APPENDIX

Theorem A.1: $\mathcal{V}(\cdot)$ is the variance. If $L_A = \{a_i\}_{i=1}^n$ and $L_B = \{b_i\}_{i=1}^m$ are two disjoint set of positive real numbers. Let $L = \{a_i\} \cup \{b_i\}$, then

$$\Delta \mathcal{V}(L) = \mathcal{V}(L) - \frac{|L_A| \mathcal{V}(L_A) + |L_B| \mathcal{V}(L_B)}{|L|} \geq 0.$$ 

If $a_1 \leq a_2 \ldots a_n \leq b_1 \ldots \leq b_m$, then the equality holds only if $a_1 = a_2 = \ldots = a_n = b_1 = \ldots = b_m$.

Proof: Let

$$\sum a_i = a, \sum a_i^2 = A, \sum b_i = b, \sum b_i^2 = B$$

We have

$$\Delta \mathcal{V}(L) = \left[ \frac{A + B}{m + n} - \left( \frac{an + bm}{m + n} \right)^2 \right] - \left[ \frac{A - na^2 + B - mb^2}{m + n} \right]$$

$$= \frac{mn}{(m + n)^2} \cdot (a - b)^2 \quad (14)$$

the statement follows.

Theorem A.2 (Theorem 3.6): $L = \{x_i\}_{i=1}^N$ is sorted, denote $L_A(i) = \{x_i\}_{i=1}^N$ and $L_B(i) = \{x_i\}_{i=N+1}^N$, let

$$\Delta \mathcal{V}(i)(L) = \mathcal{V}(L) - \frac{|L_A(i)| \mathcal{V}(L_A(i)) + |L_B(i)| \mathcal{V}(L_B(i))}{|L|}.$$ 

If $\Delta \mathcal{V}(L) = \max_i \{\Delta \mathcal{V}(i)(L)\}$, then $\Delta \mathcal{V}(L)|L| \geq \Delta \mathcal{V}(L_A(i))|L_A(i)|$ and $\Delta \mathcal{V}(L)|L| \geq \Delta \mathcal{V}(L_B(i))|L_B(i)|$ for $i = 1, 2, \ldots, N$, the equality holds only if $\Delta \mathcal{V}(L_A(i)) = 0$ and $\Delta \mathcal{V}(L_B(i)) = 0$ respectively.

Proof: Without loss of generality, let $L$ be ascending. We prove $\Delta \mathcal{V}(L)|L| \geq \Delta \mathcal{V}(L_A(i))|L_A(i)|$, the other inequality can be similarly proved. Suppose

$$\arg\max_i \{\Delta \mathcal{V}(i)(L)\} = n.$$ 

let $m = N - n$ then according to the proof of Theorem A.1,

$$\Delta \mathcal{V}(L)|L| = \frac{mn}{(m + n)^2} \cdot (a - b)^2. \quad (15)$$

Denote

$$\Delta \mathcal{V}(L_A(i))|L_A(i)| = \frac{m'n'}{(m' + n')^2} \cdot (a' - b')^2.$$ 

$$\implies \Delta \mathcal{V}(L) = \max \{\Delta \mathcal{V}(i)(L)\} = \Delta \mathcal{V}(n)(L)$$

$$\therefore \frac{m'(m + n - m')}{(m' + (m + n - m'))^2} \cdot (a' - c')^2 \leq \frac{mn}{(m + n)^2} \cdot (a - b)^2$$

$$\therefore \frac{m'(m + n - m')}{(m' + (m + n - m'))} \cdot (a' - c')^2 \leq \frac{mn}{(m + n)} \cdot (a - b)^2 \quad (16)$$

where

$$c' = \frac{ma + nb - m' a'}{m + n - m'} \geq b' \quad (L \text{ is ascending}).$$

We now show

$$\frac{m'n'}{(m' + n')} \cdot (a' - b')^2 \leq \frac{m'(m + n - m')}{(m' + (m + n - m'))} \cdot (a' - c')^2. \quad (17)$$

Note the function $(x - a')^2$ is increasing w.r.t $x$ when $x > a'$, then

$$(b' - d')^2 \leq (c' - a')^2 \quad (18)$$

And function

$$\frac{m'x}{(m' + x)} = \frac{m'}{x + 1}$$

is increasing w.r.t $x$. Since $n' < m + n - m'$, we have

$$\frac{m'n'}{(m' + n')} \leq \frac{m'(m + n - m')}{(m' + (m + n - m'))} \quad (19)$$

Combined with Eq. 18, Eq. 17 holds. According to Eq. 16, then $\Delta \mathcal{V}(L)|L| \geq \Delta \mathcal{V}(L_A(i))|L_A(i)|$, the equality holds only if $a' = b'$ thus $\Delta \mathcal{V}(L_A(i)) = 0$.

Definition A.3 (FIFO property): Given a network $G = (V, E)$, where the travel time of each edge in $G$ is time-dependent, we say $G$ is FIFO if for any arc $(i, j)$ in $E$, given A leaves node $i$ starting at time $t_1$ and B leaves node $i$ at time $t_2 \geq t_1$, then B cannot arrive at $j$ before $A$.

Theorem A.4: Let $c_{ij}(t)$ be a strictly positive function defined for a time interval $[0, T]$, which specifies how much time it takes to travel from $i$ to $j$ if departing at time $t$. The graph is FIFO $\Leftrightarrow t + c_{ij}(t)$ is non-decreasing for any $(i, j) \in E$, $t \in [0, T]$.

Proof: Note that $t + c_{ij}(t)$ is the earliest arriving time at node $j$ for one leaving node $i$ at time $t$. Then the correctness of this theorem is a direct consequence of the FIFO’s definition.

Theorem A.5: If $c_{ij}(t)$ is piecewise linear, then $G$ is FIFO if and only if the right derivative $c_{ij+}(t) \geq -1$, $\forall t \in [0, T]$.

Proof: Since $c_{ij}(t)$ is piecewise linear, then according to Theorem A.4, we have

$G$ is FIFO if and only if $t + c_{ij}(t)+ \geq 0, \forall t \in [0, T]$

$G$ is FIFO if and only if $c_{ij+}(t) \geq -1, \forall t \in [0, T] \quad (20)$

Then the theorem follows.

Theorem A.6: If the range of a single time slot $\triangle t$ satisfies:

$$\triangle t \geq t_{max} \quad (21)$$

we can reconstruct an continuous travel time function from a step travel time function to obey the FIFO rule (given the user’s custom factor).

Proof: For each time interval $(t_1, t_2) \subseteq [0, T]$ containing a discontinuity point $t_M$, e.g.

$$s_{ij}(t) = \begin{cases} f_1 & \text{if } t_1 < t \leq t_M \\ f_2 & \text{if } t_M < t < t_2. \end{cases} \quad (22)$$
Fig. 18. Travel Time Function

denote \( t_{M'} = t_M - |f_1 - f_2| \). The refined function \( c_{ij}(t) \) can be defined as:

\[
\begin{align*}
c_{ij}(t) &= \begin{cases} 
    f_1 & \text{if } t_1 < t \leq t_{M'} \\
    f_1 + \frac{f_2-f_1}{|f_2-f_1|} (t-t_{M'}) & \text{if } t_{M'} < t \leq t_M \\
    f_2 & \text{if } t_M < t < t_2.
\end{cases}
\end{align*}
\]

Note that \( |s_{ij}(t)| \leq t_{max} \) so that \( |f_1 - f_2| \leq t_{max} \), then according to inequality (21), \( t_{M'} \) is in the same time interval with \( t_M \). Fig. 18(b) is the refined travel time function of Fig. 18(a). Since the gradient of the piecewise linear function \( c_{ij}(t) \) can only be -1,0,1, it is clear that the refined travel time function satisfies the inequity (20). □