Prior-free Auctions without Reserve Prices

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Abstract – We consider the optimal prior-free auction design for allocating a set of indistinguishable items to a set of buyers where the objective is to maximize seller's revenue. The key design requirement is that the auction must always fully allocate all items to buyers, which prevents using the traditional approach based on reserve prices. This is an important class of auctions both in theory and practice.

We study two types of auctions: generalized first-price and truthful. We show a revenue upper bound $o(v_1/\log(v_1/v_2))$ for both classes of auctions. This resolves an open question of Mirrokni, Muthukrishnan and Navad. We also give optimal auctions that match the upper bound and extend the results to the case of multiple items.

Our results answer the basic question of how much revenue can an auction generate without reserve prices. We also found that in our setting even by relaxing the truthfulness condition of the auction design to a Nash implementation, one cannot gain much larger revenue.

1 Introduction

Design of prior-free auctions has been of much research focus and has found numerous applications in practice for allocation of various kinds of resources including network resource allocation [11, 10], allocation of computing resources [18], and sponsored search auctions [4, 22]. The main motivation for prior-free framework is the need for robust auctions that do not depend on the underlying distributions of buyers' valuations, which in many cases are hard or impossible to obtain.

In this paper, we study the problem of maximizing revenue in prior-free auctions under an extra condition: *not to use reserve prices*. In the prior-free framework, most of the revenue maximizing auctions [6, 8, 14, 3] use reserve prices, where the seller announces a fixed *reserve price* and a sale occurs only at a price larger or equal to this reserve price; otherwise, the item remains unsold. In many settings, buyers do not find an auction transparent if reserve prices differ from one buyer to another or the reserve prices change over time. Moreover, having an item unsold might be problematic in some settings, for example, in display advertisements as discussed by Mirrokni, Muthukrishnan and Nadav [15].

Designing auctions without reserve prices has been a basic question in economics literature. Auction with reserve prices are considered a close analog to a monopoly with bargaining power; on the contrary, auctions without reserve prices provide a framework to capture the power of competition in the market. This phenomenon was studied in a classical work by Bulow and Klemperer [1], further developed in [12]. These results are for auctions in standard Bayesian setting where distributions of buyers' valuations are assumed to be a priori known.

Our paper investigates this basic question of "how much revenue can be achieved without reserve prices?" in prior-free settings. Consider the following classical example of single item auction with two buyers: one buyer with a much larger valuation than the other buyer, $v_1 \gg v_2$. It is well known that both classical first-price and the second-price auctions only generate a revenue of v_2. The question of designing a better auction in the prior-free framework has been considered in [14, 16, 15]. The result of Lu, Teng and Yu [14] gives an optimal truthful mechanism but based on reserve prices. In [15], Mirrokni, Muthukrishnan and Nadav proposed a first-price auction whose revenue can be larger than v_2 (but less than the revenue obtained in [14]) and leave several open questions. The main open question in this line of work, also suggested in [16], is to design an optimal prior-free auction (not necessary truthful) that *always allocate items to buyers*. This is a general class of auctions, often called absolute auctions, that prevents using of reserve prices and overcomes the disadvantages of reserve prices.

The questions. Motivated by this line of research, in this paper, we study two types of auctions, generalized first-price (a formal definition is given in Section 2) and truthful, in the following setting. We consider auctions for selling a set of $k \geq 1$ indistinguishable items to $n > k$ buyers that *always must allocate all items to buyers*. We consider the case of unit-demand buyers where each buyer wants to buy at most one item and allow for arbitrary buyers' valuations, where v_i is value per unit of an item for buyer i. We ask the following questions:

- What is the optimal prior-free first-price auction? A special case of interest is the optimal revenue of a natural class of quasi-proportional sharing mechanisms (formally defined in Section 2). This is a particular open

question in [15].

- A deeper question along this lines is: can we generate a larger revenue by first-price auctions with Nash equilibrium as a solution concept than by truthful auctions? Note that in our setting, Nash outcomes can also be implemented by truthful auctions, but in contrast to the Bayesian setting, the revenue equivalence theorem does not hold in the prior-free framework. An example of a quasi-proportional sharing is given in Appendix A, showing that the revenue of generalized first-price auction is larger than that of the corresponding direct truthful auction.

Our results. The answers that we found for the questions above can be summarized as follows:

- We establish an upper bound of $o(v_1/\log(v_1/v_2))$, where $v_1 \geq v_2 > 0$ are the two highest valuations, on the revenue at Nash equilibrium. This bound holds for any generalized first-price auction for selling a single item where the seller commits on always allocating the item to buyers. This result resolves an open problem in Mirrokni et. al. [15] on the existence of a mechanism whose revenue would be competitive to the highest buyer's valuation v_1.

  We show that the revenue upper bound is essentially achievable for the case of two buyers. Specifically, we show that there exists a generalized first-price auction that always allocates the item to buyers whose revenue at Nash equilibrium is at least $C_\varepsilon \cdot v_1/\log^{1+\varepsilon}(v_1/v_2+1)$, for every $\varepsilon > 0$ and a constant $C_\varepsilon > 0$.

  Our upper bound is essentially the same as the bound of the optimal auction *with reserve prices* in Lu, Teng and Yu [14], $o(v_1/\log(v_1))$, because in their paper, they use the normalization $v_2 = 1$. We note that our upper bound depends on the second highest valuation v_2. This is an important fact because in auctions that fully allocate the item, if we fix the highest valuation v_1 and let the second highest valuation v_2 tend to zero, the optimal revenue tends to zero as well. Without imposing the constraint that the seller must always allocate items to buyers, it is easy to construct an auction whose revenue only depends on the largest valuation v_1.

- For selling a single item to more than two buyers using quasi-proportional sharing, we show an even *stronger* impossibility result. We find that the revenue at equilibrium of such auctions is *not* competitive to any function $R_2(v_1, v_2)$ such that $R_2(v_1, v_2)$ tends to infinity as the highest valuation v_1 tends to infinity. Therefore, for these auctions, the revenue is not competitive to the revenue that would have been obtained if the only buyers were two buyers with highest valuations.

  We also establish a number of revenue characterization results for the class of Tullock auctions (formally defined in Section 2). These results are of general interest [21, 9, 19] and complement those in [15]. For example, we obtained an exact maximum revenue characterization at Nash equilibrium for the case of two buyers and two-sided bounds for the general case of multiple buyers.

- We show that there also exists a truthful auction that always fully allocates the item to buyers, achieving a revenue of $C_\varepsilon \cdot v_1/\log^{1+\varepsilon}(v_1/v_2+1)$. Therefore, asymptotically we do not gain more revenue by removing the truthfulness constraint for revenue-maximizing prior-free auctions. The truthful auction is designed based on the intuition gained by considering optimal first-price auctions.

- The negative result for the quasi-proportional sharing also gives insight on how to design a truthful auction for the general case of selling a set of $k \geq 1$ indistinguishable items to $n > k$ buyers. We construct a truthful auction whose revenue is at least $C \cdot \sum_{i=1}^{k} v_i/\log^{1+\varepsilon}(v_i/v_{k+1}+1)$, for small enough $\varepsilon > 0$ and C a positive constant that only depends on ε if $n \geq 2k$. The key idea is to select a subset of buyers with highest valuations and allocate the items as if pair-wise competitions were run between these buyers for appropriately defined portions of items.

Remark Our paper focuses on the revenue of prior-free auctions. For most of auctions designed in this paper, the social-welfare efficiency is lower bounded by a constant. A more detailed discussion of the social welfare of our auctions will be available in the full version of this paper.

Related work. The most closely related work to ours are the papers by Mirrokni, Muthukrishnan, and Nadav [15] and Muthukrishnan [16]. These are among first papers to consider the question of designing auctions in the prior-free framework where every item must always be fully allocated to buyers. Our paper answers some of the open questions posed in these works and extend to the case of multiple items.

Another related work is that of Lu, Teng, and Yu [14] who considered prior-free auction design where the objective is to maximize seller's revenue but without the constraint that every item must be always fully allocated to buyers. They showed that for every truthful auction, the revenue at equilibrium is $o(v_1/\log(v_1))$ where v_1 is the highest buyer's valuation and showed that this can be arbitrarily approximated using a reserve price.

The generalized first-price auctions considered in this paper accommodate as special cases those considered in various contexts including public choice and design of contests, e.g. [21, 9, 19], and proportional allocation that is commonly used in allocation of networking and computing resources [11, 10]. Recent and innovative "pay per bid auctions" that have been gaining much of momentum in the context of online services, e.g. `www.swoopo.com`, see [2], can also be understood as special cases of generalized first-price auctions.

In a broader picture, our work belongs to the line of work on optimal auction design that has been an object of intense recent research, e.g. [6, 17, 7, 13, 8, 14, 3], taking a theoretical computer science approach in designing prior-free auctions and competitive analysis of revenue against various benchmarks. In all these previous work, a typical approach is to use reserve prices that are inferred from a subset of buyers' bids. Thus, with these auction designs, an item may remain unsold.

Structure of the paper. In the next section, we give basic notations and assumptions that we use in the rest of our paper. In Section 3, we prove the revenue upper bound for generalized first-price auctions. We present an optimal auction for the case of two buyers in Section 4. In Section 5, we consider the case of multiple buyers and show a negative result on the competitiveness of quasi-proportional sharing. We build on the insight of this negative result to develop an auction for the general case of multiple items and multiple buyers in Section 6. Some proofs are deferred to Appendix.

2 Notation and Assumptions

In this section we introduce basic notation and definitions for generalized first-price auctions and the special case of quasi-proportional sharing, which we use in the rest of the paper.

Throughout the paper, we assume that there are n unit-demand buyers, whose private valuations are $v_1, \ldots, v_n$, and without loss of generality, unless otherwise indicated, we assume that buyers are ordered in decreasing order of their valuations: $v_1 \geq \ldots \geq v_n > 0$. The buyers's bid vector, the allocation vector and payment vector are denoted by $\vec{w} = (w_1, \ldots, w_n)$, $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{p} = (p_1, \ldots, p_n)$, respectively.

The allocation vector is a function of the bid vector. For our main results, we do not impose specific assumptions on this allocation function other than some weak regularity conditions, e.g. that the allocation function is *continuous and differentiable*. Note that in classical first-price auction, the allocation function is not differentiable, but we can approximate this function by a differentiable function that essentially does not effect the mechanism.

We assume that all $0 \leq x_i \leq 1$ are real numbers. This framework accommodates allocating either infinitely divisible or indivisible items. For the case of infinitely divisible items, $\vec{x}$ determines portions of items allocated across buyers. For the case of indivisible items, we consider a randomized allocation for an integral assignment of k items to k buyers such that x_i is the probability that buyer i is allocated an item. We consider auctions with the property that the seller always must allocate all items to buyers, thus if the seller sells k items, then the condition on the allocation

vector is $\sum_i x_i = k$.

A *Generalized First-Price Auction* is an auction where a buyer's payment is equal to this buyer's bid. Considering x_i as a probability that buyer i obtains an item, there are two different versions of this payment scheme:

- *All-pay auctions* where buyer i pays $p_i = w_i$, regardless of whether she obtains the item or not. In this case, the total revenue is $\sum_i w_i$.
- *Winner-pay auctions* where buyer i pays w_i only if the item is allocated to this buyer, hence $p_i = w_i x_i$. Therefore, the total revenue in this case is $\sum_i w_i x_i$.

A Quasi-proportional Sharing auction is a special case of the generalized first-price auction defined for the case of selling a single item. For given non-decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$, the allocation function is given by

$$x_i = \frac{f(w_i)}{\sum_j f(w_j)},\ 1 \le i \le n.$$

Functions of this form are commonly referred in literature as *success functions* with various regularity properties established in previous work, e.g. [9, 19, 20].
Finally, *Tullock auctions* are special cases of Quasi-proportional Sharing, where $f(w) = w^r$ for $r \ge 0$.

3 Revenue Bounds for Generalized First-Price Auctions

In this section, we prove an upper bound on the revenue for generalized first-price auction. The result resolves the open problem in Mirrokni, Muthukrishnan and Nadav [15] by showing that there exist no auctions that are competitive to highest buyer's valuation.

In comparison with Lu, Teng and Yu [14], our result differs in two main points. First, we analyze the solution concept of Nash equilibrium and do not restrict to the class truthful mechanisms. This requires a different proof technique, as in our case, in general, buyers' bids do not correspond to buyers' valuations. Second, although the revenue bound turns out to be very similar to that in [14], there is an important difference. In [14], the bound is of the order $v_1/\log(v_1)$ while in our case it is of the order $v_1/\log(v_1/v_2)$ where $v_1 \ge v_2$ are valuations of two buyers with highest valuations. The revenue $v_1/\log(v_1)$ cannot be achieved for the class of auctions that always allocate the item to buyers. For example, assuming that the highest valuation v_1 is fixed and the second highest valuation v_2 tends to zero, the revenue necessarily tends to zero. This happens because of a lack of competition!

Theorem 1 *For every generalized first-price auction (using a continuous and differentiable allocation function) that always allocates the item to buyers, we have that for every fixed constant $C > 0$, there exists a valuation vector $\vec{v}$ such that the revenue R at Nash equilibrium satisfies*

$$R < C \cdot \frac{v_1}{\log(v_1/v_2 + e - 1)}$$

where $v_1 \ge v_2 > 0$ are the two highest valuations.

Proof. We will consider the valuation vector with only two positive valuations $\vec{v} = (v_1, v_2)$. Let (x_1, x_2) and (w_1, w_2) be the allocation and payment of buyers 1 and 2. Assume that the allocation is given by $x_1 = 1 - x_2 = \phi(w_1, w_2)$, where $\phi(\cdot)$ is a function taking values in $[0, 1]$. We assume that $\phi(\cdot)$ is continuous and differentiable. We will consider both all-pay and winner-pay auctions.

For the all-pay auction, buyer 1 maximizes the payoff $v_1\phi(w_1, w_2) - w_1$ over $w_1 \ge 0$ and buyer 2 maximizes the payoff $v_2(1 - \phi(w_1, w_2)) - w_2$ over $w_2 \ge 0$. Hence, the condition for (w_1, w_2) at Nash equilibrium is the following

$$v_1 \frac{\partial\phi(w_1, w_2)}{\partial w_1} = 1 \text{ and } \ v_2 \frac{\partial\phi(w_1, w_2)}{\partial w_2} = -1. \tag{1}$$

For the winner-pay auction, buyer 1 payoff is $\phi(w_1,w_2)(v_1-w_1)$ and buyer 2 payoff is $(1-\phi(w_1,w_2)(v_2-w_2)$. The condition for (w_1,w_2) at Nash equilibrium is

$$\frac{\partial\phi(w_1,w_2)}{\partial w_1}(v_1-w_1)=\phi(w_1,w_2) \text{ and } \frac{\partial\phi(w_1,w_2)}{\partial w_2}(v_2-w_2)=\phi(w_1,w_2)-1. \tag{2}$$

We will prove the theorem by assuming that there exists a constant $A>0$ such that the revenue obtained at Nash equilibrium is at least $A\cdot\frac{v_1}{\log(v_1/v_2+c)}$, where c is a constant, independent of A, to be chosen later. We then derive an inequality between v_1 and w_1, and use this inequality together with the Nash conditions above to show the contradiction.

The revenue is w_1+w_2 for the all-pay auction and $w_1x_1+w_2x_2$ for the winner-pay auction. It is straightforward to see that if $w_1\geq w_2$ then for the case of all-pay auction $w_1\geq 1/2(w_1+w_2)$ and for the winner-pay auction $w_1\geq w_1x_1+w_2x_2$. Therefore, in both cases, there exists a constant $C>0$ such that

$$\text{if } w_1\geq w_2 \text{ then } w_1\geq C\cdot\frac{v_1}{\log(v_1/v_2+c)}.$$

From this, we lower bound w_1/v_1 as follows. As we will see, the reason that we need the constant c is that when v_1 and v_2 are approximately equal, the value of $\log(v_1/v_2)$ can be too small to derive a useful inequality. We now have

$$\begin{aligned}\frac{w_1}{C}\log\left(\frac{w_1}{C\cdot v_2}+c\right) &\geq \frac{v_1}{\log(v_1/v_2+c)}\log\left(\frac{v_1/v_2}{\log(v_1/v_2+c)}+c\right)\\ &= v_1\left(\frac{\log(v_1/v_2+c\log(v_1/v_2+c))}{\log(v_1/v_2+c)}-\frac{\log\log(v_1/v_2+c)}{\log(v_1/v_2+c)}\right).\end{aligned}$$

For sufficiently large c, the following two inequalities hold for every $v_1/v_2\geq 1$,

$$\frac{\log(v_1/v_2+c\log(v_1/v_2+c))}{\log(v_1/v_2+c)}\geq 1 \text{ and } \frac{\log\log(v_1/v_2+c)}{\log(v_1/v_2+c)}\leq\frac{1}{2}.$$

Indeed, it is easy to check that the first inequality holds provided that $c\geq e-1$ and the second inequality holds for every $c\geq 0$.

Therefore, we obtain

$$\frac{w_1}{C}\log\left(\frac{w_1}{C\cdot v_2}+e-1\right)\geq\frac{v_1}{2}.$$

Replacing v_2 on the left-hand side with w_2, we still have a valid inequality because $v_2\geq w_2$, thus, we derive

$$\frac{w_1}{C}\log\left(\frac{w_1}{C\cdot w_2}+e-1\right)\geq\frac{v_1}{2}.$$

Consider now the all-pay auction and note

$$\frac{\partial\phi(w_1,w_2)}{\partial w_1}=\frac{1}{v_1}\geq\frac{1}{2\frac{w_1}{C}\log(\frac{w_1}{C\cdot w_2}+e-1)}, \text{ for every } w_1>w_2.$$

Integrating with respect to w_1, from w_2 to infinity, we have: $1\geq\phi(\infty,w_2)-\phi(w_2,w_2)>\infty$ which is a contradiction.

Consider now the winner-pay auction and note that we can derive similar contradiction. From (2), we have

$$\frac{\partial\phi(w_1,w_2)}{\partial w_1}=\frac{\phi(w_1,w_2)}{v_1-w_1}\geq\frac{1/2}{v_1-w_1}\geq\frac{1}{2v_1}.$$

In the last inequality, we assumed that under $v_1\geq v_2$, at Nash equilibrium, buyer 1 obtains at least $1/2$. This assumption is without loss of generality because the auction always allocates the item and thus we can always change the roles of the two buyers to ensure this condition. From this point, the proof follows exactly as in the case of all-pay auction presented above. ∎

Remark. We note that using a similar technique as in [14], we can also show that $o(\frac{v_1}{\log(v_1/v_2)})$ is an upper bound on the revenue of any *truthful* auction that fully allocates the item to buyers. For future work, it is of interest to gain a deeper understanding of the connection between the two concepts.

4 Achievability for Two Buyers

4.1 Optimal Generalized First-Price Auction

We identify an auction for two buyers whose revenue is matching the upper bound established in Theorem 1. This gives an asymptotically optimal auction for the case of two buyers whose revenue R at Nash equilibrium satisfies $R/v_1 = \Omega(1/\log^{1+\varepsilon}(v_1/v_2))$, for every fixed $\varepsilon > 0$. The auction is a simple all-pay auction for two buyers where the allocation is a function of the ratio of buyers' bids, which we refer to as *the bid-ratio auction*.

BID-RATIO AUCTION:

Input: buyers' bids w_1 and w_2
Output: allocation $(x_1, x_2) = (\phi(w_1/w_2), 1-\phi(w_1/w_2))$
where $\phi(t) = \frac{1+f(t)-f(1/t)}{2}$ and $f(t) = 1 - \frac{1}{\log(\log(t+1)+1)+1}$
payment (w_1, w_2)

Theorem 2 *For the case of two buyers, there exists a generalized first-price auction such that, for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that the revenue R at Nash equilibrium satisfies*

$$R \geq C_\varepsilon \frac{v_1}{\log^{1+\varepsilon}(v_1/v_2+1)}$$

for every buyers' valuations $v_1 \geq v_2 > 0$. For example, such an auction is the bid-ratio auction showed above.

Proof. Consider the all-pay auction with two buyers with the allocation rule $x_1 = 1 - x_2 = \phi(w_1, w_2) = f(w_1/w_2)$ where $f : \mathbb{R}_+ \to [0,1]$ is an increasing, continuous and differentiable function, satisfying the conditions $f(0) = 0$, $f(1) = 1/2$ and $f(\infty) = 1$.

Buyer 1 maximizes $v_1 x_1 - w_1 = v_1 f(w_1/w_2) - w_1$ over $w_1 \geq 0$ and buyer 2 maximizes $v_2 x_2 - w_2 = v_2(1 - f(\frac{w_1}{w_2})) - w_2$ over $w_2 \geq 0$. The bids w_1 and w_2 at the Nash equilibrium satisfy the following conditions:

$$v_1 \cdot f'\left(\frac{w_1}{w_2}\right) = w_2 \text{ and } v_2 \cdot f'\left(\frac{w_1}{w_2}\right) = \frac{w_2^2}{w_1}.$$

From these two equations, we derive $v_1/v_2 = w_1/w_2$ and denote this ratio by $t := w_1/w_2$. Using the equations above, we note that at Nash equilibrium, the revenue can be written as

$$R = w_1 + w_2 = w_2(1+t) = v_1 \cdot f'(t)(1+t). \tag{3}$$

Our goal is to give a function $f(\cdot)$ that achieves the revenue close to the upper bound in Theorem 1. To this end, consider the following function

$$f(t) = 1 - \frac{1}{\log(\log(t+1)+1)+1}.$$

According to (3), using this allocation, the revenue of an all-pay auction satisfies

$$v_1 \cdot f'\left(\frac{v_1}{v_2}\right)\left(\frac{v_1}{v_2}+1\right) = \frac{v_1}{(\log(\frac{v_1}{v_2}+1)+1)\cdot(\log(\log(\frac{v_1}{v_2}+1)+1)+1)^2} \geq C_\varepsilon \frac{v_1}{\log^{1+\varepsilon}(\frac{v_1}{v_2}+1)},$$

for a constant $C_\varepsilon > 0$. For example, it is not difficult to see that one can choose C_ε such that $C_\varepsilon \leq (\log(2)/(\log(2)+1))y^\varepsilon/(\log(y+1)+1)^2$ for every $y \geq \log(2)$ and that largest such C_ε is of the order ε^2.

This would give us a desired auction. There is, however, an issue with this scheme. First, we would like the auction to be "fair" in allocating the item equally in the symmetric case where buyers' valuations are equal, i.e. $v_1 = v_2$. This does not hold by allocating according to the function $f(\cdot)$, since we have $f(1) = 1 - 1/[\log(\log(2)+1)+1] < 1/2$. This can be easily fixed by slightly modifying the function $f(\cdot)$. There is still a problem with this approach as in order to use this allocation rule, the seller would need to know which buyer has a larger valuation (v_1). Therefore, we need to use a different allocation function that is symmetric. Fortunately, there is a general way to fix this problem. Consider the following allocation rule

$$x_1 = \phi(w_1/w_2) \text{ and } x_2 = 1 - \phi(w_1/w_2) \text{ where } \phi(t) = \frac{f(t) + 1 - f(1/t)}{2}.$$

Notice that the allocation function $\phi(\cdot)$ does not depend on the identity of buyers, because $1 - \phi(t) = \phi(1/t)$, for every $t \in \mathbb{R}_+$.

Furthermore, for the function $\phi(\cdot)$, all the desired properties are satisfied: $x_1 + x_2 = 1$, $x_1 = 0$ if $w_1 = 0$, $x_1 = 1$ if $w_1 = \infty$. Finally, the revenue of this new auction is still asymptotically optimal: because of the symmetry we can assume that $v_1 \geq v_2$, and according to (3), the revenue is

$$v_1 \cdot \phi'\left(\frac{v_1}{v_2}\right)\left(\frac{v_1}{v_2}+1\right) \geq v_1 \cdot \frac{1}{2} \cdot f'\left(\frac{v_1}{v_2}\right)\left(\frac{v_1}{v_2}+1\right) \geq \frac{C_\varepsilon}{2}\frac{v_1}{\log^{1+\varepsilon}(\frac{v_1}{v_2}+1)}.$$

■

Remark. In Theorem 2, we established that there exist auctions whose revenue at Nash equilibrium is essentially tight to the upper bound in Theorem 1, asymptotically for large valuation ratio v_1/v_2. For example, it can be readily checked that $R/v_1 \geq 1/[8\log(v_1/v_2+1)^{3/2}]$, for every $v_1/v_2 \geq 1$. For asymptotically small values of v_1/v_2, in particular for the case $v_1 = v_2$, we have $R/v_1 \geq 1/8$.

We showed that one can find an auction whose revenue at Nash equilibrium compares favorably to the upper bound in Theorem 1, for large v_1/v_2. For this, it is crucial to choose a suitable allocation function and, for example, such a function is used in BID-RATIO AUCTION.

We also note that in the auction above, the buyer with higher valuation always gets more than half of the item. Thus, the efficiency of the auction is at least 1/2.

4.2 Optimal Truthful Auction

The optimal generalized first-price auction above gives us an insight how to design a truthful auction with essentially the same revenue guarantee. As discussed in the introduction and see Appendix A for an example, although the outcome of the auction BID-RATIO AUCTION can be implemented by a truthful mechanism, but the revenue is not the same as the revenue of the generalized first-price auction. However, as we will see in Theorem 3, a truthful mechanism that uses a similar allocation also gives essentially the optimal revenue.

Theorem 3 *For the case of two buyers, there exists a truthful auction such that for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that the revenue R satisfies*

$$R \geq C_\varepsilon \frac{v_1}{\log^{1+\varepsilon}(v_1/v_2+1)}$$

for every buyers' valuations $v_1 \geq v_2 > 0$. Furthermore, such an auction is TRUTHFUL AUCTION *showed below.*

TRUTHFUL AUCTION:

Input: buyers' bids w_1 and w_2

Output: allocation $(x_1, x_2) = (\phi(w_1/w_2), 1-\phi(w_1/w_2))$

where $\phi(t) = \frac{f(t)+1-f(1/t)}{2}$ and $f(t) = 1 - \frac{1}{\log^{\varepsilon}(t+e)}$

payment $p_1 = w_1\phi(w_1/w_2) - \int_0^{w_1} \phi(t/w_2)dt$

$p_2 = w_2(1-\phi(w_1/w_2)) - \int_0^{w_2}(1-\phi(w_1/t))dt.$

The proof of the theorem is given in Appendix B. The key part of the proof is to guess the right lower bound for the revenue. In our case, we use the intuition developed in the proof of Theorem 2.

In Appendix D, we also consider the optimal Tullock auction, that is $x_i = w_i^r / \sum_j w_j^r$.

5 Non Competitiveness for more than two Buyers

5.1 Limitation of Quasi-proportional Sharing Auctions

Recall that quasi-proportional sharing auctions, considered in [15] and also considered in [21], allocate the resources according to the following allocation function $x_i = f(w_i)/\sum_j f(w_j)$. In general, in order for this auction to have a pure Nash equilibrium, it suffices to restrict to concave, monotone increasing functions $f(\cdot)$. Under this condition, it is known that the pure Nash equilibrium can also be computed by a simple, natural learning algorithm [5].

Quasi-proportional sharing auctions were studied in [15] for the case of multiple buyers under premise that this class of auctions would achieve better revenue at Nash equilibrium than by using traditional second-price auction. For the case of two buyers, this is indeed the case. However, in this section, we show that this conjecture is false for the general case of $n \geq 2$ buyers. We will establish that for *any* quasi-proportional sharing, there exists a set of valuations for $n > 2$ buyers, for which the revenue is not competitive to a value that depends only on the valuations of two buyers with highest valuations.

This result provides us with insight into designing auctions that guarantee large revenue in cases when there is a large difference between the valuations of buyers with highest valuations, which we consider in the next section. There, we also extend to the case of multiple items.

Theorem 4 *For every quasi-proportional sharing auction (either all-pay or winner-pay) and every $R_2(v_1, v_2)$ such that $\lim_{v_1 \to \infty} R_2(v_1, v_2) = \infty$, there exists a set of $n > 2$ buyers with valuations $v_1 \geq v_2 \geq \ldots \geq v_n$, such that the revenue at Nash equilibrium is smaller than $R_2(v_1, v_2)$.*

Proof of the theorem is given in Appendix C. The proof's main idea is to use contradiction by assuming that the revenue at Nash equilibrium is $R_2(v_1, v_2)$-competitive and then showing that this contradicts conditions of Nash equilibrium.

5.2 Tullock Auctions

In the remainder of this section, we characterize the revenue at Nash equilibrium for Tullock auctions for the general case of $n \geq 2$ buyers. Recall that Tullock auctions are special cases of quasi-proportional sharing with the allocation function $x_i = w_i^r / \sum_j w_j^r$, where $0 < r < 1$. This will demonstrate cases that support the impossibility result of the previous theorem. These revenue characterization results are of general interest of the class of Tullock auctions. We show that the revenue at Nash equilibrium is well characterized by a norm of the vector $\underline{\vec{v}} := ((v_2/v_1)^r, (v_3/v_1)^r, \ldots, (v_n/v_1)^r)$. In particular, the norm of interest is the p-norm

$$||\underline{\vec{v}}||_p = \left(\sum_{i=2}^{n} [(v_i/v_1)^r]^p \right)^{1/p} \quad \text{where } p = 1/(1-r).$$

Theorem 5 *For every Tullock auction with parameter $0 < r < 1$, there exist positive constants C_1, C_2, and C_3 such that for $n \geq 2$ buyers with valuations $v_1 \geq v_2 \geq \ldots \geq v_n$, the revenue w_1 exerted from the highest bidder and total revenue R at Nash equilibrium satisfy the following:*

1. *If $||\underline{\vec{v}}||_p$ is sufficiently large (e.g. $||\underline{\vec{v}}||_p^p \geq 2$), then*

$$C_1 \cdot r||\underline{\vec{v}}||_p^{-p} \leq \frac{w_1}{v_1} \leq C_2 \cdot r||\underline{\vec{v}}||_p^{-p}$$

   and $R \leq v_1 \cdot \frac{w_1}{v_1} + C_3 \cdot v_2$. Furthermore, $\frac{w_1}{v_1} \sim r||\underline{\vec{v}}||_p^{-p}$, for large $||\underline{\vec{v}}||_p$.

2. *If $||\underline{\vec{v}}||_p$ is sufficiently small (e.g. $||\underline{\vec{v}}||_p < 1/2$), then*

$$C_1 \cdot r||\underline{\vec{v}}||_p \leq \frac{w_1}{v_1} \leq C_2 \cdot r||\underline{\vec{v}}||_p$$

   and $R \leq \frac{w_1}{v_1}(v_1 + C_3 \cdot v_2)$. Furthermore, $\frac{w_1}{v_1} \sim r||\underline{\vec{v}}||_p$, for small $||\underline{\vec{v}}||_p$.

Proof of the theorem is provided in Appendix F.

Remark From item 1 in the theorem, it is easily observed that for the revenue R to be competitive to v_1 it is necessary that $||\underline{\vec{v}}||_p$ is smaller than a positive constant. This cannot hold for every valuation vector $\vec{v}$; for example, if every v_i, $i \neq 1$, is a constant factor of the highest valuation v_1, as the number of buyers tends to infinity, $||\underline{\vec{v}}||_p$ tends to infinity and thus the revenue R diminishes to zero. This conforms with the general result in Theorem 4 which tells that the revenue is not competitive to $R_2(v_1, v_2)$, in general. In particular, for the set of valuation vectors under item 2 of the theorem, we have that the revenue is competitive to $R_2(v_1, v_2)$ in Theorem 4 as $R \geq v_1 \cdot w_1/v_1 \geq v_1 \cdot C_1 r\,(v_2/v_1)^r$.

6 Optimal Auctions for Multiple Items and Multiple Buyers

So far, we established an upper bound on the revenue $o(v_1/\log(v_1/v_2))$ and showed that there exists an auction that essentially achieves this revenue for the case of *two buyers*, achieving the revenue of $\Omega(v_1/\log^{1+\varepsilon}(v_1/v_2))$. We also showed that a natural auction format for multiple buyers cannot provide the same revenue guarantee.

In our framework, we note that there is the following interesting trade-off. On the one hand, the auction must allocate all items to buyers, and thus, if the number of buyers is too small, there is not enough of competition to exert revenue from buyers. On the other hand, as in the previous section, we have seen that the revenue of natural classes of auctions can decrease as the number of buyers increases. In this section, we will use this insight to design an auction for the case of multiple buyers. In our auction, we first select a subset of highest bidders and use the truthful auction in Section 4 to allocate the item between pairs of buyers. The result can be extended to the general case of selling k indistinguishable items to $n > k$ buyers. Again, our main constraint is that the seller has to allocate all items to buyers, which rules out the use of reserve prices.

Illustrative example of two items. Consider the case of two items. First if there are only 2 buyers, the revenue that can be obtained at Nash equilibrium is 0 because the auction has to allocate all the items. Consider now the case of three buyers. Similar to the single-item case, we can use the valuation of the smallest bidder for the payment of the two larger ones. However, this mechanism can yield small revenue if the valuation of the two other bidders are large. We use the following method: allocate freely 1/3 of an item to each buyer, and set up an auction between every pair of buyers to allocate 1/3 of an item. This is illustrated in Figure 1. One can show that this auction will yield a revenue of $\Omega\left(v_1/\log^{1+\varepsilon}(v_1/v_3) + v_2/\log^{1+\varepsilon}(v_2/v_3)\right)$. Now, if the number of buyers n is larger than three and we use similar approach by running an auction between every pair of buyers for $\frac{1}{n-1}$ fraction of the item, then the resulting revenue can be arbitrarily low when n is large. Thus, when $n \geq 4$ is large, if we only select 4 highest buyers and allocate the items among them, then one can prove a good bound on the revenue of the mechanism. This is the main underlying idea of the result stated in the following theorem.

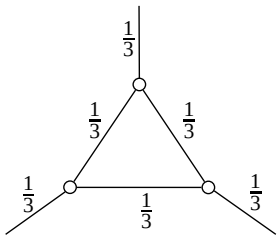

Figure 1: An illustration of multi-unit allocation for $n = 3$ buyers and $k = 2$ items.

Theorem 6 *Consider selling of $k \geq 1$ items to $n > k$ unit-demand buyers and let $\underline{n} = \min\{n, 2k\}$. There exists a truthful auction and $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$, there exists $C_\varepsilon > 0$ such that the revenue R satisfies*

$$R \geq \frac{\underline{n}-k}{\underline{n}-1} \cdot C_\varepsilon \sum_{i=1}^{k} \frac{v_i}{\log^{1+\varepsilon}(v_i/v_{k+1}+1)}$$

for every buyers' valuations $\vec{v}$. Such an auction is MULTI-UNIT AUCTION *described below.*

MULTI-UNIT AUCTION: k items, n buyers, $n > k$

Input: buyers bids $w_1, w_2, \cdots, w_n$.
Select $\underline{n} = \min\{n, 2k\}$ buyers with highest bids
with arbitrary tie break.

Output: For each pair of selected buyers, run TRUTHFUL to allocate
$\beta = 2(1-k/\underline{n})(\underline{n}-1)$ over this pair of buyers.
For each selected buyer, allocate, in addition, $2k/\underline{n} - 1$.

In case of allocating a single item, the multi-unit auction selects two buyers with highest bids and allocates the item according to TRUTHFUL for this two buyers. In the general case of $n \geq 2$ buyers, if the number of items is fewer than a half of the number of buyers, i.e. $k < n/2$, then the number of selected buyers is twice the number of items $\underline{n} = 2k$. In this case, each buyer is allocated exclusively through the pair-wise competitions. Otherwise, all the buyers are selected and every buyer is guaranteed an allocation of $2k/n - 1$.

Remark The mechanism above outputs an allocation $\vec{x}$ with $0 \leq x_i \leq 1$ and $\sum x_i = k$, where x_i is the probability that buyer i gets an item. It is known that such a distribution can be sampled among a set of at most n $\{0,1\}$ valued allocation vectors with exactly k coordinates equal to 1. It is also known that we can compute such a set of allocations and the corresponding sampling probabilities in polynomial time.

Proof. We first need to show that the auction always fully allocates k items to buyers and that each buyer is allocated at most 1. This follows by easy calculations and is thus omitted.

To prove the revenue bound, we use Theorem 3 and observe that for a constant $C_\varepsilon > 0$, the revenue extracted from buyer i, for $1 \leq i \leq \lceil k/2 \rceil$, is at least

$$\begin{aligned}
\beta \sum_{j=i+1}^{k+1} C_\varepsilon \frac{v_i}{\log^{1+\varepsilon}(v_i/v_j+1)} &\geq \beta \frac{k}{2} \cdot C_\varepsilon \frac{v_i}{\log^{1+\varepsilon}(v_i/v_{k+1}+1)} \\
&\geq \beta \frac{\underline{n}}{4} \cdot C_\varepsilon \frac{v_i}{\log^{1+\varepsilon}(v_i/v_{k+1}+1)} = \frac{\underline{n}-k}{\underline{n}-1} \cdot C_\varepsilon \frac{v_i}{\log^{1+\varepsilon}(v_i/v_{k+1}+1)}
\end{aligned}$$

where β is given in the definition of MULTI-UNIT AUCTION. Therefore, the revenue R satisfies

$$R \geq \sum_{i=1}^{k/2} \frac{\underline{n}-k}{\underline{n}-1} \cdot C_\varepsilon \frac{v_i}{\log^{1+\varepsilon}(v_i/v_{k+1}+1)} \geq \frac{\underline{n}-k}{\underline{n}-1} \cdot \frac{C_\varepsilon}{2} \sum_{i=1}^{k} \frac{v_i}{\log^{1+\varepsilon}(v_i/v_{k+1}+1)}$$

where the last inequality holds if either: (1) ε is sufficiently small (for example, one can choose $\varepsilon \leq \varepsilon_0$ where $\varepsilon_0 = 2\log(2) - 1 \approx 0.38$) or (2) $\varepsilon > 0$ is arbitrarily fixed and v_k/v_{k+1} is large enough. Under either of these two conditions, the asserted inequality holds as $v_i/\log^{1+\varepsilon}(v_i/v_{k+1}+1) \geq v_j/\log^{1+\varepsilon}(v_j/v_{k+1}+1)$, for every $1 \leq i \leq j \leq k$. ■

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A Revenue Equivalence does not hold in Prior-free Framework

The revelation principle is the most basic principle in mechanism design, which tells that we only need to focus on truthful mechanism because for every incentive compatibility mechanism, there exists a truthful one yielding the same outcome and revenue.

Unfortunately, this principle is only correct in the Bayesian setting. We will show now a simple example in prior-free framework, where the revenue equivalence theorem is not true.

Consider the case of two buyers in a single item auction with valuations v_1 and v_2. The allocation is assumed to be proportional sharing, i.e. $x_i = w_i/(w_1+w_2)$ for $i = 1,2$. It is easy to see that the Nash equilibrium of this game satisfies:

$$x_i = \frac{v_i}{v_1+v_2} \text{ and the revenue is } R(v_1,v_2) = \frac{v_1 v_2}{v_1+v_2}.$$

However, for the corresponding truthful mechanism, at Nash equilibrium, the buyers submit their valuations v_1 and v_2, and the allocation is

$$x_i = \frac{v_i}{v_1+v_2}, \text{ and payment is } p_i = v_i x_i - \int_0^{v_i} \frac{t}{t+v_j} dt.$$

Simple calculation shows that if $v_1 = v_2 = 1$, the revenue of the first auction is 0.5, while the revenue of the truthful mechanism is $2\log 2 - 1 < 0.4$.

B Proof of Theorem 3

Let us consider TRUTHFUL AUCTION. It is straightforward to establish that at Nash equilibrium buyers truthfully report their valuations, i.e. $\vec{w} = \vec{v}$. Notice that for the revenue R at Nash equilibrium, the following inequality holds

$$\begin{aligned} R &\geq p_1 = v_1\phi(v_1/v_2) - \int_0^{v_1} \phi(x/v_2)dx \\ &= v_1\frac{f(v_1/v_2)}{2} - \int_0^{v_1} \frac{f(x/v_2)}{2}dx + v_1\frac{1-f(v_2/v_1)}{2} - \int_0^{v_1}\frac{1-f(v_2/x)}{2}dx. \end{aligned}$$

Now, note that $1-f(v_2/v_1)$ is increasing with v_1/v_2, and hence $v_1\frac{1-f(v_2/v_1)}{2}-\int_0^{v_1}\frac{1-f(v_2/x)}{2}dx\geq 0$. Therefore, it follows

$$\begin{aligned} R &\geq v_1\frac{f(v_1/v_2)}{2}-\int_0^{v_1}\frac{f(x/v_2)}{2}dx \\ &= \frac{1}{2}v_1\left(1-\frac{1}{\log^{\varepsilon}(\frac{v_1}{v_2}+e)}\right)-\frac{1}{2}\int_0^{v_1}\left(1-\frac{1}{\log^{\varepsilon}(\frac{x}{v_2}+e)}\right)dx \\ &= \frac{1}{2}\int_0^{v_1}\frac{1}{\log^{\varepsilon}(\frac{x}{v_2}+e)}dx-\frac{1}{2}\frac{v_1}{\log^{\varepsilon}(\frac{v_1}{v_2}+e)}:=F(v_1). \end{aligned}$$

We are going to show that for every $\varepsilon>0$, there exists a constant $C_\varepsilon>0$ such that for every $v_1\geq v_2$, $F(v_1)\geq G(v_1)$ where

$$G(v_1)=C_\varepsilon\cdot\frac{v_1}{\log^{1+\varepsilon}(\frac{v_1}{v_2}+e)}.$$

To show this, we show that $F(v_2)\geq G(v_2)$ and $F'(v)\geq G'(v)$, for every $v>v_2$.

First, note that $F(v_2)\geq G(v_2)$ is equivalent to saying that

$$\int_0^1\frac{dx}{\log^{\varepsilon}(x+e)}-\frac{1}{\log^{\varepsilon}(1+e)}\geq 2C_\varepsilon\cdot\frac{1}{\log^{1+\varepsilon}(1+e)}.$$

Now, by partial integration, we have

$$\begin{aligned} \int_0^1\frac{dx}{\log^{\varepsilon}(x+e)} &= \frac{1}{\log^{\varepsilon}(1+e)}+\varepsilon\int_0^1\frac{x}{x+e}\frac{1}{\log^{1+\varepsilon}(x+e)}dx \\ &= \frac{1}{\log^{\varepsilon}(1+e)}+\varepsilon\int_0^1\frac{dx}{\log^{1+\varepsilon}(x+e)}+e\left(1-\frac{1}{\log^{\varepsilon}(1+e)}\right). \end{aligned}$$

Hence, the condition is

$$\varepsilon\int_0^1\frac{dx}{\log^{1+\varepsilon}(x+e)}+e\left(1-\frac{1}{\log^{\varepsilon}(1+e)}\right)\geq 2C_\varepsilon\cdot\frac{1}{\log^{1+\varepsilon}(1+e)}.$$

Indeed, the later inequality is true for C_ε such that

$$e\left(1-\frac{1}{\log^{\varepsilon}(1+e)}\right)\geq 2C_\varepsilon\cdot\frac{1}{\log^{1+\varepsilon}(1+e)}.$$

i.e. $C_\varepsilon\leq\frac{e}{2}\log(1+e)\,(\log^{\varepsilon}(1+e)-1)=\frac{e}{2}\log(1+e)\log(\log(1+e))\cdot\varepsilon+O(\varepsilon^2)$.

Second, we show that $F'(v)\geq G'(v)$, for every $v\geq v_2$. To this end, note that for $v\geq v_2$,

$$F'(v)=\frac{\varepsilon}{2}\cdot\frac{\frac{v}{v_2}}{\log^{1+\varepsilon}\left(\frac{v}{v_2}+e\right)\cdot\left(\frac{v}{v_2}+e\right)}$$

and

$$G'(v)=C_\varepsilon\cdot\frac{1}{\log^{1+\varepsilon}\left(\frac{v}{v_2}+e\right)}-C_\varepsilon(1+\varepsilon)\cdot\frac{\frac{v}{v_2}}{\log^{2+\varepsilon}\left(\frac{v}{v_2}+e\right)\left(\frac{v}{v_2}+e\right)}.$$

Hence, $F'(v)\geq G'(v)$ is equivalent to saying that for every $t\geq 1$,

$$\frac{\varepsilon}{2}\cdot\frac{t}{(t+e)\log^{1+\varepsilon}(t+e)}\geq C_\varepsilon\frac{1}{\log^{1+\varepsilon}(t+e)}-C_\varepsilon\frac{(1+\varepsilon)t}{(t+e)\log^{2+\varepsilon}(t+e)}$$

i.e.

$$\frac{\varepsilon}{2}\geq C_\varepsilon\cdot\left(1+\frac{e}{t}-\frac{1+\varepsilon}{\log(t+e)}\right).$$

Observe that the right-hand side is at most $C_\varepsilon\cdot(1+e)$, for every $t\geq 1$, and thus, for the condition to hold it suffices that C_ε is chosen such that $C_\varepsilon\leq\varepsilon/[2(1+e)]$. ∎

C Proof of Theorem 4

In order to contradict, let us assume that there exists $R_2(v_1,v_2) > 0$ and a function $f(\cdot)$ that is concave, monotone and increasing such that the revenue at Nash equilibrium is at least $R_2(v_1,v_2)$. We will separately consider the cases of all-pay and winner-pay auctions. The main idea of the proof is to first derive a relation that holds at Nash equilibrium, and then come up with a valuation profile for which the revenue cannot be larger than $R_2(v_1,v_2)$.

The proof for all-pay auctions is somewhat more complicated and involves considering three cases with respect to the function $f(\cdot)$, than for winner-pay auctions.

All-pay auctions. The payoff of buyer i is $v_i \frac{f(w_i)}{\sum_j f(w_j)} - w_i$. Taking the partial derivative with respect to w_i and noting that the payoff must be non-negative, we have that the following relations hold at Nash equilibrium:

$$\begin{aligned} v_i(1-x_i)f'(w_i) &= \sum_j f(w_j) \text{ if } x_i = \frac{f(w_i)}{\sum_j f(w_j)} > 0 \\ v_i x_i - w_i &\geq 0. \end{aligned} \tag{4}$$

In the following, we consider three distinct cases with respect to function $f(\cdot)$ and in each use different arguments to show that the auction is not competitive to $R_2(v_1,v_2)$.

Case 1: $f'(0) = \infty$ (for example $f(w) = w^r$, $0 < r < 1$). In this case, we will consider buyers' valuations $v, 1, 1, \ldots, 1$. Because of the symmetry, at Nash equilibrium, buyer 1 is allocated a positive value x_1 and all other buyers are allocated a positive value x_2. Therefore, $x_i < 1/2$ for $i \neq 1$. Together with the condition (4), we have

$$\frac{1}{2}f'(w_2) \leq 1 \cdot (1-x_2)f'(w_2) = v(1-x_1)f'(w_1) < vf'(w_1).$$

Therefore,

$$f'(w_2) < 2vf'(w_1).$$

Because we assume that the auction is competitive to $R_2(v_1,v_2)$ and because we can collect at most a revenue of 1 from every buyer except buyer 1, there exists a v^* such that if we take $v > v^*$, then $w_1 > 1$. It follows, $f'(w_2) < 2vf'(1)$, and thus $w_2 > (f')^{-1}(2vf'(1)) > 0$. The last inequality is true because of the assumption $f'(0) = \infty$.

Now, we have derived that w_2 is greater than a positive constant that is independent of the number of buyers n. This leads to a contradiction, because as n goes to infinity, the payoff of buyer 2, $x_2 - w_2$ tends becomes negative, which is a contradiction.

Case 2: $f'(0) = c < \infty$, $f(\infty) = \infty$ (for example $f(w) = w$ or $f(w) = \log(w+1)$). In this case, we also consider the valuation vector of the form $(v, 1, 1, \ldots, 1)$. From condition (4), we have

$$\sum_j f(w_j) = (1-x_2)f'(w_2) < f'(0) = c.$$

The last inequality follows from the fact that $f(\cdot)$ is a concave function. Thus, we have $f(w_1) < c$, and therefore, w_1 cannot go to infinity as v tends to infinity, which is a contradiction.

Case 3: $f'(0) = c < \infty$, $f(\infty) < \infty$, without loss of generality, we can assume that $f(\infty) = 1$ (for example $f(w) = w/(w+1)$). In this case, we will consider the valuation vector $(v_0, v_0, \ldots, v_0)$, for $v_0 > 0$. Then, at Nash equilibrium, every buyer is allocated $1/n$, and pays w_0 that satisfies

$$v_0(1-1/n)f'(w_0) = n/2.$$

From this, we obtain

$$v_0 f'(w_0) = \frac{n^2}{2(n-1)}. \tag{5}$$

Let w^* be the value such that $f(w^*) = 1/16$. We note that w^* only depends on $f(\cdot)$, not on the number of buyers n. We will show that if we increase the valuation of buyer 1 and keep all other buyer's valuations fixed to v_0, then at Nash equilibrium, the payment of buyers with valuation v_0 will be at least w^*. For now, assume that this is true. Then, this will lead to a contradiction, because

$$x_1 = \frac{f(w_1)}{\sum_j f(w_i)} \leq \frac{1}{1+(n-1)f(w^*)}.$$

Therefore, as $n \to \infty$, then $x_1 \to 0$. This means that given the valuation v_1 of buyer 1, if the number of buyers with valuation $v_0 < v_1$ increases, the revenue obtained from buyer 1 is at most $v_1 x_1$, which tends to 0. From this we conclude that the revenue that is extracted from buyer 1 cannot be a function that only depends on v_1 and v_0, a contradiction to the assumption that the auction is competitive to $R_2(v_1, v_2)$.

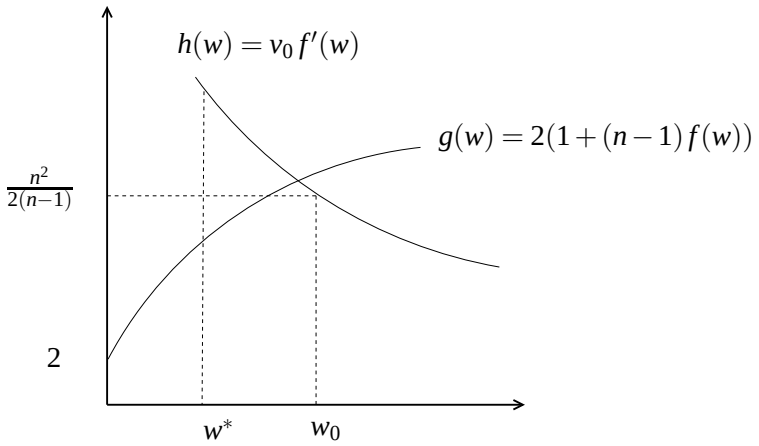

Figure 2: $w > w^*$

It remains only to show that at Nash equilibrium, each buyer $2, \ldots, n$, pays w such that $w > w^*$. Because of the Nash condition (4), we have

$$v_0(1-x_i) f'(w) = f(w_1) + (n-1) f(w).$$

We have $x_i < 1/2$ and $f(w_1) \leq 1$, because of the condition on the present subclass of functions f that we consider.) Thus, $\frac{1}{2} v_0 f'(w) < 1 + (n-1) f(w)$, which is equivalent to

$$v_0 f'(w) < 2(1+(n-1) f(w)) \tag{6}$$

We will prove $w > w^*$ from (5) and (6). Consider two functions $h(w) := v_0 f'(w)$ and $g(w) := 2(1+(n-1) f(w))$. (See Figure 2). $v_0 f'(w)$ is decreasing, while $2(1+(n-1) f(w))$ is increasing, thus in order for (6) to be satisfied, w is at least the value where the two curves intersect. But we have

$$h(w^*) > h(w_0) = \frac{n^2}{2(n-1)}$$

and

$$g(w^*) = 2(1+(n-1) f(w^*) = 2\left(1 + \frac{n-1}{16}\right) \leq \frac{n^2}{2(n-1)} \text{ if } n > 2.$$

Therefore, the intersection of the two curves has to be at a point larger than w^*.

Winner-pay auctions. We now prove that the winner-pay case is not competitive either.

The payoff of each buyer is $\frac{f(w_i)}{\sum f(w_j)}(v_i - w_i)$. Taking the partial derivative with respect to w_i, we have the relation at Nash equilibrium:

$$f'(w_i)\frac{\sum_{j\neq i} f(w_j)}{(\sum_j f(w_j))^2}(v_i - w_j) - \frac{f(w_i)}{\sum f(w_j)} = 0$$

which is equivalent to

$$f'(w_i)(1-x_i)(v_i - w_i) = f(w_i). \tag{7}$$

This is a similar condition as for the case of all-pay auctions. On the left-hand side, we have a decreasing function of w_i and on the right-hand side, we have an increasing function of w_i. Given x_i and v_i, w_i is the unique solution of (7). It is straightforward to see that if x_i decreases, then the function on the left-hand side increases, and as a result, the unique solution w_i also increases.

Now, if we consider n buyers with valuations $(v,1,\ldots,1)$ and let n increases, we will show that the bid w_i, for every buyer $i = 2,\ldots,n$, is at least a positive number w_0 that does not depend on n.

Let w_0 be the solution of the equation $f'(w_0)(1-1/2)(1-w_0) = f(w_0)$. Clearly, when n increases, $x_i, i \neq 1$ will be at most $1/2$, and thus $w_i \geq w_0$.

But then

$$x_1 = \frac{f(w_1)}{\sum_j f(w_j)} \leq \frac{f(v)}{f(v)+(n-1)f(w_0)} \overset{n\to\infty}{\to} 0.$$

Thus, the revenue satisfies

$$\sum_i w_i x_i \leq \sum v_i x_i \leq v x_1 + 1 \overset{n\to\infty}{\to} 1.$$

By this, we show that the winner-pay auctions are not competitive either. ∎

D Optimal Tullock Auction for Two Buyers

An interesting question is whether commonly considered allocation functions can achieve the upper bound in Theorem 1 or there is an inefficiency gap. We answer this question for the well-known class of Tullock auctions.

We will establish a tight upper bound on the revenue at Nash equilibrium for Tullock auctions with two buyers. This shows that there exist Tullock auctions for which the upper bound in Theorem 1 is achieved, for sufficiently large v_1/v_2 and, moreover, this holds already for small values of v_1/v_2. Note, however, that identifying such a Tullock auction amounts to finding a revenue-maximizing parameter r, which requires a priori knowing the valuation ratio v_1/v_2. Therefore, such an approach is *not* prior free. In contrast, the lower bound in Theorem 2 was established by a constructive proof that allows a prior-free design of asymptotically optimal auctions. In order to state the result, we need to introduce the following positive constants $v^* = (1+a)^{\frac{1+a}{2+a}}$ and $c = \frac{1+a}{a(2+a)}$ where $a > 0$ is the unique solution of $\log(1+a) = 1 + 2/a$. The constants v^*, c, and a are approximately equal to 3.567, 0.224, 3.681, respectively.

Theorem 7 *For every given buyers' valuations $v_1 \geq v_2 > 0$, there exists a Tullock auction with parameter r for which a unique Nash equilibrium exists and the revenue R in this equilibrium satisfies*

$$\frac{R}{v_1+v_2} \leq \begin{cases} 1 - \frac{1}{\bar{r}(v_1/v_2)} & \text{if } v_1/v_2 \leq v^*, \\ \frac{c}{\log(v_1/v_2)} & \text{otherwise}, \end{cases}$$

where $\bar{r}(v_1/v_2)$ is given by $\bar{r} = 1 + (v_2/v_1)^{\bar{r}}$.

Furthermore, the bound is tight. For the case $v_1/v_2 \leq v^*$, it is achieved asymptotically as r tends to $\bar{r}(v_1/v_2)$ from below, and otherwise, it is achieved for $r = \log(1+a)/\log(v_1/v_2)$.

Proof. For every $n \geq 2$ we have bids $\vec{w}$ at Nash equilibrium that satisfy the following. Every buyer i maximizes payoff $v_i \frac{w_i^r}{\sum_j w_j^r} - w_i$ over $w_i \geq 0$. Taking the partial derivative with respect to w_i we obtain the following conditions, for every $1 \leq i \leq n$,

$$\begin{aligned} v_i r w_i^{r-1} \frac{\sum_j w_j^r - w_i^r}{\left(\sum_j w_j^r\right)^2} &= 1 \\ v_i x_i - w_i &\geq 0 \end{aligned} \tag{8}$$

where the last inequality means that each buyer's payoff is non-negative, which indeed holds as each buyer is guaranteed a zero payoff by selecting a zero bid.

For the case of two buyers $i = 1, 2$, this reads as

$$v_1 r \frac{w_1^{r-1} w_2^r}{(w_1^r + w_2^r)^2} = 1 \text{ and } v_2 r \frac{w_1^r w_2^{r-1}}{(w_1^r + w_2^r)^2} = 1. \tag{9}$$

From these equations, it follows

$$\frac{w_1}{v_1} = \frac{w_2}{v_2}.$$

Combined with (9) we have

$$w_1 = \frac{v_1}{v_2} w_2 \text{ and } w_2 = v_2 \frac{r(v_1/v_2)^r}{[1 + (v_1/v_2)^r]^2}. \tag{10}$$

Therefore, the revenue at Nash equilibrium is

$$R = (v_1 + v_2) \frac{r(v_1/v_2)^r}{[1 + (v_1/v_2)^r]^2}. \tag{11}$$

From (8), we have

$$v_1 \frac{w_1^{r-1}}{w_1^r + w_2^r} \geq 1 \text{ and } v_2 \frac{w_2^{r-1}}{w_1^r + w_2^r} \geq 1.$$

Combining with (10), we obtain that the last two inequalities are equivalent to saying $r \leq 1 + (v_2/v_1)^r$.

For every set of two buyers with valuations $v_1 \geq v_2$, maximizing the revenue amounts to solving the following problem

$$\begin{aligned} \text{maximize} \quad & R(r) \\ \text{over} \quad & r \geq 0 \\ \text{subject to} \quad & r \leq 1 + (v_2/v_1)^r. \end{aligned}$$

Let us first consider the above problem without the constraint. A solution of the problem satisfies the condition

$$\frac{d}{dr} \log R(r) = \frac{1}{r} + \log(v_1/v_2) - 2 \frac{\log(v_1/v_2)(v_1/v_2)^r}{1 + (v_1/v_2)^r} = 0$$

which is equivalent to

$$\log((v_1/v_2)^r) = 1 + \frac{2}{(v_1/v_2)^r - 1}.$$

Using the notation $a = (v_1/v_2)^r - 1$ we note that $a > 0$ is a unique solution of

$$\log(1 + a) = 1 + 2/a. \tag{12}$$

Hence, the optimum return to scale parameter r^*, without the constraint of the problem, is

$$r^* = \frac{\log(1 + a)}{\log(v_1/v_2)}.$$

The constraint of the problem $r \leq 1+(v_2/v_1)^r$ is equivalent to saying $r \leq \bar{r}$ where $\bar{r}$ is defined in the statement of the theorem. Therefore, if $r^* \leq \bar{r}$, then r^* is the maximizer for our original problem, otherwise, $\bar{r}$ is the maximizer for our original problem.

Now, notice that $\log(v_1/v_2) = -\log(\bar{r}-1)/\bar{r}$ and use this along with definition of r^* to note that $r^* \leq \bar{r}$ is equivalent to $\bar{r} \leq 1+1/(1+a)$. Since $\bar{r}$ is decreasing with v_1/v_2, we have that $r^* \leq \bar{r}$ is equivalent to $v_1/v_2 \geq (1+a)^{(1+a)/(2+a)} := v^*$. In the following we consider the following two cases:

Case 1: $v_1/v_2 \geq v^*$. In this case the maximizer r^* and we obtain

$$\frac{R(r^*)}{v_1+v_2} = \frac{(1+a)\log(1+a)}{(2+a)^2}\frac{1}{\log(v_1/v_2)}.$$

Using (12), we derive $[(1+a)/(2+a)^2]\log(1+a) = (1+a)/[a(2+a)] := c$ from which the asserted result follows.

Case 2: $v_1/v_2 < v^*$. In this case, the maximizer is $\bar{r}$ and using (11), the asserted result follows.

This completes the proof of the theorem. ∎

E A Characterization Lemma for Tullock Auctions

In this section, we characterize the Nash equilibrium allocation for Tullock auctions, i.e. for the allocation function $x_i = w_i^r/\sum_j w_j^r$, with an arbitrary number of buyers $n \geq 2$ and any $0 < r < 1$. This characterization is of general interest and is used for deriving revenue bounds in Section F. We will see that the following functions play an important role, let $a(x)$ and $b(x)$ be functions defined on $[0,1]$ as follows:

$$a(x) = \frac{x^r}{1-\sqrt{1-x}} \text{ and } b(x) = \frac{x^r}{1+\sqrt{1-x}}. \tag{13}$$

The following lemma characterizes the Nash equilibrium buyers' bids for every Tullock auction with parameter $0 < r \leq 1$.

Lemma 1 *Consider a Tullock auction with parameter $0 < r \leq 1$ and a set of $n \geq 2$ buyers with valuations $v_1 \geq v_2 \geq \cdots \geq v_n > 0$. Then, at Nash equilibrium, the buyers' bids $w_1, w_2, \ldots, w_n$ are characterized as follows.*

There are two cases: Case A if $\sum_{i=2}^n \left[1-\sqrt{1-a^{-1}((v_1/v_i)^r)}\right] \geq 1$ and, Case B otherwise.

Case A:

$$w_i = \frac{r}{4}v_i a^{-1}\left(\left(\frac{v_1}{v_i}\right)^r a(y_1)\right), \text{ for } 1 \leq i \leq n, \tag{14}$$

where y_1 is given by

$$A(y_1) := 1-\sqrt{1-y_1}+\sum_{i=2}^n\left[1-\sqrt{1-a^{-1}((v_1/v_i)^r a(y_1))}\right] = 2. \tag{15}$$

Case B:

$$w_i = \frac{r}{4}v_i a^{-1}\left(\left(\frac{v_1}{v_i}\right)^r b(y_1)\right), \text{ for } 1 \leq i \leq n, \tag{16}$$

where y_1 is given by

$$B(y_1) := 1+\sqrt{1-y_1}+\sum_{i=2}^n\left[1-\sqrt{1-a^{-1}((v_1/v_i)^r b(y_1))}\right] = 2. \tag{17}$$

Proof. The Nash equilibrium bids $w_1, w_2, \ldots, w_n$ satisfy, for every i,

$$\text{either } w_i = 0 \text{ or } \frac{\partial}{\partial w_i}\left(v_i \frac{w_i^r}{\sum_j w_j^r} - w_i\right) = 0.$$

The last identity is equivalent to

$$\frac{w_i^r}{\sum_j w_j^r}\left(1 - \frac{w_i^r}{\sum_j w_j^r}\right) = \frac{1}{r}\frac{w_i}{v_i}. \tag{18}$$

Notice that this is a quadratic equation with respect to $\frac{w_i^r}{\sum_j w_j^r}$ whose solving allows us to note that

$$\text{for every } i \text{ either } \frac{w_i^r}{\sum_j w_j^r} = \frac{1}{2}\left(1 + \sqrt{1-y_i}\right) \text{ or } \frac{w_i^r}{\sum_j w_j^r} = \frac{1}{2}\left(1 - \sqrt{1-y_i}\right) \tag{19}$$

where by definition

$$y_i = \frac{4}{r}\frac{w_i}{v_i}, \text{ for every } 1 \le i \le n. \tag{20}$$

From (18) and the assumption $v_1 \ge v_2 \ge \cdots \ge v_n$, it follows that $w_1 \ge w_2 \ge \cdots \ge w_n$. Therefore, $w_i^r / \sum_j w_j^r \ge 1/2$ may hold only for $i = 1$. Hence, it follows

$$\frac{w_i^r}{\sum_j w_j^r} = \begin{cases} \frac{1}{2}\left(1 \pm \sqrt{1-y_1}\right) & i = 1 \\ \frac{1}{2}\left(1 - \sqrt{1-y_i}\right) & 1 < i \le n. \end{cases} \tag{21}$$

From (20) and (21) we have for $1 < i \le n$,

$$\text{either Case A:} \frac{v_i^r y_i^r}{v_1^r y_1^r} = \frac{1-\sqrt{1-y_i}}{1-\sqrt{1-y_1}} \text{ or Case B:} \frac{v_i^r y_i^r}{v_1^r y_1^r} = \frac{1-\sqrt{1-y_i}}{1+\sqrt{1-y_1}}.$$

Using the definitions of functions $a(x)$ and $b(x)$ in (13), the last identities can be rewritten as, for every $1 < i \le n$, either Case A:

$$a(y_i) = \left(\frac{v_1}{v_i}\right)^r a(y_1) \tag{22}$$

or Case B:

$$a(y_i) = \left(\frac{v_1}{v_i}\right)^r b(y_1). \tag{23}$$

Equations (14) and (16) follow, respectively, from (22) and (23). Notice that given an equilibrium value of y_1, equilibrium values y_i, $1 < i \le n$ are determined by (22) and (23). It remains to determine conditions that determines whether we have either Case A or Case B and determine equilibrium value of y_1, which we do in the following.

Summing (21) over i we have

$$1 \pm \sqrt{1-y_1} + \sum_{i=2}^{n}\left(1 - \sqrt{1-y_i}\right) = 2 \tag{24}$$

where the sign of the second element in the left-hand side is negative under Case A and is positive under Case B. Combined with (22) we obtain that y_1 is a solution of $A(y_1) = 2$ under Case A and $B(y_1) = 2$ under Case B, where function $A(x)$ and $B(x)$ are defined in the assertion of the proposition.

On the one hand, notice that $A(x)$ is an increasing function on $[0,1]$ with values at the endpoints $A(0) = 0$ and $A(1) = 1 + \sum_{i=2}^{n}\left(1 - a^{-1}((v_1/v_i)^r)\right)$. Therefore, there exists a solution to $A(y_1) = 2$ if and only if $A(1) \ge 2$, i.e.

$$\sum_{i=2}^{n}\left(1 - a^{-1}((v_1/v_i)^r)\right) \ge 1.$$

On the other hand, $B(x)$ is a decreasing function on $[\underline{y}_1, 1]$ where $b(\underline{y}_1) = (v_2/v_1)^r$, with values at the endpoints $B(\underline{y}_1) \ge B(1)$ and $B(1) = A(1)$. Therefore, Case B holds if $A(1) < 2$. ∎

F Proof of Theorem 5

The proof makes use of the equilibrium characterization in Appendix E. In particular, we repeatedly use the following bounds, which are straightforwardly derived from (13), for every $0 < r < 1$,

$$\frac{1}{x^{1-r}} \leq a(x) \leq \frac{2}{x^{1-r}}, \text{ for } 0 \leq x \leq 1, \tag{25}$$

$$\left(\frac{1}{x}\right)^{\frac{1}{1-r}} \leq a^{-1}(x) \leq \left(\frac{2}{x}\right)^{\frac{1}{1-r}}, \text{ for } x > 0, \tag{26}$$

and

$$\frac{1}{2}x^r \leq b(x) \leq x^r, \text{ for } 0 \leq x \leq 1. \tag{27}$$

Proof of Item 1

We first show that under this item, Case A in Lemma 1 holds, i.e.

$$\sum_{i=2}^{n}\left(1-\sqrt{1-a^{-1}\left((v_1/v_i)^r\right)}\right) \geq 1. \tag{28}$$

Indeed,

$$\sum_{i=2}^{n}\left(1-\sqrt{1-a^{-1}\left((v_1/v_i)^r\right)}\right) \geq \frac{1}{2}\sum_{i=2}^{n}a^{-1}\left((v_1/v_i)^r\right) \geq \frac{1}{2}\sum_{i=2}^{n}\left(\frac{v_i}{v_1}\right)^{\frac{r}{1-r}} = \frac{1}{2}||\underline{\vec{v}}||_p^p.$$

Therefore, if $||\underline{\vec{v}}||_p^p \geq 2$, then Case A holds.

Under Case A, y_1 is given by $A(y_1) = 2$, where $A(\cdot)$ is defined in (17), which is equivalent to

$$1+\sqrt{1-y_1} = \sum_{i=2}^{n}\left[1-\sqrt{1-a^{-1}((v_1/v_i)^r a(y_1))}\right]. \tag{29}$$

Upper bound for w_1/v_1. From (29),

$$\begin{aligned}
1+\sqrt{1-y_1} &= \sum_{i=2}^{n}\left[1-\sqrt{1-a^{-1}((v_1/v_i)^r a(y_1))}\right] \\
&\geq \frac{1}{2}\sum_{i=2}^{n}a^{-1}((v_1/v_i)^r a(y_1)) \\
&\geq \frac{1}{2a(y_1)^{\frac{1}{1-r}}}\sum_{i=2}^{n}\left(\frac{v_i}{v_1}\right)^{\frac{r}{1-r}} = \frac{1}{2a(y_1)^{\frac{1}{1-r}}}||\underline{\vec{v}}||_p^p.
\end{aligned}$$

From this, the following relations hold

$$2^{\frac{1}{1-r}}r\frac{v_1}{w_1} = 2\cdot 2\cdot 2^{\frac{1}{1-r}}\frac{1}{y_1} \geq (1+\sqrt{1-y_1})2a(y_1)^{\frac{1}{1-r}} \geq ||\underline{\vec{v}}||_p^p.$$

Therefore,

$$\frac{w_1}{v_1} \leq 2^{\frac{1}{1-r}}r||\underline{\vec{v}}||_p^{-p}. \tag{30}$$

Lower bound for w_1/v_1. We follow similar steps as for the upper bound. First, from (29),

$$\begin{aligned} 1+\sqrt{1-y_1} &= \sum_{i=2}^{n}\left[1-\sqrt{1-a^{-1}((v_1/v_i)^r a(y_1))}\right] \\ &\leq \sum_{i=2}^{n} a^{-1}((v_1/v_i)^r a(y_1)) \\ &\leq \frac{2^{\frac{1}{1-r}}}{a(y_1)^{\frac{1}{1-r}}}\sum_{i=2}^{n}\left(\frac{v_i}{v_1}\right)^{\frac{r}{1-r}} = \frac{2^{\frac{1}{1-r}}}{a(y_1)^{\frac{1}{1-r}}}||\underline{\vec{v}}||_p^p. \end{aligned}$$

From this,

$$2^{-\frac{3-2r}{1-r}} r\frac{v_1}{w_1} = 2^{-\frac{1}{1-r}}\frac{1}{y_1} \leq (1+\sqrt{1-y_1})2^{-\frac{1}{1-r}} a(y_1)^{\frac{1}{1-r}} \leq ||\underline{\vec{v}}||_p^p.$$

Therefore,

$$\frac{w_1}{v_1} \geq 2^{-\frac{3-2r}{1-r}} r||\underline{\vec{v}}||_p^{-p}. \tag{31}$$

Revenue upper bound. First, we note

$$R = \sum_i w_i = v_1 \cdot \frac{w_1}{v_1} + v_1 \cdot \frac{w_1}{v_1}\sum_{i=2}^{n}\frac{v_i}{v_1}\frac{y_i}{y_1}.$$

It remains to show that for a positive constant C_3,

$$v_1 \cdot \frac{w_1}{v_1}\sum_{i=2}^{n}\frac{v_i}{v_1}\frac{y_i}{y_1} \leq C_3 v_2.$$

From (22) and (25),

$$\frac{2}{y_i^{1-r}} \geq a(y_i) = \left(\frac{v_1}{v_i}\right)^r a(y_1) \geq \left(\frac{v_1}{v_i}\right)^r \frac{1}{y_1^{1-r}}$$

and, thus,

$$\frac{y_i}{y_1} \leq 2^{\frac{1}{1-r}}\left(\frac{v_i}{v_1}\right)^{\frac{r}{1-r}}. \tag{32}$$

Therefore,

$$v_1 \cdot \frac{w_1}{v_1}\sum_{i=2}^{n}\frac{v_i}{v_1}\frac{y_i}{y_1} \leq v_1 \cdot C_2||\underline{\vec{v}}||_p^{-p} \cdot 2^{\frac{1}{1-r}}\sum_{i=2}^{n}\frac{v_i}{v_1}\left(\frac{v_i}{v_1}\right)^{r/(1-r)} = C_2 2^{\frac{1}{1-r}} v_1 \cdot \frac{\sum_{i=2}^{n}\frac{v_i}{v_1}\left(\frac{v_i}{v_1}\right)^{r/(1-r)}}{\sum_{i=2}^{n}\left(\frac{v_i}{v_1}\right)^{r/(1-r)}} \leq C_2 2^{\frac{1}{1-r}} v_2.$$

Asymptotic w_1/v_1. We consider asymptotically dominant terms in (28) as $||\underline{\vec{v}}||_p$ tends to infinity. In view of the established upper bound on w_1/v_1, we have that y_1 tends to 0 and so does $a(y_1)$ and thus, $a^{-1}((v_1/v_i)^r a(y_1))$ tends to zero, for every i. Therefore, from (28),

$$2 \sim \frac{1}{2}\sum_{i=2}^{n} a^{-1}\left(\left(\frac{v_1}{v_i}\right)^r a(y_1)\right), \text{ for small } ||\underline{\vec{v}}||_p.$$

Furthermore, using the asymptotes

$$a(x) \sim \frac{2}{x^{1-r}}, \text{ small } x \text{ and } a^{-1}(x) \sim \left(\frac{2}{x}\right)^{\frac{1}{1-r}}, \text{ large } x,$$

we obtain

$$y_1 \sim 4||\underline{\vec{v}}||_p^{-p}, \text{ for small } ||\underline{\vec{v}}||_p,$$

which establishes the asserted asymptotic in view of the fact $y_1 = (4w_1)/(rv_1)$.

Proof of Item 2

We first show that under item 2, Case B in Lemma 1 holds, i.e.

$$\sum_{i=2}^{n}\left(1-\sqrt{1-a^{-1}\left((v_1/v_i)^r\right)}\right)<1.$$

Indeed,

$$\sum_{i=2}^{n}\left(1-\sqrt{1-a^{-1}\left((v_1/v_i)^r\right)}\right)\leq\sum_{i=2}^{n}a^{-1}\left((v_1/v_i)^r\right)\leq\sum_{i=2}^{n}\left[2\left(\frac{v_i}{v_1}\right)^r\right]^{\frac{1}{1-r}}=2^p||\underline{\vec{v}}||_p^p$$

where the first inequality is elementary, $1-\sqrt{1-x}\leq x$, for every $0\leq x\leq 1$, and the second inequality follows from (26). Therefore, if $||\underline{\vec{v}}||_p<1/2$, then Case B holds.

Under Case B, y_1 is given by $B(y_1)=2$, where $B(\cdot)$ is defined in (17), which is equivalent to

$$1-\sqrt{1-y_1}=\sum_{i=2}^{n}\left[1-\sqrt{1-a^{-1}((v_1/v_i)^r b(y_1))}\right]. \tag{33}$$

Upper bound for w_1/v_1. From (33), we have

$$\begin{aligned}1-\sqrt{1-y_1} &\leq \sum_{i=2}^{n}a^{-1}\left(\left(\frac{v_1}{v_i}\right)^r b(y_1)\right)\\ &\leq 2^{\frac{1}{1-r}}\frac{1}{b(y_1)^{\frac{1}{1-r}}}\sum_{i=2}^{n}\left(\frac{v_i}{v_1}\right)^{\frac{r}{1-r}}=2^{\frac{1}{1-r}}\frac{1}{b(y_1)^{\frac{1}{1-r}}}||\underline{\vec{v}}||_p\end{aligned}$$

where the first inequality is elementary, $1-\sqrt{1-x}\leq x$, for every $0\leq x\leq 1$, and the second inequality follows from (26). Using the fact $1-\sqrt{1-y_1}\geq\frac{1}{2}y_1$ and (27), we have

$$2^{-\frac{2-r}{1-r}}y_1^{\frac{1}{1-r}}\frac{1}{2}y_1\cdot 2^{-\frac{1}{1-r}}y_1^{\frac{r}{1-r}}\leq(1-\sqrt{1-y_1})\cdot b(y_1)^{\frac{1}{1-r}}\leq 2^{\frac{1}{1-r}}||\underline{\vec{v}}||_p.$$

Thus, $y_1\leq 2^{3-r}||\underline{\vec{v}}||_p$, i.e.

$$\frac{w_1}{v_1}\leq 2^{1-r}r||\underline{\vec{v}}||_p.$$

Lower bound for w_1/v_1. Lower bound follows by similar arguments. From (33),

$$\begin{aligned}1-\sqrt{1-y_1} &\geq \frac{1}{2}\sum_{i=2}^{n}a^{-1}\left(\left(\frac{v_1}{v_i}\right)^r b(y_1)\right)\\ &\geq \frac{1}{2}\frac{1}{b(y_1)^{\frac{1}{1-r}}}\sum_{i=2}^{n}\left(\frac{v_i}{v_1}\right)^{\frac{r}{1-r}}=\frac{1}{2}\frac{1}{b(y_1)^{\frac{1}{1-r}}}||\underline{\vec{v}}||_p\end{aligned}$$

where we used the elementary fact $1-\sqrt{1-x}\geq\frac{1}{2}x$, for every $0\leq x\leq 1$, and (26). Using (27), it follows

$$y_1^{\frac{1}{1-r}}=y_1\cdot y_1^{\frac{r}{1-r}}\geq(1-\sqrt{1-y_1})\cdot b(y_1)^{\frac{1}{1-r}}\geq\frac{1}{2}||\underline{\vec{v}}||_p.$$

Hence, $y_1\geq 2^{-(1-r)}||\underline{\vec{v}}||_p$, i.e.

$$\frac{w_1}{v_1}\geq 2^{-(3-r)}r||\underline{\vec{v}}||_p.$$

Revenue upper bound. First, we note

$$R = \sum_i w_i = \frac{w_1}{v_1}\left(v_1 + v_1 \cdot \sum_{i=2}^{n} \frac{v_i}{v_1}\frac{y_i}{y_1}\right).$$

It remains to show that for a constant $C_3 > 0$,

$$v_1 \cdot \sum_{i=2}^{n} \frac{v_i}{v_1}\frac{y_i}{y_1} \le C_3 v_2.$$

From (23) and (27),

$$\frac{2}{y_i^{1-r}} \ge a(y_i) = \left(\frac{v_1}{v_i}\right)^r b(y_1) \ge \left(\frac{v_1}{v_i}\right)^r \frac{1}{2} y_1^r,$$

and thus,

$$\frac{y_i}{y_1} \le 4^{\frac{1}{1-r}} \left(\frac{v_i}{v_1}\right)^{\frac{r}{1-r}} \cdot y_1^{-\frac{1}{1-r}}. \tag{34}$$

Therefore,

$$v_1 \cdot \sum_{i=2}^{n} \frac{v_i}{v_1}\frac{y_i}{y_1} \le v_1 \cdot \sum_{i=2}^{n} \frac{v_i}{v_1}\left(\frac{v_i}{v_1}\right)^{\frac{r}{1-r}} \cdot \frac{4^p}{y_1^p} \le v_1 \cdot \frac{\sum_{i=2}^{n} \frac{v_i}{v_1}\left(\frac{v_i}{v_1}\right)^{\frac{r}{1-r}}}{||\vec{\underline{v}}||_p^p} \frac{1}{C_1^p} \le C_1^{-p} \cdot v_2$$

where the second inequality is by $y_1 = (4/r)w_1/v_1$ and the lower-bound on w_1/v_1 in the assertion of the theorem.

Asymptotic w_1/v_1. The assertion follows by considering asymptotically dominant terms in (33). We will show that for every i, $a^{-1}((v_1/v_i)^r b(y_1))$ tends to zero as $||\vec{\underline{v}}||_p$ tends to zero. Then,

$$\sum_{i=2}^{n} \left[1 - \sqrt{1 - a^{-1}((v_1/v_i)^r b(y_1))}\right] \sim \frac{1}{2}\sum_{i=2}^{n} a^{-1}((v_1/v_i)^r b(y_1)).$$

Now, from the definition of $a(\cdot)$, it is easy to see that

$$a^{-1}(x) \sim \left(\frac{2}{x}\right)^{\frac{1}{1-r}}, \text{ large } x.$$

Using these in (33), we obtain

$$b(y_1)^{\frac{1}{1-r}}(1 - \sqrt{1-y_1}) \sim \frac{1}{2} 2^{\frac{1}{1-r}} ||\vec{\underline{v}}||_p^p.$$

Noting that $b(y_1) = \frac{1}{2}y_1^r[1 + O(y_1)]$ and $1 - \sqrt{1-y_1} = \frac{1}{2}y_1 + O(y_1^2)$, we obtain

$$y_1 \sim 4||\vec{\underline{v}}||_p$$

which combined with $y_1 = 4w_1/(rv_1)$ yields the asserted asymptotic.

It remains to show that $a^{-1}((v_1/v_i)^r b(y_1))$ tends to 0 as $||\vec{\underline{v}}||_p$ tends to 0, for every i. It is easily seen that if $||\vec{\underline{v}}||_p$, then necessarily v_2/v_1 tends to 0. It suffices to show that for every i, $(v_1/v_i)^r b(y_1)$ tends to infinity as v_2/v_1 tends to 0. Note

$$\left(\frac{v_1}{v_i}\right)^r b(y_1) \ge \left(\frac{v_1}{v_i}\right)^r \cdot \frac{1}{2} y_1^r \ge \frac{1}{2}\left(\frac{v_1}{v_i} 4C_1 ||\vec{\underline{v}}||_p\right)^r \ge \frac{1}{2}\left(4C_1\left(\frac{v_1}{v_2}\right)^{1-r}\right)^r$$

which clearly converges to infinity as v_2/v_1 tends to zero.