U-Prove Range Proof Extension

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Summary

This document extends the U-Prove Cryptographic Specification [UPCS] by specifying set membership proofs. This allows proving that a committed value is less than, less than or equal to, greater than, or greater than or equal to another (committed) value.
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Change history

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<th>Version</th>
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<tr>
<td>Revision 1</td>
<td>Initial draft</td>
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1 Introduction

This document extends the U-Prove Cryptographic Specification [UPCS] by specifying range proofs. The Prover will prove to the Verifier that a committed value is less than, less than or equal to, greater than, or greater than or equal to another (committed) value.

The Prover knows a secret value $a$, and will prove to the Verifier an inequality relation between $a$ and another value $b$ that may or may not be known to the Verifier. The Prover and Verifier have as common input a pair of generators $g, h \in G_q$. The Prover will create one of the following proofs:

$$\pi_\square = PK\{\alpha, \beta, \gamma, \delta | C_A = g^\alpha h^\gamma \cap C_B = g^\beta h^\delta \cap \alpha \square \beta\}$$

or

$$\pi_\square = PK\{\alpha, \gamma | C_A = g^\alpha h^\gamma \cap \alpha \square b\}$$

where $\square \in \{<, \leq, >, \geq\}$. The Prover knows assignments for $(\alpha, \beta, \gamma, \delta)$.

The proof relies on comparing the bit decompositions of $a$ and $b$. The Prover computes Pedersen commitments to the bit decompositions and then proves they are formed correctly. Then, the Prover compares each $i$ bit prefix of $a$ and $b$; the results of the comparisons are stored in helper commitments $D_i$. The Prover creates an Equality Proof to show that the $D_i$ are computed correctly. The committed value in $D_{n-1}$ is equal to $\{-1,0,1\}$ depending on the relationship between $a$ and $b$. The Prover adds an auxiliary proof showing that the committed value in $D_{n-1}$ is equal to the appropriate value given $\square$.

The U-Prove Cryptographic Specification [UPCS] allows the Prover, during the token presentation protocol, to create a Pedersen Commitment and show that the committed value is the equal to a particular token attribute. The Prover MAY use this Pedersen Commitment as either $C_A$ or $C_B$. The Issuance and Token Presentation protocols are unaffected by this extension. The Prover may choose to create a range proof after these two protocols complete.

The committed values in $C_A$ and $C_B$ MUST NOT be hashed. If any of these values are U-Prove token attributes, the attributes also MUST NOT be hashed.

The Range Proof protocol makes use of the following U-Prove Extensions: Set Membership Proof Extension [EXSM], Bit Decomposition Extension [EXBD], and Equality Proof Extension [EXEQ].

1.1 Notation

In addition to the notation defined in [UPCS], the following notation is used throughout the document. The range proof consists of many sub-protocols; local variables are omitted from this list unless they consistently appear with the same meaning/value.

- $a$ Value to be compared to $b$, known only to Prover.
- $b$ Value to be compared to $a$, MAY be known to Verifier.
- $C_A$ Pedersen Commitment to $a$. Only Prover knows opening.
- $C_B$ Pedersen Commitment to $b$, or null if Verifier knows $b$.
- $bIsKnown$ True if Verifier knows $b$. 
\( O, \text{proofType} \) A value in the set\{<, \leq, >, \geq\} indicating the relationship between \( a \) and \( b \) that needs to be proven.

\( \min \) Minimum possible value for \( a \) and \( b \).

\( \max \) Maximum possible value for \( a \) and \( b \).

\( \mathcal{M} \) An equality map, as defined in U-Prove Equality Proof Extension [EXEQ]. Range proofs require multiple different equality maps; this document uses local variable \( \mathcal{M} \) to refer to a map.

\( \tilde{A}_i \) The value of a DL Equation, as defined in U-Prove Equality Proof Extension [EXEQ]. Range proofs create multiple different equality proofs; this document uses local variable \( \tilde{A}_i \) to refer to the DL Equation values.

\( \bar{g}_{i,j} \) The bases of a DL Equation, as defined in U-Prove Equality Proof Extension [EXEQ]. Range proofs create multiple different equality proofs; this document uses local variable \( \bar{g}_{i,j} \) to refer to the DL Equation bases.

\( \tilde{x}_{i,j} \) The witnesses (exponents) for a DL Equation, as defined in U-Prove Equality Proof Extension [EXEQ]. Range proofs create multiple different equality proofs; this document uses local variable \( \tilde{x}_{i,j} \) to refer to the DL Equation witnesses.

\( \bar{a} = (a_0, r_0), (a_1, r_1), ..., (a_{n-1}, r_{n-1}) \) The opening information for Pedersen Commitments \( \tilde{A} \). The \( a_i \) contain the bit decomposition of \( a - \min \), while the \( r_i \) are the second exponent.

\( \bar{b} = (b_0, s_0), (b_1, s_1), ..., (b_{n-1}, s_{n-1}) \) The opening information for Pedersen Commitments \( \tilde{B} \). The \( b_i \) contain the bit decomposition of \( b - \min \), while the \( s_i \) are the second exponent. If the Verifier knows \( b \), then \( s_i = 0 \).

\( \tilde{c} = (c_0, y_0), (c_1, y_1), ..., (c_{n-1}, y_{n-1}) \) The opening information for Pedersen Commitments \( \tilde{C} \). The \( c_i \) contain the difference between \( \bar{a} \) and \( \bar{b} \): \( c_i = a_i - b_i \), while the \( y_i \) are the second exponent.

\( \bar{d} = (d_1, t_1), ..., (d_{n-1}, t_{n-1}) \) The opening information for Pedersen Commitments \( \tilde{D} \). Each \( d_i \) stores the inequality relationship between the \( i \) least significant bits of \( a \) and \( b \), represented as a value in \([-1, 0, 1]\). The \( t_i \) are the second exponent.

\( \bar{e} = (e_1, v_1), ..., (e_{n-1}, v_{n-1}) \) The opening information for Pedersen Commitments \( \tilde{E} \). Each \( e_i \) is actually equal to \( d_{i-1} \), while the \( v_i \) are the second exponent.

\( \bar{x} = (c_1, m_1), ..., (c_{n-1}, m_{n-1}) \) The opening information for Pedersen Commitments \( \tilde{X} \). Each \( c_i \) is actually equal to the \( c_i \) in \( \tilde{C} \), while the \( m_i \) are the second exponent.

\( \tilde{A} = A_0, A_1, ..., A_{n-1} \) Pedersen Commitments to \( \bar{a} \).

\( \tilde{B} = B_0, B_1, ..., B_{n-1} \) Pedersen Commitments to \( \bar{b} \).

\( \tilde{C} = C_0, C_1, ..., C_{n-1} \) Pedersen Commitments to \( \bar{c} \).

\( \tilde{D} = D_1, ..., D_{n-1} \) Pedersen Commitment to \( \bar{d} \).

\( \tilde{E} = E_1, ..., E_{n-1} \) Pedersen Commitment to \( \bar{e} \).

\( \tilde{X} = X_1, ..., X_{n-1} \) Pedersen Commitment to \( \bar{x} \).

\( \pi_A \) Proof that \( \tilde{A} \) is a valid commitment to the bit decomposition of \( a - \min \).

\( \pi_B \) Proof that \( \tilde{B} \) is a valid commitment to the bit decomposition of \( b - \min \). Null if the Verifier knows \( b \).
\( \pi_c \) Main equality proof showing that \( D \) and \( X \) are formed correctly.

\( \pi_d \) Auxiliary proof showing that \( D_n-1 \) contains the correct value; either and equality proof or a set membership proof.

\( a \leftarrow A \) Choose a uniformly at random from set \( A \).

The key words “MUST”, “MUST NOT”, “SHOULD”, “RECOMMENDED”, “MAY”, and “OPTIONAL” in this document are to be interpreted as described in [RFC 2119].

### 1.2 Feature overview

The Prover knows the opening of a Pedersen Commitments \( C_A = g^a h^r \) and \( C_B = g^b h^s \) (optionally, \( b \) may be public knowledge). The Prover needs to show that the relationship \( a \odot b \) holds, where \( \odot \in \{<,\leq,\geq\} \) is also known to the Verifier. For efficiency, the Prover and Verifier both know that \( a \) and \( b \) fall inside the range \([min, max]\). The Prover will create a special-honest verifier zero-knowledge proof of knowledge that the Prover knows a tuple of values \((a, r, b, s)\) such that:

1. \( C_A = g^a h^r \).
2. \( C_B = g^b h^s \).
3. The relationship \( a \odot b \) holds, where \( \odot \in \{<,\leq,\geq\} \).

The range proof consists of the following components:

1. Pedersen commitments \( A_0, A_1, ..., A_{n-1} \) to the bit decomposition of \( a - min \), as well as a Bit Decomposition Proof [EXBD] showing the \( A_i \) are constructed correctly.
2. (Optional) Pedersen commitments \( B_0, B_1, ..., B_{n-1} \) to the bit decomposition of \( b - min \), as well as a Bit Decomposition Proof [EXBD] showing the \( B_i \) are constructed correctly.
3. Pedersen commitments \( X_0, ..., X_{n-1} \) to \( c_i = (a_i - b_i)^2 \). These are helper values.
4. Pedersen commitments \( D_1, ..., D_{n-1} \) to \( d_i \in \{-1,0,1\} \), which represents the inequality relationship between the \( i \) least significant bits of \( a \) and \( b \). We compute it as follows:

\[
d_i = \begin{cases} 
  a_i - b_i & i = 0 \\
  d_{i-1} - d_{i-1}(a_i - b_i)^2 + (a_i - b_i) & i > 0
\end{cases}
\]

5. An Equality Proof [EXEQ] showing the \( X_i \) and \( D_i \) are formed correctly.
6. An auxiliary proof showing that \( D_{n-1} \) is a commitment to the appropriate value in \([-1,0,1]\) given the type of inequality relationship the Prover is trying to prove.

### 2 Protocol specification

As the range proof can be performed independently of the U-Prove token presentation protocols, the common parameters consist simply of the group \( G_q \), two generators \( g \) and \( h \), and a cryptographic function \( \mathcal{H} \). The commitments \( C_A \) and \( C_B \) MAY be generated by the Prover.

The remaining parameters may be chosen by either the Prover or Verifier: The values \( min \) and \( max \) indicate the maximum span for secret values \( a \) and \( b \). The variable \( blsKnown \) indicates whether the Verifier knows \( b \). The \( proofType \) indicates the inequality relationship between \( a \) and \( b \) that the Prover wishes to demonstrate.
2.1 Common Protocols
The main body of the range proof is an Equality Proof defined in the U-Prove Equality Proof Extension [EXEQ] EQProofParams() returns the common parameters for the main proof. It generates an equality map $\mathcal{M}$ and sets up the DL equations $\tilde{A}_i = \prod_{j=0}^{n_i-1} \tilde{g}_{i,j}$ where the $\tilde{A}_i$ and $\tilde{g}_{i,j}$ are public values returned by this protocol, while the $\alpha_{i,j}$ are secret values known only to the Prover.
\( \text{EQProofParams}(\ ) \)

Input

Parameters: \( \text{desc}(G_q), \text{UID}_\mathcal{K}, g, h \)
Commitment to \( a/b: \bar{C} = C_0, C_1, \ldots, C_{n-1} \)
Commitment to \( d: \bar{D} = D_1, \ldots, D_{n-1} \)
Commitment to \( (a/b)^2: \bar{X} = X_1, \ldots, X_{n-1} \)
Commitment to \( e: \bar{E} = E_1, \ldots, E_{n-1} \)

Computation

\[
\begin{align*}
\mathcal{M} & := \emptyset \\
eq & := 0
\end{align*}
\]

\( \text{// } D_i = g^{\delta_i} \cdot h^{\tau_i} \)

For \( i := 0 \) to \( n - 1 \)

\[
\begin{align*}
\mathcal{M}.\text{Add}(\text{"delta"}, i, (\eq, 0)) \\
\bar{A}_\eq & := D_i \\
\bar{g}_{\eq, 0} & := g \\
\bar{g}_{\eq, 1} & := h \\
\eq & := \eq + 1
\end{align*}
\]

End

\( \text{// } A_i/B_i = g^{\chi_i} \cdot h^{\zeta_i} \)

For \( i := 1 \) to \( n - 1 \)

\[
\begin{align*}
\mathcal{M}.\text{Add}(\text{"chi"}, i, (\eq, 0)) \\
\bar{A}_\eq & := C_i \\
\bar{g}_{\eq, 0} & := g \\
\bar{g}_{\eq, 1} & := h \\
\eq & := \eq + 1
\end{align*}
\]

End

\( \text{// } X_i = (A_i/B_i)^{\chi_i} \cdot h^{\mu_i} \)

For \( i := 1 \) to \( n - 1 \)

\[
\begin{align*}
\mathcal{M}.\text{Add}(\text{"chi"}, i, (\eq, 0)) \\
\bar{A}_\eq & := X_i \\
\bar{g}_{\eq, 0} & := C_i \\
1 & := h \\
\eq & := \eq + 1
\end{align*}
\]

End

\( \text{// } E_i = (X_i^{-1})^{\delta_{i-1}} \cdot h^{v_i} \)

For \( i := 0 \) to \( n - 1 \)

\[
\begin{align*}
\mathcal{M}.\text{Add}(\text{"delta"}, i - 1, (\eq, 0)) \\
\bar{A}_\eq & := E_i \\
\bar{g}_{\eq, 0} & := X_i^{-1} \\
\bar{g}_{\eq, 1} & := h \\
\eq & := \eq + 1
\end{align*}
\]

End

Output

Return \( \mathcal{M}, \bar{A}, \bar{g} \)

Figure 1: EQProofParams.
2.2 Presentation

The Prover calls RangeProve to generate a range proof. We break up the range proof presentation protocol into various sub-protocols for ease of exposition. The range proof also requires calling protocols from Bit Decomposition Proof [EXBD], Set Membership Proof [EXSM], and Equality Proof [EXEQ].

<table>
<thead>
<tr>
<th>RangeProve( )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input</strong></td>
</tr>
<tr>
<td>Parameters: desc($G_q$), UID$_H$, $g, h, min, max, blsKnown, b, proofType</td>
</tr>
<tr>
<td>Commitment to $a$: $C_a$</td>
</tr>
<tr>
<td>Opening information to $C_a$: $a, r$</td>
</tr>
<tr>
<td>Commitment to $b$: $C_b$</td>
</tr>
<tr>
<td>Opening information to $C_b$: $b, s$</td>
</tr>
<tr>
<td><strong>Computation</strong></td>
</tr>
<tr>
<td>$\vec{A}, \vec{a}, \pi_A, \vec{B}, \vec{b}, \pi_B$</td>
</tr>
<tr>
<td>$\vec{c}, \vec{\bar{c}} := \text{ComputeC}(\vec{A}, \vec{a}, \vec{B}, \vec{b})$</td>
</tr>
<tr>
<td>$\vec{D}, \vec{d} := \text{ComputeD}(\text{desc}(G_q), g, h, \vec{c}, \vec{\bar{c}})$</td>
</tr>
<tr>
<td>$\vec{X}, \vec{x} := \text{ComputeX}($desc$(G_q), g, h, \vec{c}, \vec{\bar{c}})$</td>
</tr>
<tr>
<td>$\vec{E}, \vec{\bar{e}} := \text{ComputeE}($desc$(G_q), g, h, \vec{c}, \vec{\bar{c}}, \vec{D}, \vec{a}, \vec{X}, \vec{x})$</td>
</tr>
<tr>
<td>$\vec{M}, \vec{\bar{A}, \bar{g}} := \text{EQProofParams}($desc$(G_q), g, h, \vec{c}, \vec{\bar{c}}, \vec{D}, \vec{X}, \vec{E})$</td>
</tr>
<tr>
<td>$\pi_c := \text{MainProof}($desc$(G_q), UID$_H, g, h, n, \vec{M}, \vec{\bar{A}, \bar{g}, \vec{c}, \vec{\bar{c}}, \vec{d}, \vec{x}, \vec{\bar{e}}})$</td>
</tr>
<tr>
<td><strong>If</strong> proofType <strong>is</strong> &gt; <strong>then</strong></td>
</tr>
<tr>
<td>$\pi_D := \text{EqualityOfDL}($desc$(G_q), UID$<em>H, g, h, 1, D</em>{n-1}, (d_{n-1}, t_{n-1}))$</td>
</tr>
<tr>
<td><strong>Else if</strong> proofType <strong>is</strong> &lt; <strong>then</strong></td>
</tr>
<tr>
<td>$\pi_D := \text{EqualityOfDL}($desc$(G_q), UID$<em>H, g, h, -1, D</em>{n-1}, (d_{n-1}, t_{n-1}))$</td>
</tr>
<tr>
<td><strong>Else if</strong> proofType <strong>is</strong> ≥ <strong>then</strong></td>
</tr>
<tr>
<td>$\pi_D := \text{SetMembershipProve}($desc$(G_q), UID$<em>H, g, h, {0, 1}, D</em>{n-1}, (d_{n-1}, t_{n-1}))$</td>
</tr>
<tr>
<td><strong>Else</strong></td>
</tr>
<tr>
<td>$\pi_D := \text{SetMembershipProve}($desc$(G_q), UID$<em>H, g, h, {0, 1}, D</em>{n-1}, (d_{n-1}, t_{n-1}))$</td>
</tr>
<tr>
<td><strong>End</strong></td>
</tr>
<tr>
<td><strong>If</strong> blsKnown <strong>then</strong></td>
</tr>
<tr>
<td>$\vec{\bar{B}} := \emptyset$</td>
</tr>
<tr>
<td><strong>End</strong></td>
</tr>
<tr>
<td><strong>Output</strong></td>
</tr>
<tr>
<td>Return $\vec{A}, \vec{B}, \vec{D}, \vec{X}, \pi_A, \pi_B, \pi_C, \pi_D$</td>
</tr>
</tbody>
</table>

**Figure 2: RangeProve**

The range proof requires dividing the bit decomposition of $A$ by the bit decomposition of $B$ to get an array of Pedersen commitments $\vec{C}$ and their openings $\vec{c}$. This step is performed in the function ComputeC().
ComputeC( )

Input  
Parameters: desc($G_q$)  
Commitment to $a$: $\vec{A} = A_0, A_1, ..., A_{n-1}$  
Opening information to $A_i$: $\vec{a} = (a_0, r_0), (a_1, r_1), ..., (a_{n-1}, r_{n-1})$  
Commitment to $b$: $\vec{B} = B_0, B_1, ..., B_{n-1}$  
Opening information to $B_i$: $\vec{b} = (b_0, s_0), (b_1, s_1), ..., (b_{n-1}, s_{n-1})$

Computation  
For $i = 0$ to $n - 1$  
$c_i := a_i - b_i$  
$y_i := r_i - s_i$  
$C_i := A_i / B_i$

End  
$\vec{c} := c_0, c_1, ..., c_{n-1}$  
$\vec{y} := (c_0, y_0), (c_1, y_1), ..., (c, y_{n-1})$

Output  
Return $\vec{c}, \vec{y}$

Figure 3: ComputeC

The range proof performs bit decompositions of $a$ and $b$ with the help of protocols from U-Prove Bit Decomposition Extension [EXBD]. For efficiency, it normalizes the range from $[\min, \max]$ to $[0, \max - \min]$. This step is important since the length of the range proof depends on the length of the bit decomposition. If the value of $b$ is known to the Verifier, the Prover will generate default Pedersen Commitments to the bit decomposition of $b$ and omit the bit decomposition proof.
The following two protocols generate a bit decomposition of an integer \( x \) and return Pedersen Commitments and their openings to this decomposition. \( \text{GenerateBitDecomposition}() \) generates random Pedersen Commitments, while \( \text{DefaultBitDecomposition}() \) sets the second exponent to 0.
**GenerateBitDecomposition()**

**Input**
- Parameters: desc($G_q$), $g$, $h$, $n$
- Commitment to $x$: $C$
- Opening information to $C$: $x, y$

**Computation**
- $x_0, x_1, ..., x_{n-1}$ ← bit decomposition of $x$
- $y_0, y_1, ..., y_{n-1}$ ← $\mathbb{Z}_q^*$
- For $i := 0$ to $n - 1$
  - $C_i := g^{x_i} h^{y_i}$
- $\vec{C} := C_0, C_1, ..., C_{n-1}$
- $\vec{x} := (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1})$

**Output**
- Return $\vec{C}, \vec{x}$

---

**DefaultBitDecomposition()**

**Input**
- Parameters: desc($G_q$), $g$, $h$, $n$
- Integer: $x$

**Computation**
- $x_0, x_1, ..., x_{n-1}$ ← bit decomposition of $x$
- $y_0, y_1, ..., y_{n-1} := 0, 0, ..., 0$
- For $i := 0$ to $n - 1$
  - $C_i := g^{x_i}$
- $\vec{C} := C_0, C_1, ..., C_{n-1}$
- $\vec{x} := (x_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1})$

**Output**
- Return $\vec{C}, \vec{x}$

---

The range proof compares $A$ to $B$ bit by bit. It does so by computing Pedersen commitments $D_1, ..., D_{n-1}$ to $d_i \in \{-1, 0, 1\}$, which represents the inequality relationship between the $i$ least significant bits of $a$ and $b$. We compute the $d_i$ as follows:

$$d_i = \begin{cases} a_i - b_i & i = 0 \\ (d_{i-1} - d_{i-1} (a_i - b_i)^2 + (a_i - b_i) & i > 0 \end{cases}$$

The function ComputeD() takes as input $c_i = a_i - b_i$, which is substituted into the above formula.
ComputeD ( )

Input
Parameters: \( \text{desc}(G_q), g, h \)
Commitment to \( a/b \): \( \vec{C} = C_0, C_1, \ldots, C_{n-1} \)
Opening information to \( C_i \): \( \vec{c} = (c_0, y_0), (c_1, y_1), \ldots, (c_{n-1}, y_{n-1}) \)

Computation
\[
d_0 := c_0 \\
\text{For } i := 1 \text{ to } n - 1 \\
\quad d_i := d_{i-1} - d_{i-1}c_i^2 + c_i \\
\quad t_i \leftarrow \mathbb{Z}_q^* \\
\quad D_i := g^{d_i}h^{t_i} \\
\text{End} \\
\vec{D} := D_1, \ldots, D_{n-1} \\
\vec{d} := (d_1, t_1), \ldots, (d_{n-1}, t_{n-1}) \\
\]

Output
Return \( \vec{D}, \vec{d} \)

Figure 7: ComputeD.

Proving that the \( D_i \) are formed correctly requires helper values \( X_i = c_i^e h^{m_i} \).

ComputeX ( )

Input
Parameters: \( \text{desc}(G_q), g, h \)
Commitment to \( a/b \): \( \vec{C} = C_0, C_1, \ldots, C_{n-1} \)
Opening information to \( C_i \): \( \vec{c} = (c_0, y_0), (c_1, y_1), \ldots, (c_{n-1}, y_{n-1}) \)

Computation
\[
\text{For } i := 1 \text{ to } n - 1 \\
\quad m_i \leftarrow \mathbb{Z}_q^* \\
\quad X_i := c_i^e h^{m_i} \\
\text{End} \\
\vec{X} := X_1, \ldots, X_{n-1} \\
\vec{x} := (x_1, m_1), \ldots, (x_{n-1}, m_{n-1}) \\
\]

Output
Return \( \vec{X}, \vec{x} \)

Figure 8: ComputeX.
Proving that the $D_i$ are formed correctly also requires helper values $E_i = (X_i^{-1})^{d_i-1}h^{\nu_i} = D_i \cdot (D_{i-1})^{-1} \cdot (C_i)^{-1}$.

**ComputeE**

**Input**
- Parameters: $\text{desc}(G_q), g, h$
- Commitment to $a/b$: $\vec{C} = c_0, c_1, ..., c_{n-1}$
- Opening information to $\vec{C}$: $\vec{c} = (c_0, y_0), (c_1, y_1), ..., (c_{n-1}, y_{n-1})$
- Commitment to $d$: $\vec{D} = D_1, ..., D_{n-1}$
- Opening information to $\vec{D}$: $\vec{d} = (d_1, t_1), ..., (d_{n-1}, t_{n-1})$
- Commitment to $(a/b)^2$: $\vec{X} = x_1, ..., x_{n-1}$
- Opening information to $\vec{X}$: $\vec{x} = (c_1, m_1), ..., (c_{n-1}, m_{n-1})$

**Computation**

For $i := 1$ to $n - 1$

\[
\begin{align*}
\nu_i & := t_i - t_{i-1} + y_i + (d_{i-1} \cdot y_i \cdot c_i) + (d_{i-1} \cdot m_i) \\
E_i & := (X_i^{-1})^{d_i-1}h^{\nu_i}
\end{align*}
\]

End

\[
\begin{align*}
\vec{E} & := E_1, ..., E_{n-1} \\
\vec{e} & := (e_1, \nu_1), ..., (e_{n-1}, \nu_{n-1})
\end{align*}
\]

**Output**

Return $\vec{E}, \vec{e}$

**Figure 9: ComputeE**

The main body of the range proof is an Equality Proof [EXEQ] showing that $\vec{D}, \vec{X}, \vec{E}$ are formed correctly.
MainProof() 

Input
Parameters: desc(\(G_q\)), UID\(_R\), g, h,
EQ Proof parameters: \(\mathcal{M}, \bar{A}, \bar{\bar{g}}\)
Opening information to \(\bar{C}\): \(\bar{c} = (c_0, y_0), (c_1, y_1), ..., (c_{n-1}, y_{n-1})\)
Opening information to \(\bar{D}\): \(\bar{d} = (d_1, t_1), ..., (d_{n-1}, t_{n-1})\)
Opening information to \(\bar{X}\): \(\bar{x} = (c_1, m_1), ..., (c_{n-1}, m_{n-1})\)
Opening information to \(\bar{E}\): \(\bar{e} = (e_1, \nu_1), ..., (e_{n-1}, \nu_{n-1})\)

Computation
\(\bar{x} := \emptyset\)
\(eq := 0\)

// \(D_i = g^{\delta_i} \cdot h^{t_i}\)
For \(i := 0\) to \(n-1\)
\(\bar{x}_{eq,0} := d_i\)
\(\bar{x}_{eq,1} := t_i\)
\(eq := eq + 1\)
End

// \(A_i / B_i = g^{\chi_i} \cdot h^{\zeta_i}\)
For \(i := 1\) to \(n-1\)
\(\bar{x}_{eq,0} := c_i\)
\(\bar{x}_{eq,1} := y_i\)
\(eq := eq + 1\)
End

// \(X_i = (A_i / B_i)^{\chi_i} \cdot h^{\mu_i}\)
For \(i := 1\) to \(n-1\)
\(\bar{x}_{eq,0} := c_i\)
\(\bar{x}_{eq,1} := m_i\)
\(eq := eq + 1\)
End

// \(E_i = (X_i^{-1})^{\delta_{i-1}} \cdot h^{\nu_i}\)
For \(i := 0\) to \(n-1\)
\(\bar{x}_{eq,0} := e_i\)
\(\bar{x}_{eq,1} := \nu_i\)
\(eq := eq + 1\)
End

\(\pi_C := \text{EqualityProve}(desc(G_q), UID_R, \bar{A}, \bar{\bar{g}}, \mathcal{M}, \bar{x})\)

Output
Return \(\pi_C\)

Figure 10: MainProof.

EqualityOfDL is a small helper proof that shows that \(D = g^{\delta} \cdot h^t\) is a Pedersen Commitment to some integer \(x\) known to the Verifier. The protocol generates an Equality Proof [EXEQ].
EqualityOfDL

Input
Parameters: desc(G_q), UID_H, g, h, x
Commitment to d: D
Opening information to D: (d, t)

Computation
M := ∅
\bar{A}_0 = D \cdot g^{-x}
\bar{g}_{0,0} := h
\bar{x}_{0,0} := t
\pi := EqualityProve(desc(G_q), UID_H, \bar{A}, \bar{g}, M, \bar{x})

Output
Return \pi

Figure 11: EqualityOfDL.

2.3 Verification
The Verifier receives the common parameters, as well as commitments to a and b and the proof. The Verifier returns true if the verification passes, false otherwise. Verification requires checking the bit decomposition proofs \pi_A and \pi_B, the main equality proof \pi_C, and the auxiliary proof \pi_D that depends on the proof type.
The Verifier uses the function `ComputeClosedC()` to compute $C_i = A_i/B_i$, which are needed to verify $\pi_c$. 
ComputeClosedC( )

Input
Parameters: desc(Gq)
Commitment to a: \( \vec{A} = A_0, A_1, ..., A_{n-1} \)
Commitment to b: \( \vec{B} = B_0, B_1, ..., B_{n-1} \)

Computation
For \( i := 0 \) to \( n - 1 \)
\[ C_i := A_i / B_i \]
End
\[ \vec{C} := C_0, C_1, ..., C_{n-1} \]

Output
Return \( \vec{C} \)

The Verifier calls function ComputeClosedE() to compute \( E_i = D_i \cdot (D_{i-1})^{-1} \cdot C_i^{-1} \), which are needed to verify \( \pi_C \).

ComputeClosedE( )

Input
Parameters: desc(Gq)
Commitment to d: \( \vec{D} = D_1, ..., D_{n-1} \)
Commitment to b: \( \vec{C} = C_0, C_1, ..., C_{n-1} \)

Computation
\[ D_0 := C_0 \]
For \( i := 1 \) to \( n - 1 \)
\[ E_i := D_i \cdot (D_{i-1})^{-1} \cdot C_i^{-1} \]
End
\[ \vec{E} := E_0, E_1, ..., E_{n-1} \]

Output
Return \( \vec{E} \)

The Verifier calls EqualityOfDLVerify to check that \( D \) is a Pedersen Commitment to \( x \).
EqualityOfDLVerify()  

Input  
Parameters: desc(G_q), UID_H, g, h, x  
Commitment to d: D  
Proof: π  

Computation  
M ≔ ∅  
\( \tilde{A}_0 \) ≔ D \cdot g^{-x}  
\( \tilde{g}_{0,0} \) ≔ h  
pass ≔ EqualityVerify(desc(G_q), UID_H, \( \tilde{A}, \tilde{g}, M, π \))  

Output  
Return pass  

3 Security Considerations  
The range proof invokes protocols from U-Prove Equality Proof Extension [EXEQ], U-Prove Bit Decomposition Extension [EXBD], and U-Prove Set Membership Proof Extension [EXSM]. Its security relies on their security. The following restriction apply:  

- The Prover and the Verifier MUST NOT know the relative discrete logarithm \( \log g h \) of the generators \( g \) and \( h \). This is not an issue if the generators are chosen from the list of U-Prove recommended parameters.  

References  