# Discrete Localization and Correlation Inequalities for Set Functions

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February 2003
Technical Report
MSR-TR-2003-07

Three theorems for set functions, closely related to the Ahlswede-Daykin 4-function theorem (4FT), are proved. First, the conclusion of the 4FT is generalized to norms other than the $L_1$ norm. Secondly, a refinement of the 4FT is proved showing that the hypothesis of the 4FT implies a family of inequalities whose sum is the conclusion of the 4FT. Finally, it is also shown that the hypothesis of the four function theorem is preserved under a form of convolution. All of these theorems are deduced from another theorem proven here: given two real valued set functions $f_1, f_2$ defined on the subsets of a finite set $S$ satisfying $\sum_{X \subseteq S} f_i(X) \geq 0$ for $i \in \{1,2\}$, there exists a strictly positive multiplicative set function $\mu$ over $S$ and two subsets $A, B \subseteq S$ such that for $i \in \{1,2\}$ $\mu(A)f_i(A)+\mu(B)f_i(B)+\mu(A\cup B)f_i(A\cup B)+\mu(A\cap B)f_i(A\cap B) \geq 0$. This theorem is an analog for discrete set functions of a geometric result of Lovász and Simonovits.

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[1]This work was done while the author was visiting Microsoft Research and was also supported in part by NSF grant CCR-9988526.

# 1 Introduction

## 1.1 Discrete and continuous localization inequalities

By a set function over a set $S$ we mean a function whose domain is the set $2^S$ of all subsets of $S$. A 4-tuple $(\alpha, \beta, \gamma, \delta)$ of nonnegative real valued set functions over the finite set $S$ is said to satisfy the *Ahlswede-Daykin* condition, $\mathrm{A}D(\alpha, \beta, \gamma, \delta)$, if:

$$\text{For all } A, B \subseteq S,\ \alpha(A)\beta(B) \leq \gamma(A \cup B)\delta(A \cap B).$$

In the case that $\alpha = \beta = \gamma = \delta$, this condition says that $\alpha$ is *log-supermodular*. The 4-function theorem (4FT) of Ahlswede and Daykin says that the collection $AD(\alpha, \beta, \gamma, \delta)$ of local conditions yields a global conclusion:

**Theorem 1.** *[AD1] If $\alpha, \beta, \gamma, \delta$ are nonnegative real valued set functions over the finite set $S$ satisfying $\mathrm{A}D(\alpha, \beta, \gamma, \delta)$ then:*

$$\alpha(2^S)\beta(2^S) \leq \gamma(2^S)\delta(2^S). \tag{1}$$

(Here, for $\mathcal{H} \subseteq 2^S$, we use the notation $f(\mathcal{H}) = \sum_{E \in \mathcal{H}} f(E)$.) This well-known result generalizes a number of important inequalities in combinatorics, probability theory and statistical mechanics including the FKG inequality, and related results due to Holley ([Ho]), Harris([Ha]), and Kleitman ([Kl]). Ahlswede and Daykin ([AD2]) and Daykin ([Day2]) developed a framework for inequalities on set functions based on a general product theorem which abstracts the induction step of the proof of the 4-function theorem. An extension to sequences of $2k$ functions with $k \geq 2$ was obtained independently by Rinott and Saks [RS] and Aharoni and Keich [AK].

The contrapositive form of Theorem 1 is:

**Corollary 2.** *If $\alpha, \beta, \gamma, \delta$ are nonnegative real valued set functions over the finite set $S$ satisfying*

$$\alpha(2^S)\beta(2^S) > \gamma(2^S)\delta(2^S). \tag{2}$$

*Then there are sets $A, B \subseteq S$ satisfying $\alpha(A)\beta(B) > \gamma(A \cup B)\delta(A \cap B)$.*

In its contrapositive form, the 4FT resembles a geometric "Localization Inequality" of Kannan, Lovász and Simonovits [KLS] (which we will not state in its strongest form). A *needle with an exponential weight function*, or briefly an *exponential needle* in $\mathbb{R}^n$ is a triple $N = (a, b, \gamma)$, where $a, b \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$. We define the integral of a function $f$ on this needle as

$$\int_N f = \int_0^1 e^{\gamma t} f(a + t(b - a))\, dt.$$

**Theorem 3.** *Let $\alpha, \beta, \gamma, \delta$ be nonnegative continuous functions on $\mathbb{R}^n$, and let $c, d > 0$. Suppose that*

$$\left(\int_{\mathbb{R}^n} \alpha(x)\, dx\right)^c \left(\int_{\mathbb{R}^n} \beta(x)\, dx\right)^d > \left(\int_{\mathbb{R}^n} \gamma(x)\, dx\right)^c \left(\int_{\mathbb{R}^n} \delta(x)\, dx\right)^d$$

*Then there is an exponential needle $N$ such that*

$$\left(\int_N \alpha\right)^c \left(\int_N \beta\right)^d > \left(\int_N \gamma\right)^c \left(\int_N \delta\right)^d.$$

This was derived from the following "Localization Lemma" of Lovász and Simonovits [LS] (again, stated in a simplified form):

**Theorem 4.** *Let $f$ and $g$ be two continuous integrable functions on $\mathbb{R}^n$ so that*

$$\int_{\mathbb{R}^n} f(x)\,dx \geq 0, \quad \textit{and} \int_{\mathbb{R}^n} g(x)\,dx \geq 0.$$

*Then there is an exponential needle $N$ such that*

$$\int_N f \geq 0, \quad \textit{and} \int_N g \geq 0.$$

In this paper we prove a result that is related to the Four Function Theorem as Theorem 4 is related to Theorem 3. If we have a real valued function $f$ whose domain is a finite set $U$ (not $2^U$) so that $f(U) \geq 0$, then there is an element $u \in U$ so that $f(u) \geq 0$. This trivial localization is an important step in many proofs, in particular in most applications of the Probabilistic Method.

Now suppose we have two functions $f_1, f_2$ defined on a finite set $U$, having the property that $f_1(U)$ and $f_2(U)$ are both nonnegative. The geometric result suggests that it might always be possible to find a "small" subset $T$ of $U$ such that $f_1(T)$ and $f_2(U)$ are both nonnegative. However for any $U$, it is easy to construct an example with the property that for any proper subset $T$ of $U$, $f_1(T) < 0$ or $f_2(T) < 0$.

As in the geometric case, we can try to use weights. In other words, we can try to find a "small" subset $T$ of $U$ and "natural" weights $w_i$ ($i \in T$) such that $\sum_{i \in T} w_i f_1(i)$ and $\sum_{i \in T} w_i f_2(i)$ are both nonnegative. There are now various ways to specify "small" and "natural" and get a valid assertion. The simplest one is stated in the following proposition (whose easy proof is omitted):

**Proposition 5.** *Let $U$ be a finite set and $f_1, f_2 : U \to \mathbb{R}$ such that $f_1(U) \geq 0$ and $f_2(U) \geq 0$. Then there exists a $T \subseteq U$ with $|T| \leq 2$ and weights $w_i > 0$ ($i \in T$) such that $\sum_{i \in T} w_i f_1(i) \geq 0$ and $\sum_{i \in T} w_i f_2(i) \geq 0$.*

If instead of two functions on an unstructured set $U$, we have two set functions $f_1, f_2 : 2^S \to \mathbb{R}$, then we have the following more difficult and more useful localization result. To define a "natural" weighting, let us call a set function $\mu$ *multiplicative* if for any subset $A$, $\mu(A) = \prod_{a \in A} \mu(\{a\})$ (where the empty product is taken to be 1). Also, for a set function $f$ on $2^S$ and $A, B \subseteq S$, define:

$$\sigma(f; A, B) = f(A) + f(B) + f(A \cup B) + f(A \cap B).$$

We prove:

**Theorem 6.** *Let $f_1, f_2$ be two set functions on a finite set $S$ such that $f_1(2^S) \geq 0$ and $f_2(2^S) \geq 0$. Then there is a strictly positive multiplicative set function $\mu$ and (not necessarily distinct) sets $A, B$ such that for $i \in \{1, 2\}$,*

$$\sigma(\mu f_i; A, B) \geq 0. \tag{3}$$

(Here $\mu f_i$ is the function defined on $2^S$ by $\mu f_i(X) = \mu(X) f_i(X)$.)

We prove this theorem in section 2. We then use it to prove three new inequalities for set functions, each extending the 4FT in a different way. The formal analogue of Theorem 3 is not the 4FT, but rather the following result.

**Corollary 7.** *Let $\alpha, \beta, \gamma, \delta$ be four nonnegative set functions over $S$ and $c, d > 0$. Suppose that*

$$\alpha(2^S)^c \beta(2^S)^d > \gamma(2^S)^c \delta(2^S)^d.$$

*Then there exists a positive multiplicative set function $\mu$ and two sets $A, B \subseteq S$ such that*

$$\sigma(\mu\alpha; A, B)^c \sigma(\mu\beta; A, B)^d > \sigma(\mu\gamma; A, B)^c \sigma(\mu\delta; A, B)^d.$$

We prove this theorem in Section 3. We remark that the conclusion is not necessarily true if $\mu$ is the identically 1 function, that is, $\mu$ can not be eliminated. As a consequence of this theorem we will deduce the following Hölder-type generalization of the 4FT. For a real valued function $f$ on $2^S$ and $0 < p$, write $\|f\|_p$ for $(\sum_{X \subseteq S} |f(X)|^p)^{1/p}$ (so $\|f\|_1 = f(2^S)$).

**Theorem 8.** *If $\alpha, \beta, \gamma, \delta$ are nonnegative real valued set functions over the finite set $S$ satisfying $\mathrm{A}D(\alpha, \beta, \gamma, \delta)$ and $p, q, r, s > 0$ satisfy $\min\{p, q\} \leq \min\{r, s\}$ and $1/p + 1/q \leq 1/r + 1/s$ then:*

$$\|\alpha\|_p \|\beta\|_q \leq \|\gamma\|_r \|\delta\|_s. \tag{4}$$

We now describe our second refinement of the 4FT. One way to refine an inequality of the form $S \leq T$ is to decompose it as a sum of inequalities, i.e., find equations $S = \sum_{i \in I} S_i$ and $T = \sum_{i \in I} T_i$ so that $S_i \leq T_i$ for all $i \in I$. For a pair $(C, D)$ of sets with $C \subseteq D$ let $P(C, D)$ denote the set of pairs $A, B$ such that $A \cap B = C$ and $A \cup B = D$. In section 5, we prove:

**Theorem 9.** *Under the hypothesis of Theorem 1 the following holds for each pair of sets $(C, D)$ with $C \subseteq D \subseteq [n]$:*

$$\sum_{(A,B) \in P(C,D)} \alpha(A)\beta(B) \leq \sum_{(A,B) \in P(C,D)} \gamma(A)\delta(B) \tag{5}$$

The conclusion of the 4FT is obtained by summing (5) over all pairs $(C, D)$.

Our final theorem concerns the closure of the Ahlswede-Daykin condition under convolution. There are (at least) two reasonable notions of convolution for set functions. The first is an algebraic notion of convolution arising from discrete fourier analysis with respect to the group $\mathbb{Z}_2^n$:

$$f * g(A) = \sum_{X, Y : X \oplus Y = A} f(X) g(Y), \tag{6}$$

where $X \oplus Y = (X - Y) \cup (Y - X)$ denotes the symmetric difference. The second notion, which we call *disjoint convolution* and denote by $\dot{*}$, is set theoretic rather than algebraic:

$$f \dot{*} g(A) = \sum_{X,Y:\, \substack{X \cup Y = A \\ X \cap Y = \emptyset}} f(X)g(Y). \tag{7}$$

Here we prove that the Ahlswede-Daykin condition is closed under disjoint convolution:

**Theorem 10.** *Let $\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2$ be nonnegative functions on $2^S$ and let $\alpha = \alpha_1 \dot{*} \alpha_2$, $\beta = \beta_1 \dot{*} \beta_2$, $\gamma = \gamma_1 \dot{*} \gamma_2$ and $\delta = \delta_1 \dot{*} \delta_2$. If $\mathrm{A}D(\alpha_i, \beta_i, \gamma_i, \delta_i)$ holds for $i \in \{1, 2\}$ then $\mathrm{A}D(\alpha, \beta, \gamma, \delta)$.*

We prove this theorem in section 6. The theorem includes the 4FT: Take $\alpha_2 = \beta_2 = \gamma_2 = \delta_2$ to be the function that is identically 1. The conclusion of Theorem 10 includes the inequality $\alpha(S)\beta(S) \leq \gamma(S)\delta(S)$ which is equivalent to the conclusion of the 4FT for $\alpha_1, \beta_1, \gamma_1, \delta_1$. Taking $\alpha_1 = \beta_1 = \gamma_1 = \delta_1$ and $\alpha_2 = \beta_2 = \gamma_2 = \delta_2$ in Theorem 10 we obtain:

**Corollary 11.** *The disjoint convolution of two log-supermodular functions is log-supermodular.*

We remark that for algebraic convolution, even the corollary does not hold. For the log-supermodular set functions $f, g$ over $\{1, 2\}$ defined by $f(\emptyset) = g(\emptyset) = 1$, $f(\{1\}) = g(\{2\}) = 0$, $f(\{2\} = g(\{1\}) = 2$ and $f(\{1, 2\}) = g(\{1, 2\}) = 1$, we have $f * g(\emptyset) = 2$, $f * g(\{1\}) = f * g(\{2\}) = 4$, and $f * g(\{1, 2\}) = 6$, so $f * g$ is not log-supermodular.

This corollary was also motivated by a continuous-discrete analogy. Submodular set functions are analogous to convex functions (see e.g. [L]), and so log-supermodular set functions are analogous to logconcave functions. Two important and nontrivial properties of logconcave functions are that their class is closed under integrating out some variables, and also under convolution [Din, P]. A discrete analogue of the first property was proved for log-supermodular set functions by Rinott and Saks [RS]. In the continuous case, closedness under convolution is implied by closedness under integration, but in the discrete case this does not seem to be the case, and we need a more elaborate argument.

# 2 Proof of Theorem 6

Suppose $f : 2^S \longrightarrow \mathbb{R}^2$ and let $f_1, f_2$ denote the set functions with range $R$ obtained by projecting $f$ onto, respectively, its first and second coordinates. A triple $(A, B, \mu)$, where $A, B \subseteq S$ and $\mu$, is a strictly positive multiplicative set function on $S$ satisfying the conclusion (3) of the theorem is called a *solution* for $f$. We prove by induction on $|S|$ that $f(2^S) \geq (0, 0)$ implies that $f$ has a solution. For convenience we assume that $S = [n]$ for some positive integer $n$, where $[n]$ denotes the set $\{1, \ldots, n\}$. For $n = 1$, $(\emptyset, \{1\}, \mathbf{1})$ is a solution and for $n = 2$, $(\{1\}, \{2\}, \mathbf{1})$ is a solution where $\mathbf{1}$ denotes the function that maps all sets to 1. The main part of the proof is the case $n = 3$; the case of general $n$ will then follow from an easy induction.

We assume that $f : 2^{[3]} \longrightarrow \mathbb{R}^2$ satisfies $f(2^S) \geq (0, 0)$ and, for contradiction, we further assume that $f$ has no solution. Without loss of generality we may assume that

$f(2^{[3]}) = (0,0)$; if not, we replace $f$ by the function $g$ with $g(\emptyset) = f(\emptyset) - f(2^{[3]})$ and $g(A) = f(A)$ for $A \neq \emptyset$ and observe that a solution for $g$ is also a solution for $f$.

It is useful to use the standard picture of $2^{[3]}$ as the set of vertices of a cube where each adjacent pair of vertices corresponds to a pairs of sets that differ by exactly one element and antipodal vertices correspond to complementary sets. We refer to a subset of $2^{[3]}$ as an *edge* or *face* if it corresponds to an edge or face of the cube. Thus an edge is a pair of sets of the form $\{A, A \cup \{i\}\}$ with $i \notin A$, and a face has the form $\{A, A \cup \{i\}, A \cup \{j\}, A \cup \{i, j\}$ where $i \neq j$ and $A = \emptyset$ or $A = [3] - \{i, j\}$. An edge is an *outer edge* if it contains $\emptyset$ or $[3]$ and is a *middle edge* otherwise. In the edge $\mathcal{E} = \{A, A \cup \{i\}\}$, $A$ is the *bottom set* of $\mathcal{E}$. Two edges $\{A, A \cup \{i\}\}$ and $\{A', A' \cup \{i'\}\}$ are *parallel* if $i = i'$.

Every face $\mathcal{F}$ consists of one outer edge and one middle edge. If $\mathcal{F}$ is a face, then its complement $\overline{\mathcal{F}}$ is also a face.

A *diamond* is a set of four vertices of the form $\{A, B, A \cap B, A \cup B\}$ where $A$ and $B$ are incomparable sets. Every face is a diamond, but not vice versa, since $\{\emptyset, \{i\}, \overline{\{i\}}, [3]\}$ is a diamond for each $i$. The following three conditions on a pair of distinct edges $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent: (D1) $\mathcal{E}_1 \cup \mathcal{E}_2$ is a diamond, (D2) $\mathcal{E}_1$ and $\mathcal{E}_2$ are parallel and at least one is an outer edge, (D3) $\mathcal{E}_1$ and $\mathcal{E}_2$ are parallel and their bottom sets are comparable under containment.

We define the *quadrant* of a point $x \in \mathbb{R}^2$ to be the set $Q(x)$ of indices $t \in \{1, 2\}$ such that $x_t \geq 0$. We say that a subset $\mathcal{B}$ of $2^{[3]}$ *belongs to quadrant* $T$ if $Q(f(\mathcal{B})) = T$. By hypothesis, $2^{[3]}$ belongs to quadrant $\{1, 2\}$.

**Claim 1.** *1. No vertex, edge or diamond belongs to quadrant* $\{1, 2\}$.

2. *For any pair of complementary faces, one belongs to quadrant* $\{1\}$ *and the other to quadrant* $\{2\}$.

3. *If* $\mathcal{E}, \mathcal{E}'$ *are edges whose union is face* $\mathcal{F}$ *then at least one of* $\mathcal{E}$ *and* $\mathcal{E}'$ *belong to the same quadrant as* $\mathcal{F}$.

*Proof.* If $X$ is a vertex that belongs to quadrant $\{1, 2\}$ then $(X, X, \mathbf{1})$ is a solution. If $\{X, X \cup \{i\}\}$ is an edge belonging to quadrant $\{1, 2\}$ then $(X, X \cup \{i\}, \mathbf{1})$ is a solution. If $A_1, A_2$ are incomparable sets and the diamond $\{A_1, A_2, A_1 \cap A_2, A_1 \cup A_2\}$ belongs to quadrant $\{1, 2\}$ then $(A_1, A_2, \mathbf{1})$ is a solution.

If $\mathcal{F}$ and $\mathcal{F}'$ are complementary faces then $f(\mathcal{F}) + f(\mathcal{F}') = (0, 0)$, and as neither one has both coordinates nonnegative, it follows that $f(\mathcal{F})$ is positive on one coordinate and negative on the other, and the coordinates of $f(\bar{\mathcal{F}})$ have signs opposite to those of $f(\mathcal{F})$. If $\mathcal{E}, \mathcal{E}'$ are edges whose union is face $\mathcal{F}$, assume without loss of generality that $Q(\mathcal{F}) = \{1\}$. Since $f(\mathcal{E}) + f(\mathcal{E}') = f(\mathcal{F})$, one of $f(\mathcal{E})$ and $f(\mathcal{E}')$ have positive first coordinate and (by the first part of the claim), negative second coordinate. □

**Claim 2.** *For any pair* $(\mathcal{E}, \mathcal{E}')$ *of edges whose union is a diamond, the line segment joining* $f(\mathcal{E})$ *and* $f(\mathcal{E}')$ *contains no nonnegative point.*

*Proof.* Suppose for contradiction that $\mathcal{E} \cup \mathcal{E}'$ is a diamond and the line segment joining them contains a nonnegative point, which means that there is a $\lambda \in (0, 1)$ such that $\lambda f(\mathcal{E}) + (1 - \lambda) f(\mathcal{E}')$ is nonnegative. By (D3), we may write $\mathcal{E} = \{B, B \cup \{i\}\}$ and $\mathcal{E}' = (B', B' \cup \{i\}\}$ and assume, without loss of generality, that $B \subseteq B' \subseteq \overline{\{i\}}$. Let $j \in B' - B$. Then $(B \cup \{i\}, B', \mu)$ is a solution where the function $\mu$ is given by $\mu(j) = (1 - \lambda)/\lambda$ and $\mu(k) = 1$ for $k \neq j$. □

For a face $\mathcal{F}$, $L(\mathcal{F})$ denotes the line determined by $(0,0)$ and $f(\mathcal{F})$. Note that $L(\mathcal{F}) = L(\overline{\mathcal{F}})$. $L(\mathcal{F})$ has negative slope since $f(\mathcal{F})$ has exactly one positive coordinate and one negative coordinate. We define $H^+(\mathcal{F})$ to be the closed halfspace bounded by $L(\mathcal{F})$ and containing the quadrant $\{1,2\}$ and $H^-(\mathcal{F})$ to be the open halfspace complementary to $H^+(\mathcal{F})$. If $\mathcal{E}, \mathcal{E}'$ are opposite edges of a face $\mathcal{F}$ then, since $f(\mathcal{F}) = f(\mathcal{E}) + f(\mathcal{E}')$, we have either $f(\mathcal{E}), f(\mathcal{E}') \in L(\mathcal{F}) \subseteq H^+(\mathcal{F})$ or one of $f(\mathcal{E}) \in H^-(\mathcal{F})$ and the other is in $H^+(\mathcal{F})$.

**Claim 3.** *Let $\mathcal{F}$ be a face and $\mathcal{E}$ an edge contained in $\mathcal{F}$. If $f(\mathcal{E}) \in H^+(\mathcal{F})$ then $\mathcal{E}$ and $\mathcal{F}$ belong to the same quadrant.*

*Proof.* Without loss of generality $Q(\mathcal{F}) = \{1\}$. Suppose, to the contrary that $f(\mathcal{E}) \in H^+(\mathcal{F})$ and $Q(\mathcal{E}) \neq \{1\}$. $Q(\mathcal{E}) \neq \{1,2\}$ by claim 1 and $Q(\mathcal{E}) \neq \{\}$ since $f(\mathcal{E}) \in H^+(L(\mathcal{F}))$, so $Q(\mathcal{E}) = \{2\}$. By claim 1, $\mathcal{E}' = \mathcal{F} - \mathcal{E}$ belongs to quadrant $\{1\}$. The line segment from $f(\mathcal{E}')$ to $f(\mathcal{E})$ passes through $f(\mathcal{F})$. The segment $S$ from $f(\mathcal{F})$ to $f(\mathcal{E})$ must go through a point $(0,y)$ since $Q(\mathcal{F}) = \{1\}$ and $Q(\mathcal{E}) = \{2\}$. Also, $S$ lies entirely in $H^+(\mathcal{F})$, which implies $y \geq 0$ and thus $(0,y)$ is a nonnegative point on the segment from $f(\mathcal{E}')$ to $f(\mathcal{E})$, contradicting claim 2. □

**Claim 4.** *If $\mathcal{F}$ is a face and $\mathcal{E}$ is an outer edge of $\mathcal{F}$ then $f(\mathcal{E}) \in H^-(\mathcal{F})$.*

*Proof.* Let $\mathcal{F}$ be a face and $\mathcal{E}$ an outer edge and suppose for contradiction that $f(\mathcal{E}) \in H^+(\mathcal{F})$. Then by claim 3, $\mathcal{E}$ and $\mathcal{F}$ belong to the same quadrant. Now the complementary face $\mathcal{F}'$ has at least one edge $\mathcal{E}'$ such that $f(\mathcal{E}') \in H^+(\mathcal{F}') = H^+(\mathcal{F})$. Again by claim 3, $\mathcal{E}'$ belongs to the same quadrant as $\mathcal{F}'$, which is the opposite quadrant to that containing $f(\mathcal{E})$. $\mathcal{E}'$ and $\mathcal{E}$ lie in opposite quadrants and above $L(\mathcal{F})$ so the segement joining them contains a nonnegative point. But since $\mathcal{E}$ is an outer edge, the union of $\mathcal{E}'$ and $\mathcal{E}$ is a diamond contradicting Claim 2. □

**Claim 5.** *If $\mathcal{F}$ is a face then each of its two middle edges belongs to the same quadrant as $\mathcal{F}$.*

*Proof.* If $\mathcal{E}$ is a middle edge then $\mathcal{F} - \mathcal{E}$ is an outer edge and $f(\mathcal{F} - \mathcal{E}) \in H^-(\mathcal{F})$ by claim 4. Thus $f(\mathcal{E}) \in H^+(\mathcal{F})$ and therefore, by claim 3, is in the same quadrant as $\mathcal{F}$. □

Claim 5 implies that two adjacent middle edges belong to the same quadrant, and hence all middle edges and all faces belong to the same quadrant. This contradicts that complementary faces belong to different quadrants, which completes the proof of the case $n = 3$ of the theorem.

For the induction step, let $n \geq 4$. Define function $g : 2^{[n-1]} \longrightarrow \mathbb{R}^2$ by $g(J) = f(J) + f(J \cup \{n\})$. By induction, there is a solution $(A, B, \lambda)$ for $g$ with $A, B \subseteq [n-1]$. Extend $\lambda$ to a multiplicative function on $2^{[n]}$ by setting $\lambda(n) = 1$. If $A \subseteq B$ then $(A \cup [n], B, \lambda)$ is a solution for $f$ so assume $A$ and $B$ are incomparable. We define sets $\{A_J : J \subseteq [3]\}$ by:

$$\begin{aligned} A_\emptyset &= A \cap B, \\ A_{\{1\}} &= A, \\ A_{\{2\}} &= B, \\ A_{\{3\}} &= (A \cap B) \cup [n] \end{aligned}$$

and for $J \subseteq [3]$ with $|J| \geq 2$ define $A_J = \bigcup_{j \in J} A_{\{j\}}$. It follows easily from the fact that the sets in $\{A_{\{i\}} - A_\emptyset : i \in [3]\}$ are pairwise disjoint that the map $J \longrightarrow A_J$ respects union and intersection.

Define the function $h : 2^{[3]} \longrightarrow \mathbb{R}^2$ by $h(J) = \lambda(A_J)f(A_J)$ for $J \subseteq [3]$. The fact that $(A_1, A_2, \lambda)$ is a solution for $g$ implies $h(2^{[3]}) \geq (0,0)$. Applying the case $n = 3$ of the theorem to $h$ gives a solution $(J_1, J_2, \mu)$ for $h$.

Now choose $a_1 \in A_1 - A_2$ and $a_2 \in A_2 - A_1$ and let $a_3 = [n]$. Define the multiplicative function $\mu'$ on $2^{[n]}$ by defining $\mu'(a_i) = \mu(\{i\})$ and $\mu'(a) = 1$ for $a \in [n] - \{a_1, a_2, a_3\}$. It follows immediately from the fact that $(J_1, J_2, \mu)$ is a solution for $h$, that $(A_{J_1}, A_{J_2}, \lambda\mu')$ is a solution for $f$, completing the proof of the induction step of Theorem 6.

We conclude this section by recording a minor variant of theorem 6 for later reference. We say that a solution $(A, B, \mu)$ for the set function $f : 2^S \longrightarrow \mathbb{R}^2$ is *strict* if the inequality (3) is strict in each coordinate.

**Corollary 12.** *Let $f$ be a set function on finite set $S$ with range $\mathbb{R}^2$ such that each coordinate of $f(2^S)$ is positive. Then $f$ has a strict solution $(A, B, \mu)$.*

*Proof.* Given $f$ satisfying the hypotheses of the corollary, let $(m_1, m_2) = 2^{-n} \sum_X f(X)$ and define the set function $g$ by $g(X) = f(X) - (m_1, m_2)$. Apply Theorem 6 to $g$ and let $(A_1, A_2, \mu)$ be a solution. Then $(A_1, A_2, \mu)$ is a strict solution for $f$. $\square$

# 3 The proofs of Corollary 7

It follows from the hypothesis that there exists a real number $A > 0$ such that

$$\frac{\alpha(2^S)^c}{\gamma(2^S)^c} < A < \frac{\beta(2^S)^d}{\delta(2^S)^d}.$$

Then

$$A^{1/c}\gamma(2^S) - \alpha(2^S) > 0, \quad \text{and} \quad \beta(2^S) - A^{1/d}\delta(2^S) > 0.$$

So we can apply Theorem 6 to the set functions $f_1(X) = A^{1/c}\gamma(X) - \alpha(X)$ and $f_2(X) = \beta(X) - A^{1/d}\delta(X)$, to get that there exists a positive multiplicative set function $\mu$ and two sets $A, B \subseteq S$ such that

$$\sigma(\mu f_1; A, B) = A^{1/c}\sigma(\mu\gamma; A, B) - \sigma(\mu\alpha; A, B) > 0,$$

and

$$\sigma(\mu f_2; A, B) = \sigma(\mu\beta; A, B) - A^{1/d}\sigma(\mu\delta; A, B) > 0.$$

But this means that

$$\frac{\sigma(\mu\alpha; A, B)^c}{\sigma(\mu\gamma; A, B)^c} < A < \frac{\sigma(\mu\beta; A, B)^d}{\sigma(\mu\delta; A, B)^d},$$

which proves the corollary.

## 4 Proof of Theorem 8

We first prove the theorem for the case $p = r$ and $q = s$. Suppose we have $\alpha, \beta, \gamma, \delta$ on $2^S$ satisfying $AD(\alpha, \beta, \gamma, \delta)$ and $p, q > 0$. Suppose for contradiction that:

$$\|\alpha\|_p \|\beta\|_q > \|\gamma\|_p \|\delta\|_q. \tag{8}$$

Let $T = S \cup \{a_1, a_2\}$, where $a_1, a_2$ are new elements. Define set functions $\alpha', \beta', \gamma', \delta'$ on $2^T$ by defining for $A \subseteq T$:

$$\begin{aligned}
\alpha'(A) &= \begin{cases} \alpha(A - \{a_1\}) & \text{if } a_1 \in A \text{ and } a_2 \notin A \\ 0 & \text{otherwise.} \end{cases} \\
\beta'(A) &= \begin{cases} \beta(A - \{a_2\}) & \text{if } a_1 \notin A \text{ and } a_2 \in A \\ 0 & \text{otherwise.} \end{cases} \\
\gamma'(A) &= \begin{cases} \gamma(A - \{a_1, a_2\}) & \text{if } a_1 \in A \text{ and } a_2 \in A \\ 0 & \text{otherwise.} \end{cases} \\
\delta'(A) &= \begin{cases} \delta(A) & \text{if } a_1 \notin A \text{ and } a_2 \notin A \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Let $c = 1/p$ and $d = 1/q$. Then (8) is equivalent to the hypothesis of Corollary 7 for $\alpha', \beta', \gamma', \delta'$ and so there is a positive multiplicative set function $\mu$ and $A, B \subseteq S$ such that:

$$\sigma(\mu\alpha'; A, B)^c \sigma(\mu\beta'; A, B)^d > \sigma(\mu\gamma'; A, B)^c \sigma(\mu\delta'; A, B)^d. \tag{9}$$

Since $\alpha'$ is nonzero only on sets containing $a_1$ and not $a_2$ and $\beta'$ is nonzero only on sets containing $a_2$ and not $a_1$, one of $A$ and $B$, say $A$, contains $a_1$ and not $a_2$, and the other, say $B$ contains $a_2$ and not $a_1$. Then (9) reduces to:

$$\mu(A)\alpha(A)\mu(B)\beta(B) > \mu(A \cup B)\gamma(A \cup B)\mu(A \cap B)\delta(A \cap B).$$

Since multiplicativity implies $\mu(A)\mu(B) = \mu(A \cup B)\mu(A \cap B)$, this contradicts $\mathrm{A}D(\alpha, \beta, \gamma, \delta)$, and the contradiction proves the theorem in the case $p = r$ and $q = s$. A virtually identical proof gives the case $p = s$ and $q = r$.

For the the general case, we will need two standard facts about $\|\cdot\|_p$, both of which are easy consequences of Hölder's inequality:

**Fact 1.** $\|\phi\|_x$ is a decreasing function of $x$.

**Fact 2.** $\|\phi\|_{1/x}$ is a log-convex function of $x$. In particular, for any positive $x, y$ and $\epsilon \in [0, 1]$:

$$(\|\phi\|_{1/x})^{\epsilon} (\|\phi_{1/y}\|)^{1-\epsilon} \geq \|\phi\|_{1/(\epsilon x + (1-\epsilon) y)}. \tag{10}$$

Since $\|\phi\|_x$ is a decreasing function of $x$, we may assume that the inquality $1/p + 1/q \leq 1/r + 1/s$ holds with equality since if not we may increase $r$ and/or $s$, while still maintaining $\min\{p, q\} \leq \min\{r, s\}$. Thus since $1/p + 1/q = 1/r + 1/s$ and $\max\{1/r, 1/s\} \geq$

$\max\{1/p, 1/q\}$ we can select $\epsilon \in [0,1]$ such that $1/p = \epsilon/r + (1-\epsilon)/s$ and $1/q = (1-\epsilon)/r + \epsilon/s$.

From the above two special cases we have:

$$\begin{aligned} \|\gamma\|_r\|\delta\|_s &= (\|\gamma\|_r\|\delta\|_s)^{\epsilon}(\|\gamma\|_r\|\delta\|_s)^{1-\epsilon} \\ &\geq (\|\alpha\|_r\|\beta\|_s)^{\epsilon}(\|\alpha\|_s\|\beta\|_r)^{1-\epsilon} \\ &= (\|\alpha\|_r)^{\epsilon}(\|\alpha\|_s)^{1-\epsilon}(\|\beta\|_s|)^{\epsilon}(\|\beta\|_r)^{1-\epsilon} \\ &\geq \|\alpha\|_p\|\delta\|_q, \end{aligned}$$

where the final inequality comes from (10).

# 5 Proof of Theorem 9

We have a 4-tuple of nonnegative functions satisfying $\mathrm{A}D(\alpha,\beta,\gamma,\delta)$, and a pair $(C,D)$ of sets satisfying $C \subseteq D \subseteq [n]$. Assume for contradiction that the conclusion (5) does not hold for $(C,D)$.

Let $S = D - C$ and define functions $\alpha_1, \alpha_2$ for $X \in 2^S$ by:

$$\begin{aligned} \alpha_1(X) &= \alpha(X \cup C) \\ \alpha_2(X) &= \alpha((S-X) \cup C) \end{aligned}$$

Define $\beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2$ analogously.

$\mathrm{A}D(\alpha_1,\beta_1,\gamma_1,\delta_1)$ and $\mathrm{A}D(\alpha_2,\beta_2,\delta_2,\gamma_2)$ both hold, where in the second condition, the roles of $\delta$ and $\gamma$ have been (intentionally and necessarily) reversed.

Let $T = S \cup \{a_1, a_2\}$, where $a_1, a_2$ are new elements. We will apply corollary 12 to the function $f : 2^T \longrightarrow \mathbb{R}^2$ defined as follows. For $A \subseteq S$,

$$\begin{aligned} f_1(A) &= 0 \\ f_1(A \cup \{a_1\}) &= \alpha_1(A)\beta_2(A) \\ f_1(A \cup \{a_2\}) &= 0 \\ f_1(A \cup \{a_1, a_2\}) &= -\gamma_1(A)\delta_2(A) \end{aligned}$$

and

$$\begin{aligned} f_2(A) &= -\gamma_2(A)\delta_1(A) \\ f_2(A \cup \{a_1\}) &= 0 \\ f_2(A \cup \{a_2\}) &= \alpha_2(A)\beta_1(A) \\ f_2(A \cup \{a_1, a_2\}) &= 0 \end{aligned}$$

The assumption that $\alpha, \beta, \gamma, \delta, C, D$ does not satisfy (5) implies $f(2^T) > (0,0)$. Therefore corollary 12 gives a strictly positive multiplicative function $\mu$ on $2^T$ and a pair of sets $A_1, A_2 \subseteq T$ such that for $j \in \{1,2\}$:

(11) $$\sigma(\mu f_j; A_1, A_2) > 0.$$

If $A_1 \cap \{a_1, a_2\} \neq \{a_1\}$ and $A_2 \cap \{a_1, a_2\} \neq \{a_1\}$ then for $j = 1$ none of the terms in the previous sum are positive. Thus, without loss of generality $A_1 = B_1 \cup \{a_1\}$ for some $B_1 \subseteq S$. By a similar argument for $j = 2$, $A_2 = B_2 \cup \{a_2\}$ for some $B_2 \subseteq S$. Inequality (11) for $j = 1$ reduces to

$$\mu(A_1)\alpha_1(B_1)\beta_2(B_1) > \mu(A_1 \cup A_2)\gamma_1(B_1 \cup B_2)\delta_2(B_1 \cup B_2)) \tag{12}$$

and for $j = 2$,

$$\mu(A_2)\beta_1(B_2)\alpha_2(B_2) > \mu(A_1 \cap A_2)\delta_1(B_1 \cap B_2)\gamma_2(B_1 \cap B_2)) \tag{13}$$

Multiplying these two inequalities, and using the multiplicativity of $\mu$ to cancel $\mu(A)\mu(B)$ with $\mu(A \cup B)\mu(A \cap B)$ yields:

$$\alpha_1(B_1)\beta_1(B_2)\alpha_2(B_2)\beta_2(B_1) > \gamma_1(B_1 \cup B_2)\delta_1(B_1 \cap B_2)\gamma_2(B_1 \cap B_2)\delta_2(B_1 \cup B_2),$$

which contradicts the assumption that both $\mathrm{A}D(\alpha_1, \beta_1, \gamma_1, \delta_1)$ and $\mathrm{A}D(\alpha_2, \beta_2, \delta_2, \gamma_2)$ hold. This completes the proof of Theorem 9.

The special case of Theorem 9 where $C = \emptyset$ will be useful in the next section and we note it as a corollary:

**Corollary 13.** *Under the hypothesis of Theorem 1 the following holds for each set $D \subseteq [n]$:*

$$\sum_{J \subseteq D} \alpha(J)\beta(D - J) \leq \sum_{J \subseteq D} \gamma(J)\delta(D - J) \tag{14}$$

# 6 The AD condition is preserved under disjoint convolution

In this section we prove Theorem 10. For $i \in \{1, 2\}$, $\alpha_i, \beta_i, \gamma_i, \delta_i$ are nonnegative set functions satisfying $AD(\alpha_i, \beta_i, \gamma_i, \delta_i)$. We define $\alpha = \alpha_1 \dot{*} \alpha_2$ and $\beta, \gamma, \delta$ analogously. We want to prove $AD(\alpha, \beta, \gamma, \delta)$, which says that for each $A, B \subseteq S$,

$$\begin{aligned} \sum_{\substack{W \subseteq A \\ X \subseteq B}} &\alpha_1(W)\alpha_2(A - W)\beta_1(X)\beta_2(B - X) \\ &\leq \sum_{\substack{Y \subseteq A \cup B \\ Z \subseteq A \cap B}} \gamma_1(Y)\gamma_2((A \cup B) - Y)\delta_1(Z)\delta_2((A \cap B) - Z). \end{aligned} \tag{15}$$

A natural approach to proving this is to fix $A$ and $B$ and define functions $\hat{\alpha}$ on $2^A$, $\hat{\beta}$ on $2^B$, $\hat{\gamma}$ on $2^{A \cup B}$ and $\hat{\delta}$ on $2^{A \cap B}$ by $\hat{\alpha}(W) = \alpha_1(W)\alpha_2(A - W)$, $\hat{\beta}(X) = \beta_1(X)\beta_2(B - X)$, $\hat{\gamma}(Y) = \gamma_1(Y)\gamma_2(A \cup B - Y)$ and $\hat{\delta}(Z) = \delta_1(Z)\delta_2(A \cap B - Z)$. If $AD(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})$ holds then the 4FT gives what we need. Unfortunately, $AD(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})$ need not hold. Take $n = 3$ and take $\alpha_1 = \beta_1 = \gamma_1 = \delta_1$ to be the constant 1 function. Take $\alpha_2 = \beta_2 = \gamma_2 = \delta_2$ to be the function that is 1 except on $\{1,2\}$ and $\{1,2,3\}$ where they are 2, Take $A = \{2,3\}$ and $B = \{1,2\}$. Then $\hat{\alpha}(\{2\})\hat{\beta}(\emptyset) = 2 > 1 = \hat{\gamma}(\{2\})\hat{\delta}(\emptyset)$.

Instead we will prove (15) by decomposing it as a sum of a family of inequalities and show that each inequality in the family is true by reducing it to Corollary 13.

*Proof.* Assume $\mathrm{A}D(\alpha_i, \beta_i, \gamma_i, \delta_i)$ for $i \in \{1,2\}$. To verify $\mathrm{A}D(\alpha, \beta, \gamma, \delta)$, we fix $A, B \subseteq S$ and prove (15). As stated above, we will decompose (15) into a sum of a family of inequalities. The family is indexed by quadruples $(A_1, B_1, C_0, C_1)$ of sets where $A_1 \subseteq A - B$, $B_1 \subseteq B - A$ and $C_0, C_1$ are disjoint subsets of $A \cap B$. We call such a quadruple *relevant*. For a relevant quadruple we define $L(A_1, B_1, C_0, C_1)$ to be the set of pairs $\{(A_1 \cup C_1 \cup V, B_1 \cup C_1 \cup (C_0 - V)) : V \subseteq C_0\}$. Each pair $(W, X)$ with $W \subseteq A$ and $X \subseteq B$ belongs to $L(A_1, B_1, C_0, C_1)$ for exactly one relevant quadruple $(A_1, B_1, C_0, C_1)$, namely the quadruple with $A_1 = W - B$, $B_1 = X - A$, $C_1 = W \cap X$ and $C_0 = ((W - X) \cup (X - W)) \cap A \cap B$. Thus the collection of sets $L(A_1, B_1, C_0, C_1)$ partitions the set of pairs $\{(W, X) : W \subseteq A, X \subseteq B\}$. Similarly, define $R(A_1, B_1, C_0, C_1)$ to be the set of pairs $\{(A_1 \cup B_1 \cup C_1 \cup U, C_1 \cup (C_0 - U)) : U \subseteq C_0\}$. Again the collection of sets $R(A_1, B_1, C_0, C_1)$ partitions the set of pairs $\{(Y, Z) : Y \subseteq A \cup B, Z \subseteq A \cap B\}$. We will prove that for each relevant quadruple $(A_1, B_1, C_0, C_1)$:

$$\sum_{\substack{(W,X)\in \\ L(A_1,B_1,C_0,C_1)}} \alpha_1(W)\alpha_2(A-W)\beta_1(X)\beta_2(B-X) \leq \sum_{\substack{(Y,Z)\in \\ R(A_1,B_1,C_0,C_1)}} \gamma_1(Y)\gamma_2((A\cup B)-Y)\delta_1(Z)\delta_2((A\cap B)-Z). \tag{16}$$

By summing (16) over all relevant quadruples we obtain (15), which proves the theorem.

To prove (16) fix a relevant quadruple, and define $A_2 = A - B - A_1$, $B_2 = B - A - B_1$ and $C_2 = (A \cap B) - (C_0 \cup C_1)$. We rewrite (16) as:

$$\sum_{V\subseteq C_0} \alpha_1(A_1\cup C_1\cup V)\alpha_2(A_2\cup C_2\cup(C_0-V))\beta_1(B_1\cup C_1\cup(C_0-V))\beta_2(B_2\cup C_2\cup V) \leq \sum_{U\subseteq C_0} \gamma_1(A_1\cup B_1\cup C_1\cup U)\gamma_2((A_2\cup B_2\cup C_2\cup(C_0-U))\delta_1(C_1\cup(C_0-U))\delta_2(C_2\cup U). \tag{17}$$

Now, with $A, B, A_1, B_1, C_0, C_1$ still fixed, define functions $\alpha_1', \beta_1', \gamma_1', \delta_1'$, $\alpha_2', \beta_2', \gamma_2', \delta_2'$, $\alpha', \beta', \gamma', \delta'$ for $J \subseteq C_0$ by:

$$\begin{array}{ll}
\alpha_1'(J) = \alpha_1(A_1 \cup C_1 \cup J), & \alpha_2'(J) = \alpha_2(A_2 \cup C_2 \cup (C_0 - J)), \\
\alpha'(J) = \alpha_1'(J)\alpha_2'(J), & \\
\beta_1'(J) = \beta_1(B_1 \cup C_1 \cup J), & \beta_2'(J) = \beta_2(B_2 \cup C_2 \cup (C_0 - J)), \\
\beta'(J) = \beta_1'(J)\beta_2'(J), & \\
\gamma_1'(J) = \gamma_1(A_1 \cup B_1 \cup C_1 \cup J), & \gamma_2'(J) = \gamma_2(A_2 \cup B_2 \cup C_2 \cup (C_0 - J)), \\
\gamma'(J) = \gamma_1'(J)\gamma_2'(J), & \\
\delta_1'(J) = \delta_1(C_1 \cup J), & \delta_2'(J) = \delta_2(C_2 \cup (C_0 - J)), \\
\delta'(J) = \delta_1'(J)\delta_2'(J). &
\end{array}$$

Then (17) becomes:

$$\sum_{V \subseteq C_0} \alpha'(V)\beta'(C_0 - V) \leq \sum_{U \subseteq C_0} \gamma'(U)\delta'(C_0 - U) \tag{18}$$

It is easy to see that $AD(\alpha_1, \beta_1, \gamma_1, \delta_1)$ implies $AD(\alpha_1', \beta_1', \gamma_1', \delta_1')$ and that $AD(\alpha_2, \beta_2, \gamma_2, \delta_2)$ implies $AD(\alpha_2', \beta_2', \delta_2', \gamma_2')$ (here $\delta_2'$ and $\gamma_2'$ are intentionally interchanged). Together these imply $AD(\alpha', \beta', \gamma', \delta')$. Now Corollary 13 with $D = C_0$ implies (18) to complete the proof of the theorem. □

# 7 Some final remarks

Theorems 8, 9 and 10 all include the 4FT. Is there a natural theorem that unifies all three of them, or some two of them?

The fact that (16) holds for all choices of $A, B \subseteq S$ and all relevant quadruples $(A_1, B_1, C_0, C_1)$ with respect to $A, B$ refines Theorem 10 and also includes Theorem 9 by taking $A = B = S$, $A_1 = B_1 = \emptyset$, $C_1 = C$ and $C_0 = D - C$. However, the statement of such a theorem would be rather cumbersome and it would be nice to find a cleaner unifying statement.

It is natural to look for generalizations of Theorem 6 to set functions whose range is $\mathbb{R}^k$ for $k \geq 2$. It is not clear how to generalize the conclusion to higher $k$. One can restate the conclusion of Theorem 6 to say that for $k = 2$ there is a sublattice generated by two sets on which the sum of $\mu \cdot f$ is nonnegative. One might conjecture that there is a sublattice generated by $k$ sets on which the sum of $\mu \cdot f$ is nonnegative. Unfortunately it would limit applicability of this statement that such a sublattice grows very fast with $k$: the rank of the largest sublattice that can be generated by $k$ sets is $\ell(k) = \binom{k}{\lfloor k/2 \rfloor}$, and its size is $2^{\ell(k)}$. We certainly cannot sharpen this assertion and put, say a stricter limit on the size of the lattice, because if each $f_i$ is positive only on a single set $A_i$, then the sublattice taken for the conclusion must include all of the $A_i$.

Another interesting direction may be to consider other models of discrete localization. For example, suppose that we have a connected graph $G = (V, E)$ and a function $f : V \to \mathbb{R}^2$ such that $\sum_{v \in V} f(v) > 0$; we would like to conclude that there is a path $v_0 v_1 \ldots v_r$ and a number $\lambda > 0$ such that $\sum_{k=0}^{r} \lambda^k f(v_k) > (0, 0)$. This is certainly not true in general. An easy example is a star with $2k + 1$ nodes. Let $f_1$ be 1 on half of the leaves, 0 on he other half, and $-(k - 1)$ in the center. Let $f_2$ be obtained by interchanging the values 0 and 1. It is easy to check that this is a counterexample. Perhaps this form of discrete localization holds if the underlying graph has a node-transitive automorphism group.

We mention the following result of Beck and Krogdahl [BK2] that is of this flavor: Let $(c_{ij})$ be a positive logsupermodular matrix, i.e., $c_{ij} \cdot c_{i+1,j+1} \geq c_{i+1,j} \cdot c_{i,j+1}$. Then there exists a path from $c_{1,1}$ to $c_{n,m}$ whose average dominates that of all the entries of the whole matrix.

## Acknowledgements

The authors thank Navin Goyal for his corrections of an earlier draft of the paper.

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