State transformations of colliding optical solitons and possible application to computation in bulk media

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Using explicit, bright-soliton solutions for the coupled Manakov system recently described by Radhakrishnan, Lakshmanan, and Hietarinta, we show that collisions of these solitons can be completely described by explicit linear fractional transformations of a complex-valued polarization state. We design sequences of solitons operating on other sequences of solitons that effect logic operations, including controlled NOT gates. Both data and logic operators have the self-restoring and reusability features of digital logic circuits. This suggests a method for implementing computation in a bulk nonlinear medium without interconnecting discrete components. [S1063-651X(98)15711-2]

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I. INTRODUCTION

Radhakrishnan *et al.* [1] have recently given an explicit bright two-soliton solution for the Manakov system that is more general than any previous, and derived explicit asymptotic results for collisions in the anomalous dispersion region. Those results are remarkable because they show large energy switching between components in an integrable vector system. Surprisingly, as we show in this paper, the parameters controlling this switching exhibit nontrivial information transformations [2], contradicting an earlier conjecture that this was not possible in integrable systems [3]. Furthermore, these transformations can be used to implement logic operations in a self-restoring digital domain, suggesting exciting possibilities for all-soliton digital information processing in nonlinear optical media without radiative losses.

In this paper we use the explicit two-soliton solutions in [1] to show that in the coupled Manakov system:

(i) An appropriately defined polarization state [4] that is a single complex number can be used to characterize a soliton in collisions. Thus, two degrees of freedom per soliton suffice to describe state transformations in collisions, instead of the six degrees of freedom in a complete description of a Manakov soliton.

(ii) The transformations of this state caused by collisions are given by explicit linear fractional transformations of the extended complex plane. These transformations depend on the total energies and velocities of the solitons (the complex k parameters), which are invariant in collisions, but which can be used to tailor desired transformations.

In order to make use of the basic state transformations, we will derive some of their features and limitations. View a particle in state ρ_1 as an operator T_{ρ_1} that transforms the state of any other particle by colliding with it. Then we show, among other things, that every such operator has a simply determined inverse, that the only fixed points of such an operator are ρ_1 and its inverse, that no such operator effects a pure rotation of the complex state for all operands, that by concatenating such operators a pure rotation operator can be achieved, that certain sequences of such operators map the unit circle to itself, and so on.

Finally, we discuss the application of these ideas to implementing all-optical digital computation without employing physically discrete components. Such a computing machine would be based on the propagation and collision of solitons, and could use conservative logic operations [5], since the collisions we consider preserve the total energy and number of solitons. Finding a sufficiently powerful set of operators and reusable particles in this regime would open the way for integrated computation in homogeneous nonlinear optical media [6,7], quite a different scheme from using soliton-dragging gates [8] as discrete components to build a computer.

II. INFORMATIONAL STATE IN THE MANAKOV SYSTEM

A. The Manakov system and its solutions

We review the integrable one-dimensional Manakov system [9–11] and its analytical solutions from [1]. The system consists of two coupled nonlinear Schrödinger (NLS) equations,

$$\begin{aligned} &iq_{1t}+q_{1xx}+2\mu(|q_1|^2+|q_2|^2)q_1=0,\\ &iq_{2t}+q_{2xx}+2\mu(|q_1|^2+|q_2|^2)q_2=0, \end{aligned} \tag{1}$$

where $q_1=q_1(x,t)$ and $q_2=q_2(x,t)$ are the complex amplitudes of two interacting components, μ is a positive parameter, and x and t are normalized space and time. Note that our variables x and t are interchanged with those of [1], in order to represent the propagation variable, the one associated with the first-order derivative in the Manakov equation, by t. [This is consistent with Manakov's original paper [9], Eq. (3).] The system admits single-soliton solutions consisting of two components,

$$q_1=\frac{\alpha}{2}\, e^{-(R/2)+i\eta_I}\operatorname{sech}\left(\eta_R+\frac{R}{2}\right),$$

$$q_2=\frac{\beta}{2}\, e^{-(R/2)+i\eta_I}\operatorname{sech}\left(\eta_R+\frac{R}{2}\right), \tag{2}$$

where

$$\eta=k(x+ikt), \tag{3}$$

$$e^R=\frac{\mu(|\alpha|^2+|\beta^2|)}{k+k^*}, \tag{4}$$

and α, β, and k are arbitrary complex parameters. Subscripts R and I on η and k indicate real and imaginary parts. Note that $k_R\neq 0$.

To study the effects of collisions on soliton states in this system, we consider the analytical two-soliton solution given in [1]. By taking limits of this solution as $t\to\pm\infty$ and $x\to\pm\infty$, we find asymptotic formulas for the widely separated solitons before and after a collision. In Sec. II B we determine the effects of collisions on solitons by comparing these formulas with Eq. (2).

B. State in the Manakov system

The three complex numbers α, β, and k (with six degrees of freedom) in Eq. (2) characterize bright solitons in the Manakov system. Since k is unchanged by collisions, two degrees of freedom can be removed immediately from an informational state characterization. We note that Manakov [9] removed an additional degree of freedom by normalizing the polarization vector determined by α and β by the total magnitude $(\alpha^2+\beta^2)^{1/2}$. However, we show that the single complex-valued polarization state $\rho=\alpha/\beta$, with only two degrees of freedom [4], suffices to characterize two-soliton collisions when the constants k of both solitons are given.

We use the tuple (ρ,k) to refer to a soliton with variable state ρ and constant parameter k:

(i) $\rho=q_1(x,t)/q_2(x,t)=\alpha/\beta$: a complex number, constant between collisions.

(ii) $k=k_R+ik_I$: a complex number, with $k_R\neq 0$.

Throughout this paper we use the complex plane extended to include the point at infinity.

Consider a two-soliton collision, and let k_1 and k_2 represent the constant soliton parameters. Let ρ_1 and ρ_L denote the respective soliton states before impact. Suppose the collision transforms ρ_1 into ρ_R, and ρ_L into ρ_2 (see Fig. 1). In the rest of this paper we always associate k_1 and ρ_1 with the right-moving particle, and k_2 and ρ_L with the left-moving particle. To specify these state transformations, we write

$$T_{\rho_1}(\rho_L)=\rho_2, \tag{5}$$

$$T_{\rho_L}(\rho_1)=\rho_R. \tag{6}$$

The soliton velocities are determined by k_{1I} and k_{2I}, and are therefore constant. With our conventions, $k_{2I}<0<k_{1I}$. Actually, all that follows also holds if both solitons are traveling in the same direction, provided they collide. That is, we need only assume that $k_{2I}<\kappa_{1I}$.

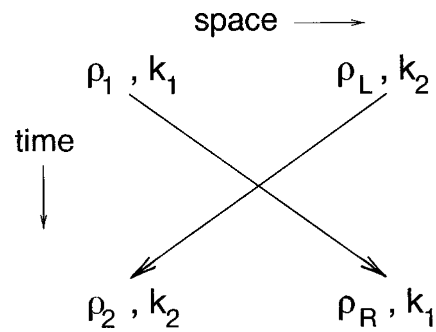

FIG. 1. A general two-soliton collision in the Manakov system. The complex numbers ρ_1, ρ_L, ρ_2, and ρ_R indicate the variable soliton states; k_1 and k_2 indicate the constant soliton parameters.

To determine the state changes undergone by the colliding solitons, we take the limits $x\to\pm\infty$ and $t\to\pm\infty$ in the two-soliton expression from [1]. These limits depend on the signs of k_{1R} and k_{2R}; there are four cases, each of which yields asymptotic formulas for both components of each soliton before and after the collision. We then find each soliton's state by computing the quotient of the soliton's two components. When $k_{1R}>0$ and $k_{2R}>0$, we obtain

$$\rho_2=\frac{[(1-g)/\rho_1^*+\rho_1]\rho_L+g\rho_1/\rho_1^*}{g\rho_L+(1-g)\rho_1+1/\rho_1^*}, \tag{7}$$

where

$$g(k_1,k_2)=\frac{k_1+k_1^*}{k_2+k_1^*}. \tag{8}$$

By a symmetry argument, we obtain

$$\rho_R=\frac{[(1-h^*)/\rho_L^*+\rho_L]\rho_1+h^*\rho_L/\rho_L^*}{h^*\rho_1+(1-h^*)\rho_L+1/\rho_L^*}, \tag{9}$$

where

$$h(k_1,k_2)=\frac{k_2+k_2^*}{k_1+k_2^*}. \tag{10}$$

Note that the state changes given by Eqs. (7) and (9) [and by Eqs. (5) and (6)] depend on the constants k_1 and k_2. We often omit these from expressions, as in Eqs. (5) and (6). However, when we need to specify the values of k_1 and k_2 explicitly, we write

$$T_{\rho_1,k_1}(\rho_L,k_2)=\rho_2. \tag{11}$$

For the remaining three cases of signs of k_{1R} and k_{2R}, we used similar methods to find six additional state-change expressions:

(a) Case 2: $k_{1R}<0$, $k_{2R}>0$,

$$\rho_2=\frac{[(g-1)\rho_1-1/\rho_1^*]\rho_L+g\rho_1/\rho_1^*}{g\rho_L+(g-1)/\rho_1^*-\rho_1}, \tag{12}$$

TABLE I. State-change factors for T_0 and T_∞ transformations. The columns for T_{0,k_1} and T_{∞,k_1} list the factors by which ρ_1 is multiplied to get ρ_R, and the columns for T_{0,k_2} and T_{∞,k_2} list the factors by which ρ_L is multiplied to get ρ_2.

Sign of k_{1R}	Sign of k_{2R}	T_{0,k_1}	T_{0,k_2}	T_{∞,k_1}	T_{∞,k_2}
+	+	$1-g$	$1-h^*$	$1/(1-g)$	$1/(1-h^*)$
−	+	$1/(1-g)$	$1/(1-h)$	$1-g$	$1-h$
+	−	$1/(1-g^*)$	$1/(1-h^*)$	$1-g^*$	$1-h^*$
−	−	$1-g^*$	$1-h$	$1/(1-g^*)$	$1/(1-h)$

$$\rho_R=\frac{[(h-1)\rho_L-1/\rho_L^*]\rho_1+h\rho_L/\rho_L^*}{h\rho_1+(h-1)/\rho_L^*-\rho_L}. \tag{13}$$

(b) Case 3: $k_{1R}>0$, $k_{2R}<0$

$$\rho_2=\frac{[(g^*-1)\rho_1-1/\rho_1^*]\rho_L+g^*\rho_1/\rho_1^*}{g^*\rho_L+(g^*-1)/\rho_1^*-\rho_1}, \tag{14}$$

$$\rho_R=\frac{[(h^*-1)\rho_L-1/\rho_L^*]\rho_1+h^*\rho_L/\rho_L^*}{h^*\rho_L+(h^*-1)/\rho_L^*-\rho_L}. \tag{15}$$

(c) Case 4: $k_{1R}<0$, $k_{2R}<0$

$$\rho_2=\frac{[(1-g^*)/\rho_1^*+\rho_1]\rho_L+g^*\rho_1/\rho_1^*}{g^*\rho_L+(1-g^*)\rho_1+1/\rho_1^*}, \tag{16}$$

$$\rho_R=\frac{[(1-h)/\rho_L^*+\rho_L]\rho_1+h\rho_L/\rho_L^*}{h\rho_1+(1-h)\rho_L+1/\rho_L^*}. \tag{17}$$

It is a matter of algebra to verify that these can be obtained from Eqs. (7) and (9) by using the following relations:

$$T_{\rho_1,k_{1R}+ik_{1I}}(\rho_L,k_{2R}+ik_{2I}) \rightarrow T_{-1/\rho_1^*,-k_{1R}+ik_{1I}}(\rho_L,k_{2R}+ik_{2I}), \tag{18}$$

$$T_{\rho_L,k_{2R}+ik_{2I}}(\rho_1,k_{1R}+ik_{1I}) \rightarrow T_{-1/\rho_L^*,k_{2R}-ik_{2I}}(\rho_1,-k_{1R}-ik_{1I}). \tag{19}$$

Relation (18) states that when the sign of k_{1R} is changed in Eq. (7), we must also replace ρ_1 with $-1/\rho_1^*$ in the same equation in order to obtain the correct formula for the state change. The particle $-1/\rho_1^*$ has a special significance—it acts as inverse operator to ρ_1 (see Sec. III). The same relation can also be used to obtain the proper state-change formula when the sign of k_{2R} is changed in Eq. (9). Relation (19) can be used to obtain the state-change expressions when the sign of k_{2R} is changed in Eq. (7), or when the sign of k_{1R} is changed in Eq. (9). Taken together, relations (18) and (19) imply

$$T_{\rho_1,k_{1R}+ik_{1I}}(\rho_L,k_{2R}+ik_{2I}) \rightarrow T_{\rho_1,-k_{1R}-ik_{1I}}(\rho_L,-k_{2R}-ik_{2I}); \tag{20}$$

that is, when the signs of both k_{1R} and k_{2R} are switched in either Eq. (7) or Eq. (9), we must also conjugate k_1 and k_2 in the equations in order to obtain the correct state-change expressions. Also, note that $g,h\neq 0$, since $k_{1R},k_{2R}\neq 0$, and that g and h cannot be pure real numbers. It can also be verified that all four cases collapse to the first if we simply use $|k_{1R}|$ and $|k_{2R}|$ in Eqs. (7)–(10), although that does not seem to be obvious at the start.

In each of the four cases mentioned above, the soliton states after collision are completely determined by the soliton states before collision, which shows that our definition of state is complete. The k parameter of a soliton remains constant, but is in general different for different solitons.

A special class of state transformations

The class of transformations given by $T_{0,k}$ and $T_{\infty,k}$, where $k=k_1$ (right-moving) or k_2 (left-moving), will be useful later. They specify state changes caused by collisions with solitons whose entire energy is contained in only one component, and are functions of k_1 and k_2. Table I shows the state-change factors due to these transformations. It is not hard to verify from this table that

$$T_{0,k_R+ik_I}\rightarrow T_{\infty,-k_R+ik_I}, \tag{21}$$

which is a special case of the state-change relations described earlier.

III. PROPERTIES OF THE COLLISION STATE TRANSFORMATION

For concreteness we will restrict attention in this section to the case k_{1R}, $k_{2R}>0$, and the transformation, Eq. (7), of the left-moving particle in state ρ_L to the left-moving particle in state ρ_2. All the results hold for other signs of k_{1R} and k_{2R} and the other collisions with appropriate changes in the particle names and the parameter that plays the role of g. When the signs of k_{1R} and k_{2R} are an issue we will mention that explicitly.

A state transformation can be viewed either as a mapping $T_{\rho_1}(\rho_L)$ from the complex ρ_L plane to the complex ρ_2 plane, or in general as a mapping from the complex plane to itself, depending on the context. The state transformation is in fact a *linear fractional transformation* (LFT) (or *bilinear* or *Möbius*) of the form

$$\rho_2=\frac{a\rho_L+b}{c\rho_L+d}, \tag{22}$$

where the coefficients are functions of the right-moving particle in state ρ_1,

$$a=(1-g)/\rho_1^*+\rho_1,$$

$$b=g\rho_1/\rho_1^*,$$

$$c=g,$$

$$d=(1-g)\rho_1+1/\rho_1^*.$$

The choice of a, b, c, and d is not unique, since we can multiply numerator and denominator by an arbitrary nonzero number, but we will use these throughout this paper. The limiting versions of Eq. (22) then give $T_\infty(\rho)=[1/(1-g)]\rho$ and $T_0(\rho)=(1-g)\rho$.

When there is no danger of confusion we will refer to particles and their states interchangeably; so, for example, we can speak of ''transforming the particle ρ_L.'' By Eq. (6), the collision above also results in the transformation $T_{\rho_L}(\rho_1)$ of ρ_1 caused by collision with ρ_L, and each result we give about the properties of the transformations of left-moving particles has its symmetrical counterpart about transformations of right-moving particles.

It is usually assumed that an LFT must have a nonzero determinant $ad-bc$, which ensures that it is nonconstant. This is always true for our class of LFT's and a straightforward calculation shows:

Property 1 (determinant). The LFT Eq. (22) has determinant

$$(1-g)(\rho_1+1/\rho_1^*)^2, \tag{23}$$

which cannot vanish since $g\neq 1$.

Property 2 (inverse). Every operator T_{ρ_1} has a unique inverse T_σ, where $\sigma=-1/\rho_1^*$ and g is the same for ρ_1 and its inverse.

Proof. Replacing ρ_1 by $-1/\rho_1^*$ in the expressions above for a, b, c, and d above results in $-d$, b, c, and $-a$, which are the coefficients in the inverse of Eq. (22). Uniqueness follows because the set of all LFT's forms a group. ■

We refer to a particle ρ followed by its inverse $-1/\rho^*$ as an *inverse pair*. It follows from the next result that collision with an inverse pair leaves any sequence of particles unchanged. This property will be especially useful in designing logic operators since data encoded as inverse particle pairs leaves operators unchanged, and the logic operators can therefore be used for subsequent logic operations on new data.

Property 3 (preservation of inverse pairs). If an inverse pair collides with any particle, the two resulting particles also form an inverse pair.

Proof. Replacing $\partial/\partial t$ by $-\partial/\partial t$ in the original Manakov system [Eq. (1)] shows that if the system is run backwards in time, the same collision rules apply if solutions are replaced by their conjugates. Thus if an inverse pair leaves a particle σ invariant, the conjugates of the collision products of the inverse pair do also, and hence the collision products must themselves be an inverse pair. ■

Property 4 (fixed points). Every operator T_{ρ_1} has exactly two distinct fixed points, ρ_1 and $-1/\rho_1^*$. It follows that a particle is transparent to itself and the particle corresponding to its inverse operator, and to no other particles.

Proof. The fixed-point condition $T_{\rho_1}(\rho_L)=\rho_L$ using Eq. (22) leads to a quadratic equation, since $c=g\neq 0$. There are therefore at most two fixed points. The stated fixed points are always distinct and it is easy to verify that they satisfy Eq. (22) by direct substitution. ■

The following property is expected from the fact that the two components of the Manakov system are incoherently coupled.

Property 5 (rotational invariance of collisions). If ρ_1 and ρ_L are both rotated by θ, then ρ_2 and ρ_R are also.

Proof. Easily verified in the transformation Eqs. (7) *et seq.* ■

Property 6 (absence of pure rotations or scalars). There is no (single-collision) operator of the form

$$T_{\rho_1}(\rho_L)=e^{i\theta}\rho_L \tag{24}$$

for any angle θ, or

$$T_{\rho_1}(\rho_L)=K\rho_L \tag{25}$$

for any real K. In particular, there is no single-collision identity operator.

Proof. First consider the possibility of pure rotations. The case $\theta=0$ corresponds to the identity operator, for which every point is a fixed point, contradicting property 4. When $\theta\neq 0$, the fixed points of a pure rotation are 0 and ∞, so any pure rotation must be a T_0 or T_∞. Since every T_∞ is the inverse of a T_0, it suffices to consider the case of a T_0, which has the operator $T_0(\rho_L)=(1-g)\rho_L$. We can write

$$1-g=\frac{k_{2R}-k_{1R}-i}{k_{2R}+k_{1R}-i}, \tag{26}$$

where we normalize by setting

$$\Delta=k_{2I}-k_{1I}=-1.$$

This normalization is allowed because the difference in k_I's represents the difference in envelope velocities, and so must be nonzero if there is to be a collision. The magnitude of $1-g$ in Eq. (26) cannot be 1 unless either $k_{1R}=0$ or $k_{2R}=0$, which is not allowed. For the possibility of a scalar multiplication, we can again restrict attention to T_0's and Eq. (26), by the same reasoning as above. The right-hand side of that equation cannot be real unless $k_{1R}=0$. ■

The composition of any number of LFT's can be written as an LFT $w=L(z)$, and if it has exactly two distinct fixed points z_1 and z_2, it can be written in the implicit form

$$\frac{w-z_1}{w-z_2}=K\,\frac{z-z_1}{z-z_2}. \tag{27}$$

In the single-collision case this becomes:

Property 7 (implicit form). The single-collision transformation $T_{\rho_1}(\rho_L)$ can be written in the implicit form

$$\frac{\rho_2-\rho_1}{\rho_2+1/\rho_1^*}=K\,\frac{\rho_L-\rho_1}{\rho_L+1/\rho_1^*}, \tag{28}$$

where $K=(1-g)$ is called the "invariant" of the LFT. By symmetry, the inverse transformation is the same except K is replaced by $1/K$.

The next result is standard [12].

Property 8 (invariant circles). The fixed points of a LFT with two distinct fixed points determine two orthogonal families of circles in the w plane: (1) $\mathcal{C}_1$, which are the circles that pass through the fixed points; and (2) $\mathcal{C}_2$, which are the circles determined by the condition that the distances to the fixed points have a constant ratio, the *circles of Apollonius*. These are images, respectively, of points through the origin and concentric circles about the origin in the w plane. The circles $\mathcal{C}_1$ are mapped onto themselves as a set, and similarly for the circles $\mathcal{C}_2$. For each circle in $\mathcal{C}_1$ to be mapped onto itself, we require that invariant $K=1-g$ be real, which is not possible for a single collision. (The mapping in this case is called *hyperbolic*.) For each circle in $\mathcal{C}_2$ to be mapped onto itself, we require that $|K|=1$, which is also not possible in the single collision case. (The mapping in this case is called *elliptic*.)

One consequence of this last result is that if we can design a sequence of particles with a real or modulus-1 invariant K, and arrange for the unit circle to be in $\mathcal{C}_1$ or $\mathcal{C}_2$, then the net effect of collisions with these particles will be to map the unit circle to itself. That is, the modulus-1 property of particles will be preserved on collision with these "operator" sequences, and we will effectively have a state variable with one degree of freedom in the "processed" particles.

Property 9 (invariant of multiple collision). Consider a composite collision with a set of particles, each of which is either a given ρ or its inverse, and each having a possibly different invariant K_j. The fixed points are the same as those of ρ (ρ and $-1/\rho^*$) and the invariant is $\Pi(K_j^{\pm 1})$, where we use K_j^{+1} for the ρ's and K_j^{-1} for the inverses.

Proof. The fact that the fixed points are those of ρ is an immediate consequence of property 4. We need to consider only the two cases of collision with two copies of ρ, and with ρ and its inverse; the general result then follows by induction on the number of collisions. The invariant for a transformation with fixed points ρ and $-1/\rho^*$ can be written [13]

$$K=\frac{a-c\rho}{a+c/\rho^*}, \tag{29}$$

using the coefficients a and c in Eq. (22). The result for these two cases follows by straightforward algebra. ■

The impossibility of finding any single particle that can act as a pure rotation operator (that is, that effects a pure rotation on the state of any other particle) suggests looking for multiple collisions that do have that effect. We do so in the next section, where we exhibit pure rotation operators realized by composite particles composed of T_0's and T_∞'s. This is achieved by carefully designing the particles' k parameters.

IV. PARTICLE DESIGN

We conclude with some examples that illustrate the design of sequences of particles that effect certain transformations that have potential application to embedded logic. Future work will explore the limits of this approach, and especially the question of the extent to which arithmetic and possibly general computation can be encoded in this system.

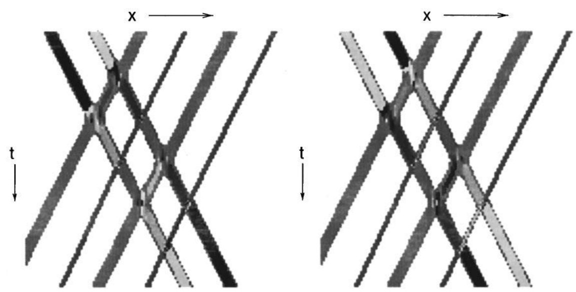

FIG. 2. Numerical simulation of a phase-switching NOT processor implemented in the Manakov system. These graphs display the color-coded phase of ρ for solitons that encode data and operators for two cases. In the initial conditions (top of graphs), the two right-moving (data) solitons are an inverse pair that can represent TRUE in the left graph, and FALSE in the right graph. In each graph, these solitons collide with the four left-moving (operator) solitons, resulting in a soliton pair representing a FALSE and TRUE, respectively. The operator solitons emerge unchanged. These graphs were obtained by numerical simulation of Eq. (1) with $\mu=1$.

A. An i operator

A simple nontrivial operator is pure rotation by $\pi/2$, or multiplication by i. This changes linearly polarized solitons to circularly polarized solitons, and vice versa. A numerical search yielded the useful transformations (see Table I)

$$T_{\rho_L}(\rho_1)=T_{0,1-i}(\rho,1+i)=1-h^*(1+i,1-i)$$

$$=\frac{1}{\sqrt{2}}e^{-(\pi/4)i}\rho, \tag{30}$$

$$T_{\rho_L}(\rho_1)=T_{\infty,5-i}(\rho,1+i)=\frac{1}{1-h^*(1+i,5-i)}=\sqrt{2}e^{(3\pi/4)i}\rho, \tag{31}$$

which, when composed, result in the transformation

$$U(\rho,1+i)=i\rho. \tag{32}$$

(Here we think of the data as right-moving and the operator as left-moving.) Note that the h^*'s in Eqs. (30) and (31) are different, corresponding as they do to different k's. We refer to U as an i operator. Its effect is achieved by first colliding a soliton $(\rho,1+i)$ with $(0,1-i)$, and then colliding the result with $(\infty,5-i)$, which yields $(i\rho,1+i)$.

B. NOT processors

Composing two i operators results in the -1 operator, which with appropriate encoding of information can be used as a logical NOT processor. Figure 2 shows a NOT processor with reusable data and operator solitons. The two right-moving particles represent data and are an inverse pair, and thus leave the operator unchanged; the left-moving sequence comprises the four components of the -1 operator. This figure was obtained by direct numerical simulation of the

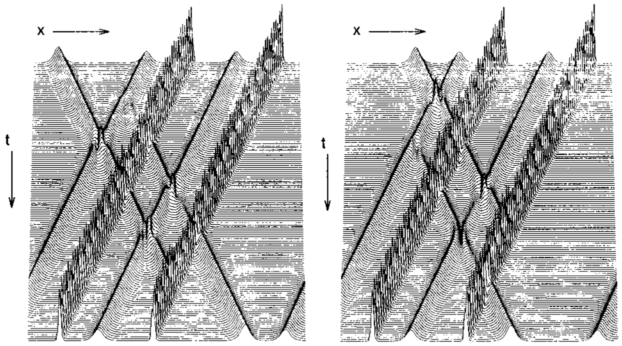

FIG. 3. Numerical simulation of an energy-switching NOT processor implemented in the Manakov system. These graphs display the magnitude of one component, for the same two cases as in the previous figure. In this gate the right-moving (data) particles are the inverse pair with states ∞,0 (left) or 0,∞ (right) and the first component is shown. As before, the left-moving (operator) particles emerge unchanged, but here have initial and final states ± 1.

Manakov system, with an initial state that contains the appropriate data and processor solitons.

We may treat this NOT processor as a controlled NOT (or XOR) by observing that the operator solitons can be selected so that both the data and operator solitons are unaffected by collisions with one another. This NOT processor switches the phase of the (right-moving ± 1) data particles, using the energy partition of the (left-moving 0 and ∞) operator particles. A kind of dual NOT gate exists, with the same composite $K = -1$, which switches the energy of data particles using only the phase of the operator particles. (See Fig. 3.) In particular, if we use the same k's as in the phase-switching NOT gate, code data as 0 and ∞, and use a sequence of four ± 1 operator particles, the effect is to switch 0 to ∞ and ∞ to 0; that is, to switch all the energy from one component of the data particles to the other. This can be checked easily by setting $K = -1$ and $\rho_1 = 1$ in the implicit form, Eq. (28), using property 9 for this composite collision.

C. Particles that map the unit circle to itself

By property 8 any composite (multiparticle) operator with an invariant K that is real will map the unit circle to itself if the operator particles are all the same and themselves have a state on the unit circle. Let the operator particles have state $\rho = e^{i\theta}$. Then by Eq. (28) the transformation from z to w is

$$\frac{w-e^{i\theta}}{w+e^{i\theta}} = K\,\frac{z-e^{i\theta}}{z+e^{i\theta}}, \tag{33}$$

and, therefore, if $z = e^{i\phi}$ and $w = e^{i\psi}$,

$$\psi = \theta + 2\arctan\left(K\tan\frac{\phi-\theta}{2}\right). \tag{34}$$

Thus, if we restrict all ''data'' particles to the unit circle, collision (''operator'') particles of this type will preserve that property.

For an example, consider the composition of eight identical operator particles with $k_2 = 1-i$, colliding with a particle having $k_1 = 1+i$. The resulting invariant is, using $1-h^* = 2^{-1/2}e^{-\pi i/4}$ from Eq. (30), $K = (2^{-1/2}e^{-\pi i/4})^8 = 1/16$. We can also mix copies of an operator particle and its inverse and use property 9 to get a wider variety of invariant values.

V. DISCUSSION

The line of inquiry followed in this paper suggests that it may be possible to perform useful computation in bulk media by using colliding solitons alone, and leads to many open questions, which we are now studying. First it would be useful to obtain a complete mathematical characterization of the state LFT's obtainable by composing either a finite number—or an infinite number—of the special ones induced by collisions in the Manakov system. Second, we should like to know whether the complex-valued polarization state used here for the Manakov system is also useful in other vector soliton systems, especially those that are near-integrable and support spatial solitons [2,14–32]. Finally, we need to study the computational power of this and related systems from the point of view of implementing logic of some generality; in particular, which, if any, such systems, in 1+1 or 2+1 dimensions, integrable or nonintegrable, are Turing equivalent and therefore universal.

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[1] R. Radhakrishnan, M. Lakshmanan, and J. Hietarinta, Phys. Rev. E **56**, 2213 (1997).
[2] M. H. Jakubowski, K. Steiglitz, and R. K. Squier, Phys. Rev. E **56**, 7267 (1997).
[3] M. H. Jakubowski, K. Steiglitz, and R. K. Squier, Complex Syst. **10**, 1 (1996).
[4] A. Yariv and P. Yeh, *Nonlinear Waves in Crystals* (Wiley, New York, 1984).
[5] E. Fredkin and T. Toffoli, Int. J. Theor. Phys. **21**, 219 (1982).
[6] R. Squier and K. Steiglitz, Complex Syst. **8**, 311 (1994).
[7] M. H. Jakubowski, K. Steiglitz, and R. K. Squier, in *Proceedings of the 1995 International Conference on Application Specific Array Processors*, Strasbourg, France, 1995, edited by P. Cappello *et al.* (IEEE Comp. Soc. Press, Los Alamitos, CA, 1995).
[8] M. N. Islam, *Ultrafast Fiber Switching Devices and Systems* (Cambridge University Press, Cambridge, 1992).
[9] S. V. Manakov, Sov. Phys. JETP **38**, 248 (1974).
[10] D. J. Kaup and B. A. Malomed, Phys. Rev. A **48**, 599 (1993).
[11] R. Radhakrishnan and M. Lakshmanan, J. Phys. A **28**, 2683 (1995).
[12] L. V. Ahlfors, *Complex Analysis*, 3rd ed. (McGraw-Hill, New York, 1979).
[13] A. Kyrala, *Applied Functions of a Complex Variable* (Wiley-Interscience, New York, 1972).

[14] J. S. Aitchison, A. M. Weiner, Y. Silberberg, M. K. Oliver, J. L. Jackel, D. E. Leaird, E. M. Vogel, and P. W. Smith, Opt. Lett. **15**, 471 (1990).
[15] H. Frauenkron, Y. S. Kivshar, and B. A. Malomed, Phys. Rev. E **54**, R2244 (1996).
[16] M. Segev, G. C. Valley, B. Crosignani, P. DiPorto, and A. Yariv, Phys. Rev. Lett. **73**, 3211 (1994).
[17] M. Shih and M. Segev, Opt. Lett. **21**, 1538 (1996).
[18] M. Shih, M. Segev, G. C. Valley, G. Salamo, B. Crosignani, and P. DiPorto, Electron. Lett. **31**, 826 (1995).
[19] A. W. Snyder and A. P. Sheppard, Opt. Lett. **18**, 482 (1993).
[20] A. W. Snyder, D. J. Mitchell, and Y. S. Kivshar, Mod. Phys. Lett. B **9**, 1479 (1995).
[21] V. Tikhonenko, J. Christou, and B. Luther-Davies, Phys. Rev. Lett. **76**, 2698 (1996).
[22] M. Shih, M. Segev, and G. Salamo, Phys. Rev. Lett. **78**, 2551 (1997).
[23] V. V. Steblina, Y. S. Kivshar, and A. V. Buryak, Opt. Lett. **23**, 156 (1998).
[24] X. D. Cao and D. D. Meyerhofer, Opt. Lett. **19**, 1711 (1994).
[25] D. N. Christodoulides and R. I. Joseph, Opt. Lett. **13**, 53 (1988).
[26] C. R. Menyuk, J. Opt. Soc. Am. B **5**, 392 (1988).
[27] D. N. Christodoulides, S. R. Singh, M. I. Carvalho, and M. Segev, Appl. Phys. Lett. **68**, 1763 (1996).
[28] Z. Chen, M. Segev, T. H. Coskun, and D. N. Christodoulides, Opt. Lett. **21**, 1436 (1996).
[29] M. Mitchell, M. Segev, and D. N. Christodoulides, Phys. Rev. Lett. **80**, 4657 (1998).
[30] J. U. Kang, G. I. Stegeman, J. S. Aitchison, and N. Akhmediev, Phys. Rev. Lett. **76**, 3699 (1996).
[31] Z. Chen, M. Segev, T. H. Coskun, D. N. Christodoulides, and Y. S. Kivshar, J. Opt. Soc. Am. B **14**, 3066 (1997).
[32] D. N. Christodoulides, Phys. Lett. A **132**, 451 (1988).