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ON THE DIFFICULTY OF DETERMINING TEARING MODE STABILITY*

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Abstract—The effect of local pressure gradients and of a local flattening of the pressure profile ($p' \to 0$) around the resonant surface of a tearing mode is investigated in toroidal geometry. It is shown that the stability index Δ', calculated from the ideal outer region, is modified by local profile changes in a way reminiscent of the favourable curvature stabilization of linear and non-linear tearing mode layer theory. If the width of the region of pressure flattening is of the order of the linear resistive layer width, the stabilization from the ideal outer region compensates for the loss of pressure gradient stabilization from the layer, and the overall stability of the mode is largely unaffected. For pressure flattening over a larger region, however, the mode can be strongly destabilized. Since the flattening region may then still be too small to resolve experimentally, this result implies the essential difficulty of determining the tearing mode stability of experimental profiles.

1. INTRODUCTION

TEARING MODE STABILITY is known to depend sensitively on the pressure gradient at the tearing layer. In cylindrical geometry, COPPI *et al.* (1966) showed that a pressure gradient within the resistive layer can drive a tearing-type instability, and GLASSER *et al.* (1975) later showed that in toroidal geometry, where the average curvature is favourable ($D_R < 0$), tearing modes can be strongly stabilized by pressure gradient effects within the resistive layer. This effect was then shown to persist (SOMON, 1984; KOTSCHENREUTHER *et al.*, 1985) in the non-linear phase of island growth as originally described by RUTHERFORD (1973). In this note we investigate the effect of small modifications to the axisymmetric equilibrium in which the pressure profile (but not the q profile) is perturbed over a small region localized around the resonant magnetic surface. In Section 2 we assume a form for the local pressure gradient profile

$$p'(X) = p'_0 \frac{X^2}{X^2+\delta^2} \tag{1}$$

which represents local flattening at the resonant surface r_s (Fig. 1). In equation (1), $X = (r-r_s)$ with r a flux surface coordinate (with dimension of length) and $\delta/r_s \equiv \varepsilon \ll 1$.

In Section 3 we consider the form

$$p'(X) = \left(\frac{4\delta_1}{\delta_2}\right) p'_0 \left(\frac{\delta_1}{X} + \frac{X}{\delta_2}\right)^{-2} \tag{2}$$

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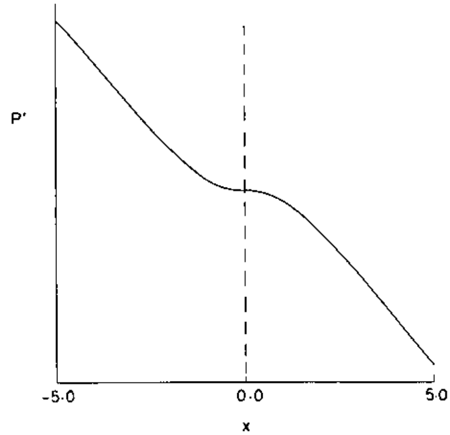

Fig. 1.—The model pressure profile of equation (1), showing the flattening of the pressure in the vicinity of the rational surface ($X = 0$).

with $\delta_1 \leqslant \delta_2 \ll r_s$. This represents local pressure gradients (Fig. 2) with a maximum value of p'_0 in the region $\delta_1 < |X| < \delta_2$ with $p' \to 0$ at the resonant surface ($|X| \ll \delta_1$), and $p' \to 0$ far from the rational surface ($|X| \gg \delta_2$).

For both profiles (1) and (2) we derive expressions for the values of the stability index Δ' calculated close to the rational surface (Δ'_0) and far from the rational surface (Δ'_∞).

2. THE EFFECT OF LOCAL FLATTENING OF THE PRESSURE PROFILE

In the cylindrical model of a Tokamak, the equation governing tearing mode stability takes the form

$$\frac{\mathrm{d}}{\mathrm{d}r} r \frac{\mathrm{d}\Psi_m}{\mathrm{d}r} - \frac{m^2}{r}\Psi_m - \frac{mq\sigma'}{m-nq}\Psi_m - \frac{2p'}{B_0^2}\frac{m^2}{(m-nq)^2}\Psi_m = 0 \tag{3}$$

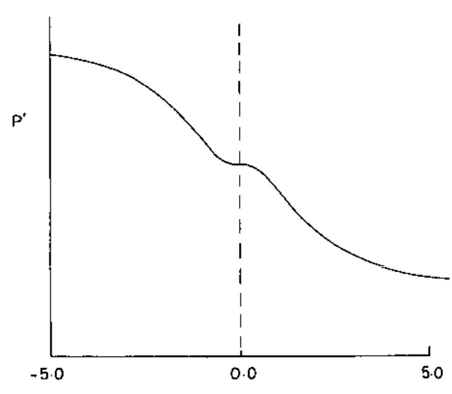

Fig. 2.—The model pressure profile of equation (2), showing the flattening of the pressure both very close to, and very far from, the rational surface.

where m and n are the poloidal and axial ($k_z = n/R$) mode numbers and Ψ_m is the perturbed flux function. Primes denote radial derivatives and $\sigma = J_\parallel/B$ is the longitudinal current. As discussed by Coppi *et al.* (1966), tearing mode stability is determined by asymptotic matching of the solutions to equation (3) to those obtained by solving more complicated equations in the resistive layer around r_s, where r_s is defined by $m - nq(r_s) = 0$. For this purpose a single stability index Δ' [see equation (10) below for a precise definition] is calculated from equation (3).

In toroidal geometry, equation (3) is modified by coupling of different poloidal harmonics and by toroidal curvature [$p' \rightarrow p'(1-q^2)$]. Close to the resonant surface, however, the pressure gradient term dominates over the current gradient term and also over toroidal coupling effects so that, locally, Ψ_m satisfies the equation

$$\frac{d^2\Psi}{dX^2} + D_I \frac{\Psi}{X^2} = 0 \tag{4}$$

where

$$D_I = -\frac{2rp'}{B_0^2}\left(\frac{1-q^2}{s^2}\right) \quad \text{with} \quad s = rq'/q,$$

and we have dropped the harmonic suffix m for simplicity.

Now introducing the form (1) for the local pressure profile

$$D_I \rightarrow D_{I_0} \frac{X^2}{X^2+\delta^2} \tag{5}$$

and using the stretched variable $x = X/\delta$, equation (4) takes the form

$$(1+x^2)\frac{d^2\Psi}{dx^2} + D_{I_0}\Psi = 0. \tag{6}$$

This equation has the general solution

$$\Psi_{R,L}(x) = (1+x^2)[P'_\nu(ix) + \lambda_{R,L} Q'_\nu(ix)] \tag{7}$$

where P_ν and Q_ν are Legendre functions, prime denotes differentiation with respect to the argument, $\lambda_{R,L}$ are arbitrary constants characterizing the solutions on the left ($x < 0$) and right ($x > 0$) and the order ν is given by

$$\nu = -\tfrac{1}{2} + (\tfrac{1}{4} - D_{I_0})^{1/2}. \tag{8}$$

Now the solutions for $|x| \gg 1$ on the left and on the right in general have the form

$$\Psi \simeq |X|^{-\nu} + \Delta_{R_\infty}|X|^{1+\nu} \quad X > 0$$

$$\simeq |X|^{-\nu}+\Delta_{L_\infty}|X|^{1+\nu} \quad X<0 \tag{9}$$

and the relevant measure of Δ' in the outer region ($|X| \gg \delta$) is

$$\Delta'_\infty = \tfrac{1}{2}(\Delta_{R_\infty}+\Delta_{L_\infty}). \tag{10}$$

Our object is to calculate the equivalent Δ' quantity (i.e. the ratio of the coefficient of the small solution to the coefficient of the large solution) in the inner region where $|X| \ll \delta$ (i.e. $x \to 0$). Note that it is still assumed that the relevant equations are those of ideal MHD in this region, and therefore that the resistive layer width δ_η is even smaller than δ ($\delta_\eta/\delta \ll 1$). In the inner region of the flattening zone, the solutions (7) have the asymptotic forms

$$\Psi \simeq 1+\Delta_{R_0}X, \qquad X>0$$
$$\Psi \simeq 1+\Delta_{L_0}|X|, \quad X<0 \tag{11}$$

so that the relevant Δ' parameter in the inner region is

$$\Delta'_0 = \tfrac{1}{2}(\Delta_{L_0}+\Delta_{R_0}). \tag{12}$$

Using the appropriate asymptotic forms (ERDELYI, 1953) of the Legendre functions at large and small arguments to relate Δ_{R_0} to Δ_{R_∞} and Δ_{L_0} to Δ_{L_∞}, we obtain the following result

$$\Delta_{\substack{R_0\\L}} = -\frac{F_2(\nu)}{\delta}\tan\frac{\pi\nu}{2}+\frac{F_2(\nu)}{F_1(\nu)}\left[1-\tan^2\frac{\pi\nu}{2}\right]\Delta_{\substack{R_\infty\\L}}\delta^{2\nu} \tag{13}$$

where

$$F_1(\nu) = \frac{2^{2\nu+1}\Gamma(1-\nu)\Gamma(\nu+\frac{1}{2})\Gamma(\nu+\frac{3}{2})\cos\pi\nu}{\pi\Gamma(\nu+2)} \tag{14}$$

$$F_2(\nu) = \frac{(1+\nu)\Gamma\left(1-\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)}{\Gamma\left(1+\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\nu}{2}\right)}. \tag{15}$$

From equation (13) we immediately derive the desired relationship, namely

$$\Delta'_0 = -\frac{F_2}{\delta}\tan\frac{\pi\nu}{2}+\delta^{2\nu}\Delta'_\infty\frac{F_2}{F_1}\left(1-\tan^2\frac{\pi\nu}{2}\right). \tag{16}$$

For a low β equilibrium, $|D_1| \ll 1$ and $\nu \approx -D_1$, this relation reduces to the form

$$r_s\Delta_0' = \varepsilon^{2\nu}(r_s^{1+2\nu}\Delta_\infty') + \frac{\pi D_I}{2\varepsilon} \tag{17}$$

where $\varepsilon = 2\delta/r_s$. To understand the content of equation (17) it is instructive to examine the tearing mode dispersion relations one would obtain in the two cases corresponding to (i) no pressure flattening around the singular surface (when Δ_∞' must be matched to the analogous quantity obtained from the tearing layer equations) and (ii) pressure flattening around the singular surface (in which case Δ_0' must be matched to the analogous quantity obtained from a layer theory in which $p' = 0$).

To do this we choose the compressible resistive MHD model of the layer (GLASSER *et al.*, 1975) and obtain for case (i)

$$r_s^{1+2\nu}\Delta_\infty' = \pi\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\frac{Q^{3/2}}{\varepsilon_\eta^{1+2\nu}} - \frac{\pi^2 D_R}{\varepsilon_\eta^{1+2\eta}}\frac{\Gamma(\frac{3}{4})}{4\Gamma(\frac{1}{4})} \tag{18}$$

and for case (ii)

$$r_s\Delta_0' = \pi\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\frac{Q^{3/2}}{\varepsilon_\eta} \tag{19}$$

where $\varepsilon_\eta = \delta_\eta/r_s$ with δ_η being the linear layer width given by $\delta_\eta = r_s[\gamma/(\gamma_A n^2 s^2 S)]^{1/4}$, $S = (\gamma_A \tau_\eta)$ with $\tau_\eta = r_s^2/\eta c^2$ and $\gamma_A = B/[R^2\rho(1+2q^2)]^{1/2}$, and $Q = \gamma/\gamma_A(n^2 s^2 S)^{-1/3}$. Note that the $Q^{5/4}$ factor familiar in the resistive layer theory of tearing modes is recovered in equations (18) or (19) when the Q dependence of the layer width ε_η is represented explicitly. With the aid of equation (17), the dispersion relation (19) can be expressed in the alternative form (containing Δ_∞'):

$$r_s^{1+2\nu}\Delta_\infty' = \pi\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\frac{Q^{3/2}}{\varepsilon_\eta\varepsilon^{2\nu}} - \pi\frac{D_I}{2\varepsilon^{1+2\nu}}. \tag{20}$$

Thus equation (18) describes the dispersion relation obtained without pressure flattening, while equation (20) gives the dispersion relation for a pressure profile flattened near the resonant surface (Fig. 1). Comparison of these two equations shows that the principle difference (since $\nu \ll 1$) is a factor $(\varepsilon_\eta/\varepsilon)$ in the second term. Thus if the width ε of the region of pressure flattening is of the order of the linear layer width ε_η then the pressure flattening has little overall effect on the stability of the mode. For $\varepsilon > \varepsilon_\eta$, however, the pressure flattening does indeed remove the stabilizing term and considerably affects the stability of the mode. This effect can occur when the width of the flattening region is still too small to be resolved by the usual experimental diagnostics. Thus it is essentially impossible to determine the tearing mode stability of an experimental profile since this would require detailed knowledge of the pressure profile over small length scales near rational surfaces. Although this result was illustrated using a resistive MHD layer theory, it is actually a consequence of equation (17) and hence is independent of the model used to describe the layer. The favourable average curvature effect noted by GLASSER *et al.* (1975) in toroidal geometry has been transferred from the resistive layer in equation (18) (where it is associated with ε_η) to the outer ideal region in equation (20) (where it is associated with ε).

The appearance of D_I rather than D_R in equation (20) needs some comment. In their non-linear island evolution calculation KOTSCHENREUTHER *et al.* (1985) obtained the form

$$\frac{1}{\varepsilon_w^{2\nu}}\frac{d}{dt}\varepsilon_w = 1.66\eta\left[r_s^{1+2\nu}\Delta'_\infty + 6.3\frac{D_K}{\varepsilon_w^{1+2\nu}}\right] \tag{21}$$

where $\varepsilon_w = W/r_s$ with W the island width. Within the large aspect ratio ordering used in this work $D_K \simeq D_R$ rather than D_I. However, as discussed by these authors, D_R, and the precise form of the curvature term, must depend on the importance of particle transport in determining the pressure profile across the island. In the unchanged topology of the ideal scenario envisaged in our model only D_I can be expected to appear.

3. EFFECT OF A LOCAL STEP IN THE PRESSURE PROFILE

In this section we consider the effect on Δ' of local pressure gradients close to the resonant surface of a tearing mode. To do this we assume a pressure gradient of the form (Fig. 2)

$$p'(X) = \left(\frac{4\delta_1}{\delta_2}\right)p'_0\left(\frac{\delta_1}{X}+\frac{X}{\delta_2}\right)^{-2} \tag{22}$$

with $\delta_1 \leqslant \delta_2 \ll r_s$ and $X = r - r_s$. Now introducing the scaled variable $x = X/\sqrt{\delta_1\delta_2}$, equation (4) for Ψ takes the form

$$(1+x^2)^2\frac{d^2\Psi}{dx^2} + 4D_{I_0}\Psi = 0 \tag{23}$$

with general solution

$$\Psi = (1+x^2)^{1/2}\{C_1\cos(\alpha\tan^{-1}x) + C_2\sin(\alpha\tan^{-1}x)\} \tag{24}$$

where $\alpha = (1+4D_{I_0})^{1/2}$ and $C_{1,2}$ are arbitrary constants. The values of

$$\Delta' \equiv \left[\frac{1}{\Psi}, \frac{d\Psi}{dX}\right]_{r_s^-}^{r_s^+}$$

are readily calculated in the limits $|X| \ll \delta_1$ and $|X| \gg \delta_2$ to obtain Δ'_0 and Δ'_∞, with the result

$$r_s\Delta_0' = \frac{\alpha}{\sqrt{\varepsilon_1\varepsilon_2}}\frac{\left[r_s\Delta_\infty'\alpha\sqrt{\varepsilon_1\varepsilon_2}\sin\left(\alpha\frac{\pi}{2}\right)-\cos\left(\alpha\frac{\pi}{2}\right)\right]}{\left[\sin\left(\alpha\frac{\pi}{2}\right)+r_s\Delta_\infty'\alpha\sqrt{\varepsilon_1\varepsilon_2}\cos\left(\alpha\frac{\pi}{2}\right)\right]} \quad (25)$$

where $\varepsilon_j = \delta_j/r_s$, $j = 1,2$.

For typical values of β and shear s, $D_I \ll 1$ and equation (25) reduces to

$$r_s\Delta_0' = r_s\Delta_\infty' + \frac{\pi}{\sqrt{\varepsilon_1\varepsilon_2}}D_{I_0} \quad (26)$$

which is similar to equation (17) of Section 2. With favourable average curvature $D_{I_0} < 0$, so that steep pressure gradients close to the resonant surface evidently have a strong stabilizing effect on tearing modes.

4. CONCLUSIONS

Conventional linear and non-linear tearing mode layer theories predict a stabilizing effect arising from local pressure gradients at the resonant surface coupled to favourable average curvature. We have shown that a flattening of the pressure profile at the resonant surface does not necessarily remove this effect since it can reappear in the ideal outer region. Indeed, if the width of the flattened region is of the order of the linear layer width the two effects roughly compensate, and the overall stability of the mode is largely unaffected. For flattening over a larger region, however, the stabilizing effect can be significantly reduced. Since the spatial resolution of diagnostics for determining experimental pressure profiles is much larger than a typical linear layer width, and since (in the absence of an adequate theory of local transport) pressure profiles cannot be reliably calculated, these results imply that the tearing mode stability of Tokamak discharges cannot reliably be ascertained.

Local flattening of the current gradient can also modify the stability of tearing modes. However, since the singularity associated with current gradients is weaker ($\sim X^{-1}$) then that associated with pressure gradients ($\sim X^{-2}$), current flattening must necessarily encompass a wider zone to affect stability significantly. A local analysis for current gradient flattening, similar to that performed in this paper for pressure flattening, would provide information on the accuracy required in measuring current profiles (poloidal fields, in practice) to compare with stability predictions.

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