Near Optimal Non-truthful Auctions

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Abstract– In several e-commerce applications, non-truthful auctions have been preferred over truthful weakly dominant strategy ones partly because of their simplicity and scalability. Although non-truthful auctions can have weaker incentive constraints than truthful ones, the question of how much more revenue they can generate than truthful auctions is not well understood. We study this question for natural and broad classes of non-truthful mechanisms, including quasi-proportional sharing and weakly monotonic auctions. Quasi-proportional sharing mechanisms allocate to each bidder i an amount of resource proportional to a monotonic and concave function $f(b_i)$ where b_i is the bid of bidder i and ask for a payment of b_i. Weakly monotonic auctions refer to a more general class of auctions which satisfy some natural continuity and monotonicity conditions.

We prove that although weakly monotonic auctions are much broader and require weaker incentive constraints than dominant strategy auctions, they are not more powerful with respect to the revenue in the setting of selling a single item. Furthermore, we show that quasi-proportional sharing with multiple bidders cannot guarantee a revenue that is larger than the second highest valuation, asymptotically as the number of bidders grows large. However, in a more general single-parameter setting modeled by a downward-closed set system, a version of the proportional sharing mechanism can obtain a constant factor of the optimal social welfare of the game where the highest valuation is replaced by the second highest valuation, which is essentially the best revenue benchmark in the prior-free framework. This is in sharp contrast to weakly dominant strategy mechanisms that cannot achieve better than $\log n$ approximation for this benchmark.

1 Introduction

Revenue maximization is a fundamental problem in mechanism design. Although maximizing net social surplus is usually considered a desirable goal of a mechanism, in practice, it is often the case that the mechanism designer is the seller, who wants to raise the greatest possible revenue from sales. Thus, profit maximization has been one of the central problems of mechanism design in the economics literature and in the new area of algorithmic mechanism design.

For revenue maximizing problems, the following two main classes of mechanisms have been considered in the literature: *truthful (weakly dominant strategy)* and *Bayesian incentive compatible* mechanisms. In the Bayesian settings, it is assumed that private information comes from a *prior* distribution, which is known to the seller (and buyers) and a revenue maximizing mechanism depends on the details of the assumed distribution [20]. This approach has raised the following two well known concerns: first, in many scenarios it is hard or impossible to know the prior distribution; second, even small uncertainty about the prior distribution can cause significant uncertainty about what auction to use. Therefore, a natural and important goal is to design mechanisms that are "detail-free". The recent approach of prior-free truthful mechanism design [8, 9] provides a solution by proposing a "reasonable" worst-case revenue benchmark and trying to design a truthful dominant strategy mechanism that *always* generates a revenue that is at least a constant factor of the benchmark. However, thus far, most known constant factor approximations in this framework are for simple auction settings, where the goods are in unlimited supply and/or the bidders are symmetric. Moreover, because of the strong incentive constraints, many negative results on the set of implementable outcomes and several worst-case upper bounds for the revenue of truthful dominant strategy mechanisms are known [7, 25, 24, 15, 21].[1]

These negative results led naturally to the consideration of different solution concepts other than truthfulness in the prior-free framework. Furthermore, in practice, many mechanisms currently used in the context of computer systems are *not* truthful. Although non-truthful auctions have weaker incentive

[1] These results are based on the dominant strategy incentive constraints only, not on computational constraints.

constraints than truthful auctions, the question of how much more revenue they can generate than truthful auctions is not well understood. We study this question for natural and broad classes of non-truthful mechanisms that accommodate auctions that are widely used in practice, namely, we consider *quasi-proportional sharing* and *weakly monotonic auctions*. Quasi-proportional sharing mechanisms generalize the traditional proportional sharing that is common in resource allocation problems ([14]). This mechanism allocates to each bidder i, an amount of resource proportional to a monotonic (concave) function $f(b_i)$, where b_i is the bid of bidder i, and asks for a payment of b_i. Weakly monotonic auctions refer to a more general class of auctions that includes quasi-proportional sharing and truthful mechanisms as special cases; this class of auctions requires an auction only to satisfy some rather weak continuity and monotonicity conditions (defined in Section 4).

We prove that although weakly monotonic auctions are more general and require weaker incentive constraints than dominant strategy auctions, they are *not* more powerful with respect to the revenue guarantee in the setting of selling a single item. Furthermore, we show that quasi-proportional sharing with multiple bidders cannot guarantee a revenue asymptotically larger than the second highest valuation. However, in a more general single parameter setting, modeled by *a downward-closed set system* (e.g. including single-item auctions, digital good auctions and some combinatorial auctions), we constructed an auction that combines a VCG mechanism and a variant of proportional sharing that guarantees a revenue that is a constant factor of the optimal social welfare of the game where the highest buyer's valuation is replaced by the second highest valuation. For this general setting, we show that no weakly dominant strategy mechanism can achieve better than $\log n$ approximation for this benchmark. Furthermore, for our proportional sharing mechanism, the equilibrium can be computed in polynomial time and there exists a learning dynamics that converges to the equilibrium.

Related Work From a broad view, our work continues the line of research on profit maximization in mechanism design. Design of prior-free mechanisms is an important topic in computer science literature. This approach was first considered by [8] and studied by many thereafter, see the survey [9] and the citations therein.

From a mathematical point of view, our approach is different from the papers cited above in the constraints of the information structure. We assume that bidders can observe and adapt to others' strategies, thus, one can think of our approach as a Nash implementation problem in full information setting. The theory of Nash implementation assumes that bidders have full information about each other's preferences, but the planner (auctioneer) only knows the set of outcomes and does not have any information about the private type of bidders. This framework was first introduced by the work of Maskin [16], who gave an almost complete characterization of the set of implementable outcomes.[2] There is a large body of literature on this topic, see the surveys and the citations in [16, 23]. However, almost all the mechanisms in the Nash implementation literature use the so called *integer game* structure. These mechanisms require bidders to report others' private information and if any two bidders report differently, they receive large penalties. This type of cross-reporting is a well known concern in the implementation theory. Moreover, in our setting these mechanisms can generate many equilibria including ones with arbitrarily small revenue. In the present paper, by focusing on a special class of mechanisms we avoid this issue. Furthermore, the justification of the full information structure in our setting comes from the fact that in repeated auctions (which are commonplace in computer systems and services), bidders can observe others' bids and *might* gradually learn their types. On the contrary, the Nash implementation literature assumes the full information structure to begin with. This is unlike to our setting, as we show that our mechanism has a learning dynamics which alleviates the need for the bidders to know each other's preferences.

[2] Maskin's result holds for the case of more than 2 agents. The complete characterization was given later by Moore and Repullo [19] and Dutta and Sen [4].

In the computer science literature, our paper is not the first one that relaxes the truthful dominant strategy constraints. Babaioff, Lavi and Pavlov [1] considered the concept of implementation in undominated strategies. The focus of their work was, however, on the social welfare and the computational issues of single value combinatorial auctions. Chen, Hassidim and Micali [3] considered the same problem as ours: maximizing revenue in full information setting, however, as in the implementation literature, they also use cross-reporting mechanisms.

The most closely related work to the present paper is [18] and [21]. In [18], Mirrokni, Muthukrishnan and Nadav show that in a single-item auction, quasi-proportional sharing rule can generate better revenue than the traditional second-price auction. They left open the question of what optimum quasi-proportional mechanisms are in the prior-free framework. Our work answers this question by two negative results. First, if there is no restriction on the number of bidders, then quasi-proportional sharing mechanism cannot generate a larger revenue than the second-price auction, asymptotically for large number of bidders. Second, the revenue upper-bound for a larger class of weakly monotonic auctions is the same as for the narrower class of weakly dominant-strategy auctions. In [21], the author proposed a similar mechanism, based on a combination of proportional sharing and a truthful mechanism, which also gives a separation result between truthful dominant strategy auctions and non-truthful auctions. The present paper supersedes [21] in two important aspects. First, the revenue benchmark in [21] is weaker than in this paper; in fact, our new benchmark is essentially the best possible that one would hope for. Second, it is not known whether the equilibrium in [21] can be computed in polynomial time, while the equilibrium of our new mechanism is the optimal point of a convex optimization problem and there is a converging distributed learning dynamics.

Our learning dynamics is derived as a natural subgradient method for solving an associated (primal) convex optimization problem to the resource allocation game studied, where the objective function of this problem incorporates the resource constraints as a cost function. This class of dynamics is in the same spirit to those introduced in [14] and was studied by many thereafter. We prove the convergence of these dynamics to an arbitrary small neighborhood of the Nash equilibrium point under some mild conditions on the resource cost functions.

Structure of The Paper We introduce notation and some preliminarily results in the next section. Section 3 presents our main results on the revenue bounds for the setting of selling a single item, specifically, providing a revenue upper-bound for the quasi-proportional sharing (Theorem 1) and showing that the revenue guarantee of weakly dominant strategy auctions cannot be improved by allowing for more general class of weakly monotonic auctions (Theorem 2). In Section 4 we consider the general setting of downward-closed set systems and present our main results on the revenue guarantees in this setting. In particular, we show that there exists a non-truthful mechanism that can guaranteed a revenue that is a constant factor of the aforementioned benchmark (Theorem 3) while truthful mechanisms cannot do better than $\log n$ approximation (Theorem 4). In Section 5 we conclude. Some missing proofs and our analysis of the learning dynamics are provided in Appendix.

2 Notations and Preliminaries

2.1 The Problem

In this paper we consider auctions for n bidders in single parameter environments modeled by downward-closed set systems previously studied in [10]. More precisely, let $\mathcal{F} \subset 2^{[n]}$ be a set system, then $\mathcal{F}$ is said to be *downward-closed* if for every $F \in \mathcal{F}$ and $G \subset F$, it holds that $G \in \mathcal{F}$. A set $F \in \mathcal{F}$, called a *feasible set*, represents a set of bidders that can be served simultaneously. We consider randomized mechanisms that can output feasible sets $F_k \in \mathcal{F}$ with probability q_k, where $\sum_k q_k = 1$. Given this outcome, let x_i be the probability the service is allocated to bidder i, that is $x_i = \sum_{k:i\in F_k} q_k$. Each bidder i's valuation for receiving the service is v_i. We assume that bidders have quasi-linear utilities expressed as $v_i x_i - p_i$, where p_i is the payment of bidder i.

This class of problems accommodates many auction settings in theory and practice; for example, in single-item auctions, the feasible set system contains singletons, while in combinatorial auctions with single-minded bidders, feasible sets correspond to subsets of bidders seeking disjoint bundles of goods.

Given a set system $\mathcal{F}$, we call vector $\vec{x} = (x_1, x_2, \ldots, x_n)$ a *feasible allocation* if there is a randomized outcome $\{F_k, q_k\}$ such that the corresponding probability of getting the service for each agent i is x_i. It is not hard to see that the set of feasible allocations can be captured by a polyhedron specified by $A\vec{x} \leq \vec{1}$ and $\vec{x} \geq \vec{0}$ where A is a non-negative matrix. See [21] for a formal proof. For example, in the single-item setting, the resource constraint is $\sum_i x_i \leq 1$ while in the setting of digital goods [8], the constraints are $0 \leq x_i \leq 1$, for every i. We can think of this auction setting as a resource allocation problem, where the auctioneer has a resource to allocate to bidders by an auction with the constraints induced by the set system. Therefore, we often call these constraints *resource constraints* in this paper.

2.2 VCG Mechanism

The VCG mechanism for downward-closed set system allocates the service to a feasible set of bidders with the highest total valuation. Each winning bidder ($x_i = 1$) pays the lowest amount such that this bidder can still win with this valuation. Bidders who loose ($x_i = 0$) pay 0.

We will use a version of the VCG mechanism, where instead of allocating the full amount of resource to the winning bidders, we only allocate $x_i = \alpha \leq 1$ to them. We will call this mechanism *VCG with scaling constant* α. Clearly, because of the quasi-linear payoff, if the price per unit good of a winning bidder i is p_i, then with scaling constant α, this bidder needs to pay αp_i.

Notice that in a VCG mechanism, the payoff of a losing bidder is 0, thus we have the following simple observation that we will need later.

LEMMA 1 *If a bidder i does not win in the VCG mechanism and would like to bid higher in order to win, then the payment of this bidder is larger or equal to her valuation v_i.*

2.3 Proportional Sharing Mechanism

As discussed above, the downward-closed set system can be seen as an auction environment where the feasible allocations form a polyhedron. This formulation allows us to use a proportional sharing mechanism for multiple resources. This is a well studied mechanism that can be applied also for general concave utility functions [12]. Here, instead of bidding a single number, each bidder submits a vector of bids, one bid for each individual resource constraint $e \in E$ where E denotes the set of constraints. The disadvantage of this approach for general polyhedron setting is that it is unknown whether the equilibrium is unique and can be found in polynomial time. However, we only use a special case of this general setting for our result.

Specifically, we consider a particular type of constraints such that for every resource constraint $e \in E$ and a subset of bidders I_e, it holds $\sum_{i \in I_e} x_i \leq C$, for the same value $C > 0$ on every resource constraint $e \in E$. Each bidder i is endowed with a linear utility function $U_i(x_i) = v_i x_i$.

To get an intuition for the general proportional sharing mechanism, consider the case of a single resource with the resource constraint $\sum_i x_i \leq C$. Here each bidder i bids and pays value b_i and receives the allocation $x_i = C\frac{b_i}{\sum_j b_j}$. The payoff of user i is then $U_i(C\frac{b_i}{\sum_j b_j}) - b_i$. Taking the derivate of this payoff function we can show that the condition for Nash equilibrium is the following, for some $p > 0$,

$$U_i'(x_i)\left(1 - \frac{x_i}{C}\right) = p \text{ if } x_i > 0 \text{ and } U_i'(0) \leq p \text{ if } x_i = 0. \tag{1}$$

Here $p = \frac{1}{C}\sum_i b_i$, and for linear utility functions, we have $U_i'(x_i) = v_i$. The value p can be understood as the price per unit good because each bidder i needs to pay px_i when receiving the allocation x_i. This mechanism can be generalized to general networks and general polyhedrons [13, 22]. In our special setting, the mechanism is described by the following definition.

DEFINITION 1 *Our environment has multiple constraints of the following form: for each resource constraint $e \in E$, there is a set of bidders I_e and the allocations need to satisfy $\sum_{i\in I_e} x_i \le C$.*

In the mechanism, each bidder i submits a bid b_i^e and a requested amount r_i^e for each constraint e where $i \in I_e$. For constraint e, we use the following allocation. For $i \in I_e$,

- *if $\sum_{k:\, k\in I_e} b_k^e > 0$ then $x_i^e = C\frac{b_i^e}{\sum_{k:\, k\in I_e} b_k^e}$;*
- *If $\sum_{k:\, k\in I_e} b_k^e = 0$ and $\sum_{k:\, k\in I_e} r_k^e \le C$ then $x_i^e = r_i^e$, else, set $x_i^e = 0$.*

A bidder i's payment is $\sum_{e:i\in I_e} b_i^e$ and her final allocation is $x_i = \min_{e:i\in I_e} x_i^e$.

In the game defined above, because all the resources have the same capacity, the condition for Nash equilibrium becomes simpler than that for more general polyhedrons. We can apply the condition of equilibrium for the general case to our environment to get the following lemma.

LEMMA 2 *There always exists a Nash equilibrium for the game defined by Definition 1. An allocation vector $\vec{x}$, a bid vector $\vec{b}$ and a request vector $\vec{r}$ is a Nash solution* if and only if*:*

$$\begin{aligned} U_i'(x_i)(1-\tfrac{x_i}{C}) &= \textstyle\sum_{e:i\in I_e} p^e \quad \text{for } x_i > 0 \\ \text{and } U_i'(0) &\le \textstyle\sum_{e:i\in I_e} p^e \quad \text{for } x_i = 0 \end{aligned} \tag{2}$$

where $p^e = \frac{1}{C}\sum_{i\in I_e} b_i^e$.

Furthermore, the condition above is exactly the condition of the following convex optimization problem

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_i \int_0^{x_i} U_i'(y)(1-\frac{y}{C})dy \\ \text{over} \quad & x_i \ge 0 \text{ for } i = 1,2,\ldots,n \\ \text{subject to} \quad & \textstyle\sum_{i\in I_e} x_i \le C \text{ for } e \in E. \end{aligned} \tag{3}$$

The allocation vector $\vec{x}$ in Nash equilibrium is unique if all utility functions $U_i(x_i)$ are nondecreasing and concave.

In Lemma 2, we stated conditions for general concave utilities U_i. In our setting, the utilities are linear, so we can replace $U_i'(x_i)$ with v_i. Even with general concave utilities, we can see that (3) is trivially implied by (2) because of the KTT conditions. The proof of Lemma 2 is a direct application of [13]. One can show this by deriving from the "first-order" condition for each bidder i's payoff in a similar way as deriving equation (1). We also refer to [12] for a formal proof.

In Section 4 we will combine this mechanism with a VCG mechanism to construct our main auction.

3 Revenue Upper Bounds for Single-Item Sales

We start our investigation of the optimal revenue of non-truthful auctions by considering the case of single-item auctions. In this section, we present our revenue upper bounds for the classes of quasi-proportional auctions and weakly monotonic auctions. This will establish fundamental bounds on the best possible revenue guarantee that may be achieved by the respective classes of auctions.

3.1 Quasi-Proportional Mechanisms

Our first result is for the class of quasi-proportional sharing auctions.

DEFINITION 2 *An auction is said to be a* quasi-proportional sharing auction *if it uses the following allocation rule: given a non-decreasing and concave function $f : \mathbb{R}_+ \to \mathbb{R}_+$, the allocation to each buyer i is given by $x_i = \frac{f(b_i)}{\sum_j f(b_j)}$ and each buyer pays his own bid, i.e. $p_i = b_i$.*

Allocation rules of this form are commonly referred in literature as *success functions* with various regularity properties established in previous work, e.g. [11, 26]. The special case of quasi-proportional sharing where $f(w) = w^r$, for $0 \le r \le 1$, is often referred to as *Tullock auctions or contests*, in recognition of an early work on such allocations [28].

THEOREM 1 *For every increasing function $R : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{v\to\infty} R(v) = \infty$ and every quasi-proportional sharing auction, there exists an integer $n > 1$ and a valuation vector $\vec{v} \in \mathbb{R}_+^n$ such that the revenue in Nash equilibrium is less than $R(v_{(1)})$, where $v_{(1)}$ is the highest buyer's valuation.*

This result shows that no quasi-proportional sharing can generate a revenue that increases with $v_{(1)}$ and tends to infinity as $v_{(1)} \to \infty$, assuming that $v_{(1)}$ is the largest valuation, and all other valuations are fixed. In particular, while Mirrokni et al [18] show that for the case of two bidders, proportional sharing can guarantee a revenue of $\Omega(\sqrt{v_{(1)}})$, using our result, it follows that this cannot be guaranteed in the general case of an unlimited number of bidders. This essentially shows that for quasi-proportional sharing mechanisms, the worst-case revenue is $O(v_{(2)})$, where $O(v_{(2)})$ denotes the second highest buyer's valuation. Interestingly, proportional sharing guarantees a revenue of at least $\frac{1}{2}v_{(2)}$ (simple fact showed in Appendix A.4), and is thus in this sense optimal.

The proof of Theorem 1 is provided in Appendix A.3.

3.2 Weakly Monotonic Auctions

We now study a more general class of mechanisms that we call weakly monotonic auctions. This is a class of auctions that includes both quasi-proportional and anonymous truthful dominant strategy auctions as special cases. An optimal revenue bound for truthful auctions was given by Lu, Teng and Yu [15]. Our result show that the same bound holds for a larger class of mechanisms with weaker incentive constraints. It is surprising that very weak and seemingly trivial properties of mechanisms are enough to give the same revenue upper-bound as for dominant-strategy auctions.

We first introduce some notations. We assume that each bidder i can bid a value $b_i \ge 0$ and the allocation and payment rules are given by non-negative functions $x_i(b_1, b_2, \ldots, b_n)$ and $p_i(b_1, b_2, \ldots, b_n)$, where $\sum_i x_i \le 1$. We assume that x_i and p_i are differentiable almost everywhere. We further assume that if $b_i = 0$, then $x_i(\vec{b}) = p_i(\vec{b}) = 0$. That is, bidders can always bid 0 to satisfy the individual rationality constraint.

Now, consider a bidder i with the valuation $v_i \ge 0$ and given bid vector of other bidders $\vec{b}_{-i}$. Bidder i's best response is the set of values b_i that maximize the payoff $v_i x_i(b_i, \vec{b}_{-i}) - p_i(b_i, \vec{b}_{-i})$. If $v_i = 0$, then $b_i = 0$ is a best response regardless of $\vec{b}_{-i}$. For each bidder i, we call the function $f_i(v_i, \vec{b}_{-i})$, the *best response function* of bidder i that depends on the valuation v_i, and the bids of others $\vec{b}_{-i}$ if $f_i(0, \vec{b}_{-i}) = 0$ and $f_i(v_i, \vec{b}_{-i})$ is one of the elements in the best response set.

DEFINITION 3 *We call a mechanism (given by functions $\{x_i, p_i\}$) weakly monotonic if the following conditions are true:*

(A1) For every $\vec{b}_{-i}$, bidder i has a best response function $f_i(v_i, \vec{b}_{-i})$ that is continuous and differentiable with respect to the valuation v_i.

(A2) Given any $\vec{b}$ and the indexes i and j such that $b_i < b_j$, there exist $\epsilon > 0$ and b'_i, both depending on $\vec{b}_{-i}$, such that $b'_i \geq b_j$ and $x_i(b'_i, \vec{b}_{-i}) > x_i(b_i, \vec{b}_{-i}) + \epsilon$.

(A3) Given a bidder i and $\vec{b}_{-i}$, then there exists $B > 0$ such that if $b_i > B$, then $p_i(\vec{b}) \geq p_j(\vec{b})$ for every $j \neq i$.

Remark Among the conditions above, condition (A1) may appear to be the strongest one. However, without the continuity property in the best response function, there may not exist a pure Nash equilibrium. Therefore, (A1) is a natural condition to guarantee existence of pure Nash equilibrium. Condition (A1) holds for many known auctions, including truthful auctions and quasi-proportional mechanisms. For truthful auctions, $f_i(v_i, \vec{b}_{-i}) = v_i$ is a valid continuous best response function; for quasi-proportional sharing mechanisms, the bidder i's payoff function is a concave function of b_i and it is not hard to see that the unique best response for each v_i gives a continuous function (and differentiable almost everywhere). More generally, if x_i and p_i satisfy some differentiability and continuity properties and the payoff function for each bidder always has a unique optimal point, then (A1) is automatically true.

Conditions (A2) and (A3) can be understood as conditions for "monotonicity" properties of a mechanism. In particular, (A2) means that if a bidder i's bid is not the highest, this bidder can always increase his bid for some amount to increase the resource he gets by at least ϵ. (A3) says that if bidder i keeps increasing his bid, he will sooner or later pay more than others.

Our revenue upper-bound for weakly monotonic auctions is stated in the following theorem.

THEOREM 2 *For every weakly monotonic auction satisfying assumptions (A1)-(A3) and every constant $\alpha > 0$, there exists a valuation vector $\vec{v}$ and a bid vector $\vec{b}$ that is in Nash equilibrium of the game and the revenue in this equilibrium is less than $\alpha \frac{v_{(1)}}{\log(v_{(1)}+1)}$, where $v_{(1)}$ is the highest buyer's valuation.*

Remark The result implies that the revenue guarantee for the class of weakly monotonic auctions is $o(\frac{v_{(1)}}{\log(v_{(1)}+1)})$, which is the same as for the narrower class of truthful auctions as the bound coincides with that of Lu, Teng and Yu [15], which was derived for the class of truthful auctions. Note that [15] also established that there exists a truthful auction that achieves the asserted revenue bound. Therefore, perhaps surprisingly, we observe that relaxing the solution concept cannot guarantee larger revenue.

It is noteworthy that the above theorem also resolves an open question posed by Mirrokni, Muthukrishnan and Nadav [18] on the existence of quasi-proportional sharing that would guarantee $\Omega(v_{(1)})$ revenue. Our result tells that this is impossible as quasi-proportional sharing is a weakly monotonic auction.

In the rest of this section we prove Theorem 2.

Proof of Theorem 2 We will prove the revenue upper-bound for the game with two bidders. The main idea is to consider a valuation vector (v_1, v_2) such that the bid vector of the form $(b_1, 1)$ is an equilibrium. We will show that for every v_1 large enough such v_2 and b_1 always exist. In this case, b_1 is a best response of bidder 1 given $b_2 = 1$ is fixed. This will allow us to establish a relation between v_1 and the payment $p_1(b_1, 1)$ and the allocation $x_1(b_1, 1)$ via the envelope theorem (Theorem 2 of [17]). The relation is similar to the formulation in truthful auctions. This is the key step in proving our upper bound.

We first need the following lemma, whose proof is provided in Appendix A.1. We note that to prove this lemma, we need to use conditions (A1) and (A2) in Definition 3.

LEMMA 3 *Given a weakly monotonic auction there exists* $T_0 > 0$ *such that the following is true. For every* $t > T_0$ *there exist* $b_1(t) \geq 1$ *and* $v_2(t)$ *such that* $(b_1, b_2) = (b_1(t), 1)$ *is in equilibrium of the auction where bidders' valuation vector is* $(v_1, v_2) = (t, v_2(t))$.

To prove Theorem 2, we will assume a contradiction that the revenue obtained in every equilibrium is at least $\alpha \frac{v_{(1)}}{\log(v_{(1)}+1)}$.

Using this assumption and condition (A3), we can show that there exists $T > T_0$ such that if $t \geq T$ then $p_1(b_1(t), 1) > p_2(b_1(t), 1)$. This is true because by condition (A3) there exists B such that if $b_1(t) > B$, then $p_1(b_1(t), 1) > p_2(b_1(t), 1)$. All we need to show now is that if t is large enough, then $b_1(t) > B$. Assume that $b_1(t) < B$ for all t, then the total revenue $p_1(b_1(t), 1) + p_2(b_1(t), 1)$ is bounded while $v_{(1)} > t$ is unbounded. Therefore, the total revenue cannot always be at least $\alpha \frac{v_{(1)}}{\log(v_{(1)}+1)}$.

We now consider $v_1 = t \geq T$. Because $p_1(b_1(t), 1) > p_2(b_1(t), 1)$, $p_1(b_1(t), 1)$ is at least the total revenue, which by the contradiction assumption is at least $\alpha \frac{v_{(1)}}{\log(v_{(1)}+1)}$. Thus,

$$p_1(b_1(t), 1) \geq \frac{\alpha}{2} \frac{v_{(1)}}{\log(v_{(1)} + 1)} \geq \frac{\alpha}{2} \frac{t}{\log(t + 1)}. \tag{4}$$

We now derive a relation between $p_1(b_1(t), 1)$, $v_1(b_1(t), 1)$ and t. We define the following two-variable function, which represents the payoff for buyer 1,

$$u(t, w) = t \cdot x_1(w, 1) - p_1(w, 1).$$

Let $V(t) = \max_w u(t, w)$ and $w^*(t) = \text{argmax}_w u(t, w)$. Because the best response of bidder 1 is $b_1(t)$, we have $w^*(t) = b_1(t)$. We will use the envelope theorem for the function $V(t) = \max_w u(t, w)$, which allows us to write[3]

$$V(s) = V(T) + \int_T^s \frac{\partial u(t, w^*(t))}{\partial t} dt.$$

Now, $w^*(t) = b_1(t)$ and, thus $\frac{\partial u(t, w^*(t))}{\partial t} = x_1(b_1(t), 1)$. In the remainder, we slightly abuse the notation by writing $x_1(t)$ and $p_1(t)$ in lieu of $p_1(b_1(t), 1)$ and $x_1(b_1(t), 1)$. From the envelope formula above, we have

$$V(s) = V(T) + \int_T^s x_1(t) dt.$$

We know that $V(s)$ is the payoff of bidder 1 when her valuation is s. Thus, $V(s) = s x_1(s) - p_1(s)$, which implies

$$p_1(s) = s x_1(s) - V(s) = s x_1(s) - \int_T^s x_1(t) dt - V(T). \tag{5}$$

One can see that (5) has a similar formulation as in truthful auctions. In fact such relation in truthful auction can also derived from the envelope theorem (see [17]). Now, functions $p_1(s)$ and $x_1(s)$ are differentiable almost everywhere, thus, one can take the derivative of (5) to obtain

$$p_1'(s) = s x_1'(s) \text{ almost everywhere, which implies } x_1'(s) = \frac{p_1'(s)}{s} \text{ almost everywhere.}$$

By taking the integral of $x'(s)$ from T to ∞ we have

$$1 > \int_T^\infty x'(s) ds = \int_T^\infty \frac{p_1'(s)}{s} ds = \frac{p_1(s)}{s}\Big|_T^\infty + \int_T^\infty \frac{p_1(s)}{s^2} dv(s) \geq \int_T^\infty \frac{p_1(s)}{s^2} dv(s). \tag{6}$$

However, because of (4) we have $p_1(s) \geq \frac{\alpha}{2} \frac{s}{\log(s+1)}$, for every $s \geq T$. Thus,

$$\int_T^\infty \frac{p_1(s)}{s^2} ds \geq \frac{\alpha}{2} \int_T^\infty \frac{1}{s \log(s + 1)} ds = \infty$$

which contradicts (6). By this, we complete the proof. ∎

[3]The function $u(\cdot, w)$ is applied to $f(x, \cdot)$ in Theorem 2 of [17].

4 General Setting

In this section, we consider a general auction setting which is captured by a downward-closed set system $\mathcal{F}$ as introduced in Subsection 2.1. Our revenue benchmark is defined as follows.

DEFINITION 4 *Given a downward-closed set system $\mathcal{F}$ and a valuation vector $\vec{v}$, let $\vec{v}^{(2)}$ be the valuation vector that is identical to $\vec{v}$ except for the highest valuation replaced by the second highest valuation. Our revenue benchmark is then defined as follows:*

$$\mathcal{S}_2(\vec{v}) = \max_{F \in \mathcal{F}} \sum_{i \in F} v_i^{(2)}.$$

In other words, the revenue benchmark is the optimal social welfare under the distorted valuation vector $\vec{v}^{(2)}$. For example, in the single-item auction, the benchmark gives the second highest valuation and in the digital goods auction setting, it essentially gives the total valuation except for the highest one valuation. Note that $\mathcal{S}_2$ is essentially the best revenue benchmark that one may hope for.[4]

Our main result in this section is to provide a mechanism that is $\mathcal{S}_2(\vec{v})$ competitive. One important feature of our mechanism is that each bidder needs to bid a roughly $n/2$ dimensional vector, in contrast to the single-item auctions discussed in the previous section, where each bidder submits a single scalar bid. In this section, we also show that truthful dominant strategy mechanisms can only give a $\log n$ approximation of $\mathcal{S}_2$. Our results are given formally in the following two theorems.

THEOREM 3 *There exists a mechanism whose allocation vector is always feasible and in the Nash equilibrium guarantees the revenue of at least $\mathcal{S}_2(\vec{v})/32$. Furthermore, the Nash equilibrium can be found in polynomial time and there exists a distributed learning that converges to an arbitrarily small neighborhood of the Nash equilibrium.*

THEOREM 4 *Truthful mechanisms cannot generate a revenue larger than $\Omega(\frac{\mathcal{S}_2(\vec{v})}{\log n})$. Furthermore, the $\log n$ factor is tight, that is, there exists a truthful mechanism that generates a revenue of $\Omega(\frac{\mathcal{S}_2(\vec{v})}{\log n})$.*

The proof of Theorem 4 is provided in Appendix A.5. The revenue upper bound is proved for the special case of digital goods in a similar way to the proof in [21].

In the rest of this section we will prove Theorem 3. We will only focus on defining the mechanism and proving the bound on the revenue while the learning issue is discussed in Appendix B.

Outline of the main idea The main idea of our mechanism leverages an observation about the revenue of proportional sharing mechanisms that can be intuitively described as follows. Given a Nash equilibrium allocation vector $\vec{x}$ of the proportional sharing mechanism for a set of resource constraints E, let N_0 be the set of bidders who obtain a "small" fraction of resource, then the revenue is at least a constant factor of the maximum social welfare for the set of bidders N_0 subject to the constraints E. The idea is then to reduce the available resource to create more competition among the bidders such that in an equilibrium of proportional sharing, "many" bidders receive a small fraction of resource. However, at the same time, we do not want to reduce the resource too much so that we can still generate large enough revenue. We will take a random partition of the bidders into 2 groups and add the extra constraint that only sets of bidders in the same group are feasible. In other words, we take a random partition of bidders and consider the new set system induced by the original set system and the partition.

[4]The only case where $\mathcal{S}_2(\vec{v})$ does not give a high value is when $v_{(1)} \gg \mathcal{S}_2(\vec{v})$ where $v_{(1)}$ is the highest valuation. However, once we have a competitive mechanism for $\mathcal{S}_2(\vec{v})$, it is easy to construct a mechanism that is $\max\{\mathcal{S}_2(\vec{v}), \frac{v_{(1)}}{\log(v_{(1)}+1)}\}$ competitive by combining with the truthful mechanism of Lu et al [15] with a positive probability.

We will prove a technical lemma showing that the expected maximum social welfare in both groups of the partition is at least a constant factor of the $\mathcal{S}_2$ benchmark. This is a probability lemma that is of interest in its own right.

Now given the above described idea and the technical lemma (Lemma 5), one can use the general proportional sharing mechanism directly for the downward-closed set system. However, it is not known that there exist converging dynamics for this general setting. We would like to design an auction with a simpler structure. The idea is to run VCG mechanism first on the two set systems induced by the partition, which gives us two winning sets. We then use proportional sharing for the two winning sets. The constraints of this environment are now simpler and can be seen as in the framework of Definition 1. However, one problem is that by running proportional sharing after VCG, truth telling is no longer an equilibrium strategy. This is because the loosing bidders of VCG can try to bid higher to enter the second stage, in which they might get the resource for cheaper prices. To fix this problem, we will modify the proportional sharing mechanism by asking each bidder to make an extra payment, equal to the product of the price per unit good (unit price) in the first-stage VCG mechanism and the amount of resource received in the second stage. This way, we can show that bidding truthfully is a Nash equilibrium in the first stage of our mechanism.

The Mechanism

As discussed above, we first define the proportional mechanism with reserve unit price that we will use in our main mechanism.

DEFINITION 5 Proportional Sharing with Reserve Unit Prices ($\vec{\tau}$). *The mechanism is for a set of resource constraints $\sum_{i \in I_e} x_i \le C$, for $e \in E$, and given reserve unit price τ_i for each bidder i. The mechanism is described as in Proportional Sharing Mechanism Definition 1, but we add an extra payment for each bidder i for the amount $\tau_i x_i$. That is, each bidder i needs to pay $\sum_{e: i \in I_e} b_i^e + \tau_i x_i$ to receive x_i.*

Remark In proportional sharing with reserve unit prices, the payoff of bidder i is $v_i x_i - \sum_e b_i^e - \tau_i x_i = (v_i - \tau_i)x_i - \sum_e b_i^e$. One can think of this game as if bidder i's valuation were $v_i - \tau_i$, thus, the Nash equilibrium condition and the result on the convergence of the dynamics in Appendix B do not change.

We are now ready to define the mechanism.

DEFINITION 6 (Two Stage Mechanism (α)) *Given a parameter $\alpha < \frac{1}{2}$, the mechanism is described as follows.*

1. *Randomly partition the set of bidders into two parts N_1 and N_2 by flipping a fair coin for each bidder.*
2. *Run VCG mechanism on two set systems induced by N_1 and N_2 with the scaling constant α.[5] Let W_1 and W_2 be the two winning sets. Let $\alpha \cdot \tau_i$ be the payment of bidder i in this mechanism.*
3. *For the set of constraints $x_i + x_j \le C = 1 - 2\alpha$, for all $i \in W_1$ and $j \in W_2$, run proportional sharing with reserve unit price τ_i for each bidder i as defined in Definition 5 .*

To give an intuition, we describe the proportional sharing for the particular set of constraints $x_i + x_j \le C = 1 - 2\alpha$, for all $i \in W_1$ and $j \in W_2$. In this case, the proportional sharing mechanism requires each bidder $i \in W_1$ to submit $b_{i,j}$ for all $j \in W_2$ and vice versa. Denote $x_{i,j} = C \frac{b_{i,j}}{b_{i,j} + b_{j,i}}$. The

[5]Recall that a winning bidder i receives $x_i = \alpha$ instead of 1. (See Subsection 2.2).

final allocation that bidder i receives is $x_i = \min_{j \in W_2} x_{i,j}$, for $i \in W_1$, and vice versa with respect to W_1 and W_2.

We first show the following lemma about the structure of the equilibrium of this mechanism.

LEMMA 4 *The outcome of the mechanism is feasible. Bidding truthfully in the first stage is an equilibrium and a bid vector $\vec{b}$ and an allocation $\vec{x}$ in the second stage are in equilibrium if and only if*

$$\begin{aligned} (v_i - \tau_i)(1 - \tfrac{x_i}{1-2\alpha}) &= \textstyle\sum_{e:i \in I_e} p^e \quad \text{if } x_i > 0 \\ \text{and } v_i - \tau_i &\leq \textstyle\sum_{e:i \in I_e} p^e \quad \text{if } x_i = 0. \end{aligned} \tag{7}$$

The revenue of the mechanism is $\sum_i \alpha\tau_i + (1-2\alpha)\sum_e p^e + \sum \tau_i x_i$.

In this lemma, the constraints are $x_i + x_j \leq C = 1 - 2\alpha$, for $i \in W_1$ and $j \in W_2$, we can think of $e = (i,j)$ and $p^{i,j}$ is the fraction $\frac{b_{i,j}+bj,i}{1-2\alpha}$ (see Definition 1 for a more formal description).

Proof. To show that the outcome allocation is feasible, we observe that in the first stage of the mechanism, bidders in the feasible sets W_1 and W_2 are given $\alpha \leq 1/2$, and in the second stage each bidder $i \in W_1$ gets x_i and each bidder $j \in W_2$ gets x_j. We need to make sure that the total amount of resource that i and j get is at most 1. But we always have $x_i + \alpha + x_j + \alpha \leq 1$ because in the second stage of the mechanism the constraints are $x_i + x_j \leq 1 - 2\alpha$, for $i \in W_1$ and $j \in W_2$.

Consider first a loosing bidder i in the first stage of the mechanism. Because in the first stage, we run a VCG mechanism, if bidder i would like to increase his bid to be in the winner set, he needs to pay a unit price that is at least his valuation. (See Lemma 1). That is, $\tau_i \geq v_i$. However, in the second stage of the mechanism, he needs to pay more than $\tau_i x_i$ for receiving x_i, thus bidder i's payoff is negative. Therefore, bidder i is better off bidding his true valuation.

Consider a winning bidder j of the first stage of the mechanism. By decreasing his bid, bidder j might become a "loser" and thus cannot participate in the second stage, which makes his payoff 0. In other cases, the reserve unit price τ_i and the set of bidders that he needs to compete with in the second stage are independent of his bid. By this, we show that bidding truthfully in the first stage is a Nash equilibrium.

The proof of the condition for Nash equilibrium in the proportional sharing with reserve prices is exactly the same as in the proof of Theorem 2. The formula for the revenue follows trivially from the mechanism. ■

The Revenue of Two Stage Mechanism

We begin with the analysis of the revenue obtained in Nash equilibrium of the mechanism by the following probability lemma, which is a key to establish our result.

LEMMA 5 *Given values $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, consider a random partition of these values into 2 groups (by flipping an independent fair coin for each a_i). Denote the sums of a_i in each group by the random variables S_1 and S_2. It is true that $E(\min\{S_1, S_2\}) \geq \frac{1}{6}\sum_{i=2}^{n} a_i$.*

Remark The proof of this lemma is provided in Appendix A.2. Note that it is important that the lower bound of $E(\min\{S_1, S_2\})$ in the formula above does not include a_1. It is easy to see that the lemma otherwise will not be true; for example, consider $a_1 = 1$ and $a_i = 0$ for $i \geq 2$. The proof of the lemma uses a subtle inductive argument, which may be of interest in its own right. In fact, the lemma is a generalization of the technical lemma in the work of Goldberg et al [8] for digital good auctions.

We are now ready to prove our main result.

Proof of Theorem 3 We will show that the revenue of Two Stage Mechanism (α) with $\alpha = 1/4$ is at least $\mathcal{S}_2(\vec{v})/32$.

In this proof we use the following notation. Given a set A of bidders and a valuation vector $\vec{v}$, we denote the total valuation $\sum_{i\in A} v_i$ by $v(A)$. We also assume that $v_1 \geq v_2 \geq \ldots \geq v_n$. Let O be a feasible set such that $v^{(2)}(O)$ is maximum, i.e. $v^{(2)}(O) = \mathcal{S}_2(\vec{v})$. As described in the definition of the mechanism, the sets N_1 and N_2 form a random partition of the set of bidders, and W_1 and W_2 are the winning sets of the set systems induced by N_1 and N_2, respectively. Assume that under this partition, the set O is divided into O_1 and O_2.

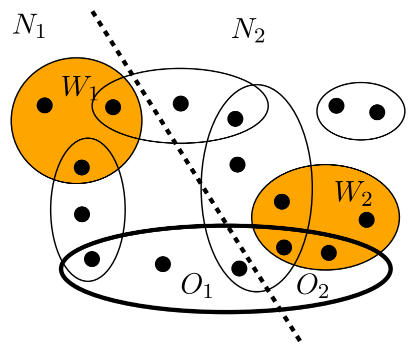

Figure 1: Partitioning of the set system.

We have

$$v(W_1) \geq v(O_1) \geq v^{(2)}(O_1) \text{ and } v(W_2) \geq v(O_2) \geq v^{(2)}(O_2).$$

Therefore,

$$\min\{v(W_1), v(W_2)\} \geq \min\{v^{(2)}(O_1), v^{(2)}(O_2)\}. \tag{8}$$

Assume that $v^* = \max_{i\in O} v_i^{(2)}$, we know that $v^* \leq v_2$ and because of Lemma 5, we have

$$E(\min\{v^{(2)}(O_1), v^{(2)}(O_2)\}) \geq \frac{1}{6}(v^{(2)}(O) - v^*) \geq \frac{1}{6}(v^{(2)}(O) - v_2). \tag{9}$$

On the other hand, it is not hard to see that

$$E(\min\{v(W_1), v(W_2)\}) \geq \frac{1}{2}v_2. \tag{10}$$

Combining (8), (9) and (10) we have

$$8E(\min\{v(W_1), v(W_2)\}) \geq v^{(2)}(O). \tag{11}$$

In the rest of the proof we show that with $\alpha = 1/4$, the mechanism defined in Definition 6 generates a revenue of at least $\frac{1}{4}\min\{v(W_1), v(W_2)\}$. With this and (11), the main theorem will be proved.

Now consider the second stage of the game. Let $\vec{x}$ be the allocation in the equilibrium, according to Lemma 4, we have

$$\begin{array}{rl} (v_i - \tau_i)(1 - \frac{x_i}{1-2\alpha}) = \sum_{e:i\in I_e} p^e & \text{if } x_i > 0 \\ \text{and } v_i - \tau_i \leq \sum_{e:i\in I_e} p^e & \text{if } x_i = 0. \end{array} \tag{12}$$

Let N_0 be the set of bidders such that $x_i \leq \frac{1-2\alpha}{2}$. For each $i \in N_0$, we have

$$(v_i - \tau_i)\frac{1}{2} \leq (v_i - \tau_i)\left(1 - \frac{x_i}{1-2\alpha}\right) \leq \sum_{e:i\in I_e} p^e. \tag{13}$$

Now, our constraints are $x_i + x_j \le 1 - 2\alpha$ for all $i \in W_1$ and $j \in W_2$, therefore, either all x_i for $i \in W_1$ or all x_j for $j \in W_2$ are less than $\frac{1-2\alpha}{2}$. Also, because our constraints are of the form $x_i + x_j \le 1 - 2\alpha$, for $i \in W_1$ and $j \in W_2$, we can identify e with a pair i, j. Therefore, we can rewrite (13) for either $i \in W_1$ or $j \in W_2$, which give us

$$\text{either } \sum_{j \in W_2} p^{i,j} \ge \frac{1}{2}(v_i - \tau_i) \ \ \forall\, i \in W_1 \text{ or } \sum_{i \in W_1} p^{i,j} \ge \frac{1}{2}(v_j - \tau_j) \ \ \forall\, j \in W_2.$$

In either case by summing over the other index we obtain

$$\text{either } \sum_{(i,j) \in W_1 \times W_2} p^{i,j} \ge \frac{1}{2} \sum_{i \in W_1} (v_i - \tau_i) \text{ or } \sum_{(i,j) \in W_1 \times W_2} p^{i,j} \ \ge \frac{1}{2} \sum_{j \in W_2} (v_j - \tau_j).$$

Using Lemma 4, the revenue R obtained in *both* stages of the mechanism satisfies

$$R = \alpha \sum_i \tau_i + (1 - 2\alpha) \sum_{(i,j) \in W_1 \times W_2} p^{i,j} \ + \sum_i \tau_i x_i \ge \alpha \sum_i \tau_i + (1 - 2\alpha) \sum_e p^e.$$

Therefore, for $\alpha = 1/4$, we have

$$R \ge \frac{1}{4} \left(\sum_i \tau_i + \min \left\{ \sum_{i \in W_1} v_i - \tau_i, \sum_{i \in W_2} v_i - \tau_i \right\} \right) \ge \frac{1}{4} \min \left\{ \sum_{i \in W_1} v_i, \sum_{i \in W_2} v_i \right\}.$$

By this and (11), we complete the proof. ■

5 Conclusion

We studied the optimal auction design in the prior-free framework. We showed that for selling a single item, allowing for optimal auction design within a wide class of weakly monotonic auctions cannot yield a larger revenue than within the narrower class of truthful auctions. On the contrary, in a more general setting of downward-closed set systems, if we allow bidders to submit vector bids, then there exists a mechanism that obtains a constant factor of the optimal social welfare in the game where the highest valuation is replaced by the second highest valuation. This mechanism is in a sense “learnable” by polynomial algorithms and does not require any cross-reporting used in the implementation theory. The benchmark is essentially the best revenue benchmark in the prior-free framework. This stands in a sharp contrast to weakly dominant strategy mechanisms that cannot achieve better than $\log n$ approximation for this benchmark.

There are several interesting directions for future work. For example, could one obtain a tight revenue upper-bound for the case of selling of *multiple* indistinguishable items using weakly monotonic auctions? Another interesting question is: What is the optimal non-truthful auction if each bidder is only allowed to submit a vector bid of dimension $k \ge 1$ that is independent of the number of bidders?

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Appendix

A Missing Proofs

A.1 Proof of Lemma 3

Recall the condition (A2) in Definition 3 for the bid $(b_1, 1)$, where $b_1 < 1$: there exist $\epsilon > 0$ and b_1', which does not depend on b_1, such that $b_1' \geq 1$ and $x_1(b_1', 1) > x_1(b_1, 1) + \epsilon$. Let

$$T_0 = \frac{p_1(b_1', 1)}{\epsilon}.$$

Given t, let $b_1(t)$ be the best response of bidder 1 when his valuation is $v_1 = t$ and bidder 2 bids 1. Assume that there exists $t > T_0$ such that $b_1(t) < 1$. Now, the payoff of bidder 1 when bidding $b_1(t)$ is

$$t \cdot x_1(b_1(t), 1) - p_1(b_1(t), 1) \leq t \cdot x_1(b_1(t), 1). \tag{14}$$

On the other hand, the payoff of bidder 1 when bidding b_1' is

$$t \cdot x_1(b_1', 1) - p_1(b_1', 1) > t \cdot (x_1(b_1(t), 1) + \epsilon) - p_1(b_1', 1). \tag{15}$$

Clearly when $t \cdot \epsilon > p_1(b_i', 1)$, then the right-hand side in (15) is larger than the right-hand side in (14), which implies a contradiction because $b_1(t)$ were the value to maximize bidder 1's payoff. Thus, for $t > T_0$, $b_1(t) \geq 1$.

Clearly, the argument above can be generalized to the prove following general argument. Let $f_1(v_1, b_2)$ is a best response function of bidder 1 when his valuation is v_1 and bidder 2 bids b_2. Assume $f_1(v_1, b_2) < b_2$, then there exist V, depending on b_2, such that if $v_1' > V$ then $f_1(v_1', b_2) \geq b_2$. Because of the symmetry, this argument holds if we exchange the role of bidder 1 and bidder 2. Thus we can apply this argument to bidder 2 best response to the bid $b_1(t) \geq 1$.

In particular, for a given v_2, consider $f_2(v_2, b_1(t))$ (the best response of bidder 2 if his valuation is v_2, and bidder 1 bids $b_1(t)$). If $f_2(v_2, b_1(t)) < b_1(t)$, then there is v_2' such that $f_2(v_2', b_1(t)) \geq b_1(t) \geq 1$. In both cases there exist v such that $f_2(v, b_1(t)) \geq 1$. But we also know that $f_2(v, b_1(t)) = 0$ and f_2 is continuous. Therefore, there exists a value, $v_2(t)$, such that $f_2(v_2(t), b_1(t)) = 1$.

Thus, also because $f_1(t, 1) = b_1(t)$, we have when $(v_1, v_2) = (t, v_2(t))$, the bid $(b_1, b_2) = (b_1(t), 1)$ is in an equilibrium. This proves the lemma. ■

A.2 Proof of Lemma 5

To prove the lemma, we will use an inductive argument. Note that we will need to consider a more general setting to make the induction argument work. Consider the following n random variables $X_1, X_2, \ldots, X_n$, where X_i is either b_i or c_i each with 1/2 probability, where b_i and c_i are assumed to be positive real values. Without loss of generality, we assume $0 \leq b_i \leq c_i$. Indeed, we have $E(\sum_i X_i) = \sum_i \frac{b_i + c_i}{2}$. We would like to bound the conditional expectation of $\sum_i X_i$ given that $\sum_i X_i$ is less than $\sum_i \frac{b_i + c_i}{2}$. Let us define

$$V(X_1, X_2, \ldots, X_n) = E(\sum_i X_i | \sum_i X_i < \sum_i E(X_i)). \tag{16}$$

Note that in the case $b_i = 0$ and $c_i = a_i$, we have $E(\min\{S_1, S_2\}) \geq V(X_1, X_2, \ldots, X_n)$ for S_1 and S_2 defined in our lemma, where the equality holds provided that $\sum_i X_i = E(\sum_i X_i)$ with probability 0.

We will now show that we can replace any two random variables, without loss of generality, say X_1 and X_2 by a single variable Y of the same type: $Y \in \{b, c\}$ each with 1/2 probability such that $E(Y) = E(X_1 + X_2)$ and $V(Y, X_3, \ldots, X_n) \leq V(X_1, X_2, \ldots, X_n)$.

To see this, we will expand $V(X_1, X_2, \ldots, X_n)$ with respect to the values of X_1 and X_2. To simplify the formulas, let $S = \sum_{i=3}^n X_i$, and $m = \sum_{i=1}^n E(X_i) = \sum_{i=1}^n \frac{b_i+c_i}{2}$. We have

$$\begin{aligned} V(X_1, X_2, \ldots, X_n) &= \frac{1}{4}E(S + c_1 + c_2 | S + c_1 + c_2 < m) + \frac{1}{4}E(S + b_1 + b_2 | S + b_1 + b_2 < m) + \\ &\quad + \frac{1}{4}E(S + b_1 + c_2 | S + b_1 + c_2 < m) + \frac{1}{4}E(S + c_1 + b_2 | S + c_1 + b_2 < m) \\ &= \frac{1}{4}\alpha_1 + \frac{1}{4}\alpha_2 + \frac{1}{4}\alpha_3 + \frac{1}{4}\alpha_4 \end{aligned}$$

where α_1, α_2, α_3 and α_4 denote the conditional expectations in the above equation in the respective order. Consider the following simple optimization problem:

$$\begin{aligned} \text{minimize} \quad & F(x) = x\alpha_1 + x\alpha_2 + (1/2 - x)\alpha_3 + (1/2 - x)\alpha_4 \\ \text{over} \quad & 0 \le x \le 1/2. \end{aligned}$$

Since $F(x)$ is a linear function, the minimum will take value at either $x^* = 0$ or $x^* = 1/2$. If $x^* = 0$, then we consider the random variable Y taking either value $b_1 + c_2$ or value $b_2 + c_1$ with $1/2$ probability. Similarly, if $x^* = 1/2$, then we consider the random variable Y taking either value $b_1 + b_2$ or value $c_1 + c_2$ with $1/2$ probability. In both cases, we have $E(Y) = E(X_1 + X_2)$ and $F(x^*) = V(Y, X_3, \ldots, X_n)$. Therefore, we can always replace X_1, X_2 with Y such that

$$V(Y, X_3, \ldots, X_n) = F(x^*) \le F(1/4) = V(X_1, X_2, \ldots, X_n).$$

We are now ready to prove the lemma. First, observe that it is without loss of generality to prove the lemma for the case $a_1 = a_2$. This is because by reducing a_1 to a_2, we can only decrease the value of $E(\min\{S_1, S_2\})$, while the value $\sum_{i=2}^n a_i$ remains unchanged. As noted above, if we let A_i be a random variable taking values 0 and a_i each with probability $1/2$, then $E(\min\{S_1, S_2\}) \ge V(A_1, A_2, \ldots, A_n)$.

We will use the inductive argument above to gradually merge the random variables A_i until we are left with only 2 variables whose means are close to each other. Now, because two random variables are merged into a new one such that the total of the means does not change, we can think of this merging process as an algorithm for putting jobs of different sizes together. We would like to obtain only 2 variables at the end, so this is a scheduling problem for two machines. It is well known (straightforward to see) that if we apply a greedy algorithm that schedules jobs in decreasing order of sizes to the machine with lower load, and with the condition that the two largest jobs are equal, $a_1 = a_2$, then the load of one machine is at most twice the load of the other. In other words, we can find an algorithm to merge the random variables such that at the end we obtain M_1 and M_2, where M_i is either α_i or β_i each with probability $1/2$ and

$$E(M_1 + M_2) = E(\sum_i A_i) = \frac{1}{2}(a_2 + \sum_{i=2}^n a_i) \ge \frac{1}{2}(\sum_{i=2}^n a_i).$$

Furthermore,

$$\frac{E(M_2)}{2} \le E(M_1) \le 2E(M_2)$$

and

$$E(\min\{S_1, S_2\}) \ge V(A_1, A_2, A_n) \ge V(M_1, M_2).$$

Consider $V(M_1, M_2)$ and, without loss of generality, assume that $\alpha_1 + \beta_2 \le \alpha_2 + \beta_1$. We have

$$V(M_1, M_2) = \frac{1}{2}(\alpha_1 + \alpha_2) + \frac{1}{2}(\alpha_1 + \beta_2) \ge \frac{1}{2}(\alpha_2 + \beta_2) = E(M_2) \ge \frac{1}{3}E(M_1 + M_2).$$

Therefore, we have

$$E(\min\{S_1, S_2\}) \geq V(A_1, A_2, \ldots, A_n) \geq V(M_1, M_2) \geq \frac{1}{6}\sum_{i=2}^{n} a_i.$$

■

A.3 Proof of Theorem 1

In order to arrive at a contradiction, let us assume that there exist a function $R(\cdot)$ and a function $f(\cdot)$ that is concave, monotonic and increasing such that the revenue at the Nash equilibrium is at least $R(v_1)$. We will separately consider the cases of all-pay and winner-pay auctions. The main idea of the proof is to first derive a relation that holds at the Nash equilibrium, and then come up with a valuation profile for which the revenue cannot be larger than $R(v_1)$.

The payoff of buyer i is $v_i \frac{f(b_i)}{\sum_j f(b_j)} - b_i$. Taking the partial derivative with respect to b_i and noting that the payoff must be non-negative, we have that the following relations hold at Nash equilibrium:

$$v_i(1 - x_i)f'(b_i) = \sum_j f(b_j) \text{ if } x_i > 0 \tag{17}$$

$$v_i x_i - b_i \geq 0. \tag{18}$$

In the following, we consider three distinct cases with respect to the function $f(\cdot)$, and in each, we use different arguments to show that the auction is not competitive to $R(v_1)$.

Case 1: $f'(0) = \infty$ (e.g., $f(w) = w^r$, $0 < r < 1$). In this case, we will consider buyers' valuations $v, 1, 1, \ldots, 1$, for $v \geq 1$. Because of the symmetry, at Nash equilibrium, buyer 1 is allocated a positive value x_1 and all other buyers are allocated a positive value x_2. Therefore, $x_i < 1/2$ for $i \neq 1$. Together with the condition (17), we have

$$\frac{1}{2}f'(b_2) \leq 1 \cdot (1 - x_2)f'(b_2) = v(1 - x_1)f'(b_1) < vf'(b_1).$$

Therefore,

$$f'(b_2) < 2vf'(b_1).$$

Since we assume that the auction is competitive to $R(v_1)$ and we can collect at most a revenue of 1 from every buyer except buyer 1, there exists a value v^* such that if we take $v > v^*$, then $b_1 > 1$. It follows, $f'(b_2) < 2vf'(1)$, and thus $b_2 > (f')^{-1}(2vf'(1)) > 0$. The last inequality is true because of the assumption $f'(0) = \infty$.

Now, we have derived that b_2 is greater than a positive constant that is independent of the number of buyers n. This leads to a contradiction, because as n goes to infinity, the payoff of buyer 2, $x_2 - b_2$ goes to a negative value, which contradicts (18).

Case 2: $f'(0) = c < \infty$ and $f(\infty) = \infty$ (e.g., $f(w) = w$ or $f(w) = \log(w + 1)$). In this case, we also consider the valuation vector of the form $(v, 1, 1, \ldots, 1)$, for $v \geq 1$. From condition (17), we have

$$\sum_j f(b_j) = (1 - x_2)f'(b_2) < f'(0) = c.$$

The last inequality follows from the fact that $f(\cdot)$ is a concave function. Thus, we have $f(b_1) < c$, and therefore, b_1 cannot go to infinity as v tends to infinity, which is a contradiction.

Case 3: $f'(0) = c < \infty$ and $f(\infty) < \infty$. Without loss of generality, we can assume $f(\infty) = 1$ (e.g. $f(w) = w/(w+1)$).

In this case, we will consider the following valuation vector $(v_0, v_0, \ldots, v_0)$, for $v_0 > 0$. Then, at Nash equilibrium, every buyer is allocated $1/n$, and pays b_0 that satisfies

$$v_0(1 - 1/n)f'(b_0) = n/2.$$

From this, we obtain

$$v_0 f'(b_0) = \frac{n^2}{2(n-1)}. \tag{19}$$

Let w^* be the value such that $f(w^*) = 1/16$. We note that w^* only depends on $f(\cdot)$, not on the number of buyers n. We will show that if we increase the valuation of buyer 1 and keep all other buyer's valuations fixed to v_0, then at Nash equilibrium, the payment of buyers with valuation v_0 will be at least w^*. For now, assume that this is true. Then, this will lead to a contradiction, because

$$x_1 = \frac{f(b_1)}{\sum_j f(b_i)} \leq \frac{1}{1 + (n-1)f(w^*)}$$

and, thus, x_1 goes to 0 as n goes to infinity. This means that given the valuation v_1 of buyer 1, if the number of buyers with valuation $v_0 < v_1$ increases, the revenue obtained from buyer 1 is at most $v_1 x_1$, which tends to 0. From this we conclude that the revenue that is extracted from buyer 1 cannot be a function that only depends on v_1 and v_0, a contradiction to the assumption that the auction is competitive to $R(v_1)$.

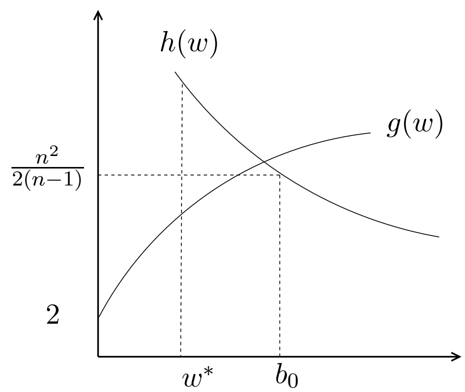

Figure 2: $w > w^*$.

It remains only to show that at Nash equilibrium, each buyer $2, \ldots, n$, pays w such that $w > w^*$. Because of the Nash condition (17), we have

$$v_0(1 - x_i)f'(w) = f(b_1) + (n-1)f(w).$$

We have $x_i < 1/2$ and $f(b_1) \leq 1$, because of the condition on the present subclass of functions f that we consider. Thus, $\frac{1}{2}v_0 f'(w) < 1 + (n-1)f(w)$, which is equivalent to

$$v_0 f'(w) < 2(1 + (n-1)f(w)). \tag{20}$$

We will prove $w > w^*$ from (19) and (20). Consider two functions $h(w) := v_0 f'(w)$ and $g(w) := 2(1 + (n-1)f(w))$. (See Figure 2). $v_0 f'(w)$ is decreasing, while $2(1 + (n-1)f(w))$ is increasing, thus, in order for (20) to be satisfied, w is at least the value where the two curves intersect. But we have

$$h(w^*) > h(b_0) = \frac{n^2}{2(n-1)}$$

and

$$g(w^*) = 2(1 + (n-1)f(w^*)) = 2\left(1 + \frac{n-1}{16}\right) \le \frac{n^2}{2(n-1)} \text{ if } n > 2.$$

Therefore, the intersection of the two curves has to be at a point larger than w^*.

■

A.4 Revenue of Proportional Sharing in Single Item Setting

We note the following revenue guarantee provided by proportional sharing.

THEOREM 5 *The revenue of proportional sharing is at least 1/2 of the second highest valuation.*

Proof. For the proportional sharing mechanism, the condition for Nash equilibrium is that each buyer i maximizes the payoff each buyer optimizes

$$v_i \frac{b_i}{\sum_j b_j} - b_i.$$

Taking the derivative respect to b_i we have

$$\text{either } v_i(1 - x_i) = \sum_j b_j \text{ or } b_i = 0.$$

On the one hand, if the first case holds for $i = 2$, since $v_1 \ge v_2 \ge \cdots \ge v_n$, we have $x_2 \le x_1$ and thus, $x_2 \le 1/2$. From this, and the condition for the Nash equilibrium, we have

$$\sum_j b_j = v_2(1 - x_2) \ge \frac{1}{2} v_2.$$

On the other hand, if the second case holds for $i = 2$, notice that by non-negativity of a buyer's payoff, we have $v_2 \le \sum_j b_j$, which completes the proof. ■

A.5 Proof of Theorem 4

We will prove the revenue upper bound for the special case of digital good auctions, i.e. for the the resource constrains $0 \le x_i \le 1$ for every i.

We will use the Yao's principle to reduce the proof for a worst case scenario of randomized truthful mechanisms to the case of deterministic truthful mechanisms for valuations that are samples from a distribution. It is well known that in single parameter settings, randomized truthful weakly dominant strategy mechanism can be decomposed to deterministic truthful mechanism, which is a reserve price auction [9]. That is, given price p, a bidder can either take the item with price p or leave it.

The distribution that we consider is $v_i = \frac{1}{k}$ with probability $\frac{1}{n}$, where $k \in \{1, 2, \ldots, n\}$. It is a straightforward calculation to see that the expected revenue from n bidders of any deterministic truthful auction for this setting is at most $n \times \frac{1}{n} = 1$.

However, we need to show that a constant revenue bound also holds for the cases where $\mathcal{S}_2(\vec{v}) = \Omega(\log n)$. The following simple argument will show that under the condition $\sum_i v_i \ge \log n - 2$, the expected revenue (denoted by R) can be at most 2. We have that the expected value and variance of valuation v_i are as follows

$$E(v_i) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{n} H_n$$

and

$$\sigma^2(v_i) = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{H_n}{n}\right)^2 = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{k^2} - \left(\frac{H_n}{n}\right)^2 \leq \frac{2}{n}.$$

Moreover,

$$Pr(\sum_{i=1}^{n} v_i < \log n - t) < Pr(\sum_{i=1}^{n} v_i < H_n - t) \leq \frac{n\sigma^2(v_i)}{t^2} \leq \frac{2}{t^2}.$$

Thus, we obtain

$$Pr(\sum_{i=1}^{n} v_i \geq \log n - 2) \geq \frac{1}{2}.$$

Now, we have

$$\begin{aligned} E(R) &= E(R|\sum v_i < \log n - 2)\cdot Pr(\sum v_i < \log n - 2) \\ &\quad + E(R|\sum v_i \geq \log n - 2)\cdot Pr(\sum v_i \geq \log n - 2). \end{aligned}$$

which implies

$$E(R|\sum v_i \geq \log n - 2) \leq \frac{E(R)}{Pr(\sum v_i \geq \log n - 2)} \leq \frac{1}{1/2} = 2.$$

Thus, we show that while $S_2(\vec{v})$ can be at least $\sum_i v_i - 1 \geq \log n - 3$, the revenue of a truthful mechanism cannot be more than 2. This show that a better than $O(\log n)$ approximation is not possible.

We now give a mechanism attaining the $\log n$ factor. We first observe that in the setting of digital good auctions, given a set of valuation $v_1 \geq v_2 \geq \ldots \geq v_n$, we consider the $\mathcal{F}_2$ benchmark introduced by [8]

$$\mathcal{F}_2(v) = \max_{i\geq 2} iv_i.$$

Observe that

$$\mathcal{F}_2(v) \geq iv_i \text{ thus, } v_i \leq \frac{1}{i}\mathcal{F}_2(v).$$

Summing over $2lei \leq n$, we obtain

$$\sum_{i=2}^{n} v_i \leq (\sum_{i=2}^{n}\frac{1}{i})\mathcal{F}_2(v) \leq \log n\mathcal{F}_2(v). \tag{21}$$

To design a mechanism obtaining a $O(\log n)$ approximation of S_2, we use similar ideas as in our mechanism in Definition 6 and [8].

We randomly partition the set of bidders into 2 groups, N_1 and N_2. Find the values $\mathcal{F}(N_1)$ and $\mathcal{F}(N_2)$ where

$$\begin{aligned} \mathcal{F}(N_i) &= \max_p\{p \times k|\ k \text{ is the largest number of bidders that form} \\ &\qquad \text{a feasible set in } N_i \text{ and each of their valuations is at least } p\}. \end{aligned}$$

We will denote with W_i the set of bidders in N_i that maximizes $\mathcal{F}(N_i)$.

Similar to the mechanism in [8], we then find the largest number of bidders in W_1 that can equally share the cost $\mathcal{F}(N_2)$ and the largest number of bidders in W_2 that can equally share the cost $\mathcal{F}(N_1)$. Note that here in order to ensure the feasibility constraint, we can only allocate a half of the item for bidders in each group. Therefore, this mechanism is truthful and generates a revenue of at least

$$\frac{1}{2}\min\{\mathcal{F}(N_1), \mathcal{F}(N_2)\}.$$

Now let O be the feasible set such that $v^{(2)}(O) = S_2(\vec{v})$. Let O_1 and O_2 be the partitions of O under the random partition. We have

$$\mathcal{F}(O_i) \leq \mathcal{F}(N_i)$$

and, thus,

$$\min\{\mathcal{F}(N_1), \mathcal{F}(N_2)\} \geq \min\{\mathcal{F}(O_1), \mathcal{F}(O_2)\}.$$

However, we also have

$$E(\min\{\mathcal{F}(O_1), \mathcal{F}(O_2)\}) \geq \frac{1}{4}\mathcal{F}_2(O).$$

Now, from (21), we have

$$\mathcal{F}_2(O) \geq \frac{1}{\log n} v^{(2)}(O) = \frac{\mathcal{S}_2(\vec{v})}{\log n}.$$

Combining the above inequalities, we have that the revenue of this truthful mechanism is at least $\frac{\mathcal{S}_2(\vec{v})}{8 \log n}$.
■

B Learning Dynamics

Dynamics of proportional sharing mechanisms were studied by Even-Dar, Mansour and Nadav in [6], where no-regret dynamics are proved to converge to a Nash equilibrium in a special case of the single resource game and linear utility functions.

In this section we take a different approach by considering a class of natural differential dynamics, where bidders change their bids in a continuous way in the direction of their own *payoffs' derivatives*. Modeling game dynamics in this way is known as differential games and is common in several research areas such as control theory, operations management, marketing and economics [5, 27, 2].

The proportional sharing game that we consider in this paper has a close relationship with a model of market clearing prices with price taking buyers, that we will explain below. The differential game for price taking buyers is intuitive and standard, we will adapt this approach for modeling the dynamic of our game. We will start our discussion with the case of single resource constraint.

Single Resource

Consider the game with the resource constraint $\sum_i x_i \leq C$ and increasing concave utility functions $U_i(x_i)$ for every bidder i. The market clearing price is a unit price p^* such that when we assume that bidders are price takers, the total demand is equal to the total supplies. More precisely, when each bidder i would buy an x_i fraction of the resource to maximize his payoff $U_i(x_i) - p^* x_i$, equivalently, $U_i'(x_i) = p^*$ if $x_i > 0$, the total demand $\sum_i x_i$ is equal to the available resource, $\sum_i x_i = C$. Thus, to find the market clearing price in the price-taking model, the seller needs to increase or decrease the unit price depending on the total demand at the current price until supply is equal to demand. A bidder i, on the other hand, will increase his x_i if his marginal utility at x_i is still larger than the current price, and decrease x_i if the marginal utility is smaller than the current price. Assuming both seller and bidders adjust the price and demand continuously, one can capture the game by the following standard dynamical system [27, 5]:

$$\begin{aligned} \tfrac{d}{dt}p(t) &= \kappa(\textstyle\sum_i x_i(t) - C) \\ \tfrac{d}{dt}x_i(t) &= \gamma(U_i'(x_i(t)) - p(t)), \ i = 1, 2, \ldots, n, \end{aligned} \tag{22}$$

where κ and γ are positive constants.

Now, in proportional sharing mechanism, each user chooses a bid b_i and because of the proportional sharing rule, we always have $\sum_i x_i = C$. Furthermore, the unit price that each bidder pays is the same and equal to $\frac{b_i}{x_i} = \frac{\sum_i b_i}{C} = p$. Furthermore, this game has a unique equilibrium. Thus, one could say that without a seller adjusting the unit price, bidders can find the market clearing price in a decentralized manner via the proportional sharing mechanism. However, this statement is not entirely correct. In proportional sharing mechanism, because the unit price depends on the bids of strategic bidders, there is a difference between p and p^*. In particular, as shown in Section 4, because of the price anticipating affect, instead of having $U_i'(x_i) = p$, we have the following condition for the Nash equilibrium

$$U_i'(x_i)\left(1 - \frac{x_i}{C}\right) = p = \frac{\sum_i b_i}{C} \text{ if } x_i > 0. \tag{23}$$

Despite the difference, one can argue that for example, in large markets $\frac{x_i}{C}$ tends to be small and thus p^* can be well approximated by p. Furthermore, similar to the dynamical system (22), we can also define a differential game for the proportional sharing game. To make it easier to understand, let us first describe this system in a discrete update scheme.

Assume at time $t + dt$ bidder i can observe the unit price in the previous round $p(t) = \frac{1}{C}\sum_j b_j(t)$ and would like to update his bid $b_i(t + dt)$, which can be understood as the amount of resource $x_i(t + dt) = \frac{b_i(t+dt)}{p(t)}$ that he would hope to receive at the unit price $p(t)$. Bidder i is aware of the fact that the unit price depends on his bid, which is translated into the condition for Nash equilibrium (23), by considering the derivative of bidder i's payoff function. Thus, we assume that bidder i follows the following dynamic:

$$x_i(t + dt) - x_i(t) = dt \cdot \kappa \cdot \left(U_i'(x_i(t))(1 - \frac{x_i(t)}{C}) - p(t)\right)$$

where

$$x_i(t + dt) = \frac{b_i(t + dt)}{p(t)} \text{ and } p(t) = \frac{\sum_i b_i(t)}{C}$$

and κ is a positive parameter. Observe that in this dynamic

$$p(t + dt) = \frac{\sum_i b_i(t + dt)}{C} = p(t)\frac{\sum_i x_i(t + dt)}{C}.$$

That is, while $x_i(t)$ is updated by an additive term, $p(t)$ is multiplied each time with $\frac{\sum_i x_i(t)}{C}$. First, consider the case when $\frac{\sum_i x_i(t)}{C} \geq 1 + \delta$, for $\delta > 0$. If we take dt small enough (compared with δ), then after a number of updating steps (depending on δ), while $x_i(t)$ does not change much, $p(t)$ becomes large. In particular, $p(t)$ approaches infinity as dt tends to 0. Similarly, when $\frac{\sum_i x_i(t)}{C} \leq 1 - \delta$, $p(t) \to 0$ as $dt \to 0$.

Thus, to make the dynamic more robust so that it can also be implemented where one can only take a *discrete* time process, we will approximate $p(t)$ by an *arbitrary* increasing function $f(\frac{\sum_i x_i(t)}{C})$ such that $f(1 + \delta) > N_\delta$ and $f(1 - \delta) < \frac{1}{N_\delta}$.

In the rest of this section for simplicity, we will choose $N_\delta = 1/\delta$, which is a technical detail and is not crucial for our result.

With this intuition, we now consider the following continuous dynamical system as a model of the updating process in our single resource price anticipating game, given a nonnegative and monotone function f satisfying $f(1 - \delta) < \delta$ and $f(1 + \delta) > 1/\delta$ for a small parameter δ, we have for each bidder i,

$$\begin{aligned} b_i(t) &= x_i(t)p(t) \\ \frac{d}{dt}x_i(t) &= \kappa \cdot \left(U_i'(x_i(t))\left(1 - \frac{x_i(t)}{C}\right) - p(t)\right) \\ \text{where } p(t) &= f\left(\frac{\sum_j x_j(t)}{C}\right). \end{aligned} \tag{24}$$

We will show that the dynamical system always converges to a unique solution and by choosing δ small enough to the limit point and the corresponding bid vector $\vec{b}$ give an ϵ-Nash equilibrium for every $\epsilon > 0$.

Multiple Resources

The dynamics (24) can be generalized to the proportional sharing mechanism for multiple resources with the constraints of the type $\sum_{i \in I_e} x_i \leq C$. Starting from the condition for Nash equilibria (2), which is derived from the derivative function of each bidder payoff, we have

$$xU_i'(x_i)\left(1 - \frac{x_i}{C}\right) = \sum_{e: i \in I_e} p^e \text{ if } x_i > 0$$

where

$$p^e = \frac{1}{C} \sum_{i \in I_e} b_i^e.$$

The unit price that a bidder i needs to pay is now $\sum_{e: i \in I_e} p^e$. Similar to the case of single resource, each time $t + dt$ bidder i would update $b_i^e(t + dt)$ such that

$$x_i(t + dt) = \frac{b_i^e(t + dt)}{p^e(t)} = \frac{\sum_e b_i^e(t + dt)}{\sum_e p^e(t)} \text{ for every } e \text{ such that } i \in I_e.$$

The same argument will hold for multiple resources as in the single resource above, when we take the update time dt small enough, we can approximate p^e as a function $f_e(\sum_{i \in I_e} x_i / C)$ such that f_e is a nonnegative and increasing function such that $f_e(1 - \delta) \leq \delta$ and $f_e(1 + \delta) \geq 1/\delta$. We can define the dynamics for the game with multiple resources as follows, for each buyer i and resource constraint e:

$$\begin{aligned} b_i^e(t) &= x_i(t) p^e(t) \\ \frac{d}{dt} x_i(t) &= \kappa \cdot \left(U_i'(x_i(t)) \left(1 - \frac{x_i(t)}{C}\right) - \sum_{e: i \in I_e} p^e(t) \right) \\ \text{where } p^e(t) &= f_e\left(\frac{\sum_{i \in I_e} x_i(t)}{C}\right). \end{aligned} \tag{25}$$

Remark Note that one can define a similar dynamics for the more general polyhedral environment based on the derivatives of bidders' payoffs. However, in our case, because all the constraints have the same capacity C, the Nash condition is much simpler and can be separated into two terms, one depends solely on x_i and the other is the sum of the unit prices on the resources. This is a crucial observation that allows us to prove the convergence of the dynamical system.

We now prove the result of the convergence of (25) and the relation between its fixed point and the approximate Nash equilibria of the proportional sharing game. We first recall that a strategy profile is an ϵ-Nash equilibrium if no agent can change strategy to improve his payoff by more than ϵ.

THEOREM 6 *The differential equation system (25) has a unique stable point to which all trajectories converge. Furthermore, given any $\epsilon > 0$ there exists $\delta > 0$ small enough, such that if all the functions f_e satisfy $f_e(1 - \delta) \leq \delta$ and $f_e(1 + \delta) \geq 1/\delta$, then the bid vector that corresponds to this stable point, i.e. $b_i^e = x_i p^e$ for every i, is an ϵ-Nash equilibrium.*

Proof. Denote the total $\sum_{i \in I_e} x_i$ by x^e, we define the following function

$$\Phi(x) = \sum_i \int_0^{x_i} U_i'(y)\left(1 - \frac{y}{C}\right) dy - \sum_{e \in E} \int_0^{\frac{x^e}{C}} f_e(z) dz.$$

It is straightforward to check that $\Phi(x)$ is strictly concave. Furthermore $\Phi(x)$ is a Lyapunov function of the system of differential equations (25). To see this observe $\frac{d}{dx_i}\Phi(x) = U_i'(x_i)\left(1 - \frac{x_i}{C}\right) - \sum_{e:i \in I_e} f_e(\frac{x^e}{C})$. Moreover,

$$\begin{aligned} \frac{d}{dt}\Phi(x(t)) &= \sum_i \frac{d}{dx_i}\Phi(x)\frac{d}{dt}x_i(t) \\ &= \kappa \sum_i \left(U_i'(x_i(t))\left(1 - \frac{x_i(t)}{C}\right) - \sum_{e:i \in I_e} f_e\left(\frac{x^e}{C}\right)\right)^2. \end{aligned}$$

This shows that $\Phi(x(t))$ is strictly increasing with t unless $x(t)$ is the unique maximizer of $\Phi(x)$. Therefore, all trajectories converge to this unique stable point.

In the remainder of this section, we will assume that $\vec{x}$ is the unique stable point of the dynamical system. Let $b_i^e = x_i p^e$ and x^* be the allocation vector of the proportional sharing mechanism given the bid vector $\vec{b}$. We need to show that given an $\epsilon > 0$, there exists a δ such that if f_e's satisfy the condition stated in the theorem, then the bid vector $\vec{b}$ is an ϵ-Nash equilibrium.

We first observe that if $f_e(1+\delta) \geq 1/\delta$ for δ small enough, then for every e we have $\sum_{i:e \in I_e} x_i \leq (1+\sqrt{\delta})C$. This is true because otherwise one would have $\int_0^{\frac{x^e}{C}} f_e(z)dz \geq (\sqrt{\delta} - \delta)\frac{1}{\delta} = \frac{1}{\sqrt{\delta}} - 1$, which can be arbitrarily large when δ is small, thus $\vec{x}$ cannot be the point that maximizes $\Phi(x)$. Therefore, given the bid vector $\vec{b}$, the resource that bidder i gets on resource e is $C\frac{b_i^e}{\sum_j b_j^e} = C\frac{x_i}{\sum_{j:j \in I_e} x_j} \geq x_i \frac{1}{1+\sqrt{\delta}}$.

Now, if there exists a resource e such that $\sum_{j:j \in I_e} x_j \geq C(1-\epsilon)$, then the amount of resource that i gets on e is at most $C\frac{b_i^e}{\sum_j b_j^e} = C\frac{x_i}{\sum_{j:j \in I_e} x_j} \leq x_i \frac{1}{1-\delta}$. The final allocation that bidder i gets x_i^* is the minimal among the allocations on the resources, therefore we have $x_i \frac{1}{1+\sqrt{\delta}} \leq x_i^* \leq x_i \frac{1}{1-\delta}$. On the other hand, we have

$$U_i'(x_i)\left(1 - \frac{x_i}{C}\right) = \sum_{e:i \in I_e} p^e. \tag{26}$$

Thus, by choosing δ small enough, we can make sure that x_i^* is close to x_i and therefore $U_i'(x_i^*)\left(1 - \frac{x_i^*}{C}\right)$ is close to $\sum_{e:i \in I_e} p^e$. This means that the derivative of bidder i's payoff is close to 0, which shows that by choosing δ properly, bidder i cannot change his bids to improve his payoff by more than ϵ.

We now consider the case where for all resources e that $i \in I_e$, we have $x^e < C(1-\delta)$. But then because of the condition that $f_e(1-\delta) \leq \delta$ and (26), the value $U_i'(x_i)(1 - \frac{x_i}{C})$ can be arbitrarily small as δ tends to 0. Therefore either $U_i'(x_i)$ is close to 0 or x_i is close to the maximum available resource C. We also have that $x_i^* \geq \frac{1}{1+\sqrt{\delta}} x_i$, which shows that either $U_i'(x_i^*)$ is close to 0 or x_i^* is close to C. (Here we use the assumption that U_i is nondecreasing and concave.) In both cases by bidding higher to gain more of the resource, bidder i cannot improve his utility $U_i(x_i^*)$ by much. On the other hand, the current payment that bidder i is paying is $\sum_{e:i \in I_e} b_i^e \leq \sum_{e:i \in I_e} p^e$, and because of the condition on f_e, this value can be arbitrarily small.

By this we have showed that $\vec{b}$ is in ϵ-Nash equilibrium. ■

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