MODEL-LEARNER PATTERN

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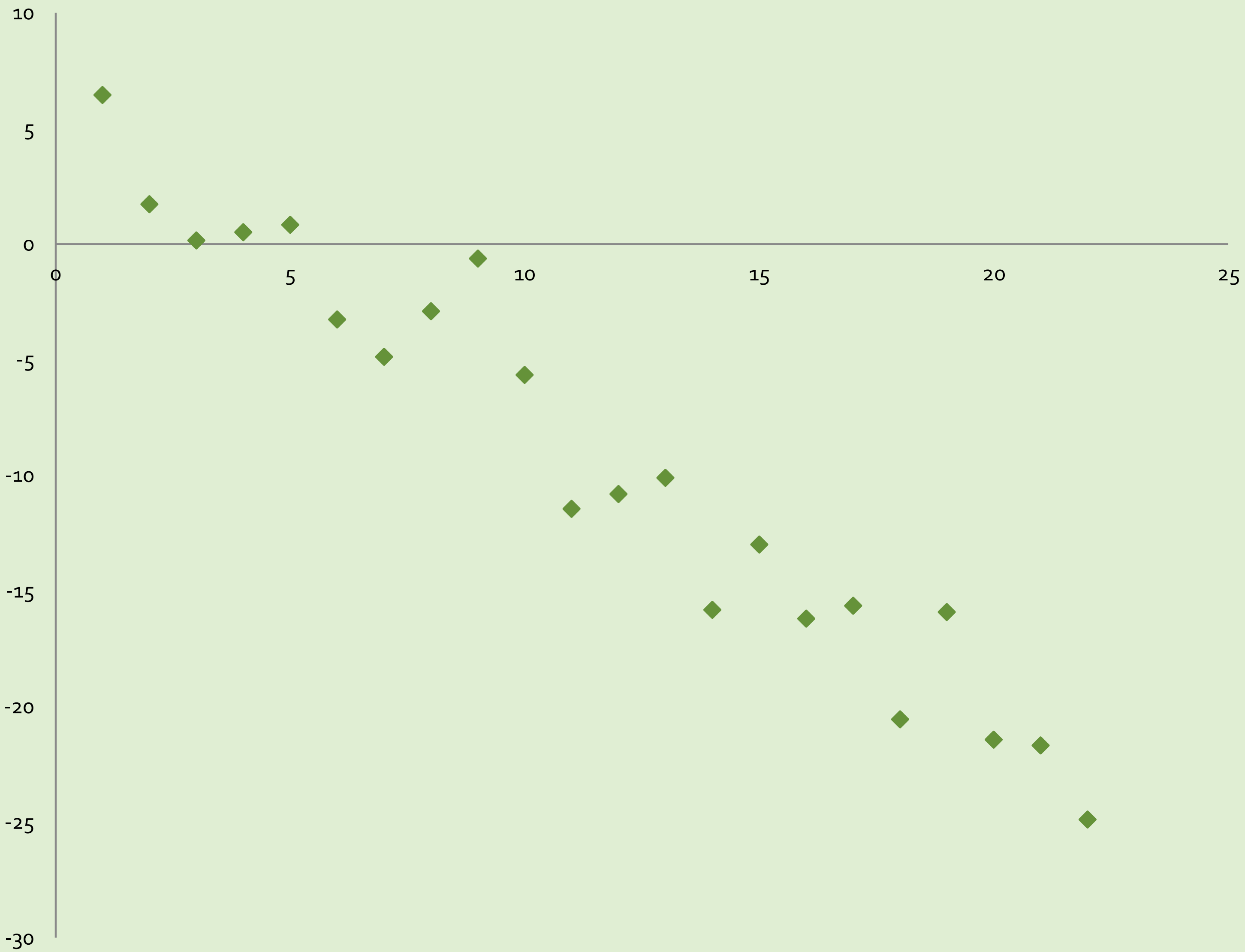
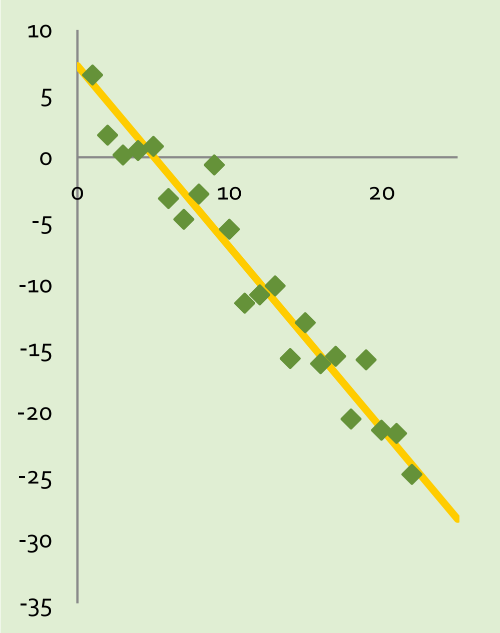
What next?



Bayesian Models



A Model

$$y = Ax + B + e$$

where **noise** $e \sim N(0, P)$

x is an **input**, y is an **output**.
A, B, P are the model **parameters**.

- We **train** on the observed inputs and outputs to **learn** the parameters, and to **predict** new outputs on unseen inputs.
- **Bayesian** models capture uncertainty about model components as probability distributions.

Five Distributions

- **Prior** distribution: p(w)
  given by $w = (A, B, P), A \sim N(0,1), B \sim N(0,1)$ and $P \sim \Gamma(1,1)$
- **Sampling** distribution: $p(y|x, w)$
  given by $y \sim N(Ax + B, P)$ for $w = (A, B, P)$
- **(Prior) Predictive** distribution:
  $$p(y|x) = \int p(y|x, w)\, p(w)\, dw$$
- **Posterior** distribution, given training data $d = (x, y)$:
  $$p(w|d) = \frac{p(y|x, w)\, p(w)}{p(y|x)}$$
- **Posterior predictive** distribution, given $d = (x, y)$:
  $$p(y'|x', d) = \int p(y'|x', w)\, p(w|d)\, dw$$

Three Classes of Bayesian Inference

$$p(w|d) = \frac{p(y|x,w)\,p(w)}{p(y|x)} \text{ where } d = (x,y)$$

- **Exact inference** for discrete distributions:
  Representation: enumerations of probabilities
  Example: $[HH, \frac{1}{10}; HT, \frac{2}{10}; TH, \frac{7}{10}; TT, 0]$
- Approximate inference: sampling eg **Markov chain Monte Carlo**:
  Representation: finite ensemble of samples
  Example: $[A = 1.7, B = 1.6; A = 9.9, B = 9.8 ; \ldots]$
- Approximate inference: **belief propagation on factor graphs**:
  Representation: parameters for marginal of each variable
  Example: $[A = N(5.1{,}10), B = N(6.0{,}5)]$

Bayesian Models are Widely Applicable

- Many machine learning tasks may be cast as Bayesian models.
- We infer functions from inputs to outputs, governed by uncertain parameters.
- Examples include:
  - A **regression function** inputs a tuple of independent variables, and produces one (or more) dependent variables (typically continuous).
  - A **classifier** inputs a vector of **features** and outputs a single value, the **class** (typically discrete).
  - A **cluster analysis** groups items so that items in each cluster are more like each other than to items in other clusters.
  - A **recommender** predicts the rating or preference that a user would give to an item (such as music, books, or movies) based on previous ratings by a set of users.
  - A **rating system** assesses a player's strength in games of skill (such as chess or Go) based on observed game outcomes.

Promise of Probabilistic Programming

- Custom inference code is hard to write, depends on mechanism
- Instead, user writes a probabilistic model for a Bayesian inference problem as a short piece of code, while the compiler turns this code into an efficient inference routine.
- Systems include BUGS, IBAL, BLOG, Church, STAN, Infer.NET, Fun, Factorie, Passage, HBC, HANSEI, and more.
- Still, no linguistic abstractions for Bayesian models.
- **Our contribution:** a new typed **model** abstraction to represent a function from X to Y, governed by W:
  - may be composed to form richer models
  - via a **sampler**, may be run to draw from predictive distribution
  - via a **learner**, may be trained to make predictions

Distributions (1-3) as Probabilistic Code

- **Prior** distribution: $p(w|h)$ for **hyperparameter** h:

```
let prior (h:TH) =
  {A = random (Gaussian(h.MeanA, h.PrecA))
   B = random (Gaussian(h.MeanB, h.PrecB))
   P = random (Gamma(h.Shape, h.Scale))} : TW
```

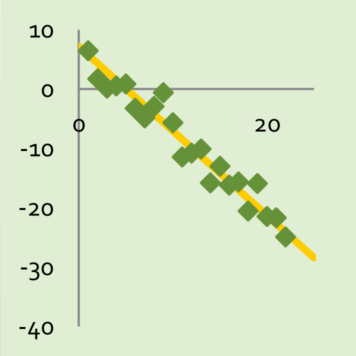
- **Sampling** distribution: $p(y|x, w)$

```
let gen(w,x) =
  [| for xi in x -> random(Gaussian(w.A * xi + w.B, w.P))|]
```

- **(Prior) Predictive** distribution:

$$p(y|x, h) = \int p(y|x, w)\, p(w|h)\, dw$$

```
let predictive(h,x) = let w = prior h in gen (w,x)
```



Distributions (4-5) as Probabilistic Code

- **Posterior** distribution, $p(w|d,h)$ where $d = (x,y)$:

$$p(w|d,h) = \frac{p(y|x,w)\,p(w,h)}{p(y|x,h)}$$

```
let posterior (h,x,y) =
  let w = prior h in observe (y = gen (w,x)); w
```

- **Posterior predictive** distribution:

$$p(y'|x',d,h) = \int p(y'|x',w)\,p(w|d,h)\,dw$$

```
let posterior_predictive (h,x,y,x') =
  let w = posterior (h,x,y) in gen (w,x')
```

Inference on Probabilistic Code

- F# quotations represent probabilistic code:

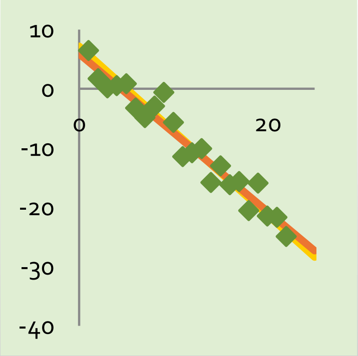
```
let d = <@ fun m -> (random(Gaussian(m,1.0), random(Bernoulli(0.5))) @>
: Expr<double -> double * bool>
```

- Infer.NET's inference invoked by a dynamically typed function, returning a marginalized representation `marginal('U)`

```
val infer : Expr<'T -> 'U> -> 'T -> marginal('U)
infer d 42.0 : Gaussian * Bernoulli
```

- Hence, we train our linear regression example:

```
let wD:{A=Gaussian;B=Gaussian;P=Gamma} =
      infer <@ posterior @> (x,y)
let yD:Gaussian[] =
      infer <@ posterior_predictive @> (x,y,x)
```



Abstraction 1: Model

- A model represents a probabilistic function from TX to TY, governed by an uncertain, learnable TW parameter, and a fixed TH hyperparameter.

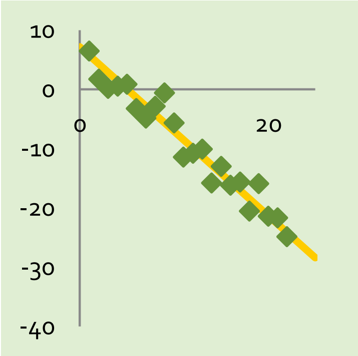
```
type Model<'TH,'TW,'TX,'TY> =
  { HyperParameter: 'TH
    Prior: Expr<'TH ->'TW>
    Gen: Expr<'TW *'TX ->'TY> }
```

```
{ HyperParameter = {MeanA=0.0; PrecA=1.0; … }
  Prior = <@ fun h ->
        { A = random(Gaussian(h.MeanA,h.PrecA))
          B = random(Gaussian(h.MeanB,h.PrecB))
          P = random(Gamma(h.ShapeN,h.ScaleN)) } @>
  Gen = <@ fun (w,x) -> [| for xi in x ->
               random(Gaussian(w.A * xi + w.B, w.P))|] @> }
```

Abstraction 2: Sampler

- A sampler is an object obtained from a model for sampling from the **prior** and **(prior) predictive** distributions, simply by running the code.

```
type ISampler<'TW,'TX,'TY> =
  interface
    abstract Parameters: 'TW
    abstract Sample: x:'TX -> 'TY
  end
```



Abstraction 3: Learner

- A learner is an object obtained from a model and an inference method, for computing the posterior and posterior predictive distributions, after training

```
type ILearner<'TDistW,'TX,'TY,'TDistY> =
  interface
    abstract Train: x:'TX * y:'TY -> unit
    abstract Posterior: unit -> 'TDistW
    abstract Predict: x:'TX -> 'TDistY
  end
```

Learner Semantics

```
type ReferenceLearner(m) =
  let mutable d = <@ (%m.Prior) (%m.HyperParameter) @>
  interface ILearner<Expr<'TW>,'TX,'TY,Expr<'TW> with
    member l.Train(x,y) =
      d <- <@ let w = %d in observe(y = (%m.Gen)(w,x)); w @>
    member l.Posterior() = d
    member l.Predict(x) = <@ let w = %d in (%m.Gen)(w,x) @>
```

- We have three efficient learners:
  - Exact (ADD/CUDD): algebraic decision diagrams
  - MCMC (Filzbach): ensembles of samples
  - Factor graphs (Infer.NET): marginal parametric distributions

Three Examples

	Linear Regression	BPM Classifier	TrueSkill
TH	{MeanA: double; PrecA: double; … }	{Ncols:int}	{Players:int}
TW	{A:double; B:double; Noise:double}	{Noise: double; Weights: Vector}	{Skills: double[]; PerfPrec: double}
TX	double	Vector	{ P1:int; P2:int }
TY	double	bool	bool
Posterior	{A:Gaussian; B:Gaussian; Noise:Gamma}	{Noise:Gaussian, Weights:VectorGaussian>	{Skills: Gaussian[]}
Predict	double -> Gaussian	Vector -> Bernoulli	{ P1:int;P2:int } -> Bernoulli

Generic Loopback Function

- Given these abstractions, we can write generic machine learning code, such as **loopback testing**

```
let test (toLearner: Model<'TH,'TW,'TX,'TY> ->
                     ILearner<'DistW,'TX,'TY,'DistY>)
         (m:Model<'TH,'TW,'TX,'TY>)
         (x:'TX) : 'TW * 'DistW =
   let S = Sampler.FromModel(m)
   let y = S.Sample(x)
   let L = toLearner(m)
   do L.Train(x,y)
   (S.Parameters,L.Posterior())
```

Array Combinator

- Allows training and prediction on IID data

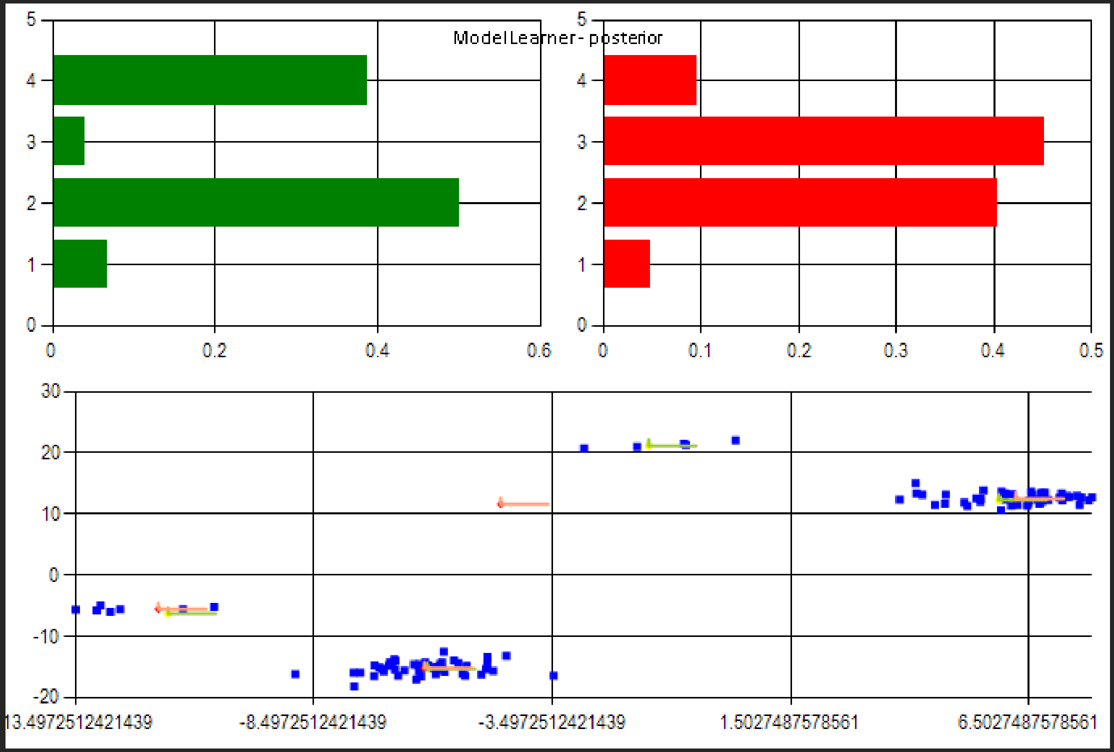
```
module IIDArray =
 let M(m:Model<'TH,'TW,'TX,'TY>)
      : Model<'TH,'TW,'TX[],'TY[]> =
     { Prior = m.Prior
       Gen = <@ fun (w,x) ->
         [| for xi in x -> (%m.Gen) (w,xi) |] @> }
```

Binary Mixture Combinator

- We code a variety of idioms as functions from models to models, eg, mixtures:

```
let Mixture(m1,m2) =
    {Prior =
      <@ fun h ->
         {Bias=random(Uniform(0.0,1.0))
          P1=(%m1.Prior) h
          P2=(%m2.Prior) h} @>
     Gen =
      <@ fun (w,x) ->
          if random(Bernoulli(w.Bias))
          then (%m1.Gen) (w.P1,x)
          else (%m2.Gen) (w.P2,x) @>}
```

Mixture Of Gaussians

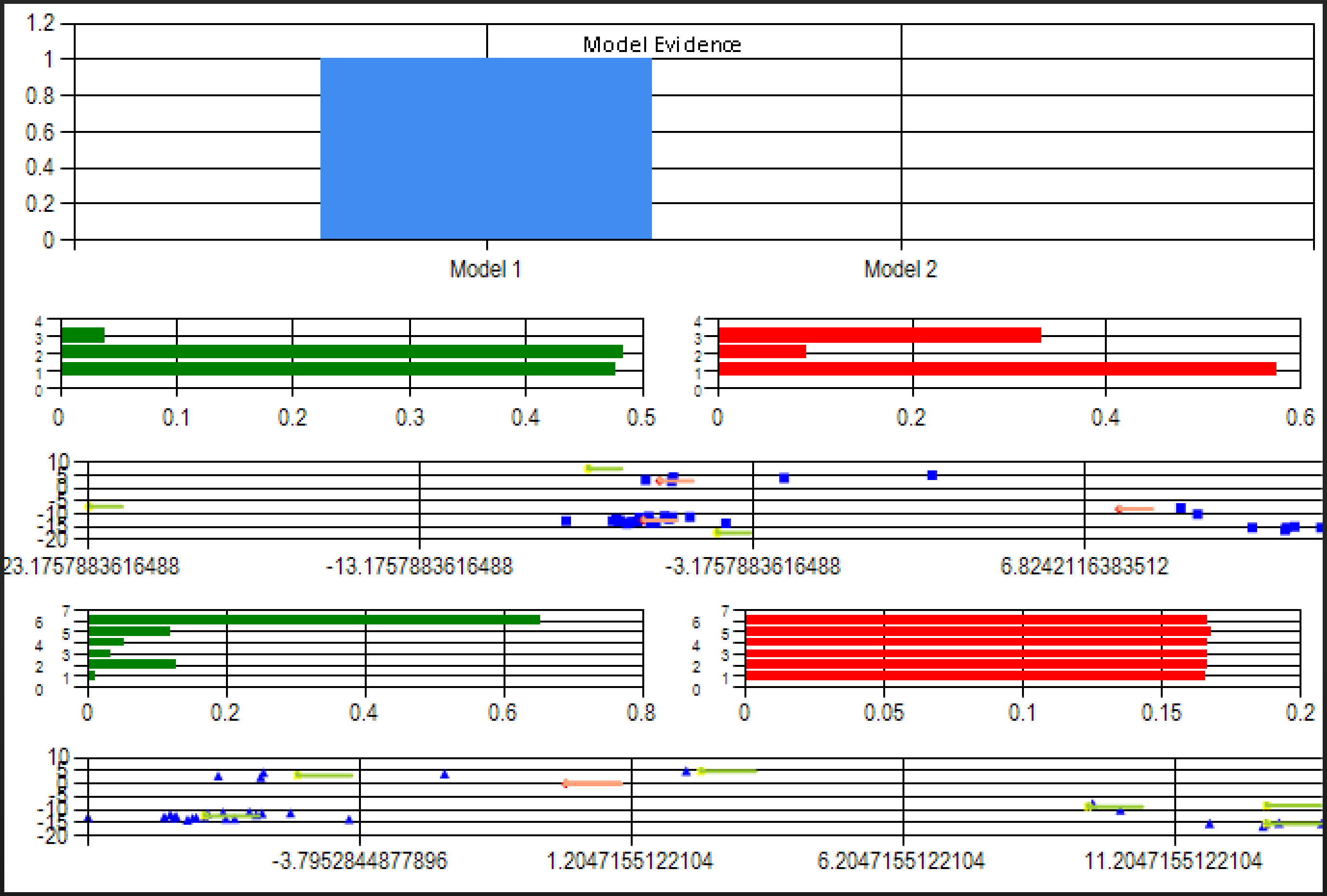



```
let k = 4 // number of clusters in the model
let M = IIDArray.M(KwayMixture.M(VectorGaussian.M,k))

let sampler1 = Sampler.FromModel(M);
let xs = [| for i in 1..100 -> () |]
let ys = sampler1.Sample(xs);

let learner1 = InferNetLearner.LearnerFromModel(M,mg0)
do  learner1.Train(xs,ys)
let (meansD2,precsD2,weightsD2) = learner1.Posterior()
```

Evidence Combinator

```
let M(m1,m2) =
  {Prior = <@ fun (bias,h1,h2) ->
              (breakSymmetry(random(Bernoulli(bias))),
               (%m1.Prior) h1, (%m2.Prior) h2) @>
   Gen = <@ fun ((switch,w1,w2),x) ->
            if switch then (%m1.Gen) (w1,x)
            else (%m2.Gen) (w2,x) @>}
```

Demo: Model Selection




```
let mx k = NwayMixture.M(VectorGaussian.M,k)
let M2 = Evidence.M(mx 3, mx 6)
```

A Dozen Models

Example / Learner	TH	TW	TDistW	TX	TY	TDistY
Sprinkler / A	SP.TH	SP.TW<bool>	ADD<SP.TW<bool>>	SP.TX	bool	ADD<bool>
TwoCoins / A	TC.TH	TC.TW<bool>	ADD<TC.TW<bool>>	TC.TX	bool	ADD<bool>
Two Coins / IN	TC.TH	TC.TW<bool>	TC.TW<Bernoulli>	TC.TX	bool	Bernoulli
Friends / A	bool[][]	bool list list	ADD<bool list list>	int * int * int	bool	ADD<bool>
Students / A	int * int	bool list list	ADD<bool list list>	int * int * int	bool	ADD<bool>
Gaussian / IN	GM.TH	GM.TW<$\mathscr{R}$,$\mathscr{R}$>	GM.TW<$\mathscr{N}$,Γ>	unit	real	$\mathscr{N}$
Gaussian Mix/ IN	MX1.TH	$\mathscr{R}$* GaussW*GaussW	β* GM.TW<$\mathscr{N}$,Γ>*GM.TW<$\mathscr{N}$,Γ>	unit	real	$\mathscr{N}$
Gaussian Mix / F	MX2.TH	(GaussW*GaussW)	(GaussW*GaussW)[]	unit	real	$\mathscr{R}$[]
PlantGrowth / F	unit	PG.TW	PG.TW[]	int	real	$\mathscr{R}$[]
TrueSkill / IN	TS.TH	TS.TW<$\mathscr{R}$>	TS.TW<$\mathscr{N}$>	TrueSkill.TX	bool	Bernoulli
Lin. Reg. / IN	LR.TH	LR.TW<$\mathscr{R}$,$\mathscr{R}$,$\mathscr{R}$>	LR.TW<$\mathscr{N}$,$\mathscr{N}$,Γ>	real	real	$\mathscr{N}$
MV Gaussian / IN	MVG.TH	MVG.TW<$\vec{\mathscr{R}}$,$\mathscr{M}$>	MVG.TW<$\vec{\mathscr{N}}$,$\mathscr{W}$>	unit	$\vec{\mathscr{R}}$	$\vec{\mathscr{N}}$

$\mathscr{R}$= real $\mathscr{N}$= Gaussian β= Beta Γ= Gamma
$\vec{\mathscr{R}}$= Vector $\mathscr{M}$= PositiveDefiniteMatrix
$\mathscr{W}$= Wishart // *generalizes* Γ *to multiple dimensions*
$\vec{\mathscr{N}}$= VectorGaussian // *multivariate Gaussian distribution*
GaussW= {Mean:$\mathscr{R}$; Precision:$\mathscr{R}$}
BetaW = {trueCount: $\mathscr{R}$; falseCount: $\mathscr{R}$}
SP.TH = {RainH: $\mathscr{R}$; SprinklerH: $\mathscr{R}$}
SP.TW<'TB> = {Rain: 'TB; Sprinkler: 'TB}
SP.TX = IsGrassWet // *a unit type*
TC.TH = {Bias1: $\mathscr{R}$; Bias2: $\mathscr{R}$}
TC.TW<'TB> = {Heads1: 'TB; Heads2: 'TB}
TC.TX = AreEitherHeads // *a unit type*
GM.TW<'TM,'TP> = {Mean:'TM; Precision:'TP}
GM.TH ={ Gaussian: GaussW, Gamma: GammaW }

MX1.TH = BetaW * GM.TH * GM.TH
MX2.TH = GM.TH * GM.TH
PG.TW = {alpha:$\mathscr{R}$;topt:$\mathscr{R}$;trho:$\mathscr{R}$;imass:$\mathscr{R}$;sigma:$\mathscr{R}$}
TS.TH = { Players: int; G: GaussW; P: GammaW }
TS.TW<'TA> = { Skills: 'TA[]}
TS.TX = { P1:int; P2: int }
LR.TH = {MeanA: $\mathscr{R}$; PrecA: $\mathscr{R}$; MeanB: $\mathscr{R}$; PrecB: $\mathscr{R}$; Shape: $\mathscr{R}$; Scale: $\mathscr{R}$}
LR.TW<'TA,'TB,'TN> = {A:'TA; B:'TB; Prec:'TN}
MVG.TH = {NCols:int; MeanVectorPrecisionCount:$\mathscr{R}$; WishartShapeConstant:$\mathscr{R}$; WishartScaleConstant:$\mathscr{R}$}
MVG.TW<'TM,'TC> = {Mean:'TM; Covariance:'TC}

Table 1. Rows show types for L : ILearner(TDistW,TX,TY,TDistY) for m : Model<TH,TW,TX,TY> (A=ADD, IN=Infer.NET, F=Filzbach)

Related and Future Work

- Roger Grosse's compositional theory of Bayesian image processing UAI 2012, plus greedy model selection algorithm – fits model-learner pattern.
- Extend our learner API to support partially observed output, eg, for Naïve Bayes or Hidden Markov Models.
- Completeness? Which Bayesian models don't fit?
- **Probabilistic metaprogramming** refers to automatic techniques for constructing probabilistic programs.
- Next, we aim to develop schema-directed probabilistic metaprogramming for inference on databases, an area in its infancy (cf Singh and Graepel's InfernoDB).

The **model-learner** pattern brings structure and types, as well as PL syntax, to probabilistic graphical models

```
module LinearRegression =
  type TH = {MeanA: double; PrecA: double; … }
  let h = {MeanA=0.0; PrecA=1.0; … }
  type TW<'a,'b,'c> = {A:'a; B:'b; Noise:'c}
  type TX = double
  type TY = double
  let M: Model<TH,TW<double,double,double>,TX,TY> =
    { Prior = <@ fun h ->
        { A = random(Gaussian(h.MeanA,h.PrecA))
          B = random(Gaussian(h.MeanB,h.PrecB))
          Noise = random(Gamma(h.ShapeN,h.ScaleN)) } @>
      Gen = <@ fun a -> let m = (a.W.A * a.X) + a.W.B
                 random(Gaussian(m, a.W.Noise)) @> }
```

```
module Classifier
  module Regression
   module TrueSkill
    module TopicModel
```

OrdNum	Delay	CusNum	SalID	Amt	Payed	State	Payment
100023	3	1030	A1	$21.80	TRUE	WA	Card
100024	2	1004	A2	$123.87	FALSE	WV	Cash
100025	5	1029	B1	[illegible]	TRUE (81%)	WV	Cash
100026	1	1030	B2	$89.69	TRUE	NY	Card
100027	2±1.2	1012	C1	$91.30	FALSE	WV (55%)	Cash
100028	5	1031	C2	$26.78	FALSE (71%)	WA	Cash
100029	1	1022	A1	$147.46	FALSE	WA	Cash
100030	4	1020	A2	$78.54	TRUE	NY	Card
100031	1	1037	B1	$139.84	FALSE	WA	Cash
100032	1	1018	B2	$134.21	FALSE	WV	Cash
100033	1	1016	C1	$21.69	FALSE (62%)	NY	Cash
100034	1	1034	C2	$131.29	FALSE	MA	Cash
100035	5	1012	A1	$12.40	FALSE (89%)	MA	Card
100036	2	1027	A2	$129.44	FALSE	NY	Cash
100037	3	1003	B1	$52.30	FALSE	WV	Cash
100038	1	1001	B2	$138.92	FALSE (66%)	NY	Cash
100039	2	1033	C1	$100.17	TRUE	MA (90%)	Card
100040	3±1.5	1008	C2	$8.90	FALSE	WA	Cash
100041	1	1034	B1	$101.91	FALSE (68%)	NY	Cash
100042	3	1032	B2	$131.82	FALSE	WA	Cash

Write your model in F# or C#

Or choose from library

Or automatically generate

```
type Model
```

Assemble multiple models

Choose algorithm (eg, EP, VMP, Gibbs, ADD, Filzbach)

Synthetic data to test learner

```
type ISampler
```

```
type ILearner
```

Train, predict, repeat

The Paper

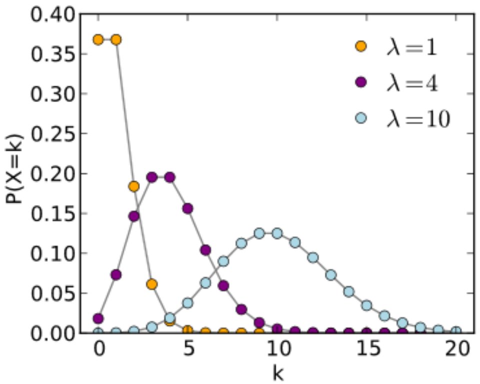
- The new conceptual insight is that code-based machine learning can be structured around **typed Bayesian models**, which are pairs of expressions representing prior and sampling distributions.
  - Definition of a type of Bayesian models, with combinators for compositionally constructing models, and operations to derive samplers and learners from an arbitrary model.
  - Many Bayesian examples expressed as models.
  - A formal semantics for models, learning, and prediction in Fun, and its semantics using measure transformers and probability monad.
  - Learners based on Algebraic Decision Diagrams, message-passing on factor graphs, and Markov chain Monte Carlo.

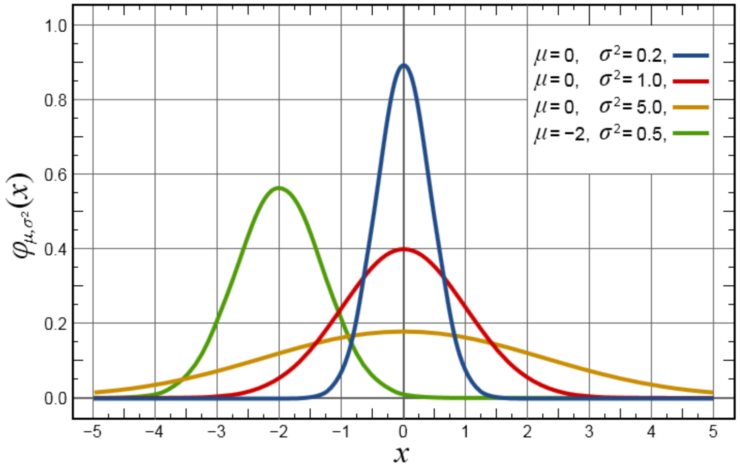
Questions?

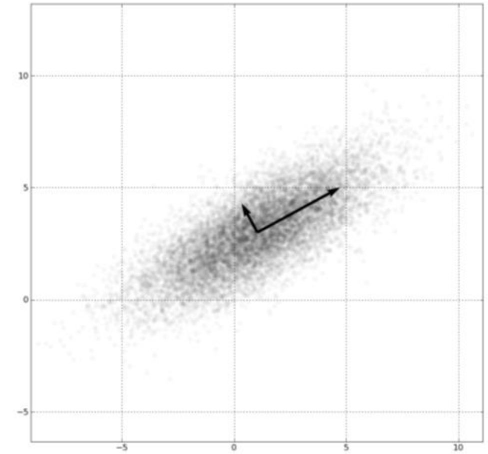
Infer.NET (since 2006)

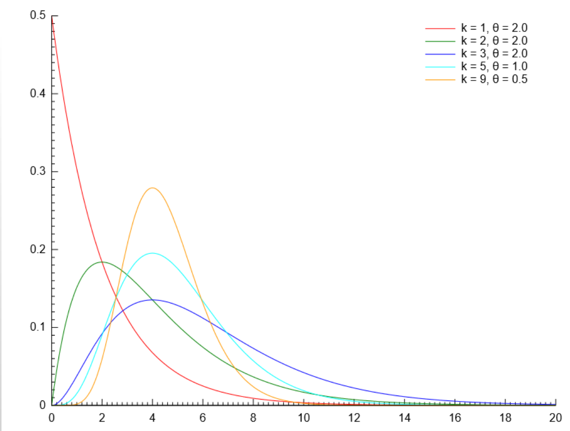
- Tom Minka, John Winn, John Guiver, and others
- A .NET library for probabilistic inference
  - Multiple inference algorithms on graphs
  - Far fewer LOC than coding inference directly
  - Designed for large scale inference
  - User extensible
- Supports rapid prototyping and deployment of Bayesian learning algorithms
  - Graphs represented by object model for pseudo code, but not as runnable code

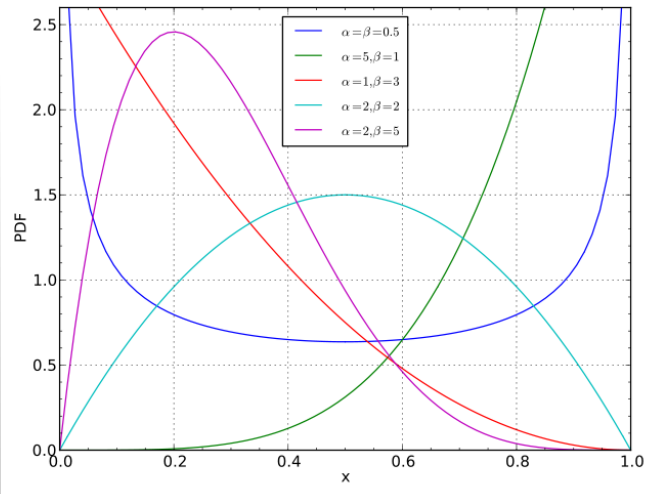
Some Probability Distributions in Fun

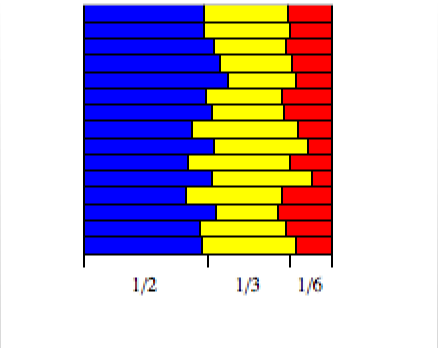












Source: Wikipedia