Efficiency and the Redistribution of Welfare

Yoram Bachrach[1] Vasilis Syrgkanis [2] Milan Vojnović [3]

May 2012

Technical Report
MSR-TR-2011-113

Microsoft Research
Microsoft Corporation
One Microsoft Way
Redmond, WA 98052
http://www.research.microsoft.com

[1]yobach@microsoft.com Microsoft Research, Cambridge, UK
[2]vasilis@cs.cornell.edu Cornell University. Part of this work was done while the author was an intern at Microsoft Research. Supported in part by ONR grant N00014-98-1-0589.
[3]milanv@microsoft.com Microsoft Research, Cambridge, UK

Abstract – We examine how the relation between individual and social utility affect the efficiency of game-theoretic solution concepts. We first provide general results for monotone utility-maximization games, showing that if each player's utility is at least his marginal contribution to the welfare, then the social welfare in any *Strong Nash Equilibrium* is at least half of the optimal. The efficiency degrades smoothly as the marginal contribution assumption is relaxed. For *non-monotone* utility maximization games, we manage to give efficiency results if the game is also a potential game. We also extend previous results on efficiency of *Nash Equilibria* for the case when social welfare is submodular.

We then focus on Effort Market Games, a general cooperative setting where players exert effort in projects and then the produced value is redistributed to them using some simple local mechanism. Our results for general utility maximization games are instrumental in reasoning about efficiency in Effort Market Games. We show that in a concave environment, splitting locally according to the Shapley value achieves globally at least half of the optimal social welfare at any *Nash Equilibrium* of the game (our proof is of a smoothness type and hence generalizes also to no-regret learning outcomes). We generally characterize the properties that a distribution mechanism has to satisfy to achieve good efficiency. Moreover, we show that equal splitting of the value works well only when the projects have small number of participants. In a convex environment, Nash Equilibria can have unbounded inefficiency regardless of the mechanism. Hence, we focus on coalitionaly robust solution concepts that allow for group deviations of the players - a natural assumption in such a cooperative setting. We show that every local mechanism is bound to have an inefficiency of at least the maximum number of participants per project at the worst Strong Nash Equilibrium. Further, both the Shapley value mechanism and the equal splitting mechanism achieve this lower bound. We also draw strong connections of our results with the axiomatic surplus sharing literature.

Keywords: Efficiency, Utility Maximization Games, Effort Market Games, Crowdsourcing, Public Projects

1 Introduction

Many game theoretic scenarios have at their essence the following feature: self-interested individuals participate in joint ventures, by investing time, effort, money or other personal resources, to produce some wealth which is then split and each one gets a share of it. Examples range from paper co-authorship settings where the wealth produced is in the form of credit in scientific projects that is implicitly split among the authors of a paper [12], to crowdsourcing scenarios that naturally arise in modern electronic markets (e.g. TopCoder, Netflix Challenge) to CEO labour markets where the wealth is shareholder value that is split among high-rank employers in the form of end-of-year bonuses [19].

In this work we are interested in understanding what are the crucial parameters of such settings that determine the overall efficiency when the players act selfishly trying to maximize their own share rather than the overall wealth produced. Specifically, we will try to identify how the relationship between social utility and individual utility affects the efficiency. The main question driving most of our results is the following: how should the wealth be split among the participants such that the incentives of an individual are approximately aligned with those of the society as a whole? Another important question is whether there actually exist ways of redistributing the wealth to accomplish this approximate alignment and what are the inherent lower bounds on the approximation that we cannot surpus.

The answers to the above questions as expected will not be universal and will depend on how player efforts affect the wealth produced. However, we show that they can be mainly categorized in two scenarios according to whether the players efforts are substitutable or complementary.

The essence of our results is the following informal theorem: *if there exists a way to redistribute wealth such each player is awarded his marginal contribution to social welfare then individual incentives are well-aligned with social ones and the overall efficiency is half the optimal one*. More generally, the inefficiency is guided by the fraction of the marginal contribution that a player is awarded.

We model the case of players' efforts being substitable or supplementary by assuming that the wealth produced has a decreasing marginal increase property: the increase in wealth that the effort of a single player causes becomes smaller and smaller as the rest of the players increase their effort. Such a property for example can capture paper co-authorship scenarios in a community of almost equally skilled individuals. In many scientific projects, the value of the paper produced is increasing as the effort of the individuals increases but it increases in a concave manner. One could capture such a setting by puting abilities to the players and then modeling the credit in scientific projects as a concave function of the weighted total effort of the players, where effort is weighted by ability.

Another setting where such a modeling is applicable could be TopCoder crowdsourcing scenarios. More and more software companies turn to TopCoder to complete programming projects rather than exclusively resorting on traditional contract-based rewarding schemes by hiring permanent employees. The quality of the code produced and the resulting revenues from commercialising it are affected by the amount of effort that the programmers put in the project. Though the efforts of the programmers are substitutes since they all produce the same product (e.g. C++ code), their abilities vary quite dramatically. This is specifically captured in the TopCoder interface through each programmer's rating. Hence, again we can model the wealth produced by the players as a concave function of the weighted total effort put in the project, or more generally by some wealth function that satisfies the decreasing marginal increase property. The framework of complete information games is quite natural in this context as players' skills are reflected in publicly available rating scores of players, and the design of contests involves a registration phase which provides information about participation of players across projects.

In such scenarios which we call *Concave Effort Markets* we are able to show that *there exist* mechanisms of redistributing wealth such that the overall efficiency at every *Nash Equilibrium* is at least half the optimal efficiency. For example, sharing the wealth of each project according to the Shapley Value is such a natural mechanism. If the value produced is a function of the weighted total effort of the players as in the previous two scenarios then sharing wealth proportional to the weighted effort of each player also achieves this high efficiency. Our proof is of a smoothness type [23] and hence the efficiency guarantee is quite robust and carries over even to the case where the players use no-regret learning strategies to find how to allocate their efforts (i.e. Coarse Correlated Equilibrium of the game). Our result is based on an adaptation of the work on monotone valid utility games introduced by Vetta [30] to games where social welfare is a function defined on a vector space rather than on sets.

Though *Concave Effort Markets* are quite applicable they don't capture settings where the efforts of the players are complementary and each player is quintessential in the success of a venture. There are several real world scenarios where to complete a project each participant must exert some positive effort. Moreover, there are settings where a project has some threshold of effort that needs to be exerted before it starts producing wealth. In such non-concave settings our results are qualitatively very different. First, it is easy to see that in such settings the Nash Equilibrium solution concept can have unbounded inefficiency. Intuitively, if we are at a state where the rest of the participants are not exerting effort then your effort is meaningless and thereby there is no incentive to exert it. Such an effect is reminiscent of mis-coordination in collective action settings.

However, in the scenarios that we want to model it makes sense to consider game theoretic solution concepts that also take into account the fact that players can coalitionally deviate. Discussing with your collaborators to agree on changing a strategy is quite natural. We try to capture such settings by studying the efficiency at Strong Nash Equilibria and at a relaxation of them where only players that share a project can coalitionally deviate, called Setwise Nash Equilibria [3].

Surprisingly, we manage to prove efficiency guarrantees of Strong Nash Equilibria for *general normal form games* and for *general potential games* that were not previously known and which we then apply to our specific model of effort market games. Specifically, we manage to show that if the social welfare of a utility maximization game is monotone, in the sense that a player by participating can only increase welfare, and if each player gets at least his marginal contribution to social welfare then each Strong Nash Equilibrium achieves at least half the optimal social welfare. To quantify the inefficiency of Strong Nash Equilibria we use the notion of Strong Price of Anarchy introduced by Andelman et al [2]. Our results extend the literature on the Strong Price of Anarchy that has mainly focused on models of cost minimization games, such as networks design games [7, 1]. In addition it heavily extends the work of Vetta [30] by showing that one can remove the assumption of submodularity of social welfare prevalent in that work and get the same results by switching to the coalitionally robust notion of the Strong Nash Equilibrium. Our results also give new insights on the connection between the analysis of the efficiency of the best nash equilibrium and that of the worst strong nash equilibrium in potential games.

Although, Strong Nash Equilibria don't have unbounded price of anarchy as Nash Equilibria do, we show that for general effort market games there is an inherent inefficiency proportional to the maximum number of participants per project that cannot be surpassed even when switching to coalitionally robust solution concepts. The reason behind this inherent inefficiency is intuitively the following: suppose that there is a project where everyone's participation is needed to produce some wealth. Then no matter how wealth is distributed there will be a player who will be getting at most a k-th fraction of the wealth, where k is the number of participants. Then if this player has an outside project that can produce wealth slightly more than this fraction he will strictly prefer it and therefore ruin social utility by implementing a much less valuable project on which he can get the whole pie.

Using our results on the efficiency of strong nash equilibria for general games we are able to show that several natural mechanisms achieve the best possible efficiency. Such mechanisms are the equal sharing mechanism or the Shapley Value mechanism.

Our results on general effort markets imply that to achieve good efficiency in settings with complementarities, the only solution is to maintain projects with small number of participants. It is interesting to observe that high complementarities in some sense imply high competition among the players. Hence, in such competitive environments, it is indeed natural for society to organize in small groups and hence keep inefficiency bounded.

1.1 Our contribution

Our formal model of the scenarios we want to study is called *Effort Market Games*, which is a restricted form of normal-form games with continuous strategy spaces. In these games, each of the players in the set N may participate in various projects of the set Ω, by exerting effort on these projects. Each player $i \in N$ may be restricted to working only on her own subset of allowed projects Ω_i, and his strategy space is a convex subset of the Euclidean space $\mathbb{R}^{|\Omega_i|}$, e.g. each player has a budget to distribute among his projects. The total utility derived from a project w is determined by the efforts exerted by the participating players using a value function $v_w : \mathbb{R}^{|N_w|} \to \mathbb{R}$, and is distributed to the participating players using a redistribution rule, such as equal sharing, effort proportional sharing or the Shapley value proportional sharing.

We now briefly describe our results, for both general Utility Maximization Games and Effort Market Games.

General Utility Maximization Games We show that in any utility maximization game where social welfare is monotone (players can only increase social welfare by participating), if each player's utility is at least his marginal contribution to the social welfare, then every Strong Nash Equilibrium achieves at least half of the optimum efficiency. Generally, if each player's utility is at least $1/k$ of his marginal contribution, then we show that the Strong Price of Anarchy is at most $k+1$. If the game is strongly monotone (players can only increase other players' utility by participating rather than staying out) then the latter bound becomes k. Moreover, if a game is strongly monotone and the social welfare is also submodular then the above result of k carries over to the PoA too. If the social welfare is non-monotone, we manage to give efficiency results in the case when it is a potential game. Specifically, we show that if the potential of a potential game is lower and upper bounded by a constant factor of the social welfare, i.e. $\alpha SW(s) \leq \Phi(s) \leq \beta SW(s)$, for some constants $\alpha > 0$ and $\beta \geq 0$, then the Strong Price of Anarchy is at most $(\beta+1)/\alpha$.

Effort Market Games We quantify the inefficiency of redistribution rules in Effort Market Games under several solution concepts: Nash equilibrium (NE), strong Nash equilibrium (SNE) and setwise Equilibrium (setwise-NE) (where only coalitions of players that share a project may deviate together). We use the concept of Price of Anarchy (PoA) (optimal social welfare over worst NE) and each corresponding Strong Price of Anarchy and setwise Price of Anarchy. We show that even when the value functions are linear functions of the total effort the equal sharing mechanism has a PoA at least k, where k is the maximum number of participants per project. On the other hand, the effort proportional sharing rule is always optimal. The latter shows that choosing the right mechanism can have a tremendous increase in efficiency.

When the value functions satisfy a decreasing marginal returns property, we show that if a mechanism gives to a player at least his marginal contribution to the value of the project then it has PoA ≤ 2. This is achieved, for example, by the Shapley Value Mechanism or the mechanism that awards proportional to the marginal contribution, as well as by a large class of mechanisms proposed in the axiomatic surplus sharing literature called fixed-path generated methods. On the other hand, we show that the equal splitting mechanism has PoA $\leq k$, partially resolving an open question of [3]. Moreover, if the value of a project is a concave function of the total effort and not of the vector of efforts then we show that also the effort proportional sharing mechanism has the marginal contribution property. In addition, we make a strong connection between the effort proportional sharing mechanism in our setting and splittable selfish routing games. Using this connection, we are able to use the local smoothness framework of [25] adapted to utility maximization games and achieve the much better bound of 1.251 on the PoA for the case when value functions have the form $v_w(x) = q_w x^\alpha$ for any $q_w > 0$ and $\alpha \in (0, 1)$.

When value functions are convex, we show that no mechanism can achieve a Strong Price of Anarchy of less than k. We show that several mechanism, such as the Shapley Value, achieve this lower bound of k, not only at Strong Nash Equilibria but also at all Setwise Nash Equilibria. In [3], it was shown that the equal sharing mechanism also achieves this bound. However, unlike the equal sharing mechanism, which has high inefficiency even for linear functions, the efficiency of Shapley Value mechanism and of similar more fair mechanisms is increased as we restrict to less convex functions.

For the case of non-monotone positive value functions, we are able to show that the equal sharing mechanism has Strong Price of Anarchy of at most $2k$ using the fact that it induces a potential game and invoking our result on efficiency of Strong Nash Equilibria in general potential games.

Finally, we provide a set of natural conditions which result in a low PoA, based on refinements of known axioms regarding reward sharing from the axiomatic surplus sharing literature [15, 17].

1.2 Related Work

The relation between individual and social welfare has been a fundamental issue in the economics literature and is prevalent in the foundations of every economic system. Various approaches have been proposed for examining and measuring the impact that the selfish behavior of agents has on efficiency of economic systems. This tension between individual and social welfare has been the base of several economic branches such as welfare economics and labor economics, dating back to early works of Arthur Pigou [21], Amartya Sen [26] and Karl Marx [14].

There have been several works on the efficiency of Nash Equilibria of utility maximization games [30, 9], also relating efficiency with the marginal contribution property. Here we extend that literature by mainly studying Strong Nash Equilibria and by studying stronger monotonicity properties. We are able to drop the assumption of submodularity prevalent in the literature. The study of the efficiency of the Strong Nash Equilibrium was introduced in [7, 2]. However, the literature focused mainly on specific models of cost minimization games like network design games (see e.g. [1, 7]). Here, we analyze efficiency of Strong Nash in general utility maximization games. Our model of effort market games is very similar to the Contribution Games model of [3]. However, in [3] they assume that all the players get the same value from a project. This corresponds to the special case of the equal sharing mechanism in our model. Moreover, they mainly focus on network games where each project has only two participants ($k = 2$). For that case, all of our results give a constant factor efficiency. Splitting scientific credit among participants has also been recently studied by Kleinberg and Oren [12] where they examine the case where players choose only a single project to work on. They also consider the equal sharing splitting rule when the game is symmetric and a splitting rule that is related to a players marginal contribution for asymmetric ability players. However, their main focus is (in our terminology) how to change globally the value functions of the projects so that optimality is achieved at some equilibrium of the resulting perturbed game. Effort Market Games also have a strong connection with the bargaining literature [11, 13, 5]. The main question in that literature is similar to what we ask here: how should a commonly produced value be split among the participants. Our approach though is very different than the bargaining literature since we focus on simple mechanisms that use only local information of a project and not global properties of the game. In many of those works it is shown that the right splitting of value achieves optimality. However, that "right splitting" is a complex function of the whole game instance. Here, we show that in many cases, simple local mechanisms are able to give approximate optimality. Recently Roughgarden [24] and independently Syrgkanis [29] proposed a new smoothness framework for the study of incomplete information games. This framework has a strong implication to our setting in that it enables our result on PoA at most 2 for the weighted proportional sharing mechanism to carry over to even incomplete information versions of effort market games where budgets and weights of players are private information. This portrays that our result is robust even when there is lack of information.

2 Efficiency in Utility Games

Consider a utility maximization game $\langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$, where N is the set of players, S_i is the strategy space of player i and $u_i : \times_i S_i \to \mathbb{R}$ his utility. We focus on utility maximization games where each player's strategy space has an s^{out} strategy, which intuitively is a "no-participation" option (i.e. player i does not take part in the game). Given a strategy profile $s \in \times_i S_i$, we define the social welfare as the sum of the players' utilities: $SW(s) = \sum_{i\in N} u_i(s)$.

A utility maximization game is said to be *monotone* if for every player i and for every strategy profile s: $SW(s) \geq SW(s_i^{out}, s_{-i})$ and is said to be *strongly monotone* if $\forall j \in N : u_j(s) \geq u_j(s_i^{out}, s_{-i})$. We will focus on utility games where if all players play their out strategy then the social welfare is 0. Moreover, a utility game satisfies the *Vickrey condition* if for every player i and strategy profile s: $u_i(s) \geq SW(s) - SW(s_i^{out}, s_{-i})$. We also state that a game satisfies the k-approximate Vickrey condition if each player gets $1/k$ his marginal contribution to the social welfare.

The monotonicity condition states that no player can hurt social welfare by participating in the game rather than staying out. Strong monotonicity states that no player can hurt other players' utility by participating. The Vickrey condition states that each player earns at least his marginal contribution to the social welfare.

The most classical solution concept in utility maximization games is the Nash Equilibrium. However, the unilateral stability of the Nash Equilibrium can be quite restrictive in several settings. Specifically, in settings where cooperation and communication among the players is natural, it is reasonable to also consider strategy profiles that are resilient to coalitional deviations. The Strong Nash Equilibrium [4] proposed by Aumann is such a concept. A strategy profile s is a Strong Nash Equilibrium (SNE) iff there is no coalitional deviation such that *each* player in the coalition receives strictly larger utility after the deviation.

Strong Nash Equilibria are a more reasonable outcome in cooperative settings but they don't always exist. However, they still remain meaningful since whenever they exist they are the most natural prediction. Therefore, it is worth studying their properties. The existence of Strong Nash Equilibria has generated a long line of research in both the economic and game theory community (see e.g. [18, 10, 7] for some latest work) and it is regarded as an independent line of research and possible future work to examine what are the general properties that utility games and our more specific model of effort market games possess a Strong Nash. In this work we focus on proving efficiency characteristics that hold for all Strong Nash Equilibria whenever they exist.

We define the Strong Price of Anarchy (SPoA) of a utility game to be the ratio of the Social Welfare at the optimal strategy profile over the social welfare at the worst Strong Nash Equilibrium. In the theorem that follows we show that the monotonicity and the Vickrey assumption are sufficient to give very high efficiency at any Strong Nash Equilibrium of a utility maximization game.

Theorem 1 *Any Strong Nash Equilibrium of a monotone utility game that satisfies the Vickrey condition achieves at least half the optimal Social Welfare. If the game satisfies k-approximate Vickrey condition then any Strong Nash Equilibrium achieves a $k+1$ fraction of the optimal social welfare.*

Proof. We focus on the case where the game satisfies the exact Vickrey condition. The approximate version can be proved with similar reasoning. Consider a strategy profile s that is a Strong Nash Equilibrium. Since it is a Strong Nash Equilibrium for any coalitional deviation from s there exists some player who is getting at least as much utility at s as he is getting in the deviating strategy profile.

Let s^* be the optimal strategy profile. If all players deviate to s^* then there is a player i who is blocking the deviation, i.e. $u_i(s) \geq u_i(s^*)$. Wlog let this person be player 1. Similarly, if players $\{2, \ldots, n\}$ deviate to playing their strategy in s^* then there exists some player, obviously different than 1 who is blocking the deviation. Wlog by reordering we can assume that this player is 2. Using similar reasoning we can reorder the players such that if players $\{k, \ldots, n\}$ deviate to their strategy in the optimal strategy profile s^* then player k is the one blocking the deviation. That is player k's utility at the Strong Nash Equilibrium is at least his utility in the deviating strategy profile. Let N_k be the set of player $\{k, \ldots, n\}$. Let $(s^*_{N_k}, s_{-N_k})$ be the strategy profile where players $\{k, \ldots, n\}$ play their optimal strategy and players $\{1, \ldots, k-1\}$ play s. By our assumption on the ordering of the players we thus have that for each player k: $u_k(s) \geq u_i(s^*_{N_k}, s_{-N_k})$.

The Vickrey condition states that each players utility at any strategy profile is at least his marginal contribution. Thereby we get:

$$u_i(s^*_{N_k}, s_{-N_k}) \geq SW(s^*_{N_k}, s_{-N_k}) - SW(s^{out}_k, s^*_{N_{k+1}}, s_{-N_k}) \tag{1}$$

In addition by the monotonicity assumption the social welfare can only increase when a player enters the game with any strategy. Thereby:

$$SW(s^{out}_k, s^*_{N_{k+1}}, s_{-N_k}) \leq SW(s_k, s^*_{N_{k+1}}, s_{-N_k}) = SW(s^*_{N_{k+1}}, s_{-N_{k+1}}) \tag{2}$$

Combining all the above we managed to get a telescoping sum which gives the desired result:

$$\begin{aligned} SW(s) = & \sum_{i=1}^{n} u_i(s) \geq \sum_{i=1}^{n} u_i(s^*_{N_k}, s_{-N_k}) \geq \sum_{i=1}^{n} SW(s^*_{N_k}, s_{-N_k}) - SW(s^{out}_k, s^*_{N_{k+1}}, s_{-N_k}) \\ \geq & \sum_{i=1}^{n} SW(s^*_{N_k}, s_{-N_k}) - SW(s^*_{N_{k+1}}, s_{-N_{k+1}}) = SW(s^*) - SW(s) \end{aligned}$$

Thus, $SW(s) \geq \frac{1}{2} SW(s^*)$. ∎

Last it is easy to alter slightly the above proof so that we get a tighter bound for the more restricted class of strongly monotone games:

Theorem 2 *Any strongly monotone utility game that satisfies the k-approximate Vickrey condition has Strong Price of Anarchy at most k. (proof in Appendix)*

On the opposite direction one could ask what efficiency guarrantees could be given for utility maximization games that don't satisfy any monotonicity property. Without any additional assumption it would be hard to imagine that any general result could be given. However, we are able to give results for non-monotone games in the case when they are potential functions and the potential is closely related with social welfare. A utility maximization game is a potential game if it admits a potential function $\Phi : \times_i S_i \to \mathbb{R}$ such that: $\forall s, s' \in \times_i S_i : u_i(s'_i, s_{-i}) - u_i(s) = \Phi(s'_i, s_{-i}) - \Phi(s)$.

Theorem 3 *If a Utility Potential Game with non-negative utilities satisfies that $\forall s \in \times_i S_i : \alpha SW(s) \leq \Phi(s) \leq \beta SW(s)$, for some $\alpha > 0$ and $\beta > 0$, then it has Strong Price of Anarchy at most $\frac{\beta+1}{\alpha}$.*

Proof. Let s be a Strong Nash Equilibrium and s^* the optimal strategy profile. Consider the players in the same order as in Theorem 1 such that for all i: $u_i(s) \geq u_i(s^*_{N_i}, s_{-N_i})$. The strategy profile$(s^*_{N_i}, s_{-N_i})$ can be thought of as resulting from $(s^*_{N_{i+1}}, s_{-N_{i+1}})$, through a unilateral deviation of player i from s_i to s^*_i. Hence, by the definition of the potential function and the fact that utilities are non-negative we get:

$$u_i(s^*_{N_i}, s_{-N_i}) = \Phi(s^*_{N_i}, s_{-N_i}) - \Phi(s^*_{N_{i+1}}, s_{-N_{i+1}}) + u_i(s^*_{N_{i+1}}, s_{-N_{i+1}}) \geq \Phi(s^*_{N_i}, s_{-N_i}) - \Phi(s^*_{N_{i+1}}, s_{-N_{i+1}})$$

Combining with our assumption on the relation between potential and social welfare we get the desired result:

$$\begin{aligned} SW(s) \geq & \sum_i u_i(s^*_{N_i}, s_{-N_i}) \geq \sum_i \Phi(s^*_{N_i}, s_{-N_i}) - \Phi(s^*_{N_{i+1}}, s_{-N_{i+1}}) \\ = & \Phi(s^*) - \Phi(s) \geq \alpha SW(s^*) - \beta SW(s) \end{aligned}$$

■

The above theorem is reminiscent of the Potential Method [20] for bounding the inefficiency of the *Best Nash Equilibrium* of a potential game. Specifically, using the potential method we would get that for games that satisfy the premises of Theorem 3 the best Nash Equilibrium achieves a β/α fraction of the optimal efficiency (the price of stability is at most β/α). Hence, in games where the Potential Method produces tight results, we see that there is a very interesting connection between the quality of the Strong Nash Equilibrium and that of the Best Nash Equilibrium.

There is a strong connection of the above results and the literature on *monotone valid utility games* [30]. A monotone valid utility game is a utility maximization game that apart from the monotonicity and the Vickrey condition also satisfies that the social welfare function is submodular (if we view social welfare as a function on sets of player strategies). Vetta [30] proved that any *Nash Equilibrium* of a monotone valid utility game achieves at least half the optimal social welfare (or a $k+1$ fraction if Vickrey is replaced by the approximate version). Later, this result was extended to Coarse Correlated Equilibria by Roughgarden [23].

Our results above complement Vetta's result by stating that if we remove the assumption of submodularity then we can recover the good efficiency by moving to the Strong Nash Equilibrium solution concept.

In addition, we manage to provide a tighter efficiency guarrantee for *Nash Equilibria* if we replace the social welfare monotonicity assumption of Vetta with strong monotonicity:

Theorem 4 *Any strongly monotone valid utility game that satisfies the k-approximate Vickrey condition has Price of Anarchy at most k. (proof in Appendix)*

In the appendix we also give a novel efficiency guarrantee for *Nash Equilibria* of potential games where the potential is a submodular function.

In the sections that follow we will use the above general theorems as quintessential tools for the analysis of the efficiency of redistribution mechanisms in our context of interest, Effort Market Games.

3 Effort Market Games

We consider the following continuous strategy space utility game. We have a set of players N and a set of projects Ω. Each player i is connected to a set of projects Ω_i on which he can participate by exerting some effort x_w^i. We denote with N_w the set of players connected to project w. Each player's strategy space S_i is a convex subset of the Euclidean space $\mathbb{R}^{|\Omega_i|}$ (our efficiency results except those that use the local smoothness framework carry over to non-convex strategy spaces too, e.g. integral versions). A very natural special case of such a convex subset is when each player has a budget of effort B_i that he has to distribute among the projects he is connected to. We denote with $\vec{x}_w = (x_w^i)_{i\in N_w}$ the effort vector associated with a project and with $\vec{x}^i = (x_w^i)_{w\in\Omega_i}$ the effort vector associated with a player. A strategy profile is denoted with $x = (\vec{x}^i)_{i\in N}$.

Each project w is associated with a value function $v_w : \mathbb{R}^{|N_w|} \to \mathbb{R}$, that takes as inputs the vector of efforts of the participants and outputs a real number. In what follows we assume that v_w is a non-decreasing function.

Moreover, the game consists of a redistribution of welfare mechanism $\mathcal{M}$. The mechanism determines how the value of a project is distributed among its participants. Hence, the mechanism outputs a share $\phi_w^i(\vec{x}_w)$ for each project $w \in \Omega$ and player $i \in N_w$. Finally, the utility of a player is: $u_i(\vec{x}) = \sum_{w\in\Omega_i} \phi_w^i(\vec{x}_w)$.

Why good mechanisms? The Linear Case We start off with a simple example of an effort market setting that portrays the importance of a good mechanism on the social welfare. Consider the scenario where there is a player a with an effort budget of 1 and the rest $n-1$ players have a small budget of ϵ. There are two projects P and C. Project P is a "low quality" private project that only player a participates in and project C is a "high quality" common project that all the players participate in. The values of the projects are linear functions of the total effort exerted on them. Specifically, project P has a value function $v_P(x) = 2x$ and project C has a value function $v_C(x) = nx$.

Suppose we were doing the trivial thing of splitting the value of each project equally among the participants. The small effort players have no choice and hence will put their ϵ effort on C. Suppose that player a puts some amount x on project P and $(1-x)$ on project C. Then his utility is $2 \cdot x + n \cdot \frac{(1-x)+(n-1)\epsilon}{n} = x + 1 + (n-1)\epsilon$. Thus, player a's utility is maximized by putting all his effort on P. This is the unique Nash Equilibrium and also the unique Strong Nash Equilibrium and it gives a social welfare of $2 + n(n-1)\epsilon$. On the other hand, the optimal would be for player a to put his effort on C leading to social welfare of $n(1+(n-1)\epsilon)$. Thus, equal sharing leads to an inefficiency of $\Omega(n)$. On the other hand, if we were awarding each player proportionally to their effort on the project, then it is easy to see that the optimal outcome would have been the unique equilibrium (as we show in Appendix C, this is always the case for linear total effort markets). Thus, we see how crucial is the choice of the mechanism on the efficiency of the outcome.

Axiomatic Surplus Sharing In searching for good mechanisms we first emphasize the connection of our setting and the classic literature on Axiomatic Surplus Sharing (see e.g. [16]). A classic problem in the cost sharing literature is that of variable demand cost and surplus sharing [16, 17, 8]. In variable demand surplus sharing each agent i from a set N provides an amount of supply x_i and the problem is to split the total surplus $V(x_1, \ldots, x_n)$ using no other information than the surplus function V.

As is easy to notice each local project in our market of efforts constitutes exactly a variable demand surplus sharing problem. Each player that participates in a project w chooses an amount of effort x_w^i to put and then the mechanism decides how to split the surplus $V_w(x_w^1, \ldots, x_w^n)$ among the participants. However, the literature on axiomatic surplus sharing focuses on a single project and has not studied the effect that each sharing scheme might have on the global welfare of the society. Thus, in some sense our setting extends the surplus sharing literature to capture characteristics of the mechanisms that could not be seen when each project sharing problem was viewed in isolation.

4 Concave Effort Markets

In this section we examine the case when the value functions v_w are concave. First we note that by [3] a Strong Nash (or even a k-wise Nash) equilibrium might not exist in a concave effort market even if the reward mechanism is the simple equal splitting mechanism. However, by [22] we know that a Nash Equilibrium in pure strategies always exists as long as the utilities of the agents, implied by the reward mechanism, are concave, continuous and twice differentiable. Hence, in the setting of concave effort markets we will mainly focus on the efficiency of Nash Equilibria rather than Strong or k-wise Nash.

We focus on the subclass of concave functions that satisfy the decreasing marginal property: $\forall \vec{y}, \vec{x}, \vec{z} : \vec{x} \geq \vec{y} \implies f(\vec{y} + \vec{z}) - f(\vec{y}) \geq f(\vec{x} + \vec{z}) - f(\vec{x})$, where $\vec{x} \geq \vec{y}$, means coordinate-wise comparison. Several natural classes of functions satisfy the above property. For example, if $v_w(\vec{x}) = f(g(\vec{x}))$, where f is a concave function and g an additive one. This class of functions could accommodate the case when each player has project specific abilities that are modelled as coefficients in his effort. Then the value of a project is a concave function of the weighted sum of efforts: $v_w(\vec{x}) = f(\sum_i a_w^i x_w^i)$. Another, case is when $v_w(\vec{x}) = \sum_i f_i(x_i)$ and f_i are concave functions.

In this case we show that as long as we pick a mechanism $\mathcal{M}$, such that the game that it induces satisfies the Vickrey condition then the PoA of any Pure Nash Equilibrium (and even Coarse Correlated) is at most 2. This requirement, due to the linearity of utilities, boils down to the fact that each player should receive at least his marginal contribution locally on each project.

Theorem 5 *Any mechanism $\mathcal{M}$ that awards each player at least his marginal contribution to the value of a project induces a monotone valid utility game, when value functions satisfy the decreasing marginal returns property. Hence, PoA ≤ 2.*

Many mechanisms proposed in the Axiomatic Surplus Sharing literature satisfy the above characteristic and hence have good global efficiency properties. An important and broad class of mechanisms are the fixed-path generated methods and convex combinations of them (see [16] for a complete analysis and appendix for a brief description). One of the most commonly used convex combination of fixed path routing methods is the Shapley-Shubik method [28], which corresponds to taking the average over all fixed path methods where players effort units are not mixed (i.e. ordering of player's arrivals), also called incremental paths. Another interesting property of the Shapley-Shubik sharing method is that the utility of a player from a project as a function of his effort is a sum of concave functions and therefore also concave. Hence, by [22] we also get that the game induced by the Shapley-Shubik mechanism always possesses a Pure Nash Equilibrium.

Apart from path generated methods several other natural mechanisms satisfy the Vickrey condition in effort markets with decreasing marginal returns. If each player gets a share proportional to his marginal contribution, i.e. $\phi_w^i(\vec{x}_w) = \frac{v_w(\vec{x}_w) - v_w(0, \vec{x}_w^{-i})}{\sum_j v_w(\vec{x}_w) - v_w(0, \vec{x}_w^{-j})} v_w(\vec{x}_w)$, then by the decreasing marginal returns property of v_w we have that $v_w(\vec{x}_w) \geq \sum_j v_w(\vec{x}_w) - v_w(0, \vec{x}_w^{-j})$ and therefore $\phi_w^i(\vec{x}_w) \geq v_w(\vec{x}_w) - v_w(0, \vec{x}_w^{-i})$.

The marginal contribution property is the crucial property that guides the efficiency. For example, as we show in the previous section, the equal sharing scheme can have a PoA of $\Omega(n)$. This is because the equal sharing scheme severely violates the property. However, if the maximum number of participants per project is k then it is easy to see that the equal sharing scheme satisfies the k-approximate Vickrey condition. Hence, by [30] we immediately get that it has PoA $\leq k+1$. Moreover, by noting that equal splitting induces a strongly monotone valid utility game, when value functions are increasing and satisfy the decreasing marginal returns property, we can apply Theorem 4 and are able to get the tighter bound of k.

4.1 Total Effort Games

In this section we study the case when the value of a project is an increasing concave function of the total effort put into it: $v_w(\vec{x}_w) = v_w(X_w)$, where $X_w = \sum_i x_w^i$. The sharing schemes proposed in the previous section apply here too. However, in this setting it is also natural to study the proportional sharing scheme, where each player's share is $\phi_w^i(\vec{x}_w) = \frac{x_w^i}{X_w} v_w(X_w)$.

Theorem 6 *The game defined by the proportional sharing scheme is a monotone valid utility game when the value functions are increasing and concave in the total effort and $v_w(0) = 0$. Hence the PoA is at most 2. (proof in Appendix)*

Similarly, one can prove that if value functions are increasing concave functions of a weighted total effort $v_w(\vec{a}_w \cdot \vec{x}_w)$, then sharing proportional to the weighted effort, $a_w^i x_w^i$, also induces a monotone valid utility game.

It is also interesting to notice that total effort games have a strong connection with splittable selfish routing games: the resulting game could be viewed as a utility version of a splittable selfish routing game where players choose only singleton strategies and the utility function of each resource is $\bar{v}_w(x) = v_w(x)/x$. We note that the above result remains to hold even when players choose sets and not singleton strategies. When $v_w(x)$ is increasing and concave, then $\bar{v}_w(x)$ is decreasing and quasi-concave. Thus, our analysis implies that the PoA of the utility version of splittable selfish routing is at most 2, when the utility resource functions $v_r(x_r)$ are decreasing and $xv_r(x)$ is increasing concave.

Moreover, we were able to apply the local smoothness framework adapted to utility maximization games to get much better bound for a natural class of value functions, as given in the theorem that follows (proof in Appendix). Also for another natural class of exponential value functions we show that the best PoA bound provable using the local smoothness framework is 2 (see Appendix) thereby coinciding with that of the above theorem.

Theorem 7 *If value functions have the form $v_w(X_w) = q_w X_w^\alpha$ for some $\alpha \in (0,1)$ then the proportional reward scheme has: PoA$(\alpha) \leq \frac{1}{2}\left(\frac{2}{1-\alpha^{\frac{\alpha}{1-\alpha}}}\right)^\alpha (2 - \alpha - \alpha^{\frac{1}{1-\alpha}}) \lessapprox 1.251$.*

It is interesting to notice that the above bound coincides with the optimality result for the case of linear functions $\alpha = 1$. The other interesting special case is that as $\alpha \to 0$ then the PoA also goes to 1. In other words if value functions are almost constant then any PNE is almost optimal.

5 Non-Concave Effort Markets

We now move to studying the case when value functions are convex or general functions. As was observed by [3] the PoA of pure equilibria in this case can be unbounded. Intuitively, convex value functions can possibly have an initial plateau phase in which no value is produced. Hence, players could be trapped in an equilibrium because no unilateral deviation could overcome this plateau phase and get to the high value region. However, in such a cooperative environment it is natural to consider the case of coalitional deviations. For example it is natural to study the inefficiency of outcomes that are stable if a subset of players that participate in a specific project don't want to do a group deviation. These setwise equilibria where studied by [3] who showed that for the case of the equal-splitting of surplus their PoA is at most $k = \max_w |N_w|$.

Anshelevich et al. [3] showed that the above bound is tight for the equal-splitting scheme. We show here that this inefficiency is not a trait of the equal-splitting scheme but is inherent of convex value functions. Specifically we show that if we allow for arbitrarily convex value functions, no local surplus sharing scheme can achieve a setwise PoA smaller than k. (proof in appendix)

Theorem 8 *No local surplus sharing scheme can have Strong PoA less than k.*

Proof. Consider the following instance. There are n players each with an effort budget of 1. They all participate in a shared project C that has the following value function $v_C(X_C) = 0$ if $X_C < n$ and $\geq n$ if $X_C \geq n$. The above example is a discontinuous function but one can think of an continuous analog where the value function continuously goes from 0 to n when X_C approaches n.

Suppose that all the players put their effort on C. The mechanism $\mathcal{M}$ must share the surplus among the participants. Hence, no matter what the mechanism is, there will be at least on player $i_{\mathcal{M}}$ that receives a share ≤ 1. Hence, in the instance $I_{\mathcal{M}}$, where player $i_{\mathcal{M}}$ also has an outside private project P with value function $v_P(X_P) = (1+\epsilon)X_P$ the unique setwise equilibrium is for player $i_{\mathcal{M}}$ to put all his effort on P. This equilibrium will then have social welfare $(1+\epsilon)$, while the optimal would be all players putting their effort on C leading to social welfare n. ■

5.1 Efficiency of Setwise and Strong Equilibria

Setwise Equilibria are a strict subset of Strong Nash Equilibria, since they don't allow for any type of group deviation. However, we are able to extend the proof of Theorem 1 to setwise equilibria for the case when the value functions are convex.

Theorem 9 *If value functions are convex then any mechanism that awards each player at least $1/\gamma$ his marginal contribution to the value of the project, has setwise PoA at most $\gamma + 1$. If the mechanism induces a strongly monotone game then it has setwise PoA at most γ.*

Proof. Since value functions are convex, the social welfare is also convex. Hence, the optimization problem is convex and therefore there must be an extreme optimal outcome where each player puts his whole effort on a single project. Denote such an optimal outcome with OPT and let N_w be the set of players that put their budget on project w at OPT. Let SNE be a setwise equilibrium. Using similar arguments as in the proof of Theorem 1 we can prove that: $\forall w : \sum_{i \in N_w} u_i(\text{SNE}) \geq \frac{1}{\gamma+1} V_w(\text{OPT})$ and if the utilities induce a strongly monotone game then: $\forall w : \sum_{i \in N_w} u_i(\text{SNE}) \geq \frac{1}{\gamma} V_w(\text{OPT})$. Summing over all projects w and since N_w are a disjoint partition of N, we get the theorem. ■

The setwise PoA of k of [3] is a special case of the above theorem, since the equal splitting mechanism awards each player at least $1/k$ his marginal contribution and also induces a strongly monotone utility game. The Shapley-Shubik sharing scheme also awards to each player at least $1/k$ his marginal contribution. Hence, has setwise PoA at most $k+1$. Moreover, it induces a strongly monotone game, when value functions have the increasing marginal returns property, Thus, for the latter case it achieves the best possible setwise PoA of k (see appendix).

Observe that unlike the submodular case the above efficiency guarantee doesn't extend to the rest of the fixed path generated methods of splitting the surplus. Moreover, the Shapley value has the property of awarding at least $1/k$ the marginal contribution for arbitrarily value functions not necessarily convex. Hence, for arbitrary value functions we are able to prove the efficiency guarantee of $k+1$ for Strong Nash Equilibria by applying Theorem 1.

Sharing surplus proportional to each players marginal contribution also provides similar guarantees to the Shapley value: it has a guarantee of $k+1$ for general convex functions (see appendix) that improves as we restrict to less and less convex functions.

Last but not least we are also able to give a bound for setwise equilibria and strong Nash equilibria for the equal splitting mechanism even for the case when the value functions are not monotone but still non-negative. To do that we use the fact that the equal splitting mechanism induces a potential game with potential $\Phi(\vec{x}) = \sum_w \frac{1}{|N_w|} v_w(\vec{x}_w)$. Thus, this game satisfies the conditions of Theorem 3 with $a = \frac{1}{k}$ and $\beta = 1$ and we get that for non-monotone value functions the equal sharing scheme has Strong PoA $\leq 2k$, which carries over to setwise NE when value functions are convex.

6 Applying Surplus Sharing Axioms

Many papers in the surplus sharing literature have studied how the effort exerted by players affects their share. Much of this work have focused on an axiomatic approach [27, 6, 17], proposing various axioms that are desirable in a sharing scheme. One fundamental axiom is the Axiom of Monotonicity [15], which states that a player's share must be monotonously increasing in his effort. Recently, Moulin [17] also introduced a group-deviation version of the above axiom and the Axiom of Solidarity which determines how a player's change in effort affects the shares of the other players.

We observe here that the latter effect is more important for global efficiency than the monotonicity axiom. The Axiom of Solidarity states that when a player changes his effort, the shares of the other players must all change in same direction (i.e. either all of them increase or all of them decrease). We now propose a refinement of the axiom of solidarity that has a natural economic interpretation and also is sufficient to guarantee reasonable global efficiency in a multiple project environment.

Definition 10 (Axiom of Elimination) *We say that a sharing scheme $(\phi_i)_{i\in N}$ satisfies the Axiom of Elimination if for any subset $C \subseteq 2^N$ and for any $i \notin C$ we have: $\phi_i^{(N,v)} \leq \phi_i^{(N-C,v)}$ if v is submodular and $\phi_i^{(N,v)} \geq \phi_i^{(N-C,v)}$ if v is supermodular.*

Several well known sharing schemes satisfy this axiom, such as the Shapley value. For the submodular case, this axiom requires that if we remove some players from the game then the imputation of each of the rest of the players can only increase. In a submodular setting, in some sense, players are substitutes. Hence, by removing players from the game we are removing competition for the remaining players. Thereby we expect the remaining players to achieve more surplus. On the contrary in convex value functions it is implied that players are complements. Hence, by removing players we are removing collaborators and hence in the resulting game each player should achieve smaller surplus.

Theorem 11 *The game defined by a sharing scheme that satisfies the Axiom of Elimination is a monotone valid utility game when value functions satisfy the decreasing marginal returns property. Hence it has PoA ≤ 2.*

7 Conclusion

We examined general and very natural settings where agents choose their engagement level when collaborating with others, and showed that in equilibrium the system is only efficient if the agents are rewarded based on their marginal contribution to social welfare. Our main result shows that if there is a way to redistribute wealth so that each player obtains at least its marginal contribution to social welfare, the incentives are well-aligned so that the social welfare in equilibrium is at least half of the optimal social welfare. In some cases it is impossible to redistribute welfare in this way, so we must resort to approximate solutions.

Our results have important implications to many real-world settings where agents may participate in teams to achieve joint goals, ranging from crowdsourcing scenarios, through scientific projects and ending with resource allocation in firms. Our main conclusion is that to improve efficiency, one must aim to compensate agents based on their marginal contribution to welfare.

Several issues remain open for future work. One key issue is the existence of strong Nash equilibria in some of our settings. Under what conditions in our domains are we guaranteed that such equilibria exist? Also, it would be interesting to studty extensions of effort market games where effort is not budgeted, but rather there is a general production cost the agents incur. Finally, are there alternative axiomatic characterizations of surpuls sharing mechanisms that result in a low inefficiency?

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A Strongly Monotone Utility Games

Proof of Theorem 2: Let s be a Strong Nash Equilibrium strategy profile and s^* the optimal strategy profile. As in the proof of Theorem 1, we reorder the players and we let N_i be the set of player $\{i, \ldots, n\}$, which implies that for all players i: $u_i(s) \geq u_i(s^*_{N_i}, s_{-N_i})$. By strong monotonicity we know that if any player goes out of the game then the utility of each other player can only decrease. Therefore, $u_i(s^*_{N_i}, s_{-N_i}) \geq u_i(s^*_{N_i}, s^{out}_{-N_i})$.

Then the k-approximate Vickrey condition implies that:

$$u_i(s^*_{N_i}, s^{out}_{-N_i}) \geq \frac{1}{k}\left(SW(s^*_{N_i}, s^{out}_{-N_i}) - SW(s^*_{N_{i+1}}, s^{out}_{-N_{i+1}})\right)$$

By combining the above we get again a telescoping sum:

$$SW(s) = \sum_{i=1}^{n} u_i(s) \geq \sum_{i=1}^{n} u_i(s^*_{N_i}, s^{out}_{-N_i}) \geq \frac{1}{k}\sum_{i=1}^{n} SW((s^*_{N_i}, s^{out}_{-N_i}) - SW(s^*_{N_{i+1}}, s^{out}_{-N_{i+1}})$$
$$= \frac{1}{k}\left(SW(s^*) - SW(s^{out})\right) = \frac{1}{k}SW(s^*)$$

■

Proof of Theorem 4: Let s be a Nash Equilibrium and s^* the optimal strategy profile. By the definition of Nash Equilibrium and by strong monotonicity we have:

$$\forall i : u_i(s) \geq u_i(s^*_i, s_{-i}) \geq u_i(s^*_i, s^{out}_{-i})$$

The assumption of submodularity on social welfare means that the marginal increase in social welfare that a player makes is smaller when more players participate in the game. More formally, consider any player i and let $S \subseteq T$ be two sets of players not including i. Then for any strategy profile s:

$$SW(s_i, s_{-S}, s^{out}_S) - SW(s^{out}_i, s_{-S}, s^{out}_S) \geq SW(s_i, s_{-T}, s^{out}_T) - SW(s^{out}_i, s_{-T}, s^{out}_T)$$

We let again N_i be the set of player $\{i, \ldots, n\}$. Combining submodularity with the approximate Vickrey condition we get:

$$SW(s) = \sum_i u_i(s) \geq \sum_i u_i(s^*_i, s^{out}_{-i}) \geq \frac{1}{k}\sum_i SW(s^*_i, s^{out}_{-i}) - SW(s^{out}_i, s^{out}_{-i})$$
$$\geq \frac{1}{k}\sum_i SW(s^*_i, s^*_{N_{i+1}}, s^{out}_{-N_i}) - SW(s^{out}_i, s^*_{N_{i+1}}, s^{out}_{-N_i})$$
$$= \frac{1}{k}\sum_i SW(s^*_{N_i}, s^{out}_{-N_i}) - SW(s^*_{N_{i+1}}, s^{out}_{-N_{i+1}})$$
$$= \frac{1}{k}(SW(s^*) - SW(s^{out})) = \frac{1}{k}SW(s^*)$$

■

B Efficiency of Nash Equilibrium in Submodular Potential Games

In this section we provide some supplementary results on the efficiency of Nash Equilibria in potential games where the potential is a submodular function.

Such an asssumption is satisfied for example for all congestion games where resources are associated with decreasing utility functions rather than cost functions. A utility congestion game consists of a set of players and a set of resources R. Each players strategy space consists of subsets of the resources. Each resource r is associated with a utility function π_r that depends only on the number of players n_r using the resource. The utility of each player is the sum of his utilities from each resource in his strategy. A utility congestion game is known to admit Rosenthal's potential: $\Phi(s) = \sum_r \sum_{k=1}^{n_r} \pi_r(k)$. Moreover, the social welfare ends up being $SW(s) = \sum_r n_r \pi_r(n_r)$. It is easy to see that if π_r are decreasing functions then $\Phi(s) \geq SW(s)$. Thus, by the submodularity of Φ we get the following theorem:

Theorem 12 *In any utility congestion game with non-negative decreasing utilities the potential at any Nash Equilibrium is at least half the maximum potential and if $\Phi(s) \leq \beta SW(s)$ then the PoA is at most $\beta + 1$.*

Proof. Since $\pi_r(k)$ are decreasing we have:

$$\Phi(s) = \sum_{r\in R}\sum_{t=1}^{n_r(s)} \pi_r(t) \geq \sum_{r\in R}\sum_{t=1}^{n_r(s)} \pi_r(n_r(s)) = \sum_{r\in R} n_r(s)\pi_r(n_r(s)) = \sum_i u_i(s)$$

Since $\pi_r(k) \geq 0$ we have $\forall s : u_i(s) \geq 0$ and therefore:

$$u_i(s) \geq \Phi(s) - \Phi(s_i', s_{-i}) + u_i(s_i', s_{-i}) \geq \Phi(s) - \Phi(s_i', s_{-i})$$

Moreover, since $\pi_r(k)$ are decreasing we also have that $\Phi(s)$ is in some sense a monotone submodular function as described below.

If we view the potential function as a function on multisets, where a player playing more than one strategies means adding to the resources the extra congestion that these strategies imply. Thus, we can think of congestion $n_r(S)$ as a set function that counts how many strategies $s \in S$ use the resource r (double counting if a strategy is more than one times in S), i.e. the multiplicity function of S. Let S, T be two multisets of strategies. Let $S \uplus T$ be the multiset sum of S and T. Then:

$$\Phi(S \uplus T) = \sum_{r\in R}\sum_{t=1}^{n_r(S\uplus T)} \pi_r(t) \geq \sum_{r\in R}\sum_{t=1}^{n_r(S)} \pi_r(t) = \Phi(S)$$

Thus $\Phi(S)$ satisfies monotonicity. Moreover, let $S \subset T$ and let W be another multiset of strategies disjoint from S, T. By $S \subseteq T$ we have $\forall r \in R : n_r(S) \geq n_r(T)$. Also it holds that $\forall r : n_r(W \uplus S) = n_r(W) + n_r(S)$. Then using the decreasing property of $\pi_r(t)$:

$$\begin{aligned} \Phi(W \uplus S) - \Phi(S) =& \sum_{r\in R}\sum_{t=n_r(S)+1}^{n_r(W\uplus S)} \pi_r(t) = \sum_{r\in R}\sum_{t=n_r(S)+1}^{n_r(S)+n_r(W)} \pi_r(t) \\ \geq& \sum_{r\in R}\sum_{t=n_r(T)+1}^{n_r(T)+n_r(W)} \pi_r(t) = \Phi(W \uplus T) - \Phi(T) \end{aligned} \tag{3}$$

Thus $\Phi(s)$ satisfies submodularity. Thus we can apply similar analysis as in monotone valid utility games. At points we use same notation for strategy profiles and sets of strategies but it is clear from the context. For example, we use notation such as NE_{-i} to denote the multiset that consists of the multiset sum of the strategies of all players except i, when strategies are viewed as sets of resources. Similarly we use $\text{NE}_{<i}$ ($\text{NE}_{>i}$) to denote the above multiset but for the strategies of all the players less (greater) than i.

$$\begin{aligned} SW(\text{NE}) \geq& \sum_i u_i(\text{NE}) \geq \sum_i u_i(\text{OPT}_i, \text{NE}_{-i}) \geq \sum_i \Phi(\text{OPT}_i, \text{NE}_{-i}) - \Phi(OUT_i, \text{NE}_{-i}) \\ =& \sum_i \Phi(\text{OPT}_i \uplus \text{NE}_{-i}) - \Phi(\text{NE}_{-i}) \\ \geq& \sum_i \Phi(\text{OPT}_i \uplus \text{NE}_{-i} \uplus \text{NE}_i \uplus \text{OPT}_{<i}) - \Phi(\text{NE}_{-i} \uplus \text{NE}_i \uplus \text{OPT}_{<i}) \\ =& \sum_i \Phi(\text{OPT}_{\leq i} \uplus \text{NE}_{\leq i} \uplus \text{NE}_{>i}) - \Phi(\text{OPT}_{\leq i-1} \uplus \text{NE}_{\leq i-1} \uplus \text{NE}_{>i-1}) \\ =& \Phi(\text{OPT}_{\leq n} \uplus \text{NE}_{\leq n}) - \Phi(\text{NE}_{>0}) \geq \Phi(\text{OPT}) - \Phi(\text{NE}) \\ \geq& SW(\text{OPT}) - \beta SW(\text{OPT}) \end{aligned}$$

Also using similar analysis with the potential maximizer instead of OPT we get:

$$\Phi(\text{NE}) \geq SW(\text{NE}) \geq \Phi(\text{POT-MAX}) - \Phi(\text{NE})$$

■

C Optimality of Effort Proportional Sharing in Linear Markets of Total Effort

Theorem 13 *In the case of proportional sharing all Nash Equilibria are optimal.*

Proof. The optimality claim stems from the fact that the optimality conditions of the social welfare optimization problem coincide with the optimality conditions of the player problem.

The utility function of a player is $\sum_{w \in \Omega_i} q_w x^i_w$. The player and social welfare problems are, respectively, given by:

$$\begin{aligned} \max \quad & \sum_{w \in \Omega_i} q_w x^i_w \\ \text{s.t.} \quad & \sum_{w \in \Omega_i} x^i_w \leq B_i \\ & x^i_w \geq 0 \end{aligned} \qquad (4) \qquad\qquad \begin{aligned} \max \quad & \sum_{w \in [m]} q_w X_w \\ \text{s.t.} \quad & \sum_{w \in \Omega_i} x^i_w \leq B_i \\ & x^i_w \geq 0 \end{aligned} \qquad (5)$$

The optimality conditions of both problems are that:

$$x^i_w > 0 \implies q_w \in \operatorname*{argmax}_{w' \in \Omega_i} q_{w'}$$

Moreover, any Nash Equilibrium is also a strong Nash Equilibrium. Each players utility is independent of what the other players do. Hence, the value of a player when he deviates coalitionally is the same as if he had deviated unilaterally. ■

D Concave Effort Games

Proof of Theorem 5: Since monotone valid utility games where initially defined for games where the social welfare was a function on sets of strategies, we present here for completeness the whole proof adapted to the case when social welfare and utilities are defined on a vector space, as are effort games.

By the decreasing marginal returns property we have:

$$\forall \vec{x}, \vec{y}, \vec{z} \geq 0 : v_w(\vec{x} + \vec{y}) - v_w(\vec{y}) \geq v_w(\vec{x} + \vec{y} + \vec{z}) - v_w(\vec{y} + \vec{z})$$

Since social welfare is the sum of such concave functions it is also concave and the above holds for social welfare too. The marginal contribution property of the mechanism implies:

$$\forall \vec{x} : u^i_w(\vec{x}) \geq v_w(\vec{x}) - v_w(0, \vec{x}_{-i})$$

and by summing up we get: $\forall \vec{x} : u_i(\vec{x}) \geq SW(\vec{x}) - SW(0, \vec{x}_{-i})$.

Let $\vec{x}$ be a Nash Equilibrium and $\vec{y}$ the optimal outcome. Also let $\vec{m}^i = (\vec{y}_1 + \vec{x}_1, \ldots, \vec{y}_i + \vec{x}_i, \vec{x}_{i+1}, \ldots, \vec{x}_n)$, i.e. players up to i implement both their Nash and OPT strategies. Although this might not be a valid strategy for them it is not a problem as this vectors are used as an intermediate step and we never consider a deviation to such a strategy. Also let $\vec{t}_i = (\vec{y}_i, \vec{0}_{-i})$, $\vec{e}_i = (\vec{0}, \vec{x}_{-i})$ and $\vec{r}_i = (\vec{y}_1, \ldots, \vec{y}_{i-1}, \vec{x}_i, 0, \ldots, 0)$. Then, we have

$$\begin{aligned}
SW(\vec{x}) = \sum_i u_i(\vec{x}) \geq & \sum_i u_i(\vec{y}_i, \vec{x}_{-i}) \\
\geq & \sum_i SW(\vec{y}_i, \vec{x}_{-i}) - SW(\vec{0}, \vec{x}_{-i}) \\
= & \sum_i SW(\vec{t}_i + \vec{e}_i) - SW(\vec{e}_i) \\
\geq & \sum_i SW(\vec{t}_i + \vec{e}_i + \vec{r}_i) - SW(\vec{e}_i + \vec{r}_i) \\
= & \sum_i SW(\vec{m}_i) - SW(\vec{m}_{i-1}) \\
= & SW(\vec{m}_n) - \sum_i SW(\vec{m}_0) \\
= & SW(\vec{x} + \vec{y}) - SW(\vec{x}) \geq SW(\vec{y}) - SW(\vec{x})
\end{aligned}$$

Thus $SW(\vec{x}) \geq \frac{1}{2} SW(\vec{y})$. ■

D.1 Brief Description of Path-Generated Methods of Sharing

One can think of a fixed-path generated method of sharing the surplus as follows: given the vector of efforts $\vec{x}_w$ break apart each player's effort in small units of effort dx_k^i. Now, order all the units of effort of all the players in an arbitrary way possibly mixing the units of different players. Notice that as the unit division goes to 0 the above corresponds to an arbitrary path in $\mathbb{R}^N$ from 0 to $\vec{x}_w$. Let T be the total number of effort units $\{dx_1^{i_1}, \ldots, dx_t^{i_T}\}$. Moreover, let $\vec{x}_t = (\sum_{k \leq t : i_k = i} dx_k^{i_k})_{i \in N}$ be the point of $\mathbb{R}^N$ that corresponds to adding up all the effort units up to unit t, vector-wise. The share of each player is then $\phi_w^i(\vec{x}_w) = \sum_{k : i_k = i} v_w(\vec{x}_t) - v_w(\vec{x}_{t-1})$. Now it is easy to see that if $v_w(\vec{x}_w)$ is a concave function then each player i gets at least $v_w(\vec{x}_w) - v_w(0, \vec{x}_w^{-i})$. This is because the latter quantity corresponds to the worst case path for player i in which all of his units are ordered last in the fixed path. If any of his units moves up in the ordering then it can only have higher marginal contribution at the value at the point it is accounted and hence this can only increase player i's share. Since, every fixed-path generated sharing method has the marginal contribution property then it is easy to see that every convex combination of fixed path routing methods also satisfies it.

Corollary 14 *Any convex combination fixed-path generated method of sharing the value of a project (e.g. Shapley-Shubik) induces a monotone valid utility game, when value functions satisfy the decreasing marginal returns property.*

E Total Effort Games

Proof of Theorem 6: We show the basic properties needed for a game to be a monotone valid utility game:

- $u_i(\vec{x}) \geq SW(\vec{x}_i, \vec{x}_{-i}) - SW(\vec{0}, \vec{x}_{-i})$: Since $v_w(X_w)$ are submodular then for any $y \in [0, X_w] : v_w(y) \geq y \frac{v_w(X_w)}{X_w}$. By setting $y = X_w - x_w^i$ we get:

$$x_w^i \frac{v_w(X_w)}{X_w} \geq v_w(X_w) - v_w(X_w - x_w^i)$$

Which implies the property since:

$$\begin{aligned}
u_i(\vec{x}) = \sum_w u_w^i(\vec{x}) = & \sum_w x_w^i \frac{v_w(X_w)}{X_w} \geq \sum_w v_w(X_w) - v_w(X_w - x_w^i) \\
= & SW(\vec{x}_i, \vec{x}_{-i}) - SW(\vec{0}, \vec{x}_{-i})
\end{aligned}$$

- $SW(\vec{x}) \geq \sum_i u_i(\vec{x})$: Holds with equality.
- $SW(\vec{x})$ is monotone submodular. Since $SW(\vec{x}) = \sum_w v_w(X_w)$ the property stems from concavity and monotonicity of value functions $v_w(X_w)$.

■

F Local Smoothness and Total Effort Games

We will present the local smoothness framework briefly, adapted to a utility maximization problem instead of a cost minimization. We also slightly generalize the framework by requiring that the player cost functions be continuously differentiable only at equilibrium points.

Consider a utility maximization game $\langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$, where each player's strategy space is a continuous convex subset of a Euclidean space $\mathbb{R}^{m_i}$. Let $u_i : \times_{i\in N} S_i \to \mathbb{R}$ be the utility function a player i, which is assumed to be continuous and concave in $\vec{x}_i$ for each fixed value of x_{-i}. The concavity and continuity assumptions lead to existence of a Pure Nash Equilibrium using the classical result of Rosen [22]. Also for an outcome $\vec{x} \in \times_{i\in N} S_i$ we denote with $SW(\vec{x}) = \sum_{i\in N} u_i(\vec{x})$.

We start by presenting the definitions of smoothness [23] and local-smoothness [25] for utility maximization games. We then examine their connection and how local smoothness is applied to our effort market setting.

Definition 15 *A utility maximization game is (λ, μ)-smooth iff for every two outcomes $\vec{x}$ and $\vec{y}$:*

$$\sum_i u_i(y_i, \vec{x}_{-i}) \geq \lambda SW(\vec{y}) - \mu SW(\vec{x})$$

Roughgarden [23] showed that any (λ, μ)-smooth game has PoA at most $(1+\mu)/\lambda$, which carries over to every coarse correlated equilibrium too. Most of the proofs about Nash Equilibria in the paper (e.g. Theorem 5) are smoothness proofs and therefore also carry over to coarse correlated equilibria. Local smoothness is a better version of smoothness for the case of continuous and convex strategy spaces, since as we'll see it has the potential of giving better PoA bounds than smoothness.

Definition 16 *A utility maximization game is locally (λ, μ)-smooth with respect to an outcome $\vec{y}$ iff for every outcome $\vec{x}$ at which $u_i(\vec{x})$ are continuously differentiable:*

$$\sum_i [u_i(\vec{x}) + \nabla_i u_i(\vec{x}) \cdot (\vec{y}^i - \vec{x}^i)] \geq \lambda SW(\vec{y}) - \mu SW(\vec{x})$$

where $\nabla_i u_i \equiv (\partial u_i/\partial x_1^i, \ldots, \partial u_i/\partial x_{m_i}^1)$.

Observe that for a concave differentiable function v it holds that $v(y) \leq v(x) + \nabla v(x) \cdot (y - x)$, hence from this observation we can derive the following lemma that connects smoothness with local smoothness:

Lemma 17 *If a concave utility maximization game is (λ, μ)-smooth then it is also locally (λ, μ)-smooth with respect to any outcome $\vec{y}$.*

Proof.

$$\sum_i [u_i(\vec{x}) + \nabla_i u_i(\vec{x}) \cdot (\vec{y}^i - \vec{x}^i)] \geq \sum_i u_i(\vec{y}^i, \vec{x}_{-i}) \geq \lambda SW(\vec{y}) - \mu SW(\vec{x})$$

■

Theorem 18 *If a utility maximization game is locally (λ, μ)-smooth with respect to an outcome $\vec{y}$ and $u_i(\vec{x})$ is continuously differentiable at every Nash equilibrium $\vec{x}$ of the game then $SW(\vec{x}) \geq \frac{\lambda}{1+\mu} SW(\vec{y})$*

Proof. The proof is the equivalent of [25] for the case of maximization games. The key claim is that if $\vec{x}$ is a Nash Equilibrium then

$$\nabla_i u_i(\vec{x}) \cdot (\vec{y}^i - \vec{x}^i) \leq 0$$

Given this claim then we get the theorem, since:

$$SW(\vec{x}) \geq \sum_i u_i(\vec{x}) + \nabla_i u_i(\vec{x}) \cdot (\vec{y}^i - \vec{x}^i) \geq \lambda SW(\vec{y}) - \mu SW(\vec{x})$$

To prove the claim define $\vec{x}^\epsilon = ((1-\epsilon)\vec{x}^i + \epsilon\vec{y}^i, \vec{x}_{-i})$ and observe that: $\lim_{\epsilon \to 0} \frac{1}{\epsilon}(u_i(\vec{x}^\epsilon) - u_i(\vec{x})) = \nabla_i u_i(\vec{x}) \cdot (\vec{y}^i - \vec{x}^i)$. Thus if $\nabla_i u_i(\vec{x}) \cdot (\vec{y}^i - \vec{x}^i) > 0$ then for some ϵ_0 it holds that $u_i(\vec{x}^{\epsilon_0}) - u_i(\vec{x}) > 0$, which means that i has a profitable deviation which contradicts the fact that $\vec{x}$ is a Nash Equilibrium. ■

Lemma 17 shows that local smoothness can possibly achieve better bounds on the PoA. If (λ_1, μ_1) where the parameters that minimized $(1+\mu)/\lambda$ subject to the game being (λ, μ) smooth, i.e. $(1+\mu_1)/\lambda_1$ was the best PoA bound we could get using smoothness, then by 17 we know that the game is also locally (λ_1, μ_1)-smooth with respect to the optimum outcome. Hence, if (λ_2, μ_2) where the parameters that minimized $(1+\mu)/\lambda$, subject to the game being locally (λ, μ)-smooth then we know that $(1+\mu_2)/\lambda_2 \leq (1+\mu_1)/\lambda_1$. Therefore, the best possible bound on the PoA achieved by local smoothness can only be better than that achieved by smoothness.

In the following theorem we will also use a lemma of [25]:

Lemma 19 *Let $x_i, y_i \geq 0$ and $x = \sum_i x_i$, $y = \sum_i y_i$ then: $\sum_i x_i(y_i - x_i) \leq k(x, y)$ where:*

$$k(x,y) = \begin{cases} \frac{y^2}{4} & x \geq y/2 \\ x(y-x) & x < y/2 \end{cases}$$

First we formulate a lemma that applies to any concave function:

Theorem 20 *Assume $v_w(X_w)$ are concave functions with $v_w(0) = 0$ and are continuously differentiable at any equilibrium point of the game defined by the proportional sharing scheme. Let $\bar{v}_w(X_w) = \frac{v_w(X_w)}{X_w}$. If:*

$$y\bar{v}_w(x) + \bar{v}'_w(x)k(x,y) \geq \lambda y \bar{v}_w(y) - \mu x \bar{v}_w(x)$$

then the game is locally (λ, μ)-smooth with respect to $\vec{y}$.

Proof. Let $\vec{x}$ be an equilibrium effort vector and $\vec{y}$ some other outcome. From the definition of the proportional sharing scheme:

$$u_i(\vec{x}) = \sum_{w \in \Omega_i} u_w^i(\vec{x}) = \sum_{w \in \Omega_i} x_w^i \frac{v_w(X_w)}{X_w} = \sum_{w \in \Omega_i} x_w^i \bar{v}_w(X_w)$$

$$\nabla_i u_i(\vec{x}) \cdot (\vec{y}^i - \vec{x}^i) = \sum_{w \in \Omega_i} \frac{\partial u_w^i(\vec{x})}{\partial x_w^i}(y_w^i - x_w^i) = \sum_{w \in \Omega_i} (\bar{v}_w(X_w) + x_w^i \bar{v}'_w(X_w))(y_w^i - x_w^i)$$

Therefore, we have

$$\begin{aligned} SW(\vec{x}) \geq\ & \sum_i u_i(\vec{x}) + \nabla_i u_i(\vec{x}) \cdot (\vec{y}^i - \vec{x}^i) \\ =\ & \sum_i \sum_{w \in \Omega_i} x_w^i \bar{v}_w(X_w) + (\bar{v}_w(X_w) + x_w^i \bar{v}'_w(X_w))(y_w^i - x_w^i) \\ =\ & \sum_i \sum_{w \in \Omega_i} y_w^i \bar{v}_w(X_w) + \bar{v}'_w(X_w) x_w^i (y_w^i - x_w^i) \\ =\ & \sum_w \left(Y_w \bar{v}_w(X_w) + \bar{v}'_w(X_w) \sum_i [x_w^i (y_w^i - x_w^i)] \right) \\ \geq\ & \sum_w \left(Y_w \bar{v}_w(X_w) + \bar{v}'_w(X_w) k(X_w, Y_w) \right) \end{aligned}$$

where the last inequality follows from the lemma and the fact that $\bar{v}_w(X_w)$ is a non-increasing function, hence $\bar{v}'_w(X_w) \leq 0$. Combining the last above inequality with the assumption of the theorem, we establish the assertion of the theorem.

Indeed, $\bar{v}_w(X_w)$ is non-increasing because $v_w(x)$ is concave with $v_x(0) = 0$. This means that the line connecting $(0,0)$ with $(x, v_w(x))$ is below $v_w(y)$ for any $y \in [0, x]$. Hence, $y\frac{v_w(x)}{x} \leq v_w(y) \implies \frac{v_w(x)}{x} \leq \frac{v_w(y)}{y}$. Thus for any $y \leq x \implies \frac{v_w(x)}{x} \leq \frac{v_w(y)}{y}$ which implies $\bar{v}_w(x)$ is decreasing. ■

F.0.1 Fractional Exponent Functions

We now cope with the natural case when the value functions have the form $q_w X_w^\alpha$ for some $\alpha \in (0,1)$. First, we need a lemma characterizing differentiability at equilibria:

Lemma 21 *If $v_w(X_w) = q_w X_w^\alpha$ for some $\alpha \in (0,1)$ then for any equilibrium $\vec{x}$ of the game it holds that $X_w > 0$ for all $w \in [m]$. Thus, $u_i(\vec{x})$ is continuously differentiable at any Nash equilibrium.*

Proof. Assume that $\vec{x}$ is an equilibrium where $X_{w_1} = 0$ for some w_1. Now consider a player that participates at w_1. Consider the deviation where he moves a small amount of effort ϵ from a project w_2 to w_1. Let $\vec{x}^\epsilon$ be the strategy vector after the deviation.

$$u_i(\vec{x}^\epsilon) - u_i(\vec{x}) = q_{w_1}\epsilon^\alpha - (u^i_{w_2}(X_{w_2}) - u^i_{w_2}((X_{w_2} - \epsilon)^\alpha)) \implies$$
$$\frac{u_i(\vec{x}^\epsilon) - u_i(\vec{x})}{\epsilon} = \frac{q_{w_1}\epsilon^\alpha}{\epsilon} - \frac{u^i_{w_2}(X_{w_2}) - u^i_{w_2}((X_{w_2} - \epsilon)^\alpha)}{\epsilon}$$

As $\epsilon \to 0$, $\frac{q_{w_1}\epsilon^\alpha}{\epsilon} \to \infty$ and $\frac{q_{w_2}(X_{w_2}^\alpha - (X_{w_2}-\epsilon)^\alpha)}{\epsilon} \to \left.\frac{\partial u^i_{w_2}(x)}{\partial x^i_{w_2}}\right|_{x=X_{w_2}} = M$. Thus for some ϵ_0 it has to be that $\frac{q_{w_1}\epsilon^\alpha}{\epsilon} > \frac{q_{w_2}(X_{w_2}^\alpha - (X_{w_2}-\epsilon)^\alpha)}{\epsilon}$. Thus x^{ϵ_0} is a profitable deviation for i. ■

Theorem 22 *If $v_w(X_w) = q_w X_w^\alpha$ for some $\alpha \in (0,1)$ then the game defined by the proportional sharing scheme is locally $(1, 2-2\alpha)$-smooth. Hence, for any Nash Equilibrium: $SW(NE) \geq \frac{1}{3-2\alpha} SW(OPT)$ and the price of anarchy is at most $3 - 2\alpha$.*

Proof. We will prove that the game defined by the proportional sharing scheme is a (λ, μ)-smooth game with $\mu = 2(1-\alpha)$ and $\lambda = 1$. By previous lemma we can assume that $X_w > 0$ for all w and hence $v_w(X_w)$ is continuously differentiable at $\vec{x}$.

If we prove that: $Y_w \bar{v}_w(X_w) + \bar{v}'_w(X_w) k(X_w, Y_w) \geq \lambda Y_w \bar{v}_w(Y_w) - \mu X_w \bar{v}_w(X_w)$ for $\mu = 2(1-\alpha)$ and $\lambda = 1$ then the theorem follows by Theorem 20.

$$Y_w \bar{v}_w(X_w) + \bar{v}'_w(X_w) k(X_w, Y_w) \geq \lambda Y_w \bar{v}_w(Y_w) - \mu X_w \bar{v}_w(X_w) \Leftrightarrow$$
$$Y_w q_w X_w^{\alpha-1} + (\alpha-1) q_w X_w^{\alpha-2} k(X_w, Y_w) \geq \lambda Y_w q_w Y_w^{\alpha-1} - \mu X_w q_w X_w^{\alpha-1} \Leftrightarrow$$
$$Y_w q_w X_w^{\alpha-1} + (\alpha-1) q_w X_w^{\alpha-2} k(X_w, Y_w) - \lambda q_w Y_w^\alpha + \mu q_w X_w^\alpha \geq 0 \Leftrightarrow$$
$$Y_w + (\alpha-1) X_w^{-1} k(X_w, Y_w) - \lambda X_w^{1-\alpha} Y_w^\alpha + \mu X_w \geq 0$$

Let $h_x(y) = y + (\alpha-1)x^{-1}k(x,y) - \lambda x^{1-\alpha}y^\alpha + \mu x$. We will show that for $\lambda = 1$ and $\mu = 2(1-\alpha)$, $h_x(y) \geq 0$ for all x, y.

We take cases about $k(x,y)$:

- **Case 1:** $x \geq y/2 \implies k(x,y) = \frac{y^2}{4}$

$$h_x(y) = y + (\alpha-1)x^{-1}\frac{y^2}{4} - \lambda x^{1-\alpha}y^\alpha + \mu x$$

Since $\alpha - 1 < 0$ and $\frac{y^2}{4} \le x^2$:

$$\begin{aligned} h_x(y) \ge & \ y + (\alpha-1)x - \lambda x^{1-\alpha} y^\alpha + \mu x \\ = & \ y + (\mu+\alpha-1)x - \lambda x^{1-\alpha} y^\alpha = g_x(y) \end{aligned}$$

$$g'_x(y) = 1 - \lambda x^{1-\alpha}\alpha y^{\alpha-1} = 0 \implies y = x(\lambda\alpha)^{\frac{1}{1-\alpha}}$$

Since $g''_x(y) = \lambda x^{1-\alpha}\alpha(1-\alpha)y^{\alpha-2} \ge 0$, g_x is convex. Hence, $g_x(y) \ge g_x(x(\lambda\alpha)^{\frac{1}{1-\alpha}})$.

$$\begin{aligned} g_x(x(\lambda\alpha)^{\frac{1}{1-\alpha}}) = & \ x(\lambda\alpha)^{\frac{1}{1-\alpha}} + (\mu+\alpha-1)x - \lambda x^{1-\alpha}x^\alpha(\lambda\alpha)^{\frac{\alpha}{1-\alpha}} \\ = & \ x(\lambda\alpha)^{\frac{1}{1-\alpha}} + (\mu+\alpha-1)x - x\lambda^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}} \\ = & \ x(\lambda\alpha)^{\frac{1}{1-\alpha}} + (\mu+\alpha-1)x - x\lambda^{\frac{1}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}\alpha^{-1} \\ = & \ (\mu+\alpha-1+(\lambda\alpha)^{\frac{1}{1-\alpha}}(1-\alpha^{-1}) \\ = & \ (\mu+\alpha-1+(\lambda\alpha)^{\frac{1}{1-\alpha}}\alpha^{-1}(\alpha-1))x \\ = & \ (\mu+(\alpha-1)(1+(\lambda\alpha)^{\frac{1}{1-\alpha}}\alpha^{-1}))x \end{aligned}$$

If $\mu \ge (1-\alpha)(1+(\lambda\alpha)^{\frac{1}{1-\alpha}}\alpha^{-1})$, then we have $g_x(y) \ge 0$.

- **Case 2:** $x \le y/2 \implies k(x,y) = x(y-x)$

$$\begin{aligned} h_x(y) = & \ y + (\alpha-1)x^{-1}x(y-x) - \lambda x^{1-\alpha}y^\alpha + \mu x \\ = & \ y + (\alpha-1)(y-x) - \lambda x^{1-\alpha}y^\alpha + \mu x \\ = & \ \alpha y + (\mu+1-\alpha)x - \lambda x^{1-\alpha}y^\alpha \end{aligned}$$

$$h'_x(y) = \alpha - \lambda x^{1-\alpha}\alpha y^{\alpha-1} = 0 \implies y^{\alpha-1} = \lambda^{-1}x^{\alpha-1} \implies y = x\lambda^{\frac{1}{1-\alpha}}$$

Since h_x is convex it is minimized at $y = x\lambda^{\frac{1}{1-\alpha}}$. Hence, $h_x(y) \ge h_x(x\lambda^{\frac{1}{1-\alpha}})$.

$$\begin{aligned} h_x(x\lambda^{\frac{1}{1-\alpha}}) = & \ \alpha x\lambda^{\frac{1}{1-\alpha}} + (\mu+1-\alpha)x - \lambda x^{1-\alpha}x^\alpha\lambda^{\frac{\alpha}{1-\alpha}} \\ = & \ \alpha x\lambda^{\frac{1}{1-\alpha}} + (\mu+1-\alpha)x - x\lambda^{\frac{1}{1-\alpha}} \\ = & \ (\mu+1-\alpha+\alpha\lambda^{\frac{1}{1-\alpha}} - \lambda^{\frac{1}{1-\alpha}})x \\ = & \ (\mu+(1-\alpha)(1-\lambda^{\frac{1}{1-\alpha}}))x \end{aligned}$$

If $\mu \ge -(1-\alpha)(1-\lambda^{\frac{1}{1-\alpha}})$ then $h_x(x\lambda^{\frac{1}{1-\alpha}}) \ge 0$ and hence $h_x(y) \ge 0$.

If we set $\lambda = 1$ then the first case requires $\mu \ge (1-\alpha)(1+\alpha^{\frac{\alpha}{1-\alpha}})$ and the second case requires $\mu \ge 0$. For $0 < \alpha < 1 : \alpha^{\frac{\alpha}{1-\alpha}} \le 1$. Hence, by setting $\mu = 2(1-\alpha)$, we satisfy the requirements. ■

The above proof used a very crude setting of the λ and μ so as to satisfy the requirements. However, to get the true bound we need to optimize. We know that for each chosen λ, μ has to satisfy:

$$\mu \ge (1-\alpha)(\lambda^{\frac{1}{1-\alpha}} - 1)$$

$$\mu \ge (1-\alpha)(\lambda^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}} + 1)$$

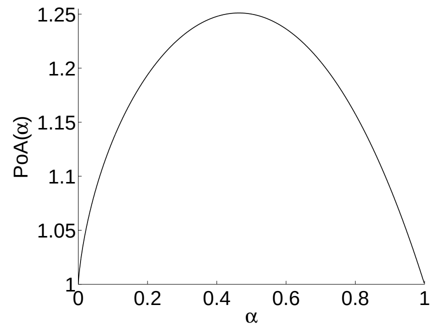

Figure 1: PoA versus the exponent α of the fractional exponent function.

and then the we have PoA $\leq \frac{1+\mu}{\lambda}$. Thus to get the best bound we need to find the λ that minimizes this quantity:

$$\text{PoA} = \inf_{\lambda>0} \frac{1 + (1-\alpha)\max\{\lambda^{\frac{1}{1-\alpha}} - 1, \lambda^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}} + 1\}}{\lambda} \tag{6}$$

In Figure 1 we depict the value of PoA as α varies. We see that as $\alpha \to 0$ and $\alpha \to 1$ the outcome becomes optimal.

The worst-case PoA can be achieved by setting λ to be the point at which the two portions in the max become identical. Thus by setting $\lambda = \left(\frac{2}{1-\alpha^{\frac{\alpha}{1-\alpha}}}\right)^{1-\alpha}$, the two portions in the max are equal and hence:

$$\begin{aligned} PoA = \frac{1 + (1-\alpha)(\lambda^{\frac{1}{1-\alpha}} - 1)}{\lambda} &= \alpha \left(\frac{1-\alpha^{\frac{\alpha}{1-\alpha}}}{2}\right)^{1-\alpha} + (1-\alpha)\left(\frac{2}{1-\alpha^{\frac{\alpha}{1-\alpha}}}\right)^{\alpha} \\ &= \frac{1}{2}\left(\frac{2}{1-\alpha^{\frac{\alpha}{1-\alpha}}}\right)^{\alpha}(2 - \alpha - \alpha^{\frac{1}{1-\alpha}}) \end{aligned}$$

Theorem 23 *If value functions have the form $v_w(X_w) = q_w X_w^\alpha$ for some $\alpha \in (0,1)$ then the proportional reward scheme has:*

$$PoA(\alpha) \leq \frac{1}{2}\left(\frac{2}{1-\alpha^{\frac{\alpha}{1-\alpha}}}\right)^{\alpha}(2 - \alpha - \alpha^{\frac{1}{1-\alpha}}) \lessapprox 1.251$$

F.0.2 Exponential Value Functions

If $v_w(X_w) = q_w(1 - p^{X_w})$ then to show that the resulting game is locally (λ, μ) smooth and hence has PoA $\leq \lambda/(1+\mu)$ we need to show the following inequality holds for all $x, y \geq 0$ and $p \in (0,1)$

$$y\frac{1-p^x}{x} + \frac{-xp^x\log(p) - (1-p^x)}{x^2}k(x,y) \geq \lambda(1-p^y) - \mu(1-p^x) \tag{7}$$

This is the condition of Theorem 20 instantiated for our specific function form. Moreover, $k(x,y) = y^2/4$ when $x \geq y/2$ and $x(y-x)$ otherwise.

Theorem 24 *For the family of exponential value functions $v(x) = q(1-p^x)$, for arbitrary constants $q > 0$ and $0 < p < 1$, we have that $PoA \leq 2$.*

We first note that function $v(x) = 1 - e^{-x}$ is *scale-invariant*, i.e. if $v(x)$ is locally (λ,μ)-smooth then $a \cdot v(b \cdot x)$ is also locally (λ,μ)- smooth, for any constants $a > 0$ and $b > 0$. Hence, it suffices to consider the function $v(x) = 1 - e^{-x}$ as our original function is given by $v_w(x) = q_w v(-\log(p_w) \cdot x)$.

We will show that the set of parameters (λ,μ) for which $v(x)$ is locally (λ,μ)-smooth is given by $\mathcal{L}=\{\mathbb{R}^2_+ : \lambda\le\mu \text{ and } \lambda\le 1\}$. That is, we will show that the following two conditions are necessary and sufficient:

$$\lambda \le 1 \tag{8}$$

$$\lambda \le \mu \tag{9}$$

We know that PoA $\le (1+\mu)/\lambda$, thus in view of the above inequalities

$$\text{PoA} \le \max\left\{1+\mu, \frac{1+\mu}{\mu}\right\}, \text{ for every } \mu\ge 0.$$

The minimum value in the right hand side is 2 achieved for $\mu=1$.

For the value function $v(x)=1-e^{-x}$ the local smoothness condition corresponds to:

$$y\frac{1-e^{-x}}{x}+\frac{xe^{-x}-(1-e^{-x})}{x^2}k(x,y)\ge\lambda(1-e^{-y})-\mu(1-e^{-x}), \text{ for every } x,y\ge 0. \tag{10}$$

In the following we separately consider two different cases.

Case 1: $x<y/2$. In this case, we have

$$\begin{aligned}\lambda(1-e^{-y})-(1+\mu) &\le y\frac{1-e^{-x}}{x}+\frac{xe^{-x}-(1-e^{-x})}{x}x(y-x)+\mu(1-e^{-x})\\ &= e^{-x}[y-x-(1+\mu)] := f(x).\end{aligned}$$

The above inequality is thus equivalent to

$$\lambda(1-e^{-y})-(1+\mu)\le \inf_{x\in[0,y/2)} f(x).$$

It is easy to notice that $f'(x)\le 0$ iff $x\le y-\mu$. Thus we can distinguish the following two subcases.
Subcase 1.a: $y\ge 2\mu$. In this case $y/2\le y-\mu$, and thus $f'(x)\le 0$, for $0\le x<y/2$. Hence,

$$\inf_{x\in[0,y/2)} f(x)=f(y/2)=e^{-y/2}[\frac{1}{2}y-(1+\mu)].$$

It follows that we must have $\lambda(1-e^{-y})-(1+\mu)\le f(y/2)$, for every $y\ge 2\mu$, i.e.

$$\lambda-(1+\mu)\le\inf_{y\ge 2\mu} g(y) \tag{11}$$

where

$$g(y)=\lambda e^{-y}+\frac{1}{2}ye^{-y/2}-(1+\mu)e^{-y/2}.$$

Since $g(0)=\lambda-(1+\mu)\le 0$, it is immediate that for the condition (11) to hold, it is necessary that $g'(0)\ge 0$, which is equivalent to $2\lambda\le\mu+2$. Sufficiency follows by the following arguments. Note that under the condition $g'(0)\ge 0$, the function $g(y)$ has a unique maximum at y^* so that $g'(y)\ge 0$ for $y\le y^*$ and $g'(y)\le 0$, for $y>y^*$. Suppose that it holds $g(y^*)\ge 0$, then we have that condition (11) holds in view of the fact that $\lim_{y\to\infty} g(y)=0$. To contradict, suppose that $g(y^*)<0$, then this would mean that $g'(y)\ge 0$, for every $y\ge 0$, which contradicts existence of y^*. We have thus showed that (11) holds iff

$$2\lambda\le\mu+2$$

which is implied by the stronger condition (8), that is showed to be necessary later.
Subcase 1.b: $y<2\mu$. Suppose first that $y\le\mu$, then $f'(x)\ge 0$, for every $x\ge 0$, and thus $\inf_{x\in[0,y/2)} f(x)=f(0)=y-(1+\mu)$. In the given case condition (10) reads as $\lambda(1-e^{-y})-(1+\mu)\le y-(1+\mu)$, i.e. $\inf_{0\le y\le\mu} g(y)\ge 0$ where $g(y)=y-\lambda(1-e^{-y})$. Since $g(0)=0$, it must hold $g'(0)\ge 0$, which is equivalent to

$$\lambda\le 1.$$

This is the above asserted condition (8), under which it obviously holds $g(y) \geq y-(1-e^{-y}) = e^{-y}-(1-y) \geq 0$, for every $y \geq 0$.

Suppose now $\mu \leq y \leq 2\mu$. In this case $f'(x) \leq 0$, for $0 \leq x \leq y-\mu$, and $f'(x) > 0$, otherwise. Since $\inf_{x\in[0,y/2)} f(x) = f(y-\mu) = -e^{-y+\mu}$, we have the condition,

$$\lambda - (1+\mu) \leq \inf_{\mu \leq y \leq 2\mu} e^{-y}[\lambda - e^{\mu}]$$

which is equivalent to $\lambda(1-e^{-\mu}) \leq \mu$ and this is implied by (9), showed to be necessary in the following.

Case 2: $x \geq y/2$. In this case it is convenient to use the following change of variables $z = y/(2x)$. Notice that $0 \leq z \leq 1$. Condition (10) can be written in the following equivalent form

$$2(1-e^{-x})z + (xe^{-x} - (1-e^{-x}))z^2 \geq \lambda(1-e^{-2xz}) - \mu(1-e^{-x}), \text{ for } x \geq 0 \text{ and } 0 \leq z \leq 1$$

or, alternatively, $f(x,z) \leq 0$, for $x \geq 0$ and $0 \leq z \leq 1$ where

$$f(x,z) = \lambda(1-e^{-2xz}) - \mu(1-e^{-x}) - 2(1-e^{-x})z + e^{-x}(e^{x} - (1+x))z^2.$$

Let us fix $0 \leq z \leq 1$ and consider $g(x) = f(x,z)$. We observe

$$\begin{aligned} g(0) &= 0 \\ g(+\infty) &= \lim_{x\to\infty} g(x) = \lambda - (1+\mu) + (1-z)^2. \end{aligned}$$

From the last fact, notice that it must hold

$$\lambda \leq \mu.$$

This is the above asserted condition (9).

We will consider the first derivative of $g(x)$ which is

$$g'(x) = e^{x}[2\lambda z e^{(1-2z)x} - (\mu + 2z - xz^2)].$$

Since $g(0) = 0$, it must hold $g'(0) \leq 0$, as otherwise $g(x) = f(x,z) > 0$, for some $x > 0$. The condition $g'(0) \leq 0$ is equivalent to $2\lambda z \leq \mu + 2z$, thus, $(\lambda - 1)z \leq \mu/2$ must hold for every $0 \leq z \leq 1$. The condition is trivially true if $\lambda \leq 1$ and otherwise holds true iff $2\lambda \leq \mu + 2$, which is implied by (8).

By inspection of the function $g'(x)$ it is readily observed that under $g'(0) \leq 0$ the function $g(x)$ is decreasing on $[0,a)$ and increasing on (a,∞) where $a \geq 0$ is such that $g'(a) = 0$. Combined with the fact $g(0) = 0$, we conclude that

$$\sup_{x\in[0,\infty)} g(x) = \max(0, g(+\infty)).$$

We already noted that $\lim_{x\to\infty} f(x,z) \leq 0$, for every $0 \leq z \leq 1$ iff $\lambda \geq \mu$. Hence, $f(x,z) \leq 0$ holds for every $x \geq 0$ and $0 \leq z \leq 1$ iff $\lambda \leq \mu$.

G Non-Concave Effort Markets

Corollary 25 *For arbitrary increasing convex project value functions, the Shapley-Shubik sharing scheme has setwise Price of Anarchy at most $k+1$. If functions also satisfy the increasing marginal returns property then the bound becomes k.*

Proof. The Shapley sharing scheme gives to each player the average over all arrival orderings of the players marginal contribution at the arrival time. Observe that there are $(|N_w|-1)!$ possible orderings where player

i arrives last. Moreover, since the value functions are increasing the marginal contributions are nonnegative. Thus:

$$\begin{aligned} u_w^i(\vec{x}_w) &= \frac{1}{|N_w|!}\sum_{\pi\in\Pi} v_w(t_{\pi(i)}) - v_w(t_{\pi(i)-1}) \\ &\geq \frac{1}{|N_w|!}(|N_w|-1)!(v_w(\vec{x}_w) - v_w(0_i,\vec{x}_w^{-i})) \\ &= \frac{1}{|N_w|}(v_w(\vec{x}_w) - v_w(0_i,\vec{x}_w^{-i})) \end{aligned}$$

Thus, the game satisfies the k-approximate Vickrey condition.

Moreover, if the functions satisfy the increasing marginal returns property then we see that if a player j switches from playing zero effort (s_{out}) to playing some positive effort on some projects then any player i that participates on those projects will have an increase in their share. This is because, on any permutation where player j comes before i, player i's marginal contribution increases, since it is a contribution on top of a larger vector. In any permutation where player j comes after player i, player i's marginal contribution is not affected. Thus every term in the summation that gives player i's share is only larger after j puts positive effort. Thus in this case the game is also strongly monotone. ■

Corollary 26 *For arbitrary increasing convex project value functions, sharing proportional to the marginal contribution induces a game with setwise Price of Anarchy at most $k+1$.*

Proof. We just need to observe that: $\sum_{i\in N_w}(v_w(\vec{x}_w) - v_w(0,\vec{x}_w^{-i})) \leq k v_w(\vec{x}_w)$. Therefore:

$$\begin{aligned} \phi_w^i(\vec{x}_w) &= \frac{v_w(\vec{x}_w) - v_w(0_i,\vec{x}_w^{-i})}{\sum_{j\in N_w}(v_w(\vec{x}_w) - v_w(0,\vec{x}_w^{-j}))} v_w(\vec{x}_w) \\ &\geq \frac{v_w(\vec{x}_w) - v_w(0_i,\vec{x}_w^{-i})}{k v_w(\vec{x}_w)} v_w(\vec{x}_w) = \frac{1}{k}(v_w(\vec{x}_w) - v_w(0_i,\vec{x}_w^{-i})) \end{aligned}$$

■

H Axiomatic Approach

Proof of Theorem 11: By the Elimination Axiom for $C = \{i\}$ we have:

$$\forall j \in N : \phi_j^{(N,v)} \leq \phi_j^{(N-i,v)}$$

Summing over all j:

$$\sum_{j\neq i}\phi_j^{(N,v)} \leq \sum_{j\neq i}\phi_j^{(N-i,v)} = v(N-i)$$

Moreover since $\phi_i^{(N,v)} = v(N) - \sum_{j\neq i}\phi_j^{(N,v)}$ we get:

$$\phi_i^{(N,v)} \geq v(N) - v(N-i)$$

Thus the utility that a player gets from each project is at least his marginal contribution. Thereby, the overall utility of a player is at least his marginal contribution to the social welfare. ■