Energy of convex sets, shortest paths and resistance

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Abstract

Let us assign independent, exponentially distributed random edge lengths to the edges of an undirected graph. Lyons, Pemantle and Peres [11] proved that the expected length of the shortest path between two given nodes is bounded from below by the resistance between these nodes, where the resistance of an edge is the expectation of its length. They remarked that instead of exponentially distributed variables, one could consider random variables with a log-concave tail.

We generalize this result in two directions. First, we note that the variables don't have to be independent: it suffices to assume that their joint distribution is log-concave. Second, the inequality can be formulated as follows: the expected length of a shortest path between two given nodes is the expected optimum of a stochastic linear program over a flow polytope, while the resistance is the minimum of a convex quadratic function over this polytope. We show that the inequality between these quantities holds true for an arbitrary polytope provided its blocker has 0-1 vertices.

1 Introduction

Let $G = (V, E)$ be an undirected graph, let $s, t \in V$, and let us assign random nonnegative edge lengths W_i $(i \in E)$ to the edges of G. The W_i are independent exponentially distributed random variables, with $w_i = \mathsf{E}(W_i)$. Lyons, Pemantle and Peres [11] proved that the expected length of the shortest path between s and t is bounded from below by the resistance between these nodes, where the resistance of an edge i is w_i. They prove a similar lower bound for the "dual" problem, namely the expected minimum cut capacity. They also show (in a slightly different form) that instead of the exponential distribution of the edge lengths, one could allow more general distributions whose survival function $\mathsf{P}(W_i \geq t)$ is log-concave. Their estimate is tight (up to a factor of 2) for a certain class of series-parallel graphs.

In this note we generalize this result in two directions. First the variables don't have to be independent; it suffices to assume that the joint distribution of the W_i are such that the joint survival function $\mathsf{P}(W_1 \geq t_1, \ldots, W_n \geq t_n)$ is log-concave.

To formulate our second generalization, we need some preliminaries from optimization. Let $K \subseteq \mathbb{R}^n_+$ be a closed convex set. We say that K is *recessive*, if $x \in K$, $y \geq x$ implies that $y \in K$. For every convex set $K_0 \subseteq \mathbb{R}^n_+$, its *dominant*

$$K = \{x \in \mathbb{R}^n_+ \ : \ x \geq y \text{ for some } y \in K\}$$

is a recessive convex set.

Among recessive convex sets, one can define a version of polarity called *blocking*, introduced by Fulkerson [3]. The *blocker* of a recessive convex set $K \subseteq \mathbb{R}^n_+$ is defined by

$$K^* = \{x \in \mathbb{R}^n_+ \ : \ y^\mathsf{T} x \geq 1 \text{ for all } y \in K\}.$$

The blocker relation has the standard properties of duality; in particular, the blocker of the blocker is the original recessive convex set. Thus it makes sense to talk about a "blocking pair".

Our starting example of a blocking pair is the following. Let P_0 consist of all (s, t)-flows in G of value 1, and let P be the dominant of P_0. Then P^* is the dominant of the convex hull of incidence vectors of (s, t)-cuts. In terms of these polyhedra, the expected length of a shortest (s, t)-path is the expected optimum of the random linear function $W^\mathsf{T} x$ over P, while the resistance between s and t is the minimum of the convex quadratic function $\sum_i w_i x_i^2$ over this polytope (see section 3).

With this notation, the inequalities of Lyons, Pemantle and Peres can be formulated as follows:

$$\mathsf{E}\left(\min\{W^{\mathsf{T}}x \;:\; x \in P\}\right) \geq \min\left\{\sum_i w_i x_i^2 \;:\; x \in P\right\} \tag{1}$$

and

$$\mathsf{E}\left(\min\{W^{\mathsf{T}}x \;:\; x \in P^*\}\right) \geq \min\left\{\sum_i w_i x_i^2 \;:\; x \in P^*\right\}$$

It would be natural to conjecture that these inequalities hold for all recessive convex sets K in place of P, but this is false; the two sides don't even scale in the same way under homothetical transformations of K. However, the two inequalities are always equivalent. Our main result (Theorem 7) is that they do hold under the following assumption: *every non-zero entry of every vertex of K^* is at least 1.* This includes the combinatorially interesting case when K^* has integer vertices. For the general case, a (weaker) inequality will also be given in section 4.

The quantity on the right hand side of 1, which we call the *energy* of P (for the justification of the name, see example 1), has some interest on its own right. It behaves nicely for blocking pairs, and generalizes some interesting combinatorial quantities (see section 3).

The result is most interesting when *both* K and K^* have 0-1 vertices. In this case, each of them can be thought of as a hypergraph, i.e., a collection of subsets of the underlying set $\{1, 2, \ldots, n\}$. In this case Z above can be thought of as the minimum weight of an edge of $\mathcal{H}$, under a random weighting of the vertices. There are many such pairs of hypergraphs arising in combinatorial optimization. Some of these will be discussed in section 7.

2 Preliminaries: blocking polyhedra and blocking clutters

Let $K \subseteq \mathbb{R}_+^n$ be a closed convex set. We call K *recessive*, if $x \in K$, $y \geq x$ implies $y \in K$. For every convex set K, its *dominant*

$$K + \mathbb{R}_+^n = \{x \in \mathbb{R}^n : \; \exists y \in K, y \leq x\}$$

is recessive. We denote by K^* the *blocker* of K:

$$K^* = \{x \in \mathbb{R}_+^n \;:\; y^{\mathsf{T}}x \geq 1 \;\; \forall y \in K\}.$$

It is clear that K^* is recessive. Furthermore, if K is recessive then $(K^*)^* = K$ [3].

There are important relations between optimization over K and its blocker K^*. To formulate them, we need the following notation. For $c \in \mathbb{R}^n$, $c > 0$ we denote by c^* the vector $(c_1^{-1}, \ldots, c_n^{-1})$. Furthermore, for $c \in \mathbb{R}^n_+$, let

$$\mathcal{L}(K, c) = \min\{c^\mathsf{T} x : \ x \in K\}.$$

Lemma 1 *For every recessive convex set $K \subseteq \mathbb{R}^n_+$ and $c, d \in \mathbb{R}^n_+$,*

$$\mathcal{L}(K, c)\mathcal{L}(K^*, d) \leq c^\mathsf{T} d. \tag{2}$$

If $c > 0$, then

$$1 \leq \mathcal{L}(K, c)\mathcal{L}(K^*, c^*) \leq n. \tag{3}$$

Furthermore,

$$\min\{c^\mathsf{T} x \ : \ x \in K\} = \max\left\{t : \ \frac{1}{t}c \in K^*\right\} = \max_{y \in K^*} \min_i \frac{c_i}{y_i}. \tag{4}$$

Proof. (2) is due to Fulkerson [3]. The upper bound in (3) is a special case of (2), while the lower bound follows since

$$(c^\mathsf{T} x)(c^{*\mathsf{T}} y) = \sum_{i,j} \frac{c_i}{c_j} x_i y_j \geq \sum_i x_i y_i \geq 1$$

for all $x \in K$ and $y \in K^*$.

(4) By definition,

$$t \leq \mathcal{L}(K, c) \iff c^\mathsf{T} x \geq t \quad \forall x \in K \iff (1/t)c \in K^*,$$

proving the first equality. The second is an immediate rewriting of the first. □

Combinatorial applications of blocking pairs of polyhedra involve blocking clutters. A *clutter* is a hypergraph in which no edge is included in another. We may restrict our attention to *clutters*, i.e.,; this is no loss of generality, since we are minimizing monotone increasing objective functions, and so edges containing another edge play no role. The *rank* $r(\mathcal{H})$ of a clutter is the minimum cardinality of its edges. The *blocking number* $\tau(\mathcal{H})$ is the minimum cardinality of a set meeting all edges. We set $n = |V|$.

The *blocking clutter* $\mathcal{H}^b$ of $\mathcal{H}$ consists of those minimal subsets of V which intersect every member of $\mathcal{H}$. It is not difficult to see that in this case $\mathcal{H}$ is the blocking clutter of $\mathcal{H}^b$. The pair of clutters $(\mathcal{H}, \mathcal{H}^b)$ is called a *blocking pair*. Clearly $r(\mathcal{H}^b) = \tau(\mathcal{H})$.

Define $D(\mathcal{H})$ as the dominant of the convex hull of incidence vectors of edges of $\mathcal{H}$. For $c \in \mathbb{R}_+^V$ and $A \subseteq V$, we use the notation $c(A) = \sum_{i \in A} c_i$. Define

$$\mathcal{L}(\mathcal{H}, c) = \min\{c(A) : A \in \mathcal{H}\} = \mathcal{L}(D(\mathcal{H}), c).$$

It is easy to see that for every clutter $\mathcal{H}$, we have

$$D(\mathcal{H}^b) \subseteq (D(\mathcal{H}))^*, \tag{5}$$

or equivalently,

$$D(\mathcal{H}) \subseteq (D(\mathcal{H}^b))^*. \tag{6}$$

We say that $\mathcal{H}$ has the *Max-Flow-Min-Cut property* if equality holds here. In more explicit terms, this means that for every weighting $w : V \to \mathbb{R}_+$, the minimum weight of an edge of $\mathcal{H}^b$ is equal to the maximum of $\sum_j y_j$, where y ranges over all weightings $y : E(\mathcal{H}^*) \to \mathbb{R}_+$ with the property that $\sum_{j \ni i} y_j \le w_i$ for each $i \in V$.

There are various characterizations of clutters with the Max-Flow-Min-Cut property. Among others, Lehmann [10] proved that if H has this property then so does its blocking clutter $\mathcal{H}^b$. See [6] for more information.

3 Energy

Let $K \subseteq \mathbb{R}^n$ be a closed convex set and $c \in \mathbb{R}^n$, $c > 0$ a vector of weights. The *energy of K with respect to c* is defined as

$$\mathcal{E}(K, c) = \min\{\sum_i c_i x_i^2 \ : \ x \in K\}.$$

In particular, the energy of K with respect to the all-1 vector $\mathbb{1}$ is the squared distance of K from the origin. Note that since the objective function is strictly convex, there is always a unique optimum solution.

The energy of a convex set and its blocker are closely related.

Lemma 2

$$\mathcal{E}(K, c)\mathcal{E}(K^*, c^*) = 1.$$

It is interesting that equality holds here, while for the minima of linear functions, only the rather week inequalities in (3) can be claimed.

Proof. First, let $x \in K$ and $y \in K^*$. Then

$$\left(\sum_i c_i x_i^2\right)\left(\sum_i \frac{1}{c_i} y_i^2\right) \ge \sum_i (\sqrt{c_i} x_i)\left(\frac{1}{\sqrt{c_i}} y_i\right) = \sum_i x_i y_i \ge 1.$$

Hence $\mathcal{E}(K,c)\mathcal{E}(K^*,c^*) \geq 1$.

Second, let $u \in K$ attain the minimum in the definition of $\mathcal{E} = \mathcal{E}(K,c)$, i.e.

$$\sum_i c_i u_i^2 = \mathcal{E}.$$

The ellipsoid defined by $\sum_i c_i x_i^2 \leq \mathcal{E}$ and the convex set K have no interior point in common, but they touch each other at u, and hence the tangent hyperplane of the ellipsoid at point u separates them. By elementary geometry, this tangent hyperplane is $v^\mathsf{T} x = 1$, where $v_i = (1/\mathcal{E})c_i u_i$. We have

$$v^\mathsf{T} x \leq 1 \qquad \text{for all } x \in K,$$

whence $v \in K^*$. Thus

$$\min\left\{\sum_i c_i^{-1} y_i^2 \ : \ y \in K^*\right\} \leq \sum_i c_i^{-1} v_i^2 = \frac{1}{\mathcal{E}^2}\sum_i c_i x_i^2 = \frac{1}{\mathcal{E}},$$

which proves that $\mathcal{E}(K,c)\mathcal{E}(K^*,c^*) \leq 1$. □

From the proof above, we can read off the following *complementary slackness conditions*:

Lemma 3 *Let $a \in K$ and $b \in K^*$. Then a is the optimizing vector for $\mathcal{E}(K,c)$ and b is the optimizing vector for $\mathcal{E}(K^*,c^*)$ if and only if $a^\mathsf{T} b = 1$, and the vector $(c_i a_i)$ is parallel to b.*

It will not be needed in this paper, but it should be noted that, if K is given by a well-guaranteed separation oracle (see [6] for details), then $\mathcal{E}(K,c)$ can be computed to arbitrary precision in oracle-polynomial time. This follows from the fact that the objective function $\sum_i c_i x_i^2$ is convex and easily computable.

There are some (easy) relations between the $\mathcal{L}$ and $\mathcal{E}$:

Lemma 4 *Let $|c|_1$ denote the ℓ_1-norm of c, then*

$$\mathcal{E}(K,c) \geq \frac{\mathcal{L}(K,c)^2}{|c|_1}.$$

If all extreme points of K are contained in $[0,1]^n$, then

$$\mathcal{E}(K,c) \leq \mathcal{L}(K,c).$$

Proof. The first inequality is just Cauchy-Schwartz, the second follows by considering an extreme point x of K minimizing $c^\mathsf{T} x$, and using that $x_i^2 \leq x_i$. □

We conclude with a few combinatorial examples where the energy of a polytope is meaningful.

Example 1 Flows in undirected graphs and electrical resistance. We start with out introductory example of paths in an undirected graph $G=(V,E)$ connecting two nodes $s,t\in V$. Let $\mathcal{H}$ be the clutter whose vertices are the edges of G, and whose edges are the $s-t$ paths in G. The blocking clutter $\mathcal{H}^b$ consists of all cuts separating s and t (and minimal with respect to inclusion). It is well known (equivalent to the Max-Flow-Min-Cut Theorem) that $D(\mathcal{H})$ is a polyhedron in $\mathbb{R}_+^V$ defined by the inequalities

$$\sum_{i\in C} x_i \ge 1$$

for every cut C separating s and t. Thus $D(\mathcal{H})^*=D(\mathcal{H}^b)$, and so $\mathcal{H}$ has the Max-Flow-Min-Cut property (this is where this name comes from).

Now let a number $c_e>0$ be associated with every edge e. Then $\mathcal{L}(D(\mathcal{H}),c)$ is the length of the shortest path between s and t (if the c_e are interpreted as "lengths"). Moreover, $\mathcal{L}(D(\mathcal{H}^b),c)$ is the minimum capacity of a cut separating s and t (if the c_i are interpreted as "capacities").

The energy $\mathcal{E}(D(\mathcal{H}),c)$ is, as remarked in the introduction, the electrical resistance between s and t (if the c_i are interpreted as "resistances"). This can be derived from Thompson's principle of minimum energy; since I don't know a good reference, let me sketch a proof. First, notice that every flow of value 1 from s to t can be written as a convex combination of $s-t$ paths; conversely, every convex combination of $s-t$ paths majorizes some flow of value 1 from s to t (it is not equal because of cancellations on edges used by different paths in different directions). So

$$\begin{aligned}\mathcal{E}(D(\mathcal{H}),c) &= \min\left\{\sum_{e\in E} c_e x_e^2 :\ x\in D(\mathcal{H})\right\}\\ &= \min\left\{\sum_{e\in E} c_e f_e^2 :\ f \text{ is an } s\text{-}t \text{ flow of value } 1\right\}.\end{aligned}$$

To be precise, we need a reference orientation for the edges to specify f, but because of the squaring, this orientation does not matter. Now $\sum_e c_e f_e^2$ is the energy of f if f is considered an electric current, and by Thompson's principle, it is minimized when f is the true current. Since the intensity of this current between s and t is 1, its energy is the resistance between s and t.

Example 2 Flows in directed paths and traffic jams. Let the directed graph $G = (V, \mathcal{A})$ describe a network, and let each edge ij have a length $c_{ij} > 0$. But unlike in the standard shortest path problem, the length has the following (perhaps more realistic) meaning: if an edge has length c_i, and x units of flow (say, x cars) enter it in unit time, then the time needed to get through is $c_{ij}x$. We want to find a flow of value 1 from s to t that minimizes the average time a car has to travel from s to t. Let us call this minimum the *congestion delay time from s to t.*

Flows from s to t of value 1 form a convex polytope $P \subset \mathbb{R}^{\mathcal{A}}$. Let $K = D(P)$ be its dominant.

Proposition 5 *The congestion delay time from s to t is given by the energy $\mathcal{E}(K, c)$. Furthermore, the minimizing vector $a \in K$ gives a flow in which every car spends exactly this time traveling.*

Proof. Let $x \in K$ be any flow of value 1 from s to t. Let $y_{ij} = c_{ij}x_{ij}$. We can write x as a convex combination of paths from s to t: $x = \sum_k \alpha_k v_k$, where $\alpha_k > 0$, $\sum_k \alpha_k = 1$, and each v_k is the incidence vector of an $s-t$ path. Think of the path v_k as the route of a car. The time this car spends on the trip is $\sum_{ij} c_{ij}x_{ij}$, where the summation extends over all edges of the path. We can write this as $v_k^{\mathsf{T}} y$. If we average this over all v_i, we get $\sum_k \alpha_k v_k^{\mathsf{T}} y = x^{\mathsf{T}} y = \sum_{ij} c_{ij} x_{ij}^2$. The minimum of this (over all choices of $x \in K$ is indeed $\mathcal{E}(K, c)$.

To prove the second statement, suppose that x above is the minimizing vector. By Lemma 3, the vector $(1/\mathcal{E}(K, c))y$ belongs to K^*. Hence $v^{\mathsf{T}} y \geq 1$ for every vertex v of K. Furthermore, we have

$$1 = x^{\mathsf{T}} y = \sum_k \alpha_k v_k^{\mathsf{T}} y \geq \sum_k \alpha_k = 1,$$

and thus $v_k^{\mathsf{T}} y = 1$ for every vertex v_k that occurs in the representation of x, i.e., which is the route of a car. □

Example 3 Multiterminal flows and hitting times in random walks. Let $G = (V, E)$ be a connected undirected graph, and choose a node $t \in V$. Assign a value $\sigma_i > 0$ to each node i so that $\sum_i \sigma_i = 1$.

Let $K \subseteq \mathbb{R}_+^E$ consist of all capacities that admit a flow with supply σ_i at each node i and demand 1 at t. It is not hard to see that K is a recessive polyhedron, and the vertices of K are vectors supported on spanning trees. Given the spanning tree, the flow along its edges with the given supplies and demand is uniquely determined, and it gives the corresponding vertex of K. It follows that for a vector $w \in \mathbb{R}_+^E$ of "lengths", $\mathcal{L}(K,w)$ is the average distance of a node (chosen from the distribution σ) from the node t.

The blocker K^* of K consists of vectors $z \in \mathbb{R}_+^V$ which can be described as follows: if we view z_{ij} as the length of an edge ij, then the average distance of a node i (chosen with probability σ_i) from node t is at least 1. It is not hard to verify that K^* consists of all vectors $z \in \mathbb{R}_+^E$ such that there exists a vector $x \in \mathbb{R}_+^V$ such that

$$\sum_i \pi_i x_i = 1 \quad \text{and} \quad |x_i - x_j| \le z_{ij}$$

for each $ij \in E$.

The vertices of K^* are of the form $\frac{1}{\sigma(S)}\chi^{\nabla(S)}$, where $S \subseteq V\setminus\{t\}$, $S \neq \emptyset$ (these vectors are obviously in K^*, but not all of them are vertices in general). Thus $\mathcal{L}(K*,w)$ is the answer to a certain isoperimetric problem: find the minimum of $w(\nabla(S))/\sigma(S)$ over all sets $S \subseteq V \setminus \{t\}$, $S \neq \emptyset$.

The reason for considering this not too exciting extension of example 1 is that the energy of K is interesting. Consider the random walk on G; the stationary probability of node i is $\pi_i = d_i/(2m)$, where d_i is the degree of the node i and m is the number of edges.

Let us choose a random node from the stationary distribution π, and walk until we hit the node t. The expected number $H(\pi,t)$ of steps is an important parameter in the theory of random walks.

Aldous and Fill [1] (Chapter 3, Proposition 37) state, as a version of Thompson's formula, that $H(\pi,t)$ is the minimum of $2m\sum_{ij\in E} f_{ij}^2$, where the minimum extends over all flows with supply π_i at each node i and demand 1 at node t. In our formulation, this means that

$$H(\pi,t) = \mathcal{E}(K, 2m\mathbb{1}). \tag{7}$$

They also state a dual form:

$$H(\pi,t) = \max\left\{\frac{2m}{\sum_{ij\in E}(x_i - x_j)^2} \,:\, x_t = 0,\ \sum_i \pi_i x_i = 1\right\}.$$

It is not hard to see that this is equivalent to

$$H(\pi,t) = \frac{1}{\mathcal{E}(K^*, (1/2m)\mathbb{1})},$$

which is equivalent to (7) by Lemma 2.

Example 4 Let $G = (V, E)$ be a bipartite graph with bipartition $V = A \cup B$, where $|A| = |B| = k$; assume that G has a perfect matching. Consider the clutter $\mathcal{H}$ of all perfect matchings. Then $D(\mathcal{H})$ is the dominant of the perfect matching polytope. By König's Theorem, the blocking clutter of $\mathcal{H}$ consists of sets of the form $E(X, Y)$, where $X \subset A$, $Y \subset B$ and $|X| + |Y| = k+1$ (here $E(X, Y)$ denotes the set of edges between X and Y).

The energy of the polyhedron $D(\mathcal{H})$ was studied by Frank and Karzanov [7]. Among others, they gave a combinatorial polynomial time algorithm to find the distance of the polyhedron from the origin.

4 A stochastic optimization problem

Let K be a recessive convex set in $\mathbb{R}^n_+$ and let the W_i, $1 \le i \le n$ be random variables whose joint survival function is log-concave. We set $W = (W_1, \ldots, W_n)$ and $w_i = \mathsf{E}(W_i)$. Consider the optimum

$$\mathcal{L}(K, W) = \min\{W^\mathsf{T} x \ : \ x \in K\};$$

This is a random variable, and we are interested in its expectation. Finding the exact value of the expectation $\mathsf{E}(\mathcal{L}(K, W))$ is difficult in general, and it belongs to the field of *stochastic optimization*. It is interesting to point out that log-concave functions play an important role in this field; see Prékopa [12] for a survey. In this paper, we are interested in obtaining bounds on $\mathsf{E}(\mathcal{L}(K, W))$.

We start with two simple bounds valid for every recessive convex set K.

Proposition 6

$$\frac{1}{\mathcal{L}(K^*, w^*)} \le \mathsf{E}(\mathcal{L}(K, W)) \le \mathcal{L}(K, w).$$

These upper and lower bounds can be rather week. Our main result is that under an assumption that includes the combinatorially interesting case when K^* has integral vertices, a better lower bound can be given:

Theorem 7 *Let $K \subseteq \mathbb{R}^n_+$ be a recessive convex polyhedron, and let W be a nonnegative random vector whose survival function $\mathsf{P}(W \ge t)$ is log-concave (as a function of $t \in \mathbb{R}^n_+$). Assume that every non-zero entry of every vertex of K^* is at least 1. Then*

$$\mathsf{E}(\mathcal{L}(K, W)) \ge \mathcal{E}(K, W).$$

The proof will be given in section 6; here we make a few remarks. By Lemmas 2 and 4,

$$\mathcal{E}(K,w) = \frac{1}{\mathcal{E}(K^*,w^*)} \geq \frac{1}{\mathcal{L}(K^*,w^*)},$$

showing that the bound in theorem 7 is always better than the bound in Lemma 6. Let us compare the bounds and the actual expectation in the case of a simple example.

Example 5 Let $n = k^2$ ($k \geq 2$) and consider the polyhedron $K \subseteq \mathbb{R}^n_+$ defined by the inequalities

$$x_1 + \ldots + x_k \geq 1,\; x_{k+1} + \ldots + x_{2k} \geq 1,\; x_{(k-1)k+1} + \ldots + x_{k^2} \geq 1.$$

The vertices of K are all 0-1 vectors containing exactly one 1 in each of the blocks $\{x_1,\ldots,x_k\}$, $\{x_{k+1},\ldots,x_{2k}\}$, $\{x_{(k-1)k+1},\ldots,x_{k^2}\}$. The vertices of K^* are all 0-1 vectors which are 1 in one of these blocks and 0 outside.

It is clear from the symmetries that the minima in both $\mathcal{L}(K,\mathbb{1})$ and $\mathcal{E}(K,\mathbb{1})$ are attained by the vector $(1/k,1/k,\ldots,1/k)$, and hence

$$\mathcal{L}(K,\mathbb{1}) = \frac{n}{k} = k, \qquad \mathcal{E}(K,\mathbb{1}) = \frac{n}{k^2} = 1.$$

Similarly, we get that

$$\mathcal{L}(K^*,\mathbb{1}) = k, \qquad \mathcal{E}(K^*,\mathbb{1}) = 1.$$

Let $W_1,\ldots,W_n$ be independent exponentially distributed random variables with expectation 1. The value of $\mathsf{E}(\mathcal{L}(K,W))$ is easy to compute:

$$\mathcal{L}(K,W) = \sum_{i=0}^{k-1} \min_{1\leq j\leq k} W_{ik+j},$$

and hence by proposition 8

$$\mathsf{E}(\mathcal{L}(K,W)) = \sum_{i=0}^{k-1} \mathsf{E}\left(\min_{1\leq j\leq k} W_{ik+j}\right) = k\cdot\frac{1}{k} = 1.$$

The expectation of $\mathcal{L}(K^*,W)$ is more difficult to compute, but for our purposes it is enough to note that

$$\mathcal{L}(K^*,W) = \min_{1\leq j\leq k} \sum_{i=0}^{k-1} W_{ik+j},$$

and (for large k) each sum on the right hand side is highly concentrated around its expectation k, showing that

$$\mathsf{E}(\mathcal{L}(K^*,W)) \sim k.$$

The lower bounds on $\mathsf{E}(\mathcal{L}(K,W))$ and on $\mathsf{E}(\mathcal{L}(K^*,W))$ provided by Lemma 6 and the Main Theorem are $1/r$ and 1, respectively, so the Main Theorem is better. The upper bound given by Lemma 6 is k. So we see that for $\mathcal{L}(K,W)$, our new lower bound is tight, but for $\mathcal{L}(K^*,W)$, it is far from the truth. We'll discuss more interesting examples in section 7.

5 A probabilistic lemma

We prove a lemma about nonnegative log-concave random variables. Examples of such variables include those uniformly distributed on $[0,a]$ for some a, or exponentially distributed with some parameter. First, let us quote the following well-known fact.

Proposition 8 *Let $W_1,\dots,W_n$ be independent, exponentially distributed random variables. Then $\min\{W_1,\dots,W_n\}$ is also exponentially distributed, with expectation*

$$\mathsf{E}(\min_i W_i) = \left(\sum_i \frac{1}{E(W_i)}\right)^{-1}.$$

The following lemma extends this fact, at least in an inequality form, to random variables that are not necessarily independent and can have a more general distribution.

Lemma 9 *Let W_i, $1 \le i \le n$ be nonnegative random variables, and suppose that their joint survival function*

$$S(t_1,\dots,t_n) = \mathsf{P}(W_1 \ge t_1,\dots,W_n \ge t_n)$$

is log-concave. Then $\min\{W_1,\dots,W_n\}$ also has a log-concave survival function, and

$$\mathsf{E}(\min_i W_i) \ge \left(\sum_i \frac{1}{E(W_i)}\right)^{-1}.$$

Proof. The first assertion is trivial, since the survival function of $\min\{W_1,\dots,W_n\}$ is $S(t,\dots,t)$, which is log-concave.

To prove the bound on the expectation, set $w_i = \mathsf{E}(W_i)$. For $n=1$ the assertion is trivial. First we settle the case $n=2$. In this case,

$$\mathsf{E}(\min\{W_1,W_2\}) = \int_0^\infty \mathsf{P}(\min\{W_1,W_2\} \ge t)\,dt = \int_0^\infty S(t,t)\,dt$$

while

$$w_1 = \mathsf{E}(W_1) = \int_0^\infty \mathsf{P}(W_1 \ge t)\,dt = \int_0^\infty S(t,0)\,dt,$$

and similarly

$$w_2 = \int_0^\infty S(0,t)\,dt$$

So the assertion is equivalent to the following inequality:

$$\left(\int_0^\infty S(t,t)\,dt\right)^{-1} \geq \left(\int_0^\infty S(t,0)\,dt\right)^{-1} + \left(\int_0^\infty S(0,t)\,dt\right)^{-1} \tag{8}$$

This inequality could be proved using the Localization Lemma from [8], but we can give a simple direct proof. We may assume that S is positive, strictly monotone decreasing and continuous. (It is easy to approximate a general S by such a function). We define a parametrization of the axes as follows: for $0 \leq x < 1$, let $a = a(x)$ and $b = b(x)$ be defined by

$$\int_0^a S(t,0)\,dt = xw_1, \qquad \int_0^b S(0,t)\,dt = xw_2.$$

Obviously,

$$\frac{d}{dx}a(x) = \frac{w_1}{S(a,0)}, \qquad \frac{d}{dx}b(x) = \frac{w_2}{S(0,b)}$$

Let $c(x) = a(x)b(x)/(a(x)+b(x))$. Then

$$(c,c) = \frac{b}{a+b}(a,0) + \frac{a}{a+b}(0,b).$$

Thus the line $x_1 = x_2$ intersects the segment connecting $(a,0)$ to $(0,b)$ at the point (c,c). Hence by log-concavity,

$$S(c,c) \geq S(a,0)^{\frac{b}{a+b}} S(0,b)^{\frac{a}{a+b}}. \tag{9}$$

Furthermore, we have

$$\frac{d}{dx}c = \frac{\frac{da}{dx}b^2 + \frac{db}{dx}a^2}{(a+b)^2} = \frac{\frac{w_1 b^2}{S(a,0)} + \frac{w_2 a^2}{S(0,b)}}{(a+b)^2}.$$

By the inequality between arithmetic and geometric means,

$$\frac{d}{dx}c \geq \frac{1}{a+b}\left(\frac{w_1 b}{S(a,0)}\right)^{\frac{b}{a+b}} \left(\frac{w_2 a}{S(0,b)}\right)^{\frac{a}{a+b}}. \tag{10}$$

We want to estimate

$$\int_0^\infty S(t,t)\,dt = \int_0^1 S(c(x),c(x))\frac{dc}{dx}\,dx. \tag{11}$$

Here the integrand can be estimated as follows, using (9) and (10):

$$\begin{aligned} S(c,c)\frac{dc}{dx} &\geq S(a,0)^{\frac{b}{a+b}} S(0,b)^{\frac{a}{a+b}} \frac{1}{a+b} \left(\frac{w_1 b}{S(a,0)}\right)^{\frac{b}{a+b}} \left(\frac{w_2 a}{S(0,b)}\right)^{\frac{a}{a+b}} \\ &= \frac{1}{a+b}(w_1 b)^{\frac{b}{a+b}} (w_2 a)^{\frac{a}{a+b}}. \end{aligned}$$

Applying the inequality between geometric and harmonic means, we get

$$S(c,c)\frac{dc}{dx} \geq \frac{1}{a+b} \frac{a+b}{b\frac{1}{w_1 b} + a\frac{1}{w_2 a}} = \frac{w_1 w_2}{w_1 + w_2}.$$

Hence we get from (11) that

$$\int_0^\infty S(t,t)\,dt \geq \frac{w_1 w_2}{w_1 + w_2},$$

which proves the lemma for $n = 2$.

Now the general case follows by induction. Let $n > 2$ and set $Y = \min\{X_2, \ldots, X_n\}$. By the induction hypothesis,

$$\mathsf{E}(Y) \geq \frac{1}{\frac{1}{w_2} + \ldots + \frac{1}{w_n}}.$$

The joint survival function

$$\mathsf{P}(X_1 \geq t_1, Y \geq s) = \mathsf{P}(X_1 \geq t_1, X_2 \geq s, \ldots, X_n \geq s) = S(t_1, s, \ldots, s)$$

of X_1 and Y is log-concave. Hence by the case $n = 2$,

$$\begin{aligned} \mathsf{E}(\min\{X_1, X_2, \ldots, X_n\}) &= \mathsf{E}(\min\{X_1, Y\}) \\ \geq \frac{1}{\frac{1}{\mathsf{E}(W_1)} + \frac{1}{\mathsf{E}(Y)}} &\geq \frac{1}{\frac{1}{w_1} + \frac{1}{w_2} + \ldots + \frac{1}{w_n}}. \end{aligned}$$

This proves the lemma. □

6 Bounds on the expected minimum: proofs

6.1 Proof of Proposition 6

First, by Lemma 1,

$$\mathcal{L}(K, W) \geq \min_i \frac{W_i}{y_i}$$

for every $y \in K^*$. Thus by Lemma 9, it follows that

$$\mathsf{E}(\mathcal{L}(K, W)) \geq \left(\sum_i \frac{y_i}{w_i}\right)^{-1} = \left(\sum_i w_i^* y_i\right)^{-1}.$$

The maximum of this lower bound is clearly $1/\mathcal{L}(K^*, w^*)$. □

By lemma 1, part (2), this lemma also implies

$$\mathsf{E}(\mathcal{L}(K,W)) \geq \frac{1}{n}\mathcal{L}(K,w).$$

6.2 Proof of Theorem 7

Let a be the vector attaining the minimum in the definition of $\mathcal{E} = \mathcal{E}(K,w)$. By Lemma 2, we have

$$\min\left\{\sum_{i=1}^{n}\frac{1}{w_i}y_i^2 \ : \ y \in K^*\right\} = \frac{1}{\mathcal{E}}.$$

Let b be the vector in K^* attaining this minimum. Then we have $b_i = w_i a_i/\mathcal{E}$.

We can write b as a convex combination of vertices v^j of K^*:

$$v = \sum_{j=1}^{p}\alpha_j v^j, \qquad \alpha_j > 0, \qquad \sum_{j=1}^{p}\alpha_j = 1.$$

Since $a^\mathsf{T} b = 1$ and $a^\mathsf{T} v^j \geq 1$, we must have $a^\mathsf{T} v^j = 1$ for all j.

For every given outcome of the W_i and any vector $x \in K$, we have

$$W^\mathsf{T} x = \sum_i W_i x_i \frac{1}{b_i}\sum_j v_i^j \alpha_j = \sum_{j=1}^{p}\alpha_j \sum_i \frac{v_i^j W_i x_i}{b_i} = \mathcal{E}\sum_{j=1}^{p}\alpha_j \sum_i \frac{v_i^j W_i x_i}{w_i a_i}$$

Since $x \in K$ and $v^j \in K^*$, we have $\sum_i x_i v_i^j \geq 1$, and hence

$$W^\mathsf{T} x \geq \mathcal{E}\sum_{j=1}^{p}\alpha_j \min_{i:\ v_i^j>0}\frac{W_i}{w_i a_i}.$$

Thus

$$\mathcal{L}(K,W) = \min\{W^\mathsf{T} x \ : \ x \in K\} \geq \sum_{j=1}^{p}\alpha_j \min_{i:\ v_i^j>0}\frac{W_i}{w_i a_i}.$$

Taking expectation and using Lemma 9, we get

$$\mathsf{E}(\mathcal{L}(K,w)) \geq \sum_{j=1}^{p}\alpha_j\Big(\sum_{i:\ v_i^j>0} a_i\Big)^{-1}.$$

Now here we have

$$\sum_{i:\ v_i^j>0} a_i \leq \sum_i a_i v_i^j = a^\mathsf{T} v^j = 1,$$

and so

$$\mathsf{E}(\mathcal{L}(K,w)) \geq \mathcal{E}\sum_{j=1}^{p}\alpha_j = \mathcal{E}.$$

□

7 Combinatorial applications

Theorem 7 applies to every clutter $\mathcal{H} = (V, E)$ (i.e., $E \subseteq 2^V$). Let (as before) $W^1, \ldots, W^n$ be non-negative random variables with log-concave joint survival function, and $w_i = \mathsf{E}(W_i)$. We are interested in estimating $\mathsf{E}(\mathcal{L}(\mathcal{H}, W))$. Unfortunately, we cannot directly apply Theorem 7 to $\mathcal{L}(D(\mathcal{H}), W)$, since the blocker of $D(\mathcal{H})$ has, in general, not only 0-1 vertices. But it does if $\mathcal{H}$ has the Max-Flow-Min-Cut property. Thus Theorem 7 implies:

Corollary 10 *Let $\mathcal{H} = (V, E)$ be a clutter with the Max-Flow-Min-Cut property. Let $W \in \mathbb{R}^V$ be non-negative random vector with log-concave survival function. Then*

$$\mathsf{E}(\mathcal{L}(\mathcal{H}, W)) \geq \mathcal{E}(D(\mathcal{H}), w).$$

There are many combinatorial examples of blocking pairs of clutters with the Max-Flow-Min-Cut Property: $s-t$ paths and $s-t$ cuts (directed or undirected), rooted arborescences and rooted cuts, perfect matchings and odd cuts, T-joins and T-cuts etc. For each of these, Theorem 7 applies, and we get a lower bound on the expected minimum weight of an edge.

We can get a corollary valid for all clutters using (6):

$$\mathcal{L}(\mathcal{H}, W) \geq \mathcal{L}(D(\mathcal{H}^b)^*, W),$$

and since the blocker of $D(\mathcal{H}^b)^*$ is $D(\mathcal{H}^b)$, which has 0-1 vertices, we get by Theorem 7:

$$\mathsf{E}(\mathcal{L}(\mathcal{H}, W)) \geq \mathsf{E}(\mathcal{L}(D(\mathcal{H}^b)^*, W)) \geq \mathcal{E}(D(\mathcal{H}^b)^*, w) = \frac{1}{\mathcal{E}(D(\mathcal{H}^b), w^*)}.$$

Thus we proved:

Corollary 11 *Let $\mathcal{H} = (V, E)$ be any clutter. Let $W \in \mathbb{R}^V$ be non-negative random vector with log-concave survival function. Then*

$$\mathsf{E}(\mathcal{L}(\mathcal{H}, W)) \geq \frac{1}{\mathcal{E}(D(\mathcal{H}^b), w^*)}.$$

One advantage of this bound is that on the right hand side, the minimum is in the denominator, so any $y \in D(\mathcal{H}^b)$ provides a lower bound. For example, assume that $\mathcal{H}$ is k-colorable, i.e., V can be partitioned into k sets none of which spans and edge of $\mathcal{H}$. Then the vector $(k-1)\mathbb{1}$ is the sum of k vertices of $\mathcal{H}^b$ (the incidence vectors of the complements of the color classes), and hence $(1-1/k)\mathbb{1} \in D(\mathcal{H}^b)$. Thus we get the bound

$$\mathsf{E}(\mathcal{L}(\mathcal{H},W)) \geq \frac{k^2}{(k-1)^2 \sum_i 1/w_i}. \tag{12}$$

It is perhaps more interesting to consider the case when V can be partitioned into k sets, each intersecting every edge of k. Then the same argument gives

$$\mathsf{E}(\mathcal{L}(\mathcal{H},W)) \geq \frac{k^2}{\sum_i 1/w_i}. \tag{13}$$

It may be interesting to elaborate upon the idea of symmetry used in Example 5. Let $\mathcal{H}$ be a clutter admitting a vertex-transitive automorphism group. Assume that $w = \mathbb{1}$. Trivially,

$$\mathcal{L}(D(\mathcal{H}),\mathbb{1}) = r(\mathcal{H}), \qquad \mathcal{L}(D(\mathcal{H}^b),\mathbb{1}) = \tau(\mathcal{H}).$$

The optimum in the definition of $\mathcal{E}(D(\mathcal{H}),\mathbb{1})$ is attained when all variables are equal, and for this case it is easy to figure out that this value is $r(\mathcal{H})/n$. So we get

$$\mathcal{E}(D(\mathcal{H}),\mathbb{1}) = \frac{r(\mathcal{H})^2}{n}, \qquad \mathcal{E}(D(\mathcal{H}),\mathbb{1}) = \frac{\tau(\mathcal{H})^2}{n}.$$

Thus we get for clutters with a transitive automorphism group:

$$\mathsf{E}(\mathcal{L}(\mathcal{H},W)) \geq \frac{n}{\tau(\mathcal{H})^2} \tag{14}$$

We note that both bounds (13) and (14) are tight for the complete multipartite hypergraph $\mathcal{H}$ on k classes of size k, which is just our example 5.

The case when $\mathcal{H}$ is the clutter of perfect matchings of a complete bipartite graph is of special interest. Let W_ij ($i \in A$, $j \in B$) be independent exponentially distributed random variables with expectation 1. There is a conjecture about the expectation of the minimum weight of a perfect matching:

$$\mathsf{E}(\mathcal{L}(D(\mathcal{H}),W)) \to \frac{\pi^2}{6} \qquad (k \to \infty).$$

In this case, (14) applies and gives that

$$\mathsf{E}(\mathcal{L}(D(\mathcal{H}),W)) \geq \frac{k^2}{\tau(\mathcal{H})^2} = 1.$$

This is known, but it shows that our estimate is only a constant factor off the target.

In section 3 we saw some examples where the energy expressions in corollaries 10 and 11 was of combinatorial interest. We have discussed the connection between shortest paths and resistance. From Example 2 we get the following inequality.

Corollary 12 *Let the edges of a directed graph have positive random "lengths", whose joint survival function is log-concave. Then the expected length of a shortest path between nodes s and t is bounded from below by the congestion delay time of the network obtained by replacing each edge length by its expectation.*

We can also apply our main theorem to Example 3, where the vertices of the dual polyhedron are not integral, but still satisfy that their non-zero entries are larger than 1. We get that in a sense we can trade randomness in the edgelengths for randomness in the paths.

Corollary 13 *Let the edges of an undirected graph have positive random "lengths", whose expectation is* 1 *and whose joint survival function is log-concave. Then average hitting time to a node t is bounded from above by $2|E|$ times the expected average distance from t.*

Finally, we note that we can also apply Theorem 7 to the polyhedron $D(\mathcal{H})^*$ (whose blocker $D(\mathcal{H})$ has 0-1 vertices), to get the inequality

$$\mathsf{E}(\mathcal{L}(D(\mathcal{H})^*, W)) \geq \mathcal{E}(D(\mathcal{H})^*, w). \tag{15}$$

However, the two sides don't seem to have any direct combinatorial significance.

8 Concluding remarks

Fulkerson [4] also introduced antiblocking pairs of convex sets (essentially by reversing the inequality sign in the definition of the blocker). Körner [9] introduced the important notion of graph entropy, which was generalized to convex bodies by Csiszár, Körner, Lovász, Marton and Simonyi [2]. Our notion of energy can be viewed as the analogue of entropy in the blocking setting.

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