

# Incentives and Efficiency in Uncertain Collaborative Environments

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**Abstract.** We consider collaborative systems where users make contributions across multiple available projects and are rewarded for their contributions in individual projects according to a local sharing of the value produced. This serves as a model of online social computing systems such as online Q&A forums and of credit sharing in scientific co-authorship settings. We show that the maximum feasible produced value can be well approximated by simple local sharing rules where users are approximately rewarded in proportion to their marginal contributions and that this holds even under incomplete information about the player’s abilities and effort constraints. For natural instances we show almost 95% optimality at equilibrium. When players incur a cost for their effort, we identify a threshold phenomenon: the efficiency is a constant fraction of the optimal when the cost is strictly convex and decreases with the number of players if the cost is linear.

## 1 Introduction

Many economic domains involve self-interested agents who participate in multiple joint ventures by investing time, effort, money or other personal resources, so as to produce some value that is then shared among the participants. Examples include traditional surplus sharing games [18, 6], co-authorship settings where the wealth produced is in the form of credit in scientific projects, that is implicitly split among the authors of a paper [14] and online services contexts where users collaborate on various projects and are rewarded by means of public reputation, achievement awards, badges or webpage attention (e.g. Q&A Forums such as Yahoo! Answers, Quora, and StackOverflow [7–9, 5, 2, 13], open source projects [19, 24, 27, 11, 28]).

We study the global efficiency of simple and prefixed rules for sharing the value locally at each project, even in the presence of incomplete information on

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the player’s abilities and private resource constraints and even if players employ learning strategies to decide how to play in the game.

The design of simple, local and predetermined mechanisms is suitable for applications such as sharing attention in online Q&A forums or scientific co-authorship scenarios, where cooperative solution concepts, that require ad-hoc negotiations and global redistribution of value, are less appropriate.

Robustness to incomplete information is essential in online applications, where players are unlikely to have full knowledge of the abilities of other players. Instead, participants have only distributional knowledge about their opponents. Additionally, public signals, such as reputation ranks, achievement boards, and history of accomplishments result in a significant asymmetry in the beliefs about a player’s abilities. Therefore, efficiency guarantees should carry over, even if player abilities are arbitrary asymmetrically distributed.

In our main result we show that if locally at each project each player is awarded at least his marginal contribution to the value, then every equilibrium is a 2-approximation to the optimal outcome. This holds even when players’ abilities and resource constraints are private information drawn from commonly known distributions and even when players use no-regret learning strategies to play the game. We portray several mechanisms that satisfy this property, such as sharing proportionally to the quality of the submission. We also give a generalization of our theorem, when players don’t have hard constraints on their resources, but rather have soft constraints in the form of convex cost functions. Finally, we give classes of instances where near optimality is achieved in equilibrium.

**Our Results.** We consider a model of collaboration where the system consists of set of players and a set of projects. Each player has a budget of time which he allocates across his projects. If a player invests some effort in some project, this results in some submission of a certain quality, which is a player and project specific increasing concave function of the effort, that depends on the player’s abilities. Each project produces some value which is a monotone submodular function of the qualities of the submissions of the different participants. This common value produced by each project is then shared among the participants of the project according to some pre-specified sharing rule, e.g. equal sharing, or sharing proportionally to quality.

**1. Marginal Contribution and Simple Sharing Rules.** We show that if each player is awarded at least his marginal contribution to the value of a project, locally, then every Nash equilibrium achieves at least half of the optimal social welfare. This holds at coarse correlated equilibria of the complete information game when player’s abilities and budget are common knowledge and at Bayes-Nash equilibria when these parameters are drawn independently from commonly known arbitrary distributions. Our result is based on showing that the resulting game is universally  $(1, 1)$ -smooth game [21, 22, 25] and corresponds to a generalization of Vetta’s [26] valid utility games to incomplete information settings. We give examples of simple sharing rules that satisfy the above condition, such as proportional to the marginal contribution or based on the Shapley value or proportional to the quality. We show that this bound is tight for very

special cases of the class of games that we study and holds even for the best pure Nash equilibrium of the complete information setting and even when the equilibrium is unique. We also analyze ranking-based sharing rules and show that they can approximately satisfy the marginal contribution condition, leading only to logarithmic in the number of players loss.

**2. Near Optimality for Constant Elasticity.** We show that for the case when the value produced at each project is of the form  $v(x) = w \cdot x^\alpha$  for  $\alpha \in (0, 1)$ , where  $x$  is the sum of the submission qualities, then the simple proportional to quality sharing rule achieves almost 95% of the optimal welfare at every pure Nash equilibrium of the game, which always exists.

**3. Soft Budget Constraints and a Threshold Phenomenon.** When the players have soft budget constraints in the form of some convex cost function of their total effort, we characterize the inefficiency as a function of the convexity of the cost functions, as captured by the standard measure of elasticity. We show that if the elasticity is strictly greater than 1 (strictly convex), then the inefficiency both in terms of produced value and in terms of social welfare (including player costs) is a constant independent of the number of players, that converges to 2 as the elasticity goes to infinity (hard budget constraint case). This stands in a stark contrast with the case when the cost functions are linear, where we show that the worst-case efficiency can decrease linearly with the number of players.

**Applications.** In the context of *social computing* each project represents a specific topic on a user-generated website such as Yahoo! Answers, Quora, and StackOverflow. Each web user has a budget of time that he spends on such a web service, which he chooses how to split among different topics/questions that arise. The quality of the response of a player is dependent on his effort and on his abilities which are most probably private information. The attention produced is implicitly split among the responders of the topic in a non-uniform manner, since the higher the slot that the response is placed in the feed, the higher the attention it gets. Hence, the website designer has the power to implicitly choose the attention-sharing mechanism locally at each topic, by strategically ordering the responses according to their quality and potentially randomizing, with the goal of maximizing the global attention on his web-service.

Another application of our work is in the context of *sharing scientific credit* in paper co-authorship scenarios. One could think of players as researchers splitting their time among different scientific projects. Given the efforts of the authors at each project there is some scientific credit produced. Local sharing rules translate to scientific credit-sharing rules among the authors of a paper, which is implicitly accomplished through the order that authors appear in the paper. Different ordering conventions in different communities correspond to different sharing mechanisms, with the alphabetical ordering corresponding to equal sharing of the credit while the contribution ordering is an instance of a sharing mechanism where a larger credit is rewarded to those who contributed more.

**Related Work.** Our model has a natural application in the context of on-line crowdsourcing mechanisms which were recently investigated by Ghosh and Hummel [7, 8], Ghosh and McAfee [9], Chawla, Hartline and Sivan [4] and Jain,

Chen, and Parkes [13]. All this prior work focuses on a *single* project. In contrast, we consider multiple projects across which a contributor can strategically invest his effort. We also allow a more general class of project value functions. Having multiple projects creates endogenous outside options that significantly affect equilibrium outcomes. DiPalantino and Vojnović [5] studied a model of crowdsourcing where users can choose exactly one project out of a set of multiple projects, each offering a fixed prize and using a “winner-take-all” sharing rule. In contrast, we allow the value shared to be increasing in the invested efforts and allow individual contributors to invest their efforts across multiple projects.

Splitting scientific credit among collaborators was recently studied by Kleinberg and Oren [14], who again examined players choosing a single project. They show how to globally change the project value functions so that optimality is achieved at some equilibrium of the perturbed game.

There have been several works on the efficiency of equilibria of utility maximization games [26, 10, 16], also relating efficiency with the marginal contribution property. However, this body of related work focused only on the complete information setting. For general games, Roughgarden [21] gave a unified framework, called smoothness, for capturing most efficiency bounds in games and showed that bounds proven via the smoothness framework automatically extend to learning outcomes. Recently, Roughgarden [22] and Syrgkanis [25] gave a variation of the smoothness framework that also extends to incomplete information settings. Additionally, Roughgarden and Schoppmann gave a version of the framework that allows for tighter bounds when the strategy space of the players is some convex set. In this work we utilize these frameworks to prove our results.

Our collaboration model is related to the contribution games of [1]. However, in [1], it is assumed that all players get the same value from a project. This corresponds to the special case of equal sharing rule in our model. Moreover, they mainly focus on network games where each project is has only two participants.

Our model is also related to the bargaining literature [12, 15, 3]. The main question in that literature is similar to what we ask here: how should a commonly produced value be split among the participants. However, our approach is very different than the bargaining literature as we focus on simple mechanisms that use only local information of a project and not global properties of the game.

## 2 Collaboration Model

Our model of collaboration is defined with respect to a set  $N$  of  $n$  players and a set  $M$  of  $m$  available projects. Each player  $i$  participates in a set of projects  $M_i$  and has a budget of effort  $B_i$ , that he chooses how to distribute among his projects. Thus the strategy of player  $i$  is specified by the amount of effort  $x_i^j \in \mathbb{R}_+$  that he invests in project  $j \in M_i$ .

*Player Abilities.* Each player  $i$  is characterized by his type  $t_i$ , which is drawn from some abstract type space  $T_i$ , and which determines his abilities on different projects as well as his budget. When player  $i$  invests an effort of  $x_i^j$  on project

$j$  this results in a submission of quality  $q_i^j(x_i^j; t_i)$ , which depends on his type, and which we assume to be some continuously differentiable, increasing concave function of his effort that is zero at zero.

For instance, the quality may be linear with respect to effort  $q_i^j(x_i^j) = a_i^j \cdot x_i^j$ , where  $a_i^j$  is some project-specific ability factor for the player that is part of his type. In the context of Q&A forums, the effort  $x_i^j$  corresponds to the amount of time spent by a participant to produce some answer at question  $j$ , the budget corresponds to the amount of time that the user spends on the forum, the ability factor  $a_i^j$  corresponds to how knowledgeable he is on topic  $j$  and  $q_i^j$  corresponds to the quality of his response.

*Project Value Functions.* Each project  $j \in M$  is associated with a value function  $v_j(q^j)$ , that maps the vector of submitted qualities  $q^j = (q_i^j)_{i \in N_j}$  into a produced value (where  $N_j$  is the set of players that participate in the project). This function, represents the profit or revenue that can be generated by utilizing the submissions. In the context, of Q&A forums  $v_j(q^j)$  could for instance correspond to the webpage attention produced by a set of responses to a question.

We assume that this value is *increasing in the quality* of each submission and satisfies the *diminishing marginal returns property*, i.e. the marginal contribution of an extra quality decreases as the existing submission qualities increase. More formally, we assume that the value is submodular with respect to the lattice defined on  $\mathbb{R}^{|N_j|}$ : for any  $z \geq y \in \mathbb{R}^{|N_j|}$  (coordinate-wise) and any  $w \in \mathbb{R}^{|N_j|}$ :

$$v_j(w \vee z) - v_j(z) \leq v_j(w \vee y) - v_j(y), \quad (1)$$

where  $\vee$  denotes the coordinate-wise maximum of two vectors. For instance, the value could be any concave function of the sum of the submitted qualities or it could be the maximum submitted quality  $v_j(q^j) = \max_{i \in N_j} q_i^j$ .

*Local Value Sharing.* We assume that the produced value  $v_j(q^j)$  is shared locally among all the participants of the project, based on some predefined redistribution mechanism. The mechanism observes the submitted qualities  $q^j$  and decides a share  $u_i^j(q^j)$  of the project value that is assigned to player  $i$ , such that  $\sum_{i \in N_j} u_i^j(q^j) = v_j(q^j)$ . The utility of a player  $i$  is the sum of his shares across his projects:  $\sum_{j \in M_i} u_i^j(q^j)$ .

In the context of Q&A forums, the latter mechanism corresponds to a local sharing rule of splitting the attention at each topic. Such a sharing rule can be achieved by ordering the submissions according to some function of their qualities and potentially randomizing to achieve the desired sharing portions.

**From Effort to Quality Space.** We start our analysis by observing that the utility of a player is essentially determined only by the submitted qualities and that there is a one-to-one correspondence between submitted quality and input effort. Hence, we can think of the players as choosing target submission qualities for each project rather than efforts. For a player to submit a quality of  $q_i^j$  he has to exert effort  $x_i^j(q_i^j; t_i)$ , which is the inverse of  $q_i^j(\cdot; t_i)$  and hence is some

increasing convex function, that depends on the player's type. Then the strategy space of a player is simply be the set:

$$\mathcal{Q}_i(t_i) = \left\{ q_i = (q_i^j)_{j \in M_i} : \sum_{j \in M_i} x_i^j(q_i^j; t_i) \leq B_i(t_i) \right\} \quad (2)$$

From here on we work with the latter representation of the game and define everything in quality space rather than the effort space. Hence, the utility of a player under a submitted quality profile  $q$ , such that  $q_i \in \mathcal{Q}_i(t_i)$  is:

$$u_i(q; t_i) = \sum_{j \in M_i} u_i^j(q^j). \quad (3)$$

and minus infinity if  $q_i \notin \mathcal{Q}_i(t_i)$ .

**Social Welfare.** We assume that the value produced is completely shared among the participants of a project, and thus, the social welfare is equal to the total value produced, assuming players choose feasible strategies for their type:

$$SW^t(q) = \sum_{i \in N} u_i(q; t_i) = \sum_{j \in M} v_j(q^j) = V(q). \quad (4)$$

We are interested in examining the social welfare achieved at the equilibria of the resulting game when compared to the optimal social welfare. For a given type profile  $t$  we denote with  $\text{OPT}(t) = \max_{q \in \mathcal{Q}(t)} SW^t(q)$  the maximum welfare.

**Equilibria, Existence and Efficiency.** We examine both the complete and the incomplete information setting. In the complete information setting, the type (e.g. abilities, budget) of all the players is fixed and common knowledge. We analyze the efficiency of Nash equilibria and of outcomes that arise from no-regret learning strategies of the players when the game is played repeatedly. A Nash equilibrium is a strategy profile where no player can increase his utility by unilaterally deviating. An outcome of a no-regret learning strategy in the limit corresponds to a coarse correlated equilibrium of the game, which is a correlated distribution over strategy profiles, such that no player wants to deviate to some fixed strategy. We note that such outcomes always exist, since no-regret learning algorithms for playing games exist. When the sharing rule induces a game where each players utility is concave with respect to his submitted quality and continuous (e.g. Shapley value) then even a pure Nash equilibrium is guaranteed to exist in our class of games, by the classic result of Rosen [20].

In the incomplete information setting the type  $t_i$  of each player is private and is drawn independently from some commonly known distribution  $F_i$  on  $T_i$ . This defines an incomplete information game where players strategies are mappings  $s_i(t_i)$ , from types to (possibly randomized) actions, which in our game corresponds to feasible quality vectors. Under this assumption we quantify the efficiency of Bayes-Nash equilibria of the resulting incomplete information game, i.e. strategy profiles where players are maximizing their utility in expectation over other player's types:

$$\mathbb{E}_{t_{-i}}[u_i(s(t))] \geq \mathbb{E}_{t_{-i}}[u_i(s'_i, s_{-i}(t_{-i}))] \quad (5)$$

We note that a mixed Bayes-Nash equilibrium in the class of games that we study always exists assuming that the type space is discretized and for a sufficiently small discretization of the strategy space. Even if the strategy and type space is not discretized, a pure Bayes-Nash equilibrium is also guaranteed to exist in the case of soft budget constraints under minimal assumptions (i.e. type space is a convex set and utility share of a player is concave with respect to his submitted quality and is differentiable with bounded slope) as was recently shown by Meirowitz [17].

We quantify the efficiency at equilibrium with respect to the ratio of the optimal social welfare over the worst equilibrium welfare, which is denoted as the *Price of Anarchy*. Equivalently, we quantify what fraction of the optimal welfare is guaranteed at equilibrium.

### 3 Approximately Efficient Sharing Rules

In this section we analyze a generic class of sharing rules that satisfy the property that locally each player is awarded at least his marginal contribution to the value:

$$u_i^j(q^j) \geq v_j(q^j) - v_j(q_{-i}^j) \quad (6)$$

where  $q_{-i}^j$  is the vector of qualities where player  $i$  submits 0 and everyone else submits  $q_i^j$ .

Several natural and simple sharing rules satisfy the above property, such as sharing proportional to the marginal contribution or according to the local Shapley value.<sup>3</sup> When the value is a concave function of the total quality submitted, then sharing proportional to the quality: i.e.  $u_i^j(q^j) = \frac{q_i^j}{\sum_{k \in N_j} q_k^j} v_j(q^j)$ , satisfies the marginal contribution property. When the value is the highest quality submission, then just awarding all the value to the highest submission (e.g. only displaying the top response in a Q&A forum) satisfies the marginal contribution property (see full version).

We show that any such sharing rule induces a game that achieves at least a 1/2 approximation to the optimal social welfare, at any no-regret learning outcome and at any Bayes-Nash equilibrium of the incomplete information setting where players' abilities and budgets are private and drawn from commonly known distributions. Our analysis is based on the recently introduced smoothness framework for games of incomplete information by Roughgarden [22] and Syrgkanis [25], which we briefly survey.

*Smoothness of Incomplete Information Games.* Consider the following class of incomplete information games: Each player  $i$  has a type  $t_i$  drawn independently from some distribution  $F_i$  on some type space  $T_i$ , which is common knowledge.

<sup>3</sup> The Shapley value corresponds to the expected contribution of a player to the value if we imagine drawing a random permutation and adding players sequentially, attributing to each player his contribution at the time that he was added.

For each type  $t_i \in T_i$  each player has a set of available actions  $A_i(t_i)$ . A player's strategy is a function  $s_i : T_i \rightarrow A_i$  that satisfies  $\forall t_i \in T_i : s_i(t_i) \in A_i(t_i)$ . A player's utility depends on his type and the action profile:  $u_i : T_i \times A \rightarrow \mathbb{R}$ .

**Definition 1 (Roughgarden [22], Syrgkanis [25]).** *An incomplete information game is universally  $(\lambda, \mu)$ -smooth if  $\forall t \in \times_i T_i$  there exists  $a^*(t) \in \times_i A_i(t)$  such that for all  $w \in \times_i T_i$  and  $a \in \times_i A_i(w)$ :*

$$\sum_{i \in N} u_i(a_i^*(t), a_{-i}; t_i) \geq \lambda \text{OPT}(t) - \mu \sum_{i \in N} u_i(a; w_i) \quad (7)$$

**Theorem 1 (Roughgarden [21, 22], Syrgkanis [25]).** *If a game is universally  $(\lambda, \mu)$ -smooth then every mixed Bayes-Nash equilibrium of the incomplete information setting and every coarse correlated equilibrium of the complete information setting achieves expected social welfare at least  $\frac{\lambda}{1+\mu}$  of the optimal.*

It is easy to observe that our collaboration model, falls into the latter class of incomplete information games, where the action of each player is his submitted quality vector  $q_i$ , and the set of feasible quality vectors depend on his private type:  $\mathcal{Q}_i(t_i)$  as defined in Equation (2). Last the utility of a player is only a function of the actions of other players and not directly of their types, since it depends only on the qualities that they submitted.

**Theorem 2.** *The game induced by any sharing rule that satisfies the marginal contribution property is universally  $(1, 1)$ -smooth.*

*Proof.* Let  $t, w$  be two type profiles, and let  $\tilde{q}(t) \in \mathcal{Q}(t)$  be the quality profile that maximizes the social welfare under type profile  $t$ , i.e.  $\tilde{q}(t) = \arg \max_{q \in \mathcal{Q}(t)} SW(q)$ . To simplify presentation we will denote  $\tilde{q} = \tilde{q}(t)$ , but remind that the vector depends on the whole type profile. Consider any quality profile  $q \in \mathcal{Q}(w)$ . By the fact that  $\tilde{q}_i \in \mathcal{Q}_i(t_i)$  is a valid strategy for player  $i$  under type profile  $t_i$ , we have:

$$\sum_{i \in N} u_i(\tilde{q}_i, q_{-i}; t_i) = \sum_{i \in N} \sum_{j \in M_i} u_i^j(\tilde{q}_i^j, q_{-i}^j)$$

By the marginal contribution property of the sharing rule we have that:

$$\sum_{i \in N} u_i(\tilde{q}_i, q_{-i}; t_i) \geq \sum_{i \in N} \sum_{j \in M_i} \left( v_j(\tilde{q}_i^j, q_{-i}^j) - v_j(q_{-i}^j) \right) = \sum_{j \in M} \sum_{i \in N_j} \left( v_j(\tilde{q}_i^j, q_{-i}^j) - v_j(q_{-i}^j) \right)$$

Following similar analysis as in Vetta [26] for the case of complete information games, by the diminishing marginal returns property of the value functions:

$$v_j(\tilde{q}_i^j, q_{-i}^j) - v_j(q_{-i}^j) \geq v_j(\tilde{q}_{\leq i}^j \vee q_{\leq i}^j, q_{> i}^j) - v_j(\tilde{q}_{< i}^j \vee q_{< i}^j, q_{\geq i}^j)$$

Where it can be seen that the right hand side is the marginal contribution of an extra quality  $\tilde{q}_i^j$  added to a larger vector than the vector on the left hand side. Specifically, the left hand side is the marginal contribution of  $\tilde{q}_i^j$  to  $q_{-i}^j$ , while the right hand side is the marginal contribution of  $\tilde{q}_i^j$  to the vector  $(q_{< i}^j + \tilde{q}_{< i}^j, q_{\geq i}^j)$ . Summing this inequality for every player in  $N_j$  we get a telescoping sum:

$$\sum_{i \in N_j} v_j(\tilde{q}_i^j, q_{-i}^j) - v_j(q_{-i}^j) \geq v_j(\tilde{q}^j \vee q^j) - v_j(q^j) \geq v_j(\tilde{q}^j) - v_j(q^j)$$



Combining this with the initial inequality and using the fact that  $q \in \mathcal{Q}(w)$ , we get the desired universal (1, 1)-smoothness property:

$$\sum_{i \in N} u_i(\tilde{q}_i, q_{-i}; t_i) \geq \sum_{j \in M} v_j(\tilde{q}^j) - \sum_{j \in M} v_j(q^j) = \text{OPT}(t) - \sum_{i \in N} u_i(q; w_i)$$

□

**Corollary 1.** *Under a local sharing rule that satisfies the marginal contribution property, every coarse correlated equilibrium of the complete information setting and every mixed Bayes-Nash equilibrium of the incomplete information game achieves at least 1/2 of the expected optimal social welfare.*

We show that this theorem is tight for the class of games that we study and more specifically, for the proportional sharing rule. The tightness holds even at pure Nash equilibria of the complete information setting, even when all players have the same ability and even when the equilibrium is unique. Intuitively what causes inefficiency is that players prefer to congest a low value project with a high rate of success (i.e. produces almost its maximal value for a very small quality), e.g. an easy topic, rather than trying their own luck on a hard project that would yield very high value but would require a lot of effort to produce it.

*Example 1.* Consider the following instance: there are  $n$  players and  $n$  projects. Every player participates in every project. Each player has a budget of effort of 1 and the quality of his submission at a project is equal to his effort. Project 1 has value function  $v_1(x) = 1 - e^{-\alpha x}$ , where  $x$  is the total submitted quality. The rest of the  $n - 1$  projects have value function  $\kappa(1 - e^{-\beta x})$ . We assume that value is shared proportional to the quality. We show that if we let  $\alpha \rightarrow \infty$ ,  $\kappa = \frac{n-1}{\beta n^2}$  and  $\beta \rightarrow 0$ , then the unique equilibrium is for all players to put their whole budget on project 1. The optimal on the other hand is for players efforts to be spread out among all the projects. A good approximation to the optimal is for each player to pick a different project and devote his whole effort on it. The ratio of the optimal social welfare and the Nash equilibrium welfare in the limit of the above values will be  $1 + (1 - 1/n)^2$ , which converges to 2 as the number of players grows large. A detailed equilibrium analysis is given in the full version. □

### 3.1 Ranking Rules and Approximate Marginal Contribution

An interesting, from both theoretical and practical standpoint, class of sharing rules is that of ranking rules. In a ranking sharing scheme, the mechanism announces a set of fixed portions  $a_1^j \geq \dots \geq a_n^j$ , such that  $\sum_t a_t^j = 1$ . After the players submit their qualities, each player is ranked based on some order that depends on the profile of qualities (e.g. in decreasing quality order or in decreasing marginal contribution order). If a player was ranked at position  $t$  then he gets a share of  $a_t^j \cdot v_j(q^j)$ . Fixed reward rules capture several real world scenarios where the only way of rewarding participants is ordering them in a deterministic manner and the designer doesn't have the freedom to award to the players arbitrary fractions of the produced value.

We show here that although such sharing rules are quite restrictive, they are expressive enough to induce games that achieve only a logarithmic in the number of players loss in efficiency. To prove this we show that by setting the fixed portions inversely proportional to the position, then every player is guaranteed at least an  $\log(n)$ -fraction of his marginal contribution. It is then easy to generalize our analysis in Theorem 2 to show that sharing rules that award each player a  $k$ -fraction of his marginal contribution induce a universally  $(1/k, 1/k)$ -smooth game and achieve at least  $1/(k+1)$  of the optimal welfare at equilibrium.

**Lemma 1.** *By setting coefficients  $a_t^j$  proportional to  $\frac{1}{t}$ , the game resulting from the ranking sharing rule, where submissions are ranked with respect to the marginal contribution order, achieves a  $\Omega(1/\log(n))$  fraction of the optimal welfare at every coarse correlated and at every Bayes-Nash equilibrium.*

If the value is a function of the sum of the quality of submissions then a similar guarantee is achieved if submissions are ordered in decreasing quality.

## 4 Almost Optimality for Uniformly Hard Projects

In this section we identify a subclass of value functions for which the social welfare at equilibrium is a much better approximation to the optimal welfare, achieving almost 95% of the optimal. We start our quest, by observing that the crucial factor that led to the tight lower bound in the previous section, is that different projects have a very different rate of success: the percentage increase in the output for a percentage increase in the input was completely different for different projects and at different qualities within a project. This discrepancy in the output sensitivity was the main force driving the lower bound. In this section we examine a broad class of functions that don't allow for such discrepancies.

The standard economic measure that captures the sensitivity of the output of a function with respect to a change in its input is that of elasticity.

**Definition 2.** *The elasticity of a function  $f(x)$  is defined as:  $\epsilon_f(x) = \left| \frac{f'(x)x}{f(x)} \right|$ .*

One can show formally that the above parameter of a function has a one-to-one correspondence with the ratio of the percentage change in the output for a percentage of change in the input. Intuitively, projects whose value has the same and constant elasticity have the same and uniform difficulty, though not necessarily the same importance. Based on this reasoning, we examine the setting where all project value functions are functions of the total quality of submissions and have constant elasticity  $\alpha$ . It can be easily seen that such functions will take the form  $v_j(q^j) = w_j \cdot Q_j^\alpha$ , where  $Q_j = \sum_{i \in N_j} q_i^j$ . The coefficient  $w_j$  can be project specific, and will correspond to the importance of a project. For such value functions we prove that the proportional to the quality sharing mechanism achieves social welfare at any pure Nash equilibrium of the complete information setting that is almost optimal.

We point that our class of games always possess a pure Nash equilibrium, since they are games defined on a convex strategy space, with continuous and concave utilities and hence the existence is implied by Rosen [20].

**Theorem 3.** *Suppose that the project value functions are of the form  $v_j(q^j) = w_j \cdot Q_j^\alpha$ , for  $0 \leq \alpha \leq 1$  and  $w_j > 0$ . Then, the proportional to the quality sharing rule achieves social welfare at least  $\frac{2^{1-\alpha}}{2-\alpha} \geq 0.94$  of the optimal social welfare at every pure Nash equilibrium of the complete information game it defines. (see full version)*

Our analysis is based on the local smoothness framework of Roughgarden and Schoppmann [23], which yields tighter bounds for games with continuous and convex strategy spaces and where utilities are continuous and differentiable. It is easy to see that the strategy spaces in our setting  $\mathcal{Q}_i(t_i)$  as defined in equation (2) are convex, by the convexity of the functions  $x_i^j(\cdot; t_i)$ .<sup>4</sup> Additionally, it is easy to check that the utilities of the players under the proportional sharing rule are going to be continuous and differentiable at any point, except potentially at 0.

However, in the full version we show that in equilibrium no project receives 0 total quality with positive probability. We also show that this relaxed condition is sufficient to apply the local smoothness framework to pure Nash equilibria. Alternatively, we can bypass this technicality by assuming there are exclusive players who participate only at a specific project and always invest an  $\epsilon$  amount of effort. Making the latter assumption, we can use the local smoothness framework in its full generality and our conclusion in Theorem 3 carries over to correlated equilibria of the game (outcomes of no-swap regret learning strategies).

## 5 Soft Budget Constraints

So far we analyzed the case where players have a hard constraint on their effort, e.g. hard time constraint. In this section we relax this assumption and study the case where instead each player incurs a cost that is a convex function of the total effort he exerts, corresponding to a soft budget constraint on his effort. We exhibit an interesting threshold phenomenon in the inefficiency of the setting: if cost is linear in the total effort then the inefficiency can grow linearly with the number of participants. However, when effort cost is strictly convex, then the inefficiency can be at most a constant independent of the number of participants.

More formally, we will assume that each player has a cost function  $c_i(x; t_i)$  that determines his cost when he exerts a total effort of  $x$ . This cost function is also dependent on his private type  $t_i$ . The total exerted effort can be expressed with respect to the quality of submission as  $X_i(q_i; t_i) = \sum_{j \in M_i} x_i^j(q_i^j; t_i)$ . Thus a player's utility as a function of the profile of chosen qualities is:

$$u_i(q) = \sum_{j \in M_i} u_i^j(q^j) - c_i(X_i(q_i; t_i); t_i). \quad (8)$$

Unlike the previous section, the social welfare is not the value produced. Instead:

$$SW^t(q) = \sum_{i \in N} u_i(q; t_i) = \sum_{j \in M} v_j(q^j) - \sum_{i \in N} c_i(X_i(q_i; t_i); t_i) = V(q) - C^t(q).$$

<sup>4</sup> If the effort required to produce two quality vectors  $q_i, \hat{q}_i$  is at most  $B_i(t_i)$ , then the effort required to produce any convex combination of them is also at most  $B_i(t_i)$

We refer to  $V(q)$  as the production of an outcome  $q$  and to  $C^t(q)$  as the cost. We first show that when a sharing rule that satisfies the marginal contribution property is used, the production plus the social welfare at equilibrium is at least the value of the optimal social welfare. We then use this result to give bounds on the equilibrium efficiency parameterized by the convexity of the cost functions.

**Lemma 2.** *Consider the game induced by any sharing rule that satisfies the marginal contribution property and where players have soft budget constraints. Then the expected social welfare plus the expected production at any coarse correlated equilibrium of the complete information setting and at any Bayes-Nash equilibrium of the incomplete information setting, is at least the expected optimal social welfare. (see full version)*

We use the latter result to derive efficiency bounds for both production and social welfare. We assume that the sharing rule used induces a utility share that is a concave function of a player's submission quality and such that a player's share at 0 quality is 0. More formally, we assume that:  $g(x) = u_i^j(x, q_{-i}^j)$  is concave, continuously differentiable and  $g(0) = 0$ . We call such sharing rules *concave sharing rules*. It is easy to see that the proportional to quality sharing rule is a concave sharing rule when the value is concave in the total quality. For general value functions, the Shapley sharing rule is also a concave sharing rule.

We show efficiency bounds parameterized by the convexity of the cost functions, using the elasticity of the cost function as the measure of convexity. An increasing convex function that is zero at zero, has an elasticity of at least 1. We will quantify the inefficiency in our game as a function of how far from 1 the elasticity of the cost functions are (e.g.  $c_i(x) = \kappa \cdot x^{1+a}$  has elasticity  $1+a$ ).

**Theorem 4.** *If a concave sharing rule is used and the elasticity of the cost functions is at least  $1 + \mu$  then: i) the expected social welfare at any coarse correlated equilibrium of the complete information setting and at any Bayes-Nash equilibrium of the incomplete information setting is at least  $\frac{\mu}{1+2\mu}$  of the optimal, ii) the total value produced in equilibrium is at least  $\frac{1}{2} \frac{\mu}{1+\mu}$  of the value produced at the social welfare maximizing outcome. (see full version)*

The proof of the theorem is based on analyzing the first order conditions that an equilibrium strategy must satisfy. Combined with the elasticity of the cost function we manage to relate the expected cost at equilibrium with the expected production. Specifically, we show that at equilibrium and at the optimal strategy profiles the expected production is at least  $(1 + \mu)$  times the expected cost, where  $1 + \mu$  is the elasticity of the cost function. Combining, the above with our Lemma 2 we get the result. Thus the theorem uses both a local deviation analyses, via the use of the first order equilibrium conditions and a global deviation analysis, via the smoothness techniques used to prove Lemma 2.

From this theorem, we obtain that as long as  $\mu > 0$ , the efficiency of any Nash equilibrium, is a constant independent of the number of players. For instance, if the cost is a quadratic function of the total effort then the social welfare at

equilibrium is a 3-approximation to the optimal and the produced value is a 4-approximation to the value produced at the welfare-maximizing outcome.

The budget constraint case that we studied in previous sections can be seen as a limit of a family of convex functions that converge to a limit function of the form  $c_i(x_i) = 0$  if  $x_i < B_i$ , and  $\infty$  otherwise. Such a limit function can be thought of as a convex function with infinite elasticity. Taking the limit as  $\mu \rightarrow \infty$  in the latter theorem, gives that the social welfare at equilibrium is at least half the optimal, which matches our analysis in the previous section.

A corner case is that of linear cost functions, where our Theorem gives no meaningful upper bound. In fact as the following example shows, for linear cost functions the inefficiency can grow linearly with the number of agents.

*Example 2.* Consider a single project with value  $v(Q) = \sqrt{Q}$  and assume that the proportional to the quality sharing rule is used. Moreover, each player pays a cost of 1 per unit of effort, i.e.  $c_i(X_i) = X_i$  and where the quality is equal to the effort. The global optimum is the solution to the unconstrained optimization problem:  $\max_{Q \in \mathbb{R}_+} \sqrt{Q} - Q$ , which leads to  $Q^* = 1/4$  and therefore  $SW(Q^*) = 1/4$ . On the other hand, each player's optimization problem is:  $\max_{q_i \in \mathbb{R}_+} q_i \frac{1}{\sqrt{Q}} - q_i$ . Symmetry and elementary calculus gives that at the unique Nash equilibrium, the total effort is  $Q = \left(\frac{n-1/2}{n}\right)^2$ , and the social welfare is  $\frac{2n-1}{4n^2} = O(1/n)$ .  $\square$

## 6 Conclusion and Future Work

We analyzed a general model of collaboration under uncertainty, capturing settings such as online social computing and scientific co-authorship. We identified simple value sharing rules that achieve good efficiency in a robust manner with respect to informational assumptions.

Some questions remain open for future research. We showed that ranking rules, which are highly popular [7, 8], achieve a logarithmic approximation, using fixed-prizes independent of the distribution of qualities (prior-free) and of the game instance. Can a constant approximation be achieved if we allow the fixed prizes associated with each position to depend on the distribution of abilities and on the instance of the game? Also, consider a *two-stage* model where in the first stage players choose the projects to participate in and then play our collaboration game in the second stage. Can any efficiency guarantee be given on the welfare achieved at the subgame-perfect equilibria of this two-stage game?

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