A determinant-based algorithm for counting perfect matchings in general graphs

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Problem Statement

- Given any graph $G$, count the number of perfect matchings it contains, $M(G)$.

- Equivalent to computing the permanent in the case of bipartite graphs, and therefore #P-complete [Valiant 1979].

- Goal is a fully polynomial randomized approximation scheme (fpras); i.e., an algorithm that produces an $\varepsilon$-approximation to $M(G)$ in time polynomial in $n$ and $\frac{1}{\varepsilon}$. 
Some Related Work

- **Markov Chain Monte Carlo**: Fully polynomial randomized approximation scheme (fpras) for bipartite case [JSV 2001].
  
  - Requires time polynomial in $N(G)/M(G)$ for general graphs, where $N(G)$ is the number of near-perfect matchings in $G$.

- **Determinants**: Achieves a good approximation in time $O((3/2)^{n/2})$ [GG 1981; KLLL 1993; Barvinok 1999,2000; CRS 2002].
Given a bipartite graph $G = (U = [n], V = [n], E)$, construct its adjacency matrix $A$:

\[
A = \begin{pmatrix}
1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}
\]
The Godsil-Gutman Estimator

From $A$, create a random matrix $B$ by replacing each 1-entry of $A$ with a uniform random element of $\{\pm 1\}$.

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Output $X_A = |\det(B)|^2$. 
Analysis of the GG Estimator

- [GG 1981] $X_A$ is unbiased; i.e., $E[X_A] = \text{per} A$.

- [KKLLL 1993] The worst-case critical ratio of $X_A$ is bounded by

$$2^{n/2} \leq \frac{E[X_A^2]}{E[X_A]^2} \leq 3^{n/2}.$$

By choosing elements of $B$ from $\{\pm 1, \pm i\}$, the bounds improve to

$$(3/2)^{n/2} \leq \frac{E[X_A^2]}{E[X_A]^2} \leq 2^{n/2}.$$

- [FJ 1995] For almost every graph, the critical ratio of $X_A$ is bounded by a polynomial.
Analysis of the GG Estimator

• [Barvinok 1999,2000] A variation of the Godsil-Gutman estimator using quaternions results in a $O(1.31^n)$-approximation w.h.p. in polynomial time.

• [CRS 2002] By choosing elements of $B$ from $\{\pm 1, \pm i, \pm j, \pm k\}$, the bounds improve again to

$$\left(\frac{5}{4}\right)^{n/2} \leq \frac{\mathbf{E}[X_A^2]}{\mathbf{E}[X_A]^2} \leq \left(\frac{3}{2}\right)^{n/2}.$$ 

The critical ratio can be reduced to a constant by using elements from high-dimensional Clifford algebras, though the resulting estimator is not known to be efficiently computable.
Our Results

We define a new determinant-based estimator $X_G$ for the number of perfect matchings $M(G)$ in a general graph $G = (V = [2n], E)$ with the following properties:

**Theorem 1:** The estimator is unbiased; i.e., $E[X_G] = M(G)$.

**Theorem 2:** The worst-case critical ratio of $X_G$ is bounded by

$$(7/3)^{n/2} \leq \frac{E[X_G^2]}{E[X_G]^2} \leq 3^{n/2}.$$ 

**Theorem 3:** Let $\omega(n)$ be any function tending to infinity. Then almost every graph $G \in G_{2n,1/2}$ satisfies $\frac{E[X_G^2]}{E[X_G]^2} \leq \alpha n \omega(n)$ for a fixed constant $\alpha$. 
The Tutte Matrix

Given a graph $G = (V = [2n], E)$, its Tutte matrix $T$ is a $2n \times 2n$ matrix defined as

$$
t_{ij} = \begin{cases} 
x_{ij} & \text{if } \{i, j\} \in E \text{ and } i < j \\
-x_{ij} & \text{if } \{i, j\} \in E \text{ and } i > j \\
0 & \text{otherwise}
\end{cases}
$$

The polynomial $\det(T)$ is not identically zero if and only if $G$ contains a perfect matching.
An Estimator for General Graphs

Given graph $G$, construct its Tutte matrix $T$.

$$G = \begin{array}{c}
1 & 2 \\
3 & 4 & 5 \\
6 & 7 & 8 & 9 \\
\end{array}$$

$$T = \begin{pmatrix}
0 & x_{12} & x_{13} & x_{14} & 0 & x_{16} \\
-x_{12} & 0 & x_{23} & 0 & 0 & 0 \\
-x_{13} & -x_{23} & 0 & 0 & x_{35} & 0 \\
-x_{14} & 0 & 0 & 0 & x_{45} & x_{46} \\
0 & 0 & -x_{35} & -x_{45} & 0 & x_{56} \\
-x_{16} & 0 & 0 & -x_{46} & -x_{56} & 0 \\
\end{pmatrix}$$
An Estimator for General Graphs

From $T$, create a random matrix $B$ by replacing each variable $x_{ij}$ in $T$ with a uniform random element of $\{\pm 1\}$.

$$T = \begin{pmatrix}
0 & x_{12} & x_{13} & x_{14} & 0 & x_{16} \\
-x_{12} & 0 & x_{23} & 0 & 0 & 0 \\
-x_{13} & -x_{23} & 0 & 0 & x_{35} & 0 \\
-x_{14} & 0 & 0 & 0 & x_{45} & x_{46} \\
0 & 0 & -x_{35} & -x_{45} & 0 & x_{56} \\
-x_{16} & 0 & 0 & -x_{46} & -x_{56} & 0
\end{pmatrix}$$

$$B = \begin{pmatrix}
0 & 1 & 1 & -1 & 0 & 1 \\
-1 & 0 & -1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 1 & 1 & 0 & -1 \\
-1 & 0 & 0 & -1 & 1 & 0
\end{pmatrix}$$

Output $X_G = \det(B)$. 


Proof of Unbiasedness

Proof:

\[ X_G = \det(B) = \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \prod_{i=1}^{2n} b_{i,\sigma_i} \]

Use equivalence between permutations in \( B \) and directed cycle covers in \( G \):

\[
\begin{align*}
&b_{12}b_{23}b_{31}b_{45}b_{56}b_{64} \\
&b_{12}b_{23}b_{31}b_{45}b_{56}b_{64}
\end{align*}
\]

\[
\begin{align*}
&b_{16}b_{23}b_{32}b_{45}b_{54}b_{61} \\
&b_{16}, b_{61}
\end{align*}
\]
Proof of Unbiasedness

Let $\mathcal{C}(G)$ be the set of cycle covers in $G$. Now we can write

$$X_G = \sum_{\kappa \in \mathcal{C}(G)} \text{sgn}(\kappa) \prod_{(i,j) \in \kappa} b_{ij}$$

and therefore

$$\mathbb{E}[X_G] = \sum_{\kappa \in \mathcal{C}(G)} \text{sgn}(\kappa) \mathbb{E}\left[ \prod_{(i,j) \in \kappa} b_{ij} \right].$$

Claim:

- Cycle covers corresponding to perfect matchings contribute 1 to the sum.
- The expected contribution of all other cycle covers is 0.
The Critical Ratio

We now need to show that the critical ratio is bounded, or \( \frac{E[X_G^2]}{E[X_G]^2} \leq 3^{n/2} \).

First,

\[
E[X_G]^2 = \sum_{H \in E(G)} 2^{c(H)}.
\]
The Critical Ratio

Lemma:

\[ X_G = \sum_{\kappa \in \mathcal{E}(G')} \text{sgn}(\kappa) \prod_{(i,j) \in \kappa} b_{ij}, \]

where \( \mathcal{E}(G) \subseteq \mathcal{C}(G) \) consists of only those cycle covers in which every cycle is of even length.

Therefore,

\[ \mathbb{E}[X_G^2] = \sum_{(\kappa, \kappa') \in \mathcal{E}(G) \times \mathcal{E}(G')} \text{sgn}(\kappa, \kappa') \mathbb{E} \left[ \prod_{(i,j) \in \kappa} b_{ij} \prod_{(i,j) \in \kappa'} b_{ij} \right] \]

\[ = \sum_{H \in \mathcal{E}(G)} 6^c(H) \]
The Critical Ratio

Hence,

\[
\frac{\mathbb{E}[X^2_G]}{\mathbb{E}[X_G]^2} = \frac{\sum_{H \in E(G)} 6^{c(H)}}{\sum_{H \in E(G)} 2^{c(H)}} \leq \max_{H \in E(G)} \frac{6^{c(G)}}{2^{c(G)}} \leq 3^{n/2}.
\]
Consider the behavior of $X_G$ on a bipartite graph $G = (U, V, E)$, where the vertices in $U$ precede those in $V$:

\[
G = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\quad
T = \begin{pmatrix}
0 & T_0 \\
-T'_0 & 0
\end{pmatrix}
\quad
B = \begin{pmatrix}
0 & B_0 \\
-B'_0 & 0
\end{pmatrix}
\]

Hence $X_G = \det(B) = |\det(B_0)|^2$, and is equivalent to the Godsil-Gutman estimator $X_A$ on bipartite graphs.
Lower Bounds on the Critical Ratio

\[ \left( \frac{7}{3} \right)^{n/2} \]
Open Problems

- Extend the new estimator to complex numbers as in [KKLLL 1993] and possibly beyond.
- Close the gap between the upper and lower bounds on the worst-case critical ratio.