Where to Start a Geometric Random Walk?

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1 Introduction

Consider a random walk in $\mathbb{R}^n$. It starts somewhere, and at each step moves to a randomly chosen "neighboring" point (which could be the current point). By a suitable choice of the "neighbor" relation, the steady state distribution of such a walk can be the uniform distribution over a convex body, or indeed any reasonable distribution in $\mathbb{R}^n$.

These walks have several nice properties. For example, they are easy to implement as algorithms. Further, they seem to approach their steady state distribution quite rapidly when the target distribution is logconcave. There has been much work in this direction, showing that the conductance of natural random walks (lattice walk, ball walk, speedy walk, and hit-and-run) is large enough that the rate of convergence is polynomial in n.

The above statement usually comes with one caveat: the walk is rapidly mixing provided it starts from a *warm* distribution (or in general something that's not too far away from stationary). In particular, there is no such guarantee if the walk starts from a single point or (say) uniformly in a small ball. As we discuss below, polynomial convergence is in fact not true for some natural walks, including the the ball walk. The problem is that the *local conductance* at some points (or some tiny regions) could extremely small, so that the walk takes a long time to escape from these regions. The *speedy walk* was invented to overcome this problem, but comes with its own set of complications, e.g. the steady state distribution is not the intended one, and it is not known if it can be implemented efficiently from an arbitrary starting point.

Random walks on a discrete subset of $\mathbb{R}^n$ (e.g. the lattice walk) avoid the local conductance problem, but have other complications that make their convergence less efficient, although still polynomial. Also, one is sampling from a discrete subset, which might be okay for applications but is a bit unsatisfactory.

So, is there a random walk that converges as quickly as the fastest known walks, and yet can start at an arbitrary distribution? *Hit-and-run* seems to be a candidate. In [4], it was shown that the conductance of large subsets is large for the case of convex sets, and this was later generalized to arbitrary logconcave functions [7]. This gives us the usual guarantee of rapid mixing from a warm start.

In this paper we describe a general condition under which the conductance of hit-and-run can be bounded for arbitrarily small sets. The condition is satisfied by the uniform density over a convex body and also by the density proportional to e^{-a^Tx} restricted to a convex body (for any vector a). Our main results are the following.

Theorem 1.1 *Let K be a convex body that contains a ball of radius r and is contained in a ball of radius R. Let σ be a starting distribution and let σ^m be the distribution of the current point after m steps of hit-and-run in K. Assume that the L_2 norm of σ w.r.t. π_f is at most M. Then for*

$$m > 10^{10}\frac{n^2R^2}{r^2}\ln\frac{Mn}{\varepsilon},$$

the total variation distance of σ^m and π_f is less than ε.

This improves on existing bounds by reducing the dependence on M and ε from polynomial to logarithmic while maintaining the (optimal) dependence on r, R and n.

Theorem 1.2 *Let f be a density proportional to $e^{-a^T x}$ such that the level set of probability $1/64$ contains a ball of radius r and $E_f(|x - z_f|^2| \le R^2$. Let σ be a starting distribution and let σ^m be the distribution of the current point after m steps of hit-and-run applied to f. Assume that the L_2 norm of σ w.r.t. π_f is at most M. Then for*

$$m > 10^{30} \frac{n^2 R^2}{r^2} \ln^5 \frac{nM}{\varepsilon},$$

the total variation distance of σ^m and π_f is less than ε.

For these density functions, starting uniformly from any subset of stationary measure π^*, we get a mixing time of proportional to $\log \frac{1}{\pi^*}$. This can be used to get a guarantee in terms of the local conductance of the starting point, and in the case of the uniform distribution in terms of the radius of the largest ball centered at the starting point. Such a guarantee is not possible for the ball walk. Our main tool is a new isoperimetric inequality.

1.1 Hit-and-run

Let f be a logconcave distribution in $\mathbb{R}^n$. For any line ℓ in $\mathbb{R}^n$, let $\mu_{\ell,f}$ be the measure induced by f on ℓ, i.e.

$$\mu_{\ell,f}(S) = \int_{p+tu \in S} f(p + tu) dt,$$

where p is any point on ℓ and u is a unit vector parallel to ℓ. We abbreviate $\mu_{\ell,f}$ by μ_ℓ if f is understood, and also $\mu_\ell(\ell)$ by μ_ℓ. The probability measure $\pi_\ell(S) = \mu_\ell(S)/\mu_\ell$ is the *restriction* of f to ℓ.

Then hit-and-run is defined as follows:

- Pick a uniformly distributed random line ℓ through the current point.
- Move to a random point y along the line ℓ chosen from the distribution π_ℓ.

2 Distance

To analyze the hit-and-run walk, we need a notion of distance according to a density function. Let f be a logconcave density function. For two points $u, v \in \mathbb{R}^n$, let $\ell(u, v)$ denote the line through them. Let $[u, v]$ denote the segment connecting u and v, and let $\ell^+(u,v)$ denote the semiline in ℓ starting at u and not containing v. Furthermore, let

$$\begin{aligned} f^+(u,v) &= \mu_{\ell,f}(\ell^+(u,v)), \\ f^-(u,v) &= \mu_{\ell,f}(\ell^+(v,u)), \\ f(u,v) &= \mu_{\ell,f}([u,v]). \end{aligned}$$

We introduce the following "distance":

$$d_f(u,v) = \frac{f(u,v)f(\ell(u,v))}{f^-(u,v)f^+(u,v)}.$$

Suppose f is the uniform distribution over a convex set K. Let u,v be two points in K and p,q be the endpoints of $\ell(u,v)\cap K$, so that the points appear in the order p,u,v,q along $\ell(u,v)$. Then,

$$d_f(u,v) = d_K(u,v) = \frac{|u-v||p-q|}{|p-u||v-q|}.$$

The two can be related in general.

Lemma 2.1 *Let f be a density function in $\mathbb{R}^n$ with support a convex body K. Let $G = \max_K f(x)/\min_K f(x)$.*

1. $d_f(u,v) \geq d_K(u,v)$.
2. *If $d_K(u,v) \leq 1/2(1+\ln G)$, then*

$$d_K(u,v) \geq \frac{1}{6(1+\ln G)} d_f(u,v).$$

The first inequality is Lemma 4.2 in [7] and the second follows from Lemma 4.4 in [7].

3 An isoperimetric inequality for concave functions over convex sets

Theorem 3.1 *Let K be a convex body in $\mathbb{R}^n$ of diameter D and $s : K \to \mathbb{R}_+$ be a concave function with $\max s(x) = S \leq D$. For $\varepsilon \in [0,1]$, let S_1, S_2, S_3 be any partition of K such that for any pair of points $u \in S_1$ and $v \in S_2$, one of the following holds:*

$$d(u,v) \geq 64\varepsilon \max\{s(u), s(v)\} \quad \text{or} \quad d_K(u,v) \geq \varepsilon \frac{S}{D}.$$

Then,

$$\mathrm{vol}(S_3) \geq \frac{\varepsilon}{D} \mathsf{E}_K(s) \frac{\mathrm{vol}(S_1)\mathrm{vol}(S_2)}{\mathrm{vol}(K)}.$$

Proof. We can rewrite the inequality as

$$\mathrm{vol}(S_3)\mathrm{vol}(K)^2 \geq \frac{\varepsilon}{D}\int_K s(x)\,dx\,\mathrm{vol}(S_1)\mathrm{vol}(S_2)$$

We can assume that $\mathrm{vol}(S_1) = A\mathrm{vol}(K)$ where $A \leq 1/2$. Suppose the theorem is false. Then there exists a partition for which

$$\begin{aligned} \mathrm{vol}(K) &= \frac{1}{A}\mathrm{vol}(S_1) \\ \mathrm{vol}(S_3) &< A\frac{\varepsilon}{D}\int_K s(x)\,dx \end{aligned}$$

By the Localization Lemma (specifically the version given in Corollary 2.4 of [2]), there exists a needle N

$$\begin{aligned} \int_N dt &= \frac{1}{A}\int_{S_1\cap N} dt \\ \int_{S_3\cap N} dt &< A\frac{\varepsilon}{D}\int_N s(t)\,dt \end{aligned}$$

In other words,

$$\int_{S_1\cap N} dt \leq \frac{1}{2}\int_N dt \Rightarrow \int_N dt \int_{S_3\cap N} dt < \frac{\varepsilon}{D}\int_N s(t)\,dt \int_{S_1\cap N} dt.$$

The needle is described by an interval $[a,b]$ and a logconcave function $f : [a,b] \to \mathbb{R}_+$. We can assume that $a = 0$. We will prove that for any such needle,

$$\int_{S_1\cap N} f(t)\,dt \leq \frac{1}{2}\int_N f(t)\,dt \Rightarrow$$

$$\int_N f(t)\,dt \int_{S_3\cap N} f(t)\,dt \geq \frac{\varepsilon}{D}\int_N s(t)f(t)\,dt \int_{S_1\cap N} f(t)\,dt$$

which is a contradiction.

By standard arguments, we can assume that $S_1 \cap N = [0,u]$, $S_2 \cap N = [v,b]$ and $S_3 \cap N = (u,v)$. Assume that

$$\int_{S_1\cap N} f(t)\,dt \leq \frac{1}{2}\int_N f(t)\,dt. \tag{1}$$

Then it suffices to prove that

$$\int_0^b f(t)\,dt \int_u^v f(t)\,dt \geq \frac{\varepsilon}{D}\int_0^b s(t)f(t)\,dt \int_0^u f(t)\,dt.$$

The hypothesis of the theorem implies that either $d(u,v) \geq 64\varepsilon \max\{s(u), s(v)\}$ or $d_K(u,v) \geq \varepsilon S/D$. Suppose the latter is true. Then since $s(t) \leq S$, it is enough to show that

$$\int_u^v f(t)\,dt \geq d_K(u,v)\int_0^u f(t)\,dt$$

which is implied by Theorem 2.2 of [7].

So let us assume that $d(u,v) \geq 64\varepsilon \max\{s(u), s(v)\}$ and $d_K(u,v) < \varepsilon S/D \leq \varepsilon$. We need to show that

$$\int_0^b f(t)\,dt \int_u^v f(t)\,dt \geq \frac{|u-v|}{64\max\{s(u),s(v)\}D} \int_0^b s(t)f(t)\,dt \int_0^u f(t)\,dt.$$

We can replace s by its tangent at u or v, since the integral on the RHS cannot decrease (since s is concave). Further if we do this at the point w (equal to u or v) which achieves $\max\{s(u), s(v)\}$ then we can maintain the value of s at that point.

Now by rotating s about w, the inequality will become stronger in one of the two directions. We do this till either $s(0) = 0$ or $s(b) = 0$. It might happen that in this process the relation between $s(u)$ and $s(v)$ changes. However, since $d_K(u,v) < \varepsilon$, we have $s(u) \leq 2s(v)$ and $s(v) \leq 2s(u)$. We can assume that the slope of the linear function s is 1. In this setting, we will prove that

$$\int_0^b f(t)\,dt \int_u^v f(t)\,dt \geq \frac{|u-v|}{64\min\{s(u),s(v)\}b} \int_0^b s(t)f(t)\,dt \int_0^u f(t)\,dt.$$

We consider the following cases.

1. $s(0) = 0$. Here $s(u) = u \leq v = s(v)$. Using the fact that $s(t) \leq b$, it suffices to prove that

$$\int_u^v f(t)\,dt \geq \frac{|u-v|}{64u} \int_0^u f(t)\,dt$$

   If $f(u) \leq f(v)$,

$$\int_u^v f(t)\,dt \geq |u-v|f(u) \text{ and } \int_0^u f(t) \leq uf(u).$$

   which implies the inequality. It could happen that $f(u) > f(v)$, but by assumption 1 and the fact that $d_K(u,v) < \varepsilon$ (in particular $|u-v| \leq u$), we get $f(u) \leq 4f(v)$, which again implies the inequality.

2. s(b)=0. This is equivalent to the situation when $s(0) = 0$, but with the roles of S_1 and S_2 exchanged, i.e. $S_1 \cap N = [v,b]$ and $S_2 \cap N = [0,u]$. We need to show that

$$\int_0^b f(t)\,dt \int_u^v f(t)\,dt \geq \frac{|u-v|}{64s(u)b} \int_0^b s(t)f(t)\,dt \int_v^b f(t)\,dt.$$

   Here we will use the logconcavity of f. Suppose $f(u) \geq f(v)$. Then,

$$\int_0^v f(t)\,dt \geq vf(v)$$

By assumption 1,

$$\int_v^b f(t)\,dt \le \frac{1}{2}\int_0^b f(t)\,dt \le \int_0^v f(t)\,dt \le vf(v)$$

we must have $f(2v) \le f(v)/2$ and by logconcavity, $f(iv) \le f(v)/2^{i-1}$ for any positive integer i. Hence,

$$\int_v^b f(t)\,dt \le 2vf(v) \text{ and } \int_0^b s(t)f(t)\,dt \le 4vs(v)f(v).$$

On the other hand,

$$\int_0^b f(t)\,dt \ge vf(v) \text{ and } \int_u^v f(t)\,dt \ge |u-v|f(v)$$

which proves the inequality.

Finally, suppose $f(u) < f(v)$. If $f(0) > f(v)/2$, then the same argument applies with an additional factor of 2. So let y be the point closest to v in the interval $[0, v]$ with $f(y) = f(v)/2$. Let $t = v - y$. Then,

$$\int_0^v f(t)\,dt \ge \frac{t}{2}f(v) \text{ and } \int_u^v f(t)\,dt \ge \frac{1}{2}\min\{t, |u-v|\}f(v)$$

By the assumption that the integral of f over S_1 is at most half the total, we must have $f(v+t) \le f(v)/2$ and $f(v+it) \le f(v)/2^i$ for any positive integer i. Hence,

$$\int_v^b f(t)\,dt \le 2tf(v) \text{ and } \int_0^b s(t)f(t)\,dt \le 4ts(v)f(v)$$

which again proves the inequality.

$\square$

The inequality can be extended to logconcave functions in $\mathbb{R}^n$.

Theorem 3.2 *Let f be a logconcave density function on $\mathbb{R}^n$ whose support is a convex body K of diameter D, and $s : K \to \mathbb{R}_+$ be a concave function with $\max s(x) = S \le D$. For $\varepsilon \in [0,1]$, let S_1, S_2, S_3 be any partition of K such that for any pair of points $u \in S_1$ and $v \in S_2$, one of the following holds:*

$$d(u,v) \ge 64\varepsilon \max\{s(u), s(v)\} \quad \textit{or} \quad d_K(u,v) \ge \varepsilon\frac{S}{D}.$$

Then,

$$\pi_f(S_3) \ge \frac{\varepsilon}{D}\mathsf{E}_K(s)\pi_f(S_1)\pi_f(S_2).$$

4 Step-size

For $x \in K$, let y be a random step from x. Following [4], we define $F(x)$ as

$$\mathsf{P}[|x-y| \le F(x)] = \frac{1}{8}. \tag{2}$$

Roughly speaking, this is the "median" step length from x. The goal of this section is to bound this function from below by a concave function. We do this separately for the uniform density of a convex body and for an exponential density.

4.1 Uniform density over a convex body

For a fixed $\gamma \ge 0$, define the *step-size* function $r : K \to \mathbb{R}_+$ as

$$s(x) = \max\{r \in \mathbb{R}_+ : \frac{\mathrm{vol}(x+rB)}{\mathrm{vol}(rB)} \ge \gamma\}.$$

(We will fix $\gamma = \frac{1}{2}$ for the convex set case).

Lemma 4.1 *For any $\gamma > 0$, the step-size $s(x)$ is a concave function.*

Proof. Let $x_1, x_2 \in K$ with $s(x_1) = r_1$ and $s(x_2) = r_2$. Let $A_1 = K \cap x_1 + r_1 B$ be the intersection of K with the ball of radius r_1 around x_1 and similary let $A_2 = K \cap x_2 + r_2 B$.

Now let $x = (x_1 + x_2)/2$ and consider $A = (A_1 + A_2)/2$. By convexity, $A \subseteq K$. Further, any point $y \in A$ can be written as

$$y = \frac{1}{2}(x_1 + z_1 \ + x_2 + z_2) = x + \frac{z_1 + z_2}{2}.$$

for some z_1, z_2 such that $|z_1| \le r_1$ and $|z_2| \le r_2$. Thus '

$$A \subseteq K \cap \left(x + \frac{r_1 + r_2}{2} B\right).$$

Next, by the Brunn-Minkowski inequality,

$$\begin{aligned} \mathrm{vol}(A)^{\frac{1}{n}} &\ge \frac{1}{2}\left(\mathrm{vol}(A_1)^{\frac{1}{n}} + \mathrm{vol}(A_2)^{\frac{1}{n}}\right) \\ &\ge \frac{1}{2}(\gamma \pi_n)^{\frac{1}{n}}(r_1 + r_2) \\ &= \gamma^{\frac{1}{n}} \mathrm{vol}(\frac{r_1 + r_2}{2} B)^{\frac{1}{n}}. \end{aligned}$$

It follows that

$$\mathrm{vol}(K \cap x + \frac{r_1 + r_2}{2} B) \ge \mathrm{vol}(A) \ge \gamma \mathrm{vol}(\frac{r_1 + r_2}{2} B)$$

and thus $s(x) \ge (r_1 + r_2)/2$. □

The average value of $s(x)$ can be bounded from below.

Lemma 4.2 *Suppose K contains a unit ball. Then,*

$$\mathsf{E}_K(r) \geq \frac{1-\gamma}{\sqrt{n}}.$$

Proof. For any fixed $\delta > 0$, and any $x \in K$, let

$$\lambda(x,t) = \frac{\mathrm{vol}(x + tB \cap K)}{\mathrm{vol}(K)},$$

the fraction of a ball of radius t around x that intersects K. In [3] (Corollary 4.6), it was shown that when K contains a unit ball,

$$\lambda(t) = \frac{1}{\mathrm{vol}(K)} \int_K \lambda(x,t)\, dx \geq 1 - \frac{t\sqrt{n}}{2}.$$

Let $p(t) = \mathsf{P}_K(\lambda(x,t) \geq \gamma)$. Then,

$$p(t) \cdot 1 + (1 - p(t)) \cdot \gamma \geq \lambda(t).$$

Hence,

$$p(t) \geq \frac{\lambda(t) - \gamma}{1-\gamma} \geq 1 - \frac{t\sqrt{n}}{2(1-\gamma)}.$$

From this we get

$$\begin{aligned}
\mathsf{E}_K(s) \geq \int_0^\infty \mathsf{P}(s \geq t)\, dt &\geq \int_0^\infty p(t)\, dt \\
&\geq \int_0^{\frac{2(1-\gamma)}{\sqrt{n}}} (1 - \frac{t\sqrt{n}}{2(1-\gamma)})\, dt \\
&= \left[t - \frac{t^2\sqrt{n}}{4(1-\gamma)} \right]_0^{\frac{2(1-\gamma)}{\sqrt{n}}} \\
&= \frac{1-\gamma}{\sqrt{n}}.
\end{aligned}$$

□

Now fix $\gamma = \frac{1}{2}$ in the definition of $s(x)$. It is easy to see the following inequality for $n > 1$.

$$F(x) \geq \frac{s(x)}{2}. \tag{3}$$

4.2 Exponential density

Consider the density function $f(x)$ with support K. For any vector $v \in \mathbb{R}^n$, define

$$B(v) = \{y : y \in B, v^T y \geq 0\}.$$

In other words, $B(v)$ is the half-ball defined by the halfspace with normal vector v.

For fixed β, γ, define

$$s(x) = \max\{r \in \mathbb{R}_+ : \min_{v \in \mathbb{R}^n} \frac{\mathrm{vol}(y \in x + rB(v) : f(y) \geq \beta f(x))}{\mathrm{vol}(x + rB(v))} \geq \gamma\}.$$

Note that for any $v \neq 0$,

$$\mathrm{vol}(x + rB(v)) = \frac{1}{2}\mathrm{vol}(rB).$$

In other words, $s(x)$ is radius of the largest ball around x such that in any "half" of this ball, the probability that a random point y has function value at least β times $f(x)$ is at least γ.

Lemma 4.3 *Suppose $f(x)$ is proportional to $e^{-a^T x}$ inside a convex body K and zero outside. Then for any fixed $\beta, \gamma > 0$, the step size $s(x)$ is a concave function.*

Proof. Let $x_1, x_2 \in K$ with $s(x_1) = r_1$ and $s(x_2) = r_2$. Fix some normal vector v. Define

$$A_1 = \{y \in x_1 + r_1 B(v) : f(y) \geq \beta f(x_1)\}$$

and

$$A_2 = \{y \in x_2 + r_2 B(v) : f(y) \geq \beta f(x_2)\}$$

Now let $x = (x_1 + x_2)/2$ and consider $A = (A_1 + A_2)/2$. Any point $y \in A$ can be written as

$$y = x + \frac{z_1 + z_2}{2}$$

for some z_1, z_2 such that $z_1 \in r_1 B(v)$ and $z_2 \in r_2 B(v)$. Thus

$$A \subseteq x + \frac{r_1 + r_2}{2} B(v).$$

Also, since $f(x)$ is proportional to $e^{-a^T x}$, we have $f(x + y) = f(x)f(y)$ and so for any $y \in A$,

$$f(y) = f(\frac{y_1 + y_2}{2}) = \sqrt{f(y_1)f(y_2)}$$

where $y_1 \in A_1$ and $y_2 \in A_2$. By the definition of these subsets, $f(y_1) \geq \beta f(x_1)$ and $f(y_2) \geq \beta f(x_2)$. Thus,

$$f(y) \geq \beta\sqrt{f(x_1)f(x_2)} = \beta f(\frac{x_1 + x_2}{2}) = \beta f(x).$$

and so

$$A \subseteq \{y \in x + \frac{r_1 + r_2}{2} B(v) : f(y) \geq \beta f(x)\}.$$

Finally, by the Brunn-Minkowski inequality,

$$\begin{aligned} \mathrm{vol}(A)^{\frac{1}{n}} &\geq \frac{1}{2}\left(\mathrm{vol}(A_1)^{\frac{1}{n}} + \mathrm{vol}(A_2)^{\frac{1}{n}}\right) \\ &\geq \frac{1}{2}(\frac{\gamma\pi_n}{2})^{\frac{1}{n}}(r_1+r_2) \\ &= \gamma^{\frac{1}{n}}\mathrm{vol}\left(\frac{r_1+r_2}{2}B(v)\right)^{\frac{1}{n}}. \end{aligned}$$

It follows that $s(x) \geq (r_1+r_2)/2$. □

To bound the expected value of $s(x)$, we will use a lemma from [6]. For a fixed t, define

$$\hat{f}(x) = \min_C \frac{1}{\mathrm{vol}(C)} \int_C f(x+u)\,du,$$

where C ranges over all convex subsets of the ball $x+tB$ with $\mathrm{vol}(C) = \mathrm{vol}(tB)/16$.

Lemma 4.4 *(Lemma 4.6 in [6]) Let f be a density function for which any level set of probability at least ε contains a ball of radius $r(\varepsilon)$. Then,*

$$\int_{\mathbb{R}^n} \hat{f}(x)\,dx \geq 1 - 16(\varepsilon + \frac{t\sqrt{n}}{r(\varepsilon)}).$$

Using this we get a lower bound on $\mathsf{E}(s(x))$. From here onwards fix $\gamma = \frac{1}{32}$ and $\beta = \frac{1}{8}$ in the definition of $s(x)$.

Lemma 4.5 *Let f be a density function such that any level set of measure $\frac{1}{64}$ contains a ball of radius r. Then,*

$$E_f(s(x)) \geq \frac{r}{2^{12}\sqrt{n}}.$$

We can also bound the maximum value of $s(x)$.

Lemma 4.6 *Let $G = \frac{\max_K f(x)}{\min_K f(x)}$ where f is proportional to $e^{-a^T x}$ with support K. Suppose K has diameter D. Then,*

$$\max_K s(x) \leq \min\{\frac{2^8\sqrt{n}D}{1+\ln G}, D\}.$$

Proof. Let $v = a$ and consider $B(v)$. □

Finally, we can relate $s(x)$ to $F(x)$.

Lemma 4.7

$$F(x) \geq \frac{\beta\gamma^{1+\frac{1}{n}}}{2}s(x) \geq \frac{s(x)}{2^{10}}.$$

5 A scale-free bound on the conductance

For a point $u \in K$, let P_u be the distribution obtained by taking one hit-and-run step from u. Then,

$$P_u(A) = \frac{2}{n\pi_n} \int_A \frac{f(x)\,dx}{\mu_f(u,x)|x-u|^{n-1}}. \tag{4}$$

5.1 Uniform density

The following lemma from [4] connects the geometric distance of two points to the variation distance of the distributions obtained by taking one hit-and-run step.

Lemma 5.1 *[4] Let $u, v \in K$. Suppose that*

$$d_K(u,v) < \frac{1}{8} \text{ and } |u-v| < \frac{2}{\sqrt{n}} \max\{F(u), F(v)\}.$$

Then

$$d(P_u, P_v) < 1 - \frac{1}{500}.$$

Theorem 5.2 *Let K be a convex body in $\mathbb{R}^n$ of diameter D, containing a unit ball. Then the conductance of hit-and-run in K is at least $\frac{1}{2^{20}nD}$.*

Proof. Let $K = S_1 \cup S_2$ be a partition into measurable sets. We will prove that

$$\mathrm{vol}(K) \int_{S_1} P_x(S_2)\,dx \quad \geq \quad \frac{1}{2^{20}nD} \min\{\mathrm{vol}(S_1), \mathrm{vol}(S_2)\} \tag{5}$$

We can read the left hand side as follows: we select a random point X from the uniform distribution and make one step to get Y. What is the probability that $X \in S_1$ and $Y \in S_2$? It is well known that this quantity remains the same if S_1 and S_2 are interchanged.

Consider the points that are deep inside these sets, i.e. unlikely to jump out of the set:

$$S_1' = \{x \in S_1 : P_x(S_2) < \frac{1}{1000}\} \text{ and } S_2' = \{x \in S_2 : P_x(S_1) < \frac{1}{1000}\}.$$

Let S_3' be the rest i.e., $S_3' = K \setminus S_1' \setminus S_2'$.

Suppose $\mathrm{vol}(S_1') < \mathrm{vol}(S_1)/2$. Then

$$\mathrm{vol}(K) \int_{S_1} P_x(S_2)\,dx \geq \frac{1}{1000}\mathrm{vol}(S_1 \setminus S_1') \geq \frac{1}{2000}\mathrm{vol}(S_1)$$

which proves (5).

So we can assume that $\mathrm{vol}(S_1') \geq \mathrm{vol}(S_1)/2$ and similarly $\mathrm{vol}(S_2') \geq \mathrm{vol}(S_2)/2$. For any $u \in S_1'$ and $v \in S_2'$,

$$d(P_u, P_v) \geq 1 - P_u(S_2) - P_v(S_1) > 1 - \frac{1}{500}.$$

Thus, by Lemma 5.1, either

$$d_K(u, v) \geq \frac{1}{8}$$

or

$$|u - v| \geq \frac{2}{\sqrt{n}} \max\{F(u), F(v)\} \geq \frac{1}{\sqrt{n}} \max\{s(u), s(v)\}.$$

Applying Theorem 3.1 to the partition S_1', S_2', S_3' and the concave function $s(x)$ with $\varepsilon = 1/64\sqrt{n}$ we get

$$\begin{aligned}
\mathrm{vol}(S_3') &\geq \frac{\varepsilon}{D} E_K(s) \frac{\mathrm{vol}(S_1')\mathrm{vol}(S_2')}{\mathrm{vol}(K)} \\
&\geq \frac{1}{512nD} \min\{\mathrm{vol}(S_1), \mathrm{vol}(S_2)\}.
\end{aligned}$$

Here we have used Lemma 4.2. Therefore,

$$\begin{aligned}
\mathrm{vol}(K) \int_{S_1} P_x(S_2)\, dx &\geq \frac{1}{2} \cdot \frac{1}{1000} \mathrm{vol}(S_3') \\
&\geq \frac{1}{2^{20} nD} \min\{\mathrm{vol}(S_1), \mathrm{vol}(S_2)\}
\end{aligned}$$

which again proves (5) $\square$

5.2 Exponential density

Define $\alpha(x)$ (as in [7]) as the smallest $s \geq 3$ for which

$$P(f(y) \geq s f(x)) \leq \frac{1}{16}.$$

Let π be the measure induced by f. Then the following lemma was proved in [7].

Lemma 5.3 *([7],Lemma 6.3)*

$$\pi(u : \alpha(u) \geq t) \leq \frac{16}{t}.$$

The next lemma is analogous to Lemma 5.1. It holds for any logconcave density f, although we know how to use it only for the exponential density. Its proof is closely related to Lemma 6.4 in [7].

Lemma 5.4 *Let $u, v \in K$. Suppose that*

$$d_f(u,v) < \frac{1}{128\ln(3+\alpha(u))} \text{ and } |u-v| < \frac{1}{4\sqrt{n}}\max\{F(u),F(v)\}.$$

Then

$$d(P_u,P_v) < 1 - \frac{1}{500}.$$

The following lemma will be used in the proof of Lemma 5.4.

Lemma 5.5 *Let ℓ be any line through a point x. Let p, q be intersection points of ℓ with the boundary of $L(F/8)$ where F is the maximum value of f along ℓ, and let $s = \max\{|x-p|/32, |x-q|/32\}$. Choose a random point y on ℓ from the distribution π_ℓ. Then*

$$\mathsf{P}(|x-y| > s) > \frac{1}{3}.$$

Proof. (of Lemma 5.4). We will show that there exists a set $A \subseteq K$ such that $P_u(A) \geq \frac{1}{4}$ and for any subset $A' \subset A$,

$$P_v(A') \geq \frac{1}{100}P_u(A').$$

To this end, we define certain "bad" lines through u. Let σ be the uniform probability measure on lines through u.

Let B_1 be the set of lines that are not almost orthogonal to $u-v$, in the sense that for any point $x \neq u$ on the line,

$$|(x-u)^T(u-v)| > \frac{2}{\sqrt{n}}|x-u||u-v|.$$

The measure of this subset can be bounded as $\sigma(B_1) \leq 1/8$.

Next, let B_2 be the set of all lines through u which contain a point y with $f(y) > 2\alpha f(u)$. By Lemma 3.5(a) in [6], if we select a line from B_2, then with probability at least $1/2$, a random step along this line takes us to a point x with $f(x) \geq \alpha f(u)$. From the definition of α, this can happen with probability at most $1/16$, which implies that $\sigma(B_2) \leq 1/8$.

Let A be the set of points x in K which are not on any of the lines in $B_1 \cap B_2$, and which are far from u in the sense of Lemma 5.5:

$$|x-u| \geq \frac{1}{32}\max\{|u-p|,|u-q|\}.$$

Applying Lemma 5.5 to each such line, we get

$$P_u(A) \geq (1 - \frac{1}{8} - \frac{1}{8})\frac{1}{3} \geq \frac{1}{4}.$$

We will show that for any subset $A' \subseteq A$,

$$P_v(A') \geq \frac{1}{100}P_u(A')$$

using the next two claims.

Claim 1. For every $x \in A$,

$$|x - v| \leq (1 + \frac{1}{n})|x - u|.$$

Claim 2. For every $x \in A$,

$$\mu_f(v, x) < 32\frac{|x - v|}{|x - u|}\mu_f(u, x).$$

Claim 1 is easy to prove (cf. [4]) and the proof of Claim 2 is identical to that given in [7]. Thus, for any $A' \subset A$,

$$\begin{aligned}
P_v(A') &= \frac{2}{n\pi_n}\int_{A'} \frac{f(x)\,dx}{\mu_f(v, x)|x - v|^{n-1}} \\
&\geq \frac{2}{32n\pi_n}\int_{A'} \frac{|x - u| f(x)\,dx}{\mu_f(u, x)|x - v|^{n}} \\
&\geq \frac{2}{32en\pi_n}\int_{A'} \frac{f(x)\,dx}{\mu_f(u, x)|x - u|^{n-1}} \\
&\geq \frac{1}{32e}P_u(A').
\end{aligned}$$

The lemma follows. □

We are now ready to state and prove the main theorem.

Theorem 5.6 *Let f be a density in $\mathbb{R}^n$ proportional to $e^{-a^T x}$ whose support is a convex body K of diameter D. Assume that any level set of measure $1/64$ contains a ball of radius r. Then for any subset S, with $\pi_f(S) = x \leq 1/2$, the conductance of hit-and-run satisfies*

$$\phi(S) \geq \frac{r}{10^{15} nD \ln(\frac{nD}{rx})}.$$

Proof.

Similar to proof of Theorem 5.2.

Let $K = S_1 \cup S_2$ be a partition into measurable sets, where $S_1 = S$ and $x = \pi_f(S_1) \leq \pi_f(S_2)$. We will prove that

$$\int_{S_1} P_x(S_2)\,dx \quad \geq \quad \frac{r}{10^{15} nD \ln \frac{nD}{rx}}\pi_f(S_1) \tag{6}$$

Consider the points that are deep inside these sets:

$$S_1' = \{x \in S_1 : P_x(S_2) < \frac{1}{1000}\} \text{ and } S_2' = \{x \in S_2 : P_x(S_1) < \frac{1}{1000}\}.$$

Let S_3' be the rest i.e., $S_3' = K \setminus S_1' \setminus S_2'$.

Suppose $\pi_f(S_1') < \pi_f(S_1)/2$. Then

$$\int_{S_1} P_x(S_2)\,dx \geq \frac{1}{1000}\pi_f(S_1 \setminus S_1') \geq \frac{1}{2000}\pi_f(S_1)$$

which proves (6).

So we can assume that $\pi_f(S_1') \geq \pi_f(S_1)/2$ and similarly $\pi_f(S_2') \geq \pi_f(S_2)/2$.

Next, define the exceptional subset W as set of points u for which $\alpha(u)$ is very large.

$$W = \{u \in S : \alpha(u) \geq \frac{10^{16} nD}{\pi_f(S_1) r}\}.$$

By Lemma 5.3,

$$\pi(W) \leq \frac{\pi_f(S_1) r}{10^{15} nD}.$$

Next, for any $u \in S_1' \setminus W$ and $v \in S_2' \setminus W$,

$$d(P_u, P_v) \geq 1 - P_u(S_2) - P_v(S_1) > 1 - \frac{1}{500}.$$

Thus, by Lemma 5.4, either

$$d_f(u,v) \geq \frac{1}{128 \ln(3 + \alpha(u))} \geq \frac{1}{2^{13} \ln \frac{nD}{rx}}$$

or

$$|u - v| \geq \frac{1}{4\sqrt{n}} \max\{F(u), F(v)\}.$$

But by Lemmas 2.1 and 4.7, this implies that either

$$d_K(u,v) \geq \frac{1}{2^{16} \ln \frac{nD}{rx}(1 + \ln G)} \quad \text{or} \quad |u - v| \geq \frac{1}{2^{12}\sqrt{n}} \max\{s(u), s(v)\}.$$

Now, by Lemma 4.6, the first condition can be written as

$$d_K(u,v) \geq \frac{1}{2^{24} \ln \frac{nD}{rx}} \frac{\max s(x)}{\sqrt{n}D}.$$

Thus, for any $u \in S_1' \setminus W$ and any $v \in S_2' \setminus W$, one of the following holds

$$d_K(u,v) \geq \varepsilon \frac{\max s(x)}{D} \quad \text{or} \quad |u - v| \geq 64\varepsilon \max\{s(u), s(v)\}.$$

with

$$\varepsilon = \frac{1}{2^{24}\sqrt{n} \ln \frac{nD}{rx}}.$$

Applying Theorem 3.1 to the partition $S_1' \setminus W, S_2' \setminus W$ and the rest, with the concave function $s(x)$ we get

$$\begin{aligned} \pi_f(S_3') &\geq \frac{\varepsilon}{D} E_K(s) \frac{\pi_f(S_1' \setminus W)\pi_f(S_2' \setminus W)}{-} \pi_f(W) \\ &\geq \frac{\varepsilon r}{2^{13}\sqrt{n}D} \pi_f(S_1). \end{aligned}$$

Here we have used Lemma 4.5 and the bound on $\pi_f(W)$. Therefore,

$$\begin{aligned}\int_{S_1} P_x(S_2)\,dx &\geq \frac{1}{2}\cdot\frac{1}{1000}\pi_f(S_3') \\ &\geq \frac{r}{10^{15} nD \ln\frac{nD}{rx}}\pi_f(S_1)\end{aligned}$$

which again proves (6) □

6 Proofs of the mixing bounds

6.1 Uniform density

If K contains a unit ball and has diameter D, then the mixing bound is immediate using Cor. 1.6 of [5]. For the general case, note that hit-and-run is invariant under a scaling of space (i.e. there is a $1-1$ mapping between the random walk in K and CK) and so the bound follows by considering K/r.

6.2 Exponential density

Since f satisfies $\mathsf{E}_f(|x-z_f|^2) \leq R^2$, we can restrict the random walk to be within the ball of radius $R\ln(2/\epsilon)$, then by Lemma 3.12 in [6], the measure of f outside this ball is at most $\varepsilon/2$. Alternatively, if we consider (only for the analysis), a ball of radius $4R\ln(nM/\varepsilon)$, then the random walk is unlikely to step outside this ball. In either case, the effective diameter of K is $D \leq 4R\ln(nM/\varepsilon)$.

The bound on the mixing time then follows by applying Theorem 5.6 along with either Cor. 1.6 in [5] or the average conductance theorem of [1].

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