoff Bound shows that:

$$\mathbb{P}(N > n^{1/16}/2) \leq 2e^{-n^{-1/8}/16},$$

and Eq.(30) follows by using the union bound. Now observe that if

$$\left\{\sup_{0\leq s\leq t}|N(ns)-[ns]|\leq n^{\alpha}\right\}$$

occurs, then the total number of jumps upto time t of the processes $Z_n(t)$ and $Y_n(t)$ combined, is no bigger than $2nt+n^{\alpha}$. Appealing to the union bound once again, we have that the second term in Eq.(28) is no bigger than

$$2n^{\alpha}(2nt+n^{\alpha})e^{-n^{1/8}/16} \leq t/n^{3/2}, \quad (31)$$

for large enough n. Combining Eqs.(31) and (29), we obtain

$$\mathbb{P}\left(\gamma_n(t) > n^{-\beta}\right) \leq \frac{12t^2+t}{n^{3/2}},$$

for large enough n, which implies that

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\gamma_n(t) > n^{-\beta}\right) < \infty,$$

and $\lim_{n\to\infty}\gamma_n(t) = 0$ $a.s.$ now follows from the first Borel-Cantelli Lemma. ■

Remark 5: Using the L^p maximum inequality for $p=4\gamma$, $\gamma > 1$, and making appropriate changes to the proof of Theorem V, one can show that $\mathbb{P}\left(\gamma_n(t) > n^{-\beta} \text{ i.o.}\right) = 0$ for all $\beta < 1/2$. Thus for any $\beta < 1/2$, there exists a corresponding integer $N_{\beta} < \infty$ such that $\gamma_n(t) < n^{-\beta}$ for $n \geq N_{\beta}$.

Combining the results in Theorems 1 and V, gives the desired result:

Theorem 3: Suppose Assumption 1 holds, $\lim_{n\to\infty} Y_n(0) = y^0$, and $Y(t)$ satisfies:

$$Y(t) = y^0 + \int_0^t F(Y(s))ds \text{ for } t\geq 0. \quad (32)$$

Then for every $t\geq 0$, we have

$$\lim_{n\to\infty}\sup_{0\leq s\leq t}\|Y_n(s)-Y(s)\| = 0 \ a.s.$$

Remark 6: The Lipschitz continuity of F guarantees that for all $y^0\in E$, there exists a unique solution to the initial value problem (IVP) corresponding to Eq.(32).

Theorem 3 shows the convergence of $\{Y_n(t)\}$ to $Y(t)$, over bounded intervals of time. For finite, but large n, Remark 5 shows that the difference between $Y_n(t)$ and $Z_n(t)$ is $O(n^{-\beta})$ for all $\beta < 1/2$. Next, we will characterize the error involved in approximating $Z_n(t)$ with $Y(t)$, following the approach given in [28].

Set $W_l^{(n)}(u) = n^{-1/2}\hat{Y}_l(nu)$ and let $V_n(t) = \sqrt{n}(Z_n(t) - X(t))$. Then, Eq.(26) can be rewritten as

$$\begin{aligned}V_n(t) &= \sum_l lW_l^{(n)}\left(\int_0^t P_l^{(n)}(nZ_n(s))ds\right)\\ &+\int_0^t\sqrt{n}(F^{(n)}(Z_n(s)) - F^{(n)}(X(s)))ds,\end{aligned}$$

which suggests the following limiting equation:

$$V(t) \triangleq \sum_l lW_l\left(\int_0^t \beta_l(X(s))ds\right) + \int_0^t \partial F(X(s))V(s)ds, \quad (33)$$

where $\beta_l(x) = \lim_{n\to\infty} P_l^{(n)}(nx^{(n)})$ for $x\in E$, and $\{x^{(n)}\}$ satisfies:

$$x^{(n)} \in E_n \text{ for } n=1,2,..., \text{ and } \lim_{n\to\infty} x^{(n)} \to x.$$

The existence and uniqueness of the above limit can easily be shown. Let Φ be the solution of the matrix equation

$$\frac{\partial}{\partial t}\Phi(t,s) = \partial F(X(t))\Phi(t,s),\ \Phi(s,s) = I, \quad (34)$$

and let

$$U(t) \triangleq \sum_l lW_l\left(\int_0^t \beta_l(X(s))ds\right).$$

Then, we have

$$V(t) = \int_0^t \Phi(t,s) dU(s).$$

Observe that since $U(t)$ is Gaussian with zero mean, $V(t)$ is Gaussian with zero mean and covariance matrix

$$Cov(V(t), V(r)) = \int_0^{t\wedge r} \Phi(t,s) C(X(s)) \Phi(r,s)^T ds,$$

where

$$C(x) = \sum_l l l^T \beta_l(x).$$

From Corollary 6, we have that $\sup_{x\in E} C(x) \le H < \infty$. Thus $V(t)$ is well defined.

Let $D_{\mathbb{R}^{M+1}}[0,\infty) = \{x : [0,\infty) \to \mathbb{R}^{M+1} |$ for all $t \ge 0$ $\lim_{s\to t+} x(s) = x(t)$ and $\lim_{s\to t-} x(s)$ exists$\}$, i.e., the space of right continuous functions having left limits. Henceforth, we will use the symbol "$\Rightarrow$" to denote the convergence in distribution in $D_{\mathbb{R}^{M+1}}[0,\infty)$, or equivalently, weak convergence in $\mathcal{P}(D_{\mathbb{R}^{M+1}}[0,\infty))$- the set of Borel probability measures on $D_{\mathbb{R}^{M+1}}[0,\infty)$. For the sake of definiteness, the metric used on $\mathcal{P}(D_{\mathbb{R}^{M+1}}[0,\infty))$ can be assumed to be the Prohorov metric (see, for example, [27, Chapter 3]); and the metric used on $D_{\mathbb{R}^{M+1}}[0,\infty)$ could be the one specified in [27, Chapter 3] that induces the Skorohod topology on $D_{\mathbb{R}^{M+1}}[0,\infty)$. For a detailed discussion of these metrics and related concepts, we refer the reader to [29].

The following theorem characterizes the error involved in approximating $Z_n(t)$ with $X(t)$:

Theorem 4: Let $V_n(t)$ and $V(t)$ be as above, then $V_n(t) \triangleq V_{1n}(t) + V_{2n}(t)$, where $V_{1n} \Rightarrow V$ and $V_{2n}(t) = O(t/\sqrt{n})$.

Remark 7: A consequence of the above result is that for large n, $X_n(t)$ can be well approximated by $nY(t/n) + n^{1/2}V(t/n)$. In view of Remark 5, the error in such an approximation is almost surely bounded by $O(n^\beta)$ for any $\beta > 1/2$. Also, since $V(t)$ has a finite variance for all t, the error in approximating $X_n(t)$ with $nY(t/n)$ is also almost surely bounded by $O(n^\beta)$ for any $\beta > 1/2$.

Proof: We have

$$\begin{aligned} V_n(t) &= \sum_l n^{-1/2} l \hat{Y}_l^{(n)} \left(n \int_0^t P_l^{(n)}(nZ_n(s)) ds \right) \\ &+ \int_0^t \sqrt{n}(F^{(n)}(Z_n(s)) - F^{(n)}(X(s))) ds \\ &= V_{1n}(t) + V_{2n}(t), \end{aligned}$$

where

$$\begin{aligned} V_{1n}(t) &\triangleq \sum_l n^{-1/2} l \hat{Y}_l^{(n)} \left(n \int_0^t P_l^{(n)}(nZ_n(s)) ds \right) \\ &+ \int_0^t \partial F(X(s)) V_n(s) ds \\ &+ \int_0^t \left[\sqrt{n} \left(F(Z_n(s)) - F(X(s)) \right) - \partial F(X(s)) V_n(s) \right] ds \end{aligned}$$

and

$$\begin{aligned} V_{2n}(t) &\triangleq \int_0^t \sqrt{n} \left(F(X(s)) - F^{(n)}(X(s)) \right) ds \\ &+ \int_0^t \sqrt{n} \left(F^{(n)}(Z_n(s)) - F(Z_n(s)) \right) ds. \end{aligned}$$

Using Eqs.(10)-(12), Eqs.(20)-(22), and noting $|np_i^{(n)} - c_i| \le K_0$ (see Remark 1), it follows that there exists a constant $K_1 < \infty$ such that for all n large enough, we have

$$\sup_{x\in E_n} \|F^{(n)}(x) - F(x)\| \le K_1/n.$$

Thus $V_{2n}(t) \le 2K_1 t/\sqrt{n} = O(t/\sqrt{n})$. Now turning to $V_{1n}(t)$, let

$$U_n(t) \triangleq \sum_l n^{-1/2} l \hat{Y}_l^{(n)} \left(n \int_0^t P_l^{(n)}(nZ_n(s)) ds \right),$$

and

$$\begin{aligned} e_n(t) &\triangleq V_{2n}(t) \\ &+ \int_0^t \left[\sqrt{n} \left(F(Z_n(s)) - F(X(s)) \right) - \partial F(X(s)) V_n(s) \right] ds \end{aligned}$$

Using the results in [27, Chapter 4], it can be shown that $U_n \Rightarrow U$, with U as above. Using the Lipschitz continuity of F, we have

$$|V_n(t)| \le |U_n(t)| + \int_0^t K |V_n(s)| ds,$$

and hence (using Gronwall's inequality)

$$\sup_{s\le t} |V_n(s)| \le \sup_{s\le t} |U_n(s)| e^{Kt}.$$

Since $U_n \Rightarrow U$ and U is continuous, it follows that $\sup_{s\le t} U_n(s) \Rightarrow \sup_{s\le t} U(s)$, and hence the V_n are stochastically bounded on bounded intervals. Furthermore, it is easy to see from Eqs.(20)-(22) that ∂F is continuous and bounded, which together with the fact that V_n are stochastically bounded on bounded intervals implies that $e_n \Rightarrow 0$. With Φ as above, we have

$$V_n(t) = U_n(t) + e_n(t) + \int_0^t \Phi(t,s) \partial F(X(s)) (U_n(s) + e_n(s)) ds.$$

Finally, noting that the mapping $J : D_{\mathbb{R}^{M+1}}[0,\infty) \to D_{\mathbb{R}^{M+1}}[0,\infty)$ given by

$$J\theta(t) = \theta(t) + \int_0^t \Phi(t,s) \partial F(X(s)) \theta(s) ds$$

is continuous, the result follows from the continuous mapping theorem (see, for example, [27, Chapter 3, pg. 103]). ■

Appendix B

In the previous section, we proved the convergence of the sequence of stochastic processes $\{Y_n(t)\}$ to the deterministic process $Y(t)$ satisfying:

$$Y(t) = Y(0) + \int_0^t F(Y(s)) ds \text{ for } t \ge 0,$$

where $F(x)$ is given by Eqs.(20)-(22). We would now like to further investigate the behavior of $Y(t)$ for large t. In particular, we would like to determine whether the vector differential equation

$$\frac{d}{dt}Y(t) = F(Y(t)), \tag{35}$$

has an equilibrium point. Supposing it does, we would like to find out whether that equilibrium point is unique. If the equilibrium point does exists and is unique, we would like to determine if the process $Y(t)$ started from an arbitrary initial state would converge to the equilibrium point.

Existence of Equilibrium Points

In this section, we will prove that the differential equation specified by (35) has at least one equilibrium point. The issue of the uniqueness will be dealt with in the next section.

Define a function $f(x)$ on E as follows:

$$f(x) = x + F(x), \text{ for } x \in E.$$

From Eqs.(20)-(22), it is easily seen that the function $f(x)$ maps E into itself. Since E is a compact subset of $\mathbb{R}^{M+1}$, Brouwer's fixed point theorem guarantees the existence of at least one fixed point of f:

Proposition 1: The fuction f has at least one fixed point in E.

Remark 8: Note that any fixed point of f is an equilibrium point of the vector differential equation specified by (35). To see this, suppose $x \in E$ is a fixed point of f. Then $f(x) = x$, implying that $F(x) = 0$; thus showing that x is indeed an equilibrium point of the vector differential equation specified by (35). Similarly, we have that any equilibrium point of the vector differential equation specified by (35) is a fixed point of f.

Uniqueness of Equilibrium Point

We will now establish the uniqueness of the equilibrium point:

Proposition 2: The vector differential equation specified by (35) has a unique equilibrium point.

Proof: Let us suppose that the vector differential equation specified by (35), has more than one equilibrium points. Then the function f must have more than one fixed points. Let x and y be two *different* fixed points of f. Then, in view of Eqs.(20)-(22), we have that x must satisfy:

$$x_0 c_0 = \sum_{i=0}^{M} c_i x_i L(x), \tag{36}$$

$$x_i c_i = x_{i-1} c_{i-1}(1 - L(x)), \text{ for } i = 1, ..., M-1, \tag{37}$$

$$x_M c_M L(x) = x_{M-1} c_{M-1}(1 - L(x)), \tag{38}$$

and y must satisfy a similar set of equations. Now the following possibilities can arise:

1) $L(x) = L(y)$. In this case, we have $\frac{x_i}{x_{i-1}} = \frac{y_i}{y_{i-1}}$ for all $i \in \{1, 2, ..., M\}$. Since $\sum_{i=0}^{M} x_i = \sum_{i=0}^{M} y_i = n$, we have $x = y$, which contradicts our initial assumption that $x \neq y$.
2) $L(x) > L(y)$. In this case, we have $\frac{x_i}{x_{i-1}} < \frac{y_i}{y_{i-1}}$ for all $i \in \{1, 2, ..., M\}$. If $x_0 \leq y_0$, then $x_i < y_i$, for all $i \in \{1, 2, ..., M\}$. However, this is not possible since $\sum_{i=0}^{M} x_i = \sum_{i=0}^{M} y_i = n$. Hence, we must have $x_0 > y_0$. Now let

$$k \triangleq \min_{i \in \{0,1,\ldots,M\}} \{i : x_i < y_i\}$$

Also, let $a_i \triangleq x_i - y_i$. From the definition of k, it follows that $a_i \geq 0$ for $i \in \{0, 1, ..., k-1\}$, and $a_i < 0$ for $i \in \{k, k+1, ..., M\}$. Since $\sum_{i=0}^{M} x_i = \sum_{i=0}^{M} y_i = n$, we have $\sum_{i=0}^{M} a_i = 0$. In particular, we have

$$\sum_{i=0}^{k-1} a_i = -\sum_{i=k}^{M} a_i$$

Using the definition of $L(x)$ and $L(y)$, we have that

$$\begin{aligned} L(x) &= \prod_{i=0}^{M} e^{-c_i x_i} = \prod_{i=0}^{M} e^{-c_i(y_i + a_i)} = L(y) \prod_{i=0}^{M} e^{-c_i a_i} \\ &= L(y) \prod_{i=0}^{k-1} e^{-c_i a_i} \prod_{i=k}^{M} e^{-c_i a_i} \\ &< L(y) e^{-c_{k-1} \sum_{i=0}^{k-1} a_i} e^{-c_k \sum_{i=k}^{m} a_i} \\ &= L(y) e^{-(c_{k-1} - c_k) \sum_{i=0}^{k-1} a_i} \\ &< L(y), \end{aligned}$$

which contradicts our initial assumption that $L(x) > L(y)$.
3) $L(x) < L(y)$. In this case also, one arrives at a contradiction, like in the previous case.

Since one of the above cases must occur, we have proved that f can have at most one fixed point, and, in view of Proposition 2, the result follows. ■

Convergence to Equilibrium Point

In this section, we will investigate whether the process $Y(t)$, started from any arbitrary initial state in E, converges to the unique equilibrium point. We have the following result for $M = 1$:

Proposition 3: Suppose $M = 1$. Then the process $Y(t)$ started from any arbitrary initial state in E, converges to the unique equilibrium point $\hat{y}$ satisfying $F(\hat{y}) = 0$.

Proof: For $M = 1$, the set of equations given by (35) simplify to

$$\frac{dy_0(t)}{dt} = c_1 y_1(t) L(y) - y_0(t) c_0 (1 - L(y)), \tag{39}$$

$$\frac{dy_1(t)}{dt} = y_0(t) c_0 (1 - L(y)) - y_1(t) c_1 L(y). \tag{40}$$

Now $\hat{y}$ satisfies $\hat{y}_0 c_0 (1 - L(\hat{y})) = c_1 y_1 L(\hat{y})$. Observe that for all $y(t)$ with $y_0(t) > \hat{y}_0$, we have $\frac{dy_0(t)}{dt} = -\frac{dy_1(t)}{dt} < 0$. Now consider the Lyapunov function $\lambda(y(t)) = (y(t) - \hat{y})^T (y(t) - \hat{y})$. It is straightforward to show that

$$\frac{d}{dt}\lambda(y(t)) = (y(t) - \hat{y})^T \frac{d}{dt} y(t) < 0 \text{ for } y(t) \neq \hat{y},$$

which implies that $\lim_{t \to \infty} y(t) = \hat{y}$. ■

Appendix C

In this section we evaluate the throughput of DCF under a special case, namely $M = 1$. For $M = 1$, the stationary distribution of the Markov chain $X_n(t)$ (see section III-A) can be computed, and thereby, one can compute the exact throughput of DCF. We start with the computation of the stationary distribution of $X_n(t)$.

Computation of Stationary Distribution

Let S_n denote the set of the system states for $M = 1$, i.e,

$$S_n = \{(x_1, x_2) : x_1, x_2 \in \mathbb{N}; x_1 + x_2 = n; x_1, x_2 \geq 0\}$$

where $\mathbb{N}$ denotes the set of integers. Observe that S_n contains $n + 1$ states. More precisely, $S_n = \{(0, n), (1, n - 1), ..., (n, 0)\}$. Let P_S^i be the steady state probability of the system being in state $(i, n - i)$. We now formulate the set of global balance equations that can be solved to obtain the stationary distribution of $X_n(t)$.

For the sake of brevity, let $\overline{p_i} \triangleq 1 - p_i$, $i \in \{0, 1, ..., M\}$. Now consider the state $(n, 0)$: The system leaves this state if there is a collision, an event of probability

$$(1 - \overline{p_0}^n - np_0\overline{p_0}^{n-1}).$$

The system can enter the state $(n, 0)$ only from the state $(n - 1, 1)$, provided the station in back-off stage 1 transmits successfully, an event of probability $p_1\overline{p_0}^n$. Balancing the probability flux entering and leaving the state $(n, 0)$, we have

$$P_S^n(1 - \overline{p_0}^n - np_0\overline{p_0}^{n-1}) = P_S^{n-1}p_1\overline{p_0}^n. \tag{41}$$

Now consider the state $(n - i, i)$: The system leaves this state if there is a successful transmission by a station in back-off stage 1, an event of probability

$$ip_1\overline{p_1}^{i-1}\overline{p_0}^{n-i};$$

or if there is an unsuccessful transmission involving at least one station in back-off stage 0, an event of probability

$$1 - \overline{p_0}^{n-i} - (n - i)p_0\overline{p_0}^{n-i-1} + (n - i)p_0\overline{p_1}^{n-i-1}(1 - \overline{p_0}^i).$$

The system can enter the state $(n - i, i)$ from the state

- $(n-i-1, i+1)$: Following a successful transmission by a station in back-off stage 1, an event of probability
$$(i + 1)p_1\overline{p_1}^i\overline{p_0}^{n-i-1};$$
- $(n-i+1, i-1)$: Following a collision involving exactly one station in back-off stage 0, and one or more stations in back-off stage 1, an event of probability
$$(n - i + 1)p_0\overline{p_0}^{n-i}(1 - \overline{p_1}^{i-1});$$
- $(n - j, j)$ for $0 \leq j \leq i - 2$: Following a collision involving $i - j$ stations in back-off stage 0, an event of probability
$$\binom{n-j}{i-j}p_0^{i-j}\overline{p_0}^{n-j}.$$

Balancing the probability flux leaving and entering the state $(n - i, i)$, we get

$$P_S^i(ip_1\overline{p_1}^{i-1}\overline{p_0}^{n-i} + 1 - \overline{p_0}^{n-i} - (n - i)p_0\overline{p_0}^{n-i-1}\overline{p_1}^i) =$$
$$P_S^{n-i-1}(i + 1)p_1\overline{p_1}^i\overline{p_0}^{n-i-1} + \sum_{j=0}^{i-2} P_S^{n-j}\binom{n-j}{i-j}p_0^{i-j}\overline{p_0}^{n-j} +$$
$$P_S^{n-i+1}(n - i + 1)p_0\overline{p_0}^{n-i}(1 - \overline{p_1}^{i-1}), \ 0 < i < n \tag{42}$$

Note that the summation term in Eq.(42) exists only for $i \geq 2$. Since the sum of stationary probabilities across all the system states must equal one, we have

$$\sum_{i=0}^{n} P_S^i = 1. \tag{43}$$

Observe that we have $n + 1$ equations in $n + 1$ unknowns. We leave it for the reader to verify that these equations are linearly independent, and therefore the stationary probabilies can be obtained by solving these equations.

Throughput Calculation

Once we have the stationary probabilities, we can calculate the throughput and other parameters of interest about the system. For $k \in \{0, 1, ..., M\}$, let:

- $T^k \triangleq$ The expected system throughput given the system is in state $(k, n - k)$.
- $P_c^k \triangleq$ The collision probability given the system is in state $(k, n - k)$.
- $I^k \triangleq$ The probability of an idle slot given the system is in state $(k, n - k)$.
- $T \triangleq$ The system throughput.
- $P_c \triangleq$ The conditional collision probability.
- $I \triangleq$ The probability of an idle slot.

Observe that $I^k = (1 - p_0)^k(1 - p_1)^{n-k}$. Arguing as in the derivation of Eq.(13), we obtain

$$P_c^k = 1 - \frac{kp_0\frac{I^k}{1-p_0} + (n - k)p_1\frac{I^k}{1-p_1}}{1 - I^k} \tag{44}$$

and the expected system throughput when the system is in state $(k, n - k)$ is given by:

$$T^k = \frac{(1 - I^k)(1 - P_c^k)P}{(1 - I^k)(1 - P_c^k)T_s + (1 - I^k)P_c^kT_c + I^k\sigma} \tag{45}$$

Since the probability that the system is in state $(k, n - k)$ is given by P_S^k, we have

$$T = \sum_{k=0}^{n} P_S^kT^k \tag{46}$$

Similarly, we have $I = \sum_{k=0}^{n} P_S^kI^k$ and $P_c = \sum_{k=0}^{n} P_S^kP_c^k$.

Performance Comparison

We now compare the exact results obtained by using the above approach, with the numerical results obtained using our technique and Bianchi's model. Note that our technique relies on the fact that for sufficiently large t, the process $X_n(t)$ stays close to the equilibrium point $x^{(n)}$ that satisfies $f^{(n)}(x^{(n)}) = 0$. To demonstrate the effectiveness of our technique, we compare the random sample paths of the system with the deterministic trajectory obtained using:

$$x(k+1) = x(k) + f^{(n)}(x(k)),$$

for $n = 50$, with $x(0) = (50, 0)$. As shown in Figure 6, not only does the system converge to a neighborhood of the equilibrium point for large t, but also the random trajectory of the system stays close to the above deterministic trajectory at all times (see Theorem 3, for a proof of such a result). Further, we see that the convergence to a neighborhood of the equilibrium point is quite rapid (within 100 slots).

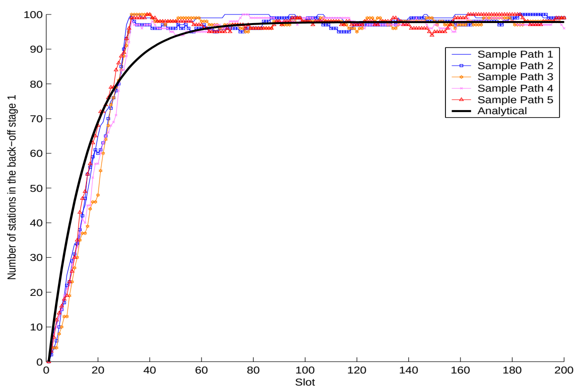

Fig. 6. Random system trajectories converging to a neighborhood of the equilibrium point.

TABLE II
THROUGHPUT: $W_0 = 32$, $M = 1$.

Stations	Throughput (T)		
	Exact	BM	OT
5	0.4664	0.4666	0.4669
15	0.4486	0.4484	0.4487
25	0.4229	0.4228	0.4230
55	0.3348	0.3348	0.3348
80	0.2543	0.2544	0.2543
100	0.1918	0.1918	0.1918

TABLE III
CONDITIONAL COLLISION PROBABILITY: $W_0 = 32$, $M = 1$.

Stations	Conditional Coll. Probability (P_c)		
	Exact	BM	OT
5	0.1008	0.1022	0.1008
15	0.2713	0.2727	0.2717
25	0.3961	0.3970	0.3965
55	0.6528	0.6530	0.6531
80	0.7879	0.7880	0.7881
100	0.8611	0.8611	0.8612

Tables II-IV show various parameters of interest obtained using the exact analysis, Bianchi's model (BM), and our technique (OT). The results shown are for RTS/CTS access mechanism with $W_0 = 32$. It is clear that both our technique and Bianchi's model are extremely accurate even for small n; and, as expected, their accuracy increases as n increases.

TABLE IV
IDLE SLOT PROBABILITY: $W_0 = 32$, $M = 1$.

Stations	Idle Slot Probability (I)		
	Exact	BM	OT
5	0.7692	0.7689	0.7681
15	0.5245	0.5244	0.5231
25	0.3782	0.3781	0.3771
55	0.1544	0.1544	0.1541
80	0.0743	0.0743	0.0742
100	0.0411	0.0411	0.0410

Appendix D

In this section, we show how the analysis in section III can be extended to a setting where heterogeneous protocol parameters are used as a means of providing service differentiation, as in the IEEE 802.11e EDCA mechanism [19]. We start with a description of our model.

Consider a similar setting as in section III-A and suppose that there are K different access categories (ACs), each maintaining its own set of back-off parameters. Each station maintains a separate transmit queue for each AC. All queues are assumed to be saturated, i.e., they always have a packet to send. As in section III-A, we make the following additional assumptions:

- **(A1)** The back-off durations are geometrically distributed, i.e., the type-k AC at a station, when in back-off stage j, transmits with probability $p_{k,j}$, where $p_{k,j} = \frac{2}{W_{k,j}+1}$, where $W_{k,j}$ is the contention window size of the type-k AC in back-off stage j.
- **(A2)** The back-off stage is reset to 0 only after a successful transmission.

For the purposes of analysis, we assume that the minimum idle duration time is the same, $DIFS$, for all ACs and no internal collision avoidance mechanism is used by the nodes. It should be noted that because of these two assumptions, the throughput obtained using our technique would not necessarily match the throughput obtained using the EDCA mechanism. A more exact analysis of the EDCA mechanism is left for future work.

Let M_k+1, $W_{k,0}$, denote the number of back-off stages and minimum contention window size, respectively, for the type-k AC, $k \in \{1, 2, ..., K\}$. Let $X_{k,j}(t)$, $j \in \{0, 1, ..., M_k\}$, denote the number of type-k ACs in back-off stage j at time t. Let

$$\mathcal{S} \triangleq \{(k, j) : k \in \{1, 2, ..., K\}; j \in \{0, 1, ..., M_k\}\}$$

and $M = \sum_{k=0}^{K} (M_k + 1)$. Then $X_n(t) = \{X_{k,j}(t)\}_{(k,j)\in\mathcal{S}}$, represents the state of the system at time t. Clearly, $X_n(t)$ for $t = 0, 1, ...,$ is a Markov chain on $\{0, 1, ..., n\}^M$, and satisfies:

$$\sum_{j=0}^{M_k} X_{k,j}(t) = n \text{ for } k \in \{1, 2, ..., K\}.$$

It can easily be shown that the Markov chain $X_n(t)$ is irreducible (see [7, Theorem 8.1], for a similar proof). Since it has only finitely many states, it follows that $X_n(t)$ is positive recurrent and possesses a stationary distribution. However, it does not appear possible to obtain a closed form expression for the stationary distribution of $X_n(t)$. Therefore, we proceed as in the previous section.

Let $\mathbb{Z}_+$ denote the set of non-negative integers, and let

$$S_n \triangleq \Big\{x = \{x_{k,j}\} : x_{k,j} \in \mathbb{Z}_+, \sum_{j=0}^{M_k} x_{k,j} = n, \; 1 \le k \le K\Big\}.$$

We denote the one-step drift of $X_n(t)$ by

$$\begin{aligned} f^{(n)}(x^{(n)}) &\triangleq \mathbb{E}\{X_n(t+1) - X_n(t) | X_n(t) = x^{(n)}\} \\ &= \sum_{l: x^{(n)}+l \in S_n} l P_l^{(n)}(x^{(n)}), \end{aligned}$$

for $x^{(n)} \in S_n$; here $P_l^{(n)}(x^{(n)})$ is the probability of making a transition from $x^{(n)}$ to $x^{(n)} + l$ over one time slot. Set $f^{(n)}(x^{(n)}) = 0$ for $x^{(n)} \notin S_n$. Arguing as in Section III-A, we obtain for $k \in \{1, 2, ..., K\}$:

$$\begin{aligned} f_{k,0}^{(n)}(x^{(n)}) &= \sum_{j=0}^{M_k} x_{k,j}^{(n)} p_{k,j} \frac{I(x^{(n)})}{1 - p_{k,j}} - x_{k,0}^{(n)} p_{k,0}, \\ f_{k,j}^{(n)}(x^{(n)}) &= x_{k,j-1}^{(n)} p_{k,j-1} \Big(1 - \frac{I(x^{(n)})}{1 - p_{k,j-1}}\Big) - x_{k,j}^{(n)} p_{k,j}, \\ &\qquad\qquad j \in \{1, 2, ..., M_k - 1\}, \\ f_{k,M_k}^{(n)}(x^{(n)}) &= x_{k,M_k-1}^{(n)} p_{k,M_k-1} \Big(1 - \frac{I(x^{(n)})}{1 - p_{k,M_k-1}}\Big) \\ &\quad - x_{k,M_k}^{(n)} p_{k,M_k} \frac{I(x^{(n)})}{1 - p_{k,M_k}}, \end{aligned} \tag{47}$$

where $I(x^{(n)}) = \prod_{(k,j) \in \mathcal{S}} (1 - p_{k,j})^{x_{k,j}^{(n)}}$. Let

$$B_n \triangleq \Big\{x = \{x_{k,j}\} : \sum_{j=0}^{M_k} x_{k,j} = n, 1 \le k \le K; x_{k,j} \ge 0\Big\},$$

and $E \triangleq B_n/n$.

The results derived in Appendix A and B can easily be extended to the current setting. In particular, under a similar set of assumptions, we can show that the sequence of scaled stochastic processes

$$Y_n(t) = X_n(\lfloor nt \rfloor)/n, \text{ for } n = 1, 2, ...,$$

converges (in the same sense, and with the same error bounds, as discussed for DCF earlier) to the deterministic limit $Y(t)$ given by the unique solution of the differential equation

$$\frac{dY(t)}{dt} = F(Y(t)) \text{ for } t \ge 0,$$

with initial condition $y^0 = \lim_{n\to\infty} Y_n(0) = X(0)/n$, and $F(x) = \lim_{n\to\infty} f^{(n)}(nx)$ for $x \in E$. The only difference from the DCF case earlier is that $f^{(n)}$ is now given by Eq.(47) instead of by Eqs.(3), (6), and (9)

Likewise, following the line of analysis in Appendix B, we can also show that there is a unique point $x \in E$ that satisfies $F(x) = 0$; we call it the equilibrium point.

Using the intuition that for large t, the process $X_n(t)$ should remain close to the point $x^{(n)} \in B_n$ satisfying $f^{(n)}(x^{(n)}) = 0$, we can carry out the throughput analysis as in section 5.

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