A determinant-based algorithm for counting perfect matchings in general graphs

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Problem Statement

- Given any graph G, count the number of perfect matchings it contains, $M(G)$.

- Equivalent to computing the permanent in the case of bipartite graphs, and therefore #P-complete [Valiant 1979].

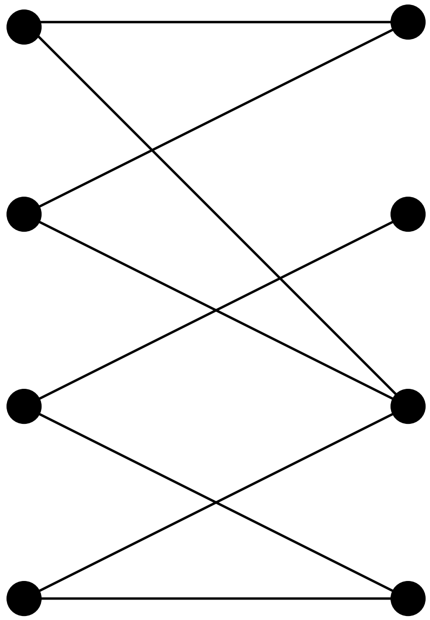
- Goal is a fully polynomial randomized approximation scheme (fpras); i.e., an algorithm that produces an ε-approximation to $M(G)$ in time polynomial in n and $\frac{1}{\varepsilon}$.

Some Related Work

- Markov Chain Monte Carlo: Fully polynomial randomized approximation scheme (fpras) for bipartite case [JSV 2001].
  - Requires time polynomial in $N(G)/M(G)$ for general graphs, where $N(G)$ is the number of near-perfect matchings in G.
- Determinants: Achieves a good approximation in time $O((3/2)^{n/2})$ [GG 1981; KKLLL 1993; Barvinok 1999,2000; CRS 2002].

The Godsil-Gutman Estimator

Given a bipartite graph $G = (U = [n], V = [n], E)$, construct its adjacency matrix A:

$G =$ 

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The Godsil-Gutman Estimator

From A, create a random matrix B by replacing each 1-entry of A with a uniform random element of $\{\pm 1\}$.

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Output $X_A = |\det(B)|^2$.

Analysis of the GG Estimator

- [GG 1981] X_A is unbiased; i.e., $\mathsf{E}[X_A] = \mathsf{per} A$.

- [KKLLL 1993] The worst-case critical ratio of X_A is bounded by

$$2^{n/2} \leq \frac{\mathsf{E}[X_A^2]}{\mathsf{E}[X_A]^2} \leq 3^{n/2}.$$

  By choosing elements of B from $\{\pm 1, \pm i\}$, the bounds improve to

$$(3/2)^{n/2} \leq \frac{\mathsf{E}[X_A^2]}{\mathsf{E}[X_A]^2} \leq 2^{n/2}.$$

- [FJ 1995] For almost every graph, the critical ratio of X_A is bounded by a polynomial.

Analysis of the GG Estimator

- [Barvinok 1999,2000] A variation of the Godsil-Gutman estimator using quaternions results in a $O(1.31^n)$-approximation w.h.p. in polynomial time.

- [CRS 2002] By choosing elements of B from $\{\pm 1, \pm i, \pm j, \pm k\}$, the bounds improve again to

$$(5/4)^{n/2} \leq \frac{\mathsf{E}[X_A^2]}{\mathsf{E}[X_A]^2} \leq (3/2)^{n/2}.$$

The critical ratio can be reduced to a constant by using elements from high-dimensional *Clifford algebras*, though the resulting estimator is not known to be efficiently computable.

Our Results

We define a new determinant-based estimator X_G for the number of perfect matchings $M(G)$ in a general graph $G = (V = [2n], E)$ with the following properties:

Theorem 1: The estimator is unbiased; i.e., $\mathsf{E}[X_G] = M(G)$.

Theorem 2: The worst-case critical ratio of X_G is bounded by

$$(7/3)^{n/2} \leq \frac{\mathsf{E}[X_G^2]}{\mathsf{E}[X_G]^2} \leq 3^{n/2}.$$

Theorem 3: Let $\omega(n)$ be any function tending to infinity. Then almost every graph $G \in \mathcal{G}_{2n,1/2}$ satisfies $\frac{\mathsf{E}[X_G^2]}{\mathsf{E}[X_G]^2} \leq \alpha n \omega(n)$ for a fixed constant α.

The Tutte Matrix

Given a graph $G = (V = [2n], E)$, its Tutte matrix T is a $2n \times 2n$ matrix defined as

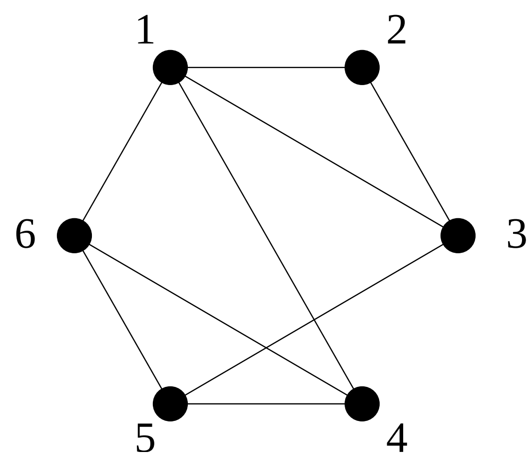
$$t_{ij} = \begin{cases} x_{ij} & \text{if } \{i,j\} \in E \text{ and } i < j \\ -x_{ij} & \text{if } \{i,j\} \in E \text{ and } i > j \\ 0 & \text{otherwise} \end{cases}$$

The polynomial $\mathtt{det}(T)$ is not identically zero if and only if G contains a perfect matching.

An Estimator for General Graphs

Given graph G, construct its Tutte matrix T.

$G =$



$$T = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & 0 & x_{16} \\ -x_{12} & 0 & x_{23} & 0 & 0 & 0 \\ -x_{13} & -x_{23} & 0 & 0 & x_{35} & 0 \\ -x_{14} & 0 & 0 & 0 & x_{45} & x_{46} \\ 0 & 0 & -x_{35} & -x_{45} & 0 & x_{56} \\ -x_{16} & 0 & 0 & -x_{46} & -x_{56} & 0 \end{pmatrix}$$

An Estimator for General Graphs

From T, create a random matrix B by replacing each variable x_{ij} in T with a uniform random element of $\{\pm 1\}$.

$$T = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & 0 & x_{16} \\ -x_{12} & 0 & x_{23} & 0 & 0 & 0 \\ -x_{13} & -x_{23} & 0 & 0 & x_{35} & 0 \\ -x_{14} & 0 & 0 & 0 & x_{45} & x_{46} \\ 0 & 0 & -x_{35} & -x_{45} & 0 & x_{56} \\ -x_{16} & 0 & 0 & -x_{46} & -x_{56} & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 & 1 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

Output $X_G = \det(B)$.

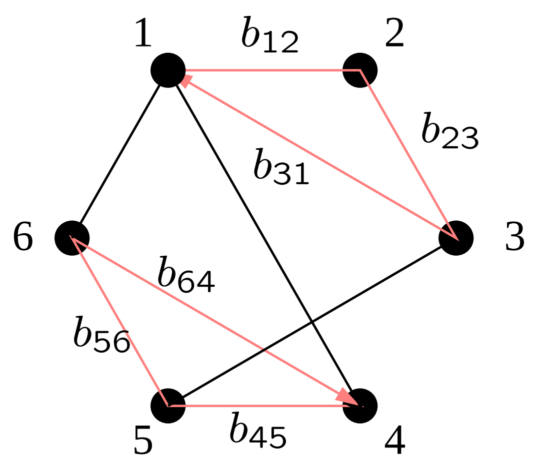
Proof of Unbiasedness

Proof:

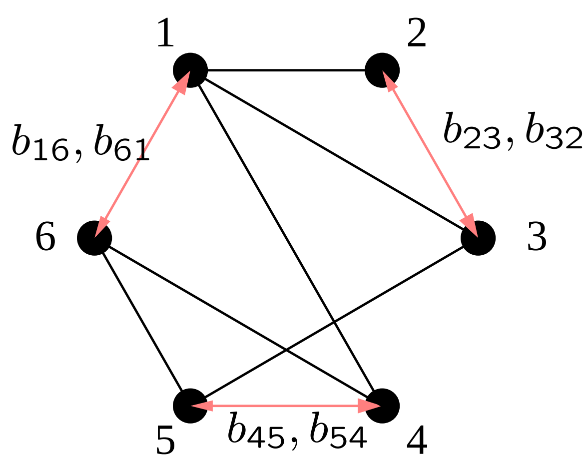
$$X_G = \det(B) = \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{2n} b_{i,\sigma i}$$

Use equivalence between permutations in B and directed cycle covers in G:

$b_{12}b_{23}b_{31}b_{45}b_{56}b_{64}$



$b_{16}b_{23}b_{32}b_{45}b_{54}b_{61}$



Proof of Unbiasedness

Let $\mathcal{C}(G)$ be the set of cycle covers in G. Now we can write

$$X_G = \sum_{\kappa \in \mathcal{C}(G)} \mathsf{sgn}(\kappa) \prod_{(i,j)\in\kappa} b_{ij}$$

and therefore

$$\mathsf{E}[X_G] = \sum_{\kappa \in \mathcal{C}(G)} \mathsf{sgn}(\kappa)\mathsf{E}[\prod_{(i,j)\in\kappa} b_{ij}].$$

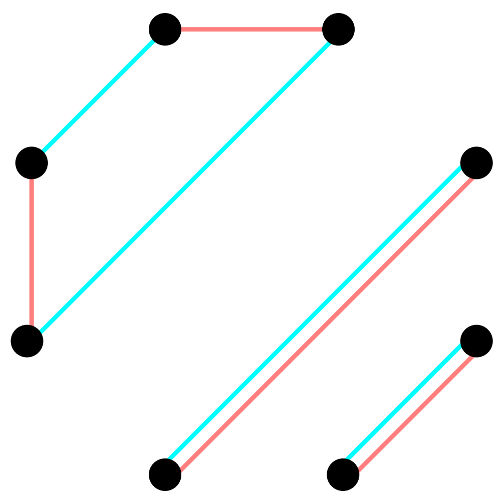
Claim:

- Cycle covers corresponding to perfect matchings contribute 1 to the sum.
- The expected contribution of all other cycle covers is 0.

The Critical Ratio

We now need to show that the critical ratio is bounded, or $\frac{\mathsf{E}[X_G^2]}{\mathsf{E}[X_G]^2} \leq 3^{n/2}$. First,

$$\mathsf{E}[X_G]^2 = \sum_{H \in E(G)} 2^{c(H)}.$$



The Critical Ratio

Lemma:

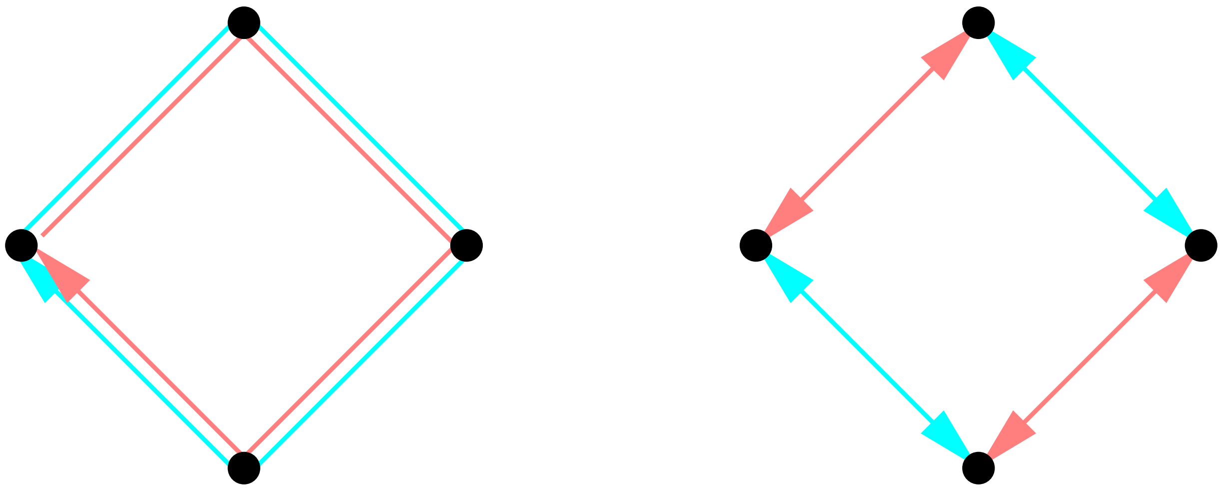
$$X_G = \sum_{\kappa \in \mathcal{E}(G)} \mathsf{sgn}(\kappa) \prod_{(i,j) \in \kappa} b_{ij},$$

where $\mathcal{E}(G) \subseteq \mathcal{C}(G)$ consists of only those cycle covers in which every cycle is of even length.

Therefore,

$$\begin{aligned} \mathsf{E}[X_G^2] &= \sum_{(\kappa,\kappa') \in \mathcal{E}(G) \times \mathcal{E}(G)} \mathsf{sgn}(\kappa, \kappa') \mathsf{E}[\prod_{(i,j) \in \kappa} b_{ij} \prod_{(i,j) \in \kappa'} b_{ij}] \\ &= \sum_{H \in E(G)} 6^{c(H)} \end{aligned}$$

The Critical Ratio

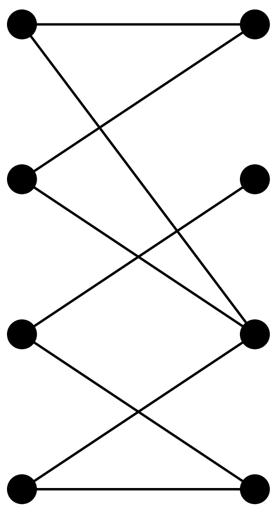


Hence,

$$\frac{\mathsf{E}[X_G^2]}{\mathsf{E}[X_G]^2} = \frac{\sum_{H\in E(G)} 6^{c(H)}}{\sum_{H\in E(G)} 2^{c(H)}} \leq \max_{H\in E(G)} \frac{6^{c(G)}}{2^{c(G)}} \leq 3^{n/2}.$$

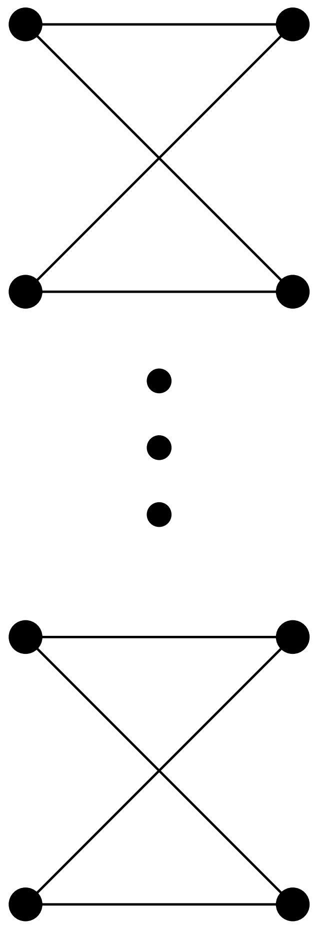
Comparison with GG Estimator

Consider the behavior of X_G on a bipartite graph $G = (U, V, E)$, where the vertices in U precede those in V:

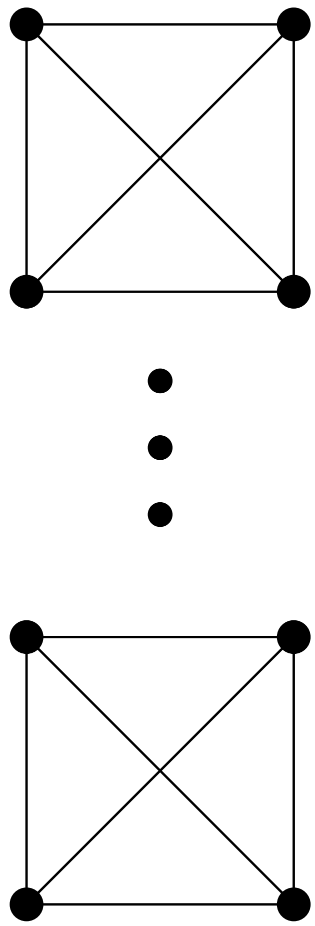
$G =$  $T = \begin{pmatrix} 0 & T_0 \\ -T_0' & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & B_0 \\ -B_0' & 0 \end{pmatrix}$

Hence $X_G = \mathtt{det}(B) = |\mathtt{det}(B_0)|^2$, and is equivalent to the Godsil-Gutman estimator X_A on bipartite graphs.

Lower Bounds on the Critical Ratio



$2^{n/2}$



$(7/3)^{n/2}$

Open Problems

- Extend the new estimator to complex numbers as in [KKLLL 1993] and possibly beyond.
- Close the gap between the upper and lower bounds on the worst-case critical ratio.