The Random Trip Model, Part I: Stability[1]

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Abstract— We define “random trip", a generic mobility model for independent random motions of nodes, which contains as special cases: the random waypoint on convex or non convex domains, random walk on torus, billiards, city section, space graph, intercity and other models. We show that, for this model, a necessary and sufficient for a stationary regime to exist is that the mean trip duration (sampled at trip endpoints) is finite. When this holds, we show that the distribution of node mobility state converges to a time-stationary distribution, starting from any initial state. For the special case of random waypoint, we provide for the first time a proof and a sufficient and necessary condition of the existence of a stationary regime, thus closing a long standing issue. We show that random walk on torus and billiards belong to the random trip class of models, and establish that the long term distribution of node location for these two models is uniform, starting from any initial distribution, even in cases where the speed vector does not have circular symmetry. In part I, we describe the random trip model, show that all examples mentioned above satisfy the assumptions, and establish the main stability result. In part II, we describe the time stationary distribution in detail, and show how to perform a simulation that eliminates the initial transients (“perfect simulation"), for any random trip model that satisfies the stability condition.

I. INTRODUCTION

RANDOM mobility models have been used extensively to evaluate performance of networking systems in both mathematical analysis and simulation based studies. The goal of our work is twofold: (Part I) provide a class of “stable” mobility models that is rich enough to accommodate a large variety of examples and (Part II–[5]) provide an algorithm to run “perfect simulation” of these models. Both goals are motivated by recent findings about the random waypoint, an apparently simple model that fits in our framework, the simulation of which was reported to pose a surprising number of challenges, such as speed decay, a change in the distribution of location and speed as the simulation progresses [21], [15], [19], [9].

A. Random trip model

We define “random trip”, a model of independent, random node movements. Such *independent* node movements are entirely defined by specifying random process of movement of a single node. The model does not directly accommodate group mobility models, which are left for further study. The random trip model is defined by a set of “stability” conditions for a node movement. These conditions guarantee existence of a time-stationary regime of node mobility state or its non existence. They also guarantee convergence of node mobility state to a time-stationary regime, whenever one exists, from origin of an arbitrary trip. The reported observations for random waypoint such as that speed vanishes to 0 as simulation progresses (“considered harmful” [20]) are *in fact all* related to the set of problems on stability of random processes that include finding conditions for existence of a stationary regime or its non existence. Stability problems also include finding conditions under which convergence to a stationary regime is guaranteed, whenever there exists one. These conditions are important to alleviate non desirable situations such as the reported vanishing of the node numerical speed to 0.

In the absence of established properties of real mobility patterns, it is not yet clear today what the requirements on mobility models should be [8]. The random trip model is a broad class of independent node movements that can be appropriately parameterised to *synthesise* an a priori assumed mobile behaviour.

B. Random trip examples

We show in Section III that many examples of random mobility models used in practice are random trip models. Our catalogue includes examples such as classical random waypoint, city driving models (“space graph” [11], “city section” or “hierarchical random waypoint” [4]), circulation models (“random waypoint on sphere”), or the special purpose “fish in a bowl” and “Swiss flag”. These are all accommodated by the “restricted random waypoint” introduced in Section III-D. These examples well illustrate the geometric diversity of mobility domains: for models such as “Swiss flag” we have a non convex area on a plane; for models such as “space graph” or “city-section”, a concatenation of line segments that represent streets; for “random waypoint on sphere”, a surface in a three dimensional space.

In some cases, it is desirable to assume that in steady-state, node location is uniformly distributed on a domain. This is provided by “random walk on torus” and “billiards”, which are defined by “bending” the paths of node movement with wrapping and billiards-like reflections, respectively, in a rectangular area on a plane. “Random waypoint on a sphere” is another such example, in three dimensions.

C. Perfect simulation

Camp, Navidi and Bauer [19] point out that if the model has a stationary regime, it is important to simulate it in this regime; otherwise, if the initial configuration is not sampled from the stationary regime, the performance evaluation of a system under study may be biased and non reproducible. In Part II, we provide an algorithm to sample time-stationary sample paths of a node movement. This is done by initialising node mobility state to a steady-state random sample which suffices to continue the simulation of node movement so that node mobility state is in steady-state throughput the simulation. This is commonly called “perfect simulation” in areas such as physics. Perfect simulation entirely eliminates the initial *transience* of node mobility state due to the convergence of its distribution to the steady-state distribution. In practice, this is of interest as for non perfect simulations, the effects of the initial transient are typically mitigated by truncating some initial part of the simulation. The difficulty with this approach is that it is many cases hard to know how long it takes for the distribution of the mobility state to converge to a given neighbourhood of the steady-state distribution.

D. Summary of the main contributions

Our main contributions are summarised as follows:

- We provide "random trip model", a generic mobility model with a framework for analysis.
- We identify a necessary and sufficient condition for existence of a time stationary regime for random trip model. This appears to be a new result even for the classical random waypoint, and fully explains the reported "harmfulness" [20].
- We show that random trip models feature convergence in distribution of node mobility state to time-stationary regime, from origin of arbitrary trip.
- In particular, we prove that a node location for "random walk on torus" and "billiards" at trip transition instants converges to the uniform distribution on a rectangular area, from any initial distribution. These results are established with complete proofs. For the "random walk on torus" model, the result requires a very mild assumption on on the distribution of the node speed vector (essentially, that it has a density) whereas previous results in [18] required the circular symmetry (speed vector is isotropic). For the "billiards" model, we require that the speed vector has a completely symmetric distribution (Section III-G), which means that it goes up or down [resp. left or right] with equal probability. This is also a weaker assumption than the circular symmetry required in [18].

In Part II [5], the main contribution is a perfect sampling algorithm that features:

- The algorithm supports random waypoint models on non convex areas.
- Sampling algorithm does not require computation of geometric integrals.

Further results in Part II are:

- Proof that for three examples (random walk on torus, billiards, random waypoint on sphere) the node location is uniform in steady-state. For the random walk on torus, the steady state is essentially the same as the naive initialisation (with uniform node placement) and there is no speed decay. In contrast, there *is* speed decay for random waypoint on a sphere.

E. Organisation of the paper

The random trip model is defined in Section II. The latter section contains a notation list. Section III shows that all examples in our catalogue are random trip models. The section contains convergence results for "random walk on torus" and "billiards" random trip models. The main result of Part I, which gives a necessary and sufficient condition for existence of a time-stationary regime and convergence to this regime when it exists, is given in Section IV. Section V provides concluding remarks. Related work is further discussed in Part II.

II. The Random Trip Model Definition

The random trip mobility model is defined by the following framework.

A. Trip, Phase, Path

1) **Domain**: The domain $\mathcal{A}$ is a subset of $\mathbb{R}^d$, for some integer $d \geq 1$.

2) **Phase**: I is a set of phases on $\mathbb{R}^m$, for some integer $m \geq 1$. A phase describes some state of the mobile, specific to the model. For example, it may indicate whether the mobile moves or pauses at a given time.

3) **Path**: $\mathcal{P}$ is a set of paths on $\mathcal{A}$. A path is a continuous mapping from $[0,1]$ to $\mathcal{A}$ that has a continuous derivative except maybe at a finite number of points (this is necessary to define the speed).

For $p \in \mathcal{P}$, $p(0)$ is the origin of p, $p(1)$ is its destination, and $p(u)$ is the point on p attained when a fraction $u \in [0,1]$ of the path is traversed.

4) **Trip**: A *trip* is specified by a path P_n and a duration S_n. The position $X(t)$ of the mobile at time t is defined iteratively as follows. There is a set $T_n \in \mathbb{R}^+$, $n \in \mathbb{Z}_+$ of *transition instants*, such that $T_0 \leq 0 < T_1 < T_2 < \ldots$. At time T_n, a phase $I_n \in I$, a path $P_n \in \mathcal{P}$ and a trip duration $S_n \in \mathbb{R}_+$ are drawn according to some specified *trip selection rule*, specific to the model. The next transition instant is $T_{n+1} = T_n + S_n$ and the position of the mobile is

$$X(t) = P_n\left(\frac{t - T_n}{S_n}\right) \quad \text{for } T_n \leq t \leq T_{n+1}.$$

The trip selection rule is constrained to choose a path P_n such that $P_n(0) = P_{n-1}(1)$. Further, we assume that, with probability 1, the duration of the trip S_n is positive (instantaneous transitions are not allowed).

Following a customary convention, whenever we consider a stationary realization of node mobility, we extend the transition instants T_n to the entire line $\mathbb{R}$, and enumerate them as $\ldots < T_{-1} < T_0 \leq 0 < T_1 < \ldots$. In these cases, 0 is an arbitrary time.

5) **Default Initialization Rule**: At time $t = 0$, a phase, path, position on the path, and remaining time until the next transition are drawn according to some specified initialization rule. We define as a *default initialization rule* that which takes time 0 as the first transition instant ($T_0 = 0$), and selects a phase, path and trip duration according to the trip selection rule. The default initialization rule has been used in simulations of many random mobility models (e.g. classical random waypoint).

We introduce additional assumptions. Some of the assumptions are either trivial to verify or always hold in real world, while some are crucial to guarantee stability of the random trip model and may not be always trivial to verify. This is discussed more concretely in Section II-D. In any case, the following assumptions accomodate a broad class of random mobility models.

B. Conditions on Phase and Path

(H1) $Y \equiv (Y_n)_{n=0}^{+\infty}$ with $Y_n := (I_n, P_n)$ defined as a couple of phase and path is a Markov chain on $I \times \mathcal{P}$.

(H2) The chain Y is *Harris recurrent* [16, Section III.9]: There exists a set $R \in I \times \mathcal{P}$, a probability measure φ with support on $I \times \mathcal{P}$, a number $\beta \in (0,1)$, and an integer $n_0 \geq 1$ such that the following two conditions hold

(i) $\mathbb{P}_y(Y_n \in R \text{ for some } n \geq 1) = 1$ for all $y \in I \times \mathcal{P}$

(ii) $\mathbb{P}_y(Y_{n_0} \in B) \geq \beta \cdot \varphi(B)$ for all $y \in R$ and any measurable B in $I \times \mathcal{P}$.

Here we use the notation $\mathbb{P}_y(\cdot) = \mathbb{P}(\cdot | Y_0 = y)$, $y \in I \times \mathcal{P}$.

Condition i implies that R is a recurrent set of the chain. Condition ii says that R is a "regeneration set" in the sense that the conditional probability that the chain hits a set B, after n_0 transitions from $y \in R$, is $\varphi(B)$ with probability β, where $\varphi(\cdot)$ is independent of y.

Condition H2 ensures the chain Y has a unique stationary measure (up to a multiplicative constant) π^0 defined by

$$\pi^0(A) = \int_{I \times \mathcal{P}} P(y, A) \pi^0(dy) \tag{1}$$

where $P(y, A) := \mathbb{P}(Y_1 \in A | Y_0 = y)$ is the transition semigroup of the chain.

(H3) The chain Y is *positive* Harris recurrent, i.e. H2 holds and the number of transitions between successive visits to the set R has a finite expectation.

Condition H3 implies the invariant measure π^0 is such that $\pi^0(I \times \mathcal{P}) < +\infty$, so that it can be normalized to a probability distribution.

C. Conditions on Trip Duration

(H4) Three hypotheses:

(i) The distribution of a trip duration S_n, given the phase I_n and path P_n, is independent of any other past and n. Formally, we have

$$\begin{aligned} & \mathbb{P}(S_n \leq s | Y_n = y, Y_{n-1}, S_{n-1}, \ldots) \\ = \; & \mathbb{P}(S_n \leq s | Y_n = y) := F(y, s), \quad \text{for all } n \in \mathbb{Z}. \end{aligned}$$

We assume that for all $y \in I \times \mathcal{P}$, $F(y, s)$ is a non defective probability distribution, that is $\lim_{s \to +\infty} F(y, s) = 1$, for all $y \in I \times \mathcal{P}$. Note that in general the trip duration S_n is dependent on the path P_n.

(ii) Each trip takes a strictly positive time, i.e.

$$\mathbb{P}(S_n > 0 | Y_n = y) = 1, \text{ all } n \geq 1 \text{ and } \pi^0 \text{ almost all } y.$$

This condition is always true in reality.

(iii) The Markov renewal process $(Y_n, S_n)_{n=0}^{\infty}$ is *non arithmetic*, i.e. there exists no $d \geq 0$ and some "shift" function $g : I \times \mathcal{P} \to [0, d)$ such that given $Y_0 = y$, S_0 takes values on the set $g(y) + d\mathbb{Z}_+$, for π^0 almost all y.

This assumption is automatically true if there is a subset $Y^0 \in I \times \mathcal{P}$ of strictly positive π^0 probability such that, given $Y_n \in Y^0$, the distribution of S_n has a density, i.e. $F(y, s) = \int_0^s f(y, t) dt$, for some function $f(\cdot)$, and $y \in Y^0$.

Condition H4.iii is needed to state the convergence in distribution to a time-stationary distribution as specified in Theorem 6–item ii, for sample paths initalized at $t = 0$ as specified by the default initialization rule (see item 5 in Section II-A)[1]

[1]Condition H4.iii is *not* needed for existence and uniqueness of a time stationary distribution (Theorem 6–item i, Section IV) and one can indeed construct time-stationary sample paths of mobility when H4.iii does not hold by appropriate initialization.

Notation Used in Section II

- $\mathcal{A} \subset \mathbb{R}^d$: model domain, connected and bounded
- $d(m, n)$ length of shortest path in $\mathcal{A}$ from $m \in \mathcal{A}$ to $n \in \mathcal{A}$; if $\mathcal{A}$ is convex $d(m, n) = \|m - n\|$
- T_n: nth transition time, at which a new trip is defined
- $I_n \in I, M_n \in \mathcal{A}, P_n \in \mathcal{P}, S_n \in (0, \infty)$: phase, starting point, path, trip duration for the trip indexed by n. The first trip has index $n = 0$.
- $I(t) \in I, M(t) \in \mathcal{A}, P(t) \in \mathcal{P}, S(t) \in (0, \infty), X(t) \in \mathcal{A}$: phase, starting point, path, trip duration for the trip used by mobile at time t, location at time t. $X(T_n) = M_n$ and if $T_n \leq t < T_{n+1}$ then $I(t) = I_n$, $M(t) = M_n$ and $S(t) = S_n$.
- $U(t) \in [0, 1]$: fraction of the current trip that was already traversed. Thus $U(t)S(t)$ is the time elapsed on the current trip and the location of the mobile at time t is $X(t) = p(U(t))$, with $p = P(t)$. We assume that the trip is done at a speed proportional to the default speed of the path, i.e. if $T_n \leq t < T_{n+1}$ then $U(t) = \frac{t - T_n}{T_{n+1} - T_n} = \frac{t - T_n}{S_n}$
- It follows that the speed vector of the mobile at a time t that is not an end of trip is $\vec{V}(t) = \frac{1}{S(t)} \frac{\partial}{\partial u} p(U(t))$, with $p = P(t)$ and the numerical speed is $V(t) = \left\| \vec{V}(t) \right\|$.
- For some random variable Z, $\mathbb{E}^0(Z)$ is the "Palm expectation", which can be interpreted as the expectation, conditional to the event that a transition occurs at time 0, when the system has a stationary regime. $\mathbb{E}^0$ denotes the event average viewpoint [3], [14]. For example $\mathbb{E}^0(S_0) = \mathbb{E}^0(S(0))$ is the average trip duration; in contrast, when the system has reached steady-state, $\mathbb{E}(S(0)) = \mathbb{E}(S(t))$ is the average duration of a trip, seen from an observer who samples the system at an arbitrary point in time. Both are usually different because the observer is more likely to sample a large trip duration.
- In order to simplify notation and at no expense of ambiguity, for a right-continuous process $X(t)$, $t \in \mathbb{R}$, and appropriately defined function $f(\cdot)$, we write $\mathbb{E}(f(X_0))$ for the Palm expectation $\mathbb{E}^0(f(X(0)))$; here $X_n := X(T_n)$ with T_n a trip transition instant.
- We say a property holds for π^0 almost all y, if it holds for all $y \in I \times \mathcal{P}$, but maybe not for some y that lies in a set of zero π^0 measure.

D. How to verify the conditions in practice ?

Condition H1 is a structural assumption on the trip selection over time and is easy to verify. The same holds for H4.i and H4.ii.

Condition H4.iii is true as soon as the trip duration has a density, for a non negligible subset of paths and phases. In practice, trip durations either have a density or are mixtures of constants. It is sufficient that, for some (non negligible) subset of path and phase conditions, the trip duration has a density. For example, H4.iii is true for a model with pauses if *either* the pause duration, *or* the (non pause) trip duration has a density.

Conditions H2 and H3 are stronger. They essentially say that the Markov chain of system states, sampled at trip endpoints, is stable, in a strong sense. The technical difficulty here is that, for many examples, we have a Markov chain on a non countable state space, for which stability conditions are mathematically complicated. However, it helps to think that for random trip with a countable state space $I \times \mathcal{P}$, conditions H2 and H3 simply mean positive recurrence. For a finite state space, they even more simply mean that the state space is connected.

We next show that conditions H1-H4 are verified by many random mobility models.

III. EXAMPLES

We give a non exhaustive catalogue of example random mobility models and show they are all random trip models.

A. Classical Random Waypoint With Pauses

This is the classical random waypoint model. $\mathcal{A}$ is assumed to be convex ($\mathcal{A}$ is a rectangle or a disk in [9], [8]). Paths are straight line segments: $p(u) = (1-u)m_0 + um_1$ for the segment with endpoints m_0 and m_1. Pauses are special cases of paths, when endpoints are equal: $p(u) = m_0$. There are two phases $I = \{pause, move\}$. At a transition instant, the trip selection rule alternates the phase from *pause* to *move* or vice versa. If the new phase is *pause*, the trip duration S_n is drawn from the distribution $F^0_{pause}(s)$; the path P_n is a pause at the current point. If the new phase is *move*, the trip selection rules picks a point M_{n+1} at random uniformly in $\mathcal{A}$, and a numerical speed V_n according to the density $f^0_V(v)$. A classical choice (uniform speed) is $f^0_V(v) = \frac{1}{v_{\max}-v_{\min}} 1_{\{v_{\min}<v<v_{\max}\}}$. The trip duration is then $S_n = \frac{\|M_{n+1}-M_n\|}{V_n}$ and the path P_n is the segment $[M_n, M_{n+1}]$. The default initialisation rule starts the model at the beginning of a pause, at a location uniformly chosen in $\mathcal{A}$.

Theorem 1: The random waypoint with pauses is a random trip model.

Proof. H1 and H4 obviously hold. By Theorem 2 shown next it is sufficient to consider the model without pauses. The driving chain is now $Y_n = (P_n) = (M_n, M_{n+1})$ and is indeed Markov. Take as recurrent set $R := \mathcal{A}^2$ so that condition H2.i obviously holds. The paths are selected such that P_n and P_m are independent for $|n-m| \geq 2$. It follows that for any $n_0 \geq 2$ and any $y \in R$,

$$\mathbb{P}_y(Y_{n_0} \in A_1 \times A_2) = \mathrm{Unif}(A_1)\mathrm{Unif}(A_2)$$

where Unif is the uniform distribution on $\mathcal{A}$, defined by $\mathrm{Unif}(A) = \int_A dx / \int_{\mathcal{A}} dx$. This shows that H2.ii holds with $\varphi = \mathrm{Unif} \otimes \mathrm{Unif}$ (product measure) and any $\beta \in (0,1)$. The recurrent set R is visited at each transition, so H3 is indeed true.

□

This model is well known; its stationary properties are studied in [19], [9], [13]. However, even for this simple model our framework provides two new results: the proof of existence of a stationary regime, and a sampling algorithm for the stationary distribution over general areas that does not require the computation of geometric integrals.

B. Adding Pauses to a Model

Assume we have a random trip model $\mathcal{M}$ with phases I_n and paths P_n. We can add pauses to this model and obtain a new model $\mathcal{M}'$ as follows. At the end of the nth trip, a pause time S'_n is drawn at random, depending only (possibly) on the current trip and phase. This means that the pause duration at the end of the nth trip, conditional on all past, depends only on P_n and I_n.

In $\mathcal{M}'$ we have phases I'_n and paths P'_n given by (for all $k \in \mathbb{N}$):

$$\begin{cases} I'_{2k} = (move, I_k), \;\; I'_{2k+1} = (pause, I_k) \\ P'_{2k} = P_k, \;\; P'_{2k+1} = \mathrm{constant}(M_{k+1}) \end{cases} \tag{2}$$

where $\mathrm{constant}(m)$ is the path that remains entirely at point M (i.e. $\mathrm{constant}(m)(u) = m$ for $0 \leq u \leq 1$).

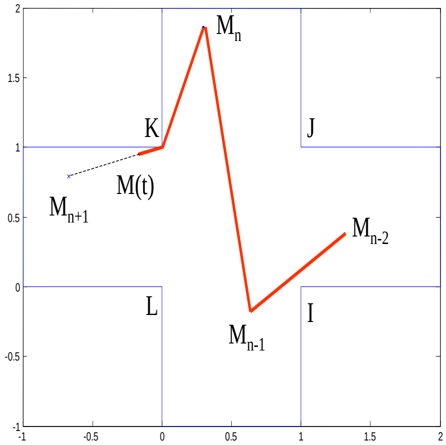

Fig. 1. Random Waypoint on a non convex domain (Swiss Flag). A trip is the shortest path inside the domain from a waypoint M_n to the next. Waypoints M_n are drawn uniformly in the domain. In figure, the shortest path M_n, M_{n+1} has two segments, with a breakpoint at K; the shortest paths M_{n-1}, M_n and M_{n-2}, M_{n-1} have one segment each. $M(t)$ is the current position.

Theorem 2: If $\mathcal{M}$ is a random trip model , then $\mathcal{M}'$ is also a random trip model.

Proof. It is straightforward to show that assumptions H1 and H4 hold. We now show H2.i. Let R be a regeneration set for model $\mathcal{M}$, which by hypothesis exists. Let

$$R' = \{move, pause\} \times R$$

Let $Z_n \in \{move, pause\}$ be the sequence that alternates between *move* and *pause*, and indicates whether the nth trip is a pause or not. The driving chain in model $\mathcal{M}'$ is Y'_n. If $Z_0 = move$ (this is implicitly assumed in Equation (2)) then $Y'_{2k} = (move, I_k, P_k)$ and thus

$$\begin{aligned} &\mathbb{P}_{(move,i,p)}\left(Y'_n \in R' \text{ for some } n\right) \\ &\quad \geq \;\; \mathbb{P}_{(move,i,p)}\left(Y'_{2k} \in R' \text{ for some } k\right) \\ &\quad = \;\; \mathbb{P}_{(i,p)}\left(Y_k \in R \text{ for some } k\right) = 1 \end{aligned}$$

and similarly

$$\begin{aligned} &\mathbb{P}_{(pause,i,p)}\left(Y'_n \in R' \text{ for some } n\right) \\ &\quad \geq \;\; \mathbb{P}_{(pause,i,p)}\left(Y'_{2k+1} \in R' \text{ for some } k\right) \\ &\quad = \;\; \mathbb{P}_{(i,p)}\left(Y_k \in R \text{ for some } k\right) = 1 \end{aligned}$$

which shows H2.i.

To show H2.ii, let n_0, β, φ be such that $\mathbb{P}_{(i,p)}(Y_{n_0} \in B) \geq \beta \cdot \varphi(B)$ for all $y \in R$ and any measurable B in $I \times \mathcal{P}$. Define the probability measure on $\{move, pause\} \times I \times \mathcal{P}$ by $\varphi'(\{move\} \times B) = \varphi(B)$ and $\varphi'(\{pause\} \times B) = 0$. We have, for $B' = \{move\} \times B$:

$$\mathbb{P}_{(move,i,p)}\left(Y'_{2n_0} \in B'\right) = \mathbb{P}_{(i,p)}\left(Y_{n_0} \in B\right) \geq \beta\varphi(B) = \beta\varphi'(B')$$

and for $B' = \{pause\} \times B$:

$$\mathbb{P}_{(move,i,p)}\left(Y'_{2n_0} \in B'\right) = 0 = \beta\varphi'(B')$$

which shows H2.ii. □

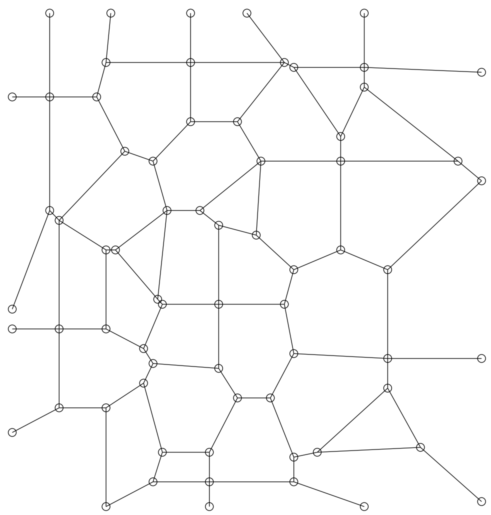

Fig. 2. Two examples of random trip mobility on the domain specified by the line segments of a graph are: (i) city section (Section III-C.2) and (ii) space graph (Section III-D.2). The distinction between the two is that in latter the trip end points are constrained to be vertices of a spatial graph. The spatial graph showed is synthetically generated as proposed in [11]. Such graphs can be constructed from real-world street maps. In particular, arbitrary numeric speeds can be assigned to the edges of a graph.

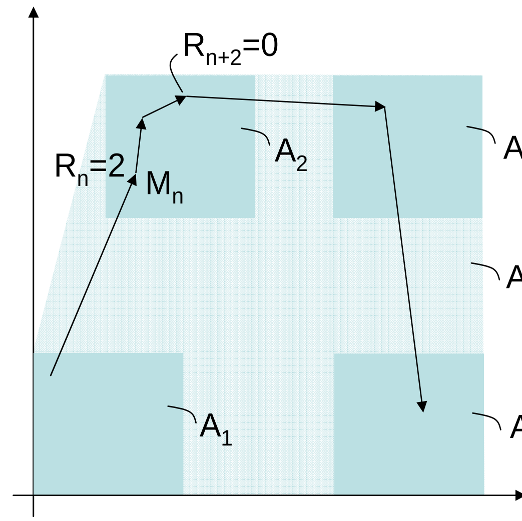

Fig. 3. Restricted random waypoint on a plane with four squares as subdomains. This model was introduced in [4] to simulate a wide-area routing protocol. It was used as an idealised view of four towns represented by squares. A mobile moves according to random waypoint within a square for a random number of visits and then picks a point uniformly at random in another randomly chosen square as a destination. The figure shows a sample path of the mobile movement. The speed on the trip is chosen according to a distribution that depends on the origin and destination squares.

C. Random Waypoint on General Connected Domain

This is a variant of the classical random waypoint (Example III-A), where we relax the assumption that $\mathcal{A}$ is convex, but assume that $\mathcal{A}$ is a connected domain over which a uniform distribution is well defined. For two points m,n in $\mathcal{A}$, we call $d(m,n)$ the distance from m to n in $\mathcal{A}$, i.e. the minimum length of a path entirely inside $\mathcal{A}$ that connects m and n. $\mathcal{P}$ is the set of shortest paths between endpoints. The trip selection rule picks a new endpoint uniformly in $\mathcal{A}$, and the next path is the shortest path to this endpoint. If there are several shortest paths, one of them is randomly chosen according to some probability distribution on the set of shortest paths. The set of phases is $I = \{pause, move\}$. This model fits in our framework for the same reasons as the former example.

1) Swiss Flag: The model is random waypoint on particular non-convex domain defined by the cross section as in Figure 1.

2) City Section: This is a special case of random waypoint on a non convex domain. The domain is the union of the segments defined by the edges of the space graph (e.g. Figure 2). Arbitrary numeric speeds can be assigned to edges of the graph. The “distance” from one location to another is the travel time.

D. Restricted Random Waypoint

This model was originally introduced by Blažević et al [4] in a special form described in Figure 3, in order to model intercity examples. We define it more generally as follows.

The trip endpoints are selected on a finite set of subdomains $\mathcal{A}_i \in \mathbb{R}^d$, $i \in \mathcal{L}$. The domain $\mathcal{A}$ is a convex closure of the subdomains $\mathcal{A}_i$, $i \in \mathcal{L}$. The trip selection rule is described as follows. To simplify, we first consider a node movement with no pauses. Suppose the node starts from a point M_n chosen uniformly at random on a subdomain i. The node picks the number of trips r to undergo with trip endpoints in the subdomain i from a distribution $F_i(\cdot)$. The next subdomain is drawn from the distribution $Q(i,\cdot)$. At each trip transition, the node decrements r by 1, as long as $r > 0$, else it sets r to a random sample from the distribution $F_j(\cdot)$. Then, if $r = 0$, the current subdomain is set to j and the next subdomain to a sample from $Q(j,\cdot)$. The trip destination is chosen uniformly at random on $\mathcal{A}_i$ if $r > 0$, else uniformly at random on $\mathcal{A}_j$. This process repeats. The model is extended to accommodate pauses in a straightforward manner by inserting pauses at the trip transition instants.

The phase is $I_n = (i, j, r, \phi)$, where i and j are respectively, the current and next subdomain, r is the number of trips with both endpoints in the subdomain i, and $\phi \in \{move, pause\}$. Given a phase $I_n = (i, j, r, move)$, the path P_n is the line segment $[M_n, M_{n+1}]$, with M_n uniformly distributed on $\mathcal{A}_i$ and M_{n+1} uniformly distributed on $\mathcal{A}_i$ or $\mathcal{A}_j$, for $r > 0$ and $r = 0$, respectively.

Theorem 3: Assume that (i) Q is an irreducible transition matrix, (ii) the number of the trips within a subdomain has a finite expectation, i.e. $\sum_{n=0}^{\infty}(1 - F_i(n)) < +\infty$, for all $i \in \mathcal{L}$. Then, the restricted random waypoint is a random trip model.

Proof. By Theorem 2, it suffices to consider a model with no pauses. $I_n \equiv (K_n, L_n, R_n)$ is indeed a Markov chain on a countable state space given by $I = \bigcup_{(i,j)\in\mathcal{L}_2}\{i\} \times \{j\} \times \mathcal{R}_i$, with $\mathcal{L}_2 = \{(i, j) \in \mathcal{L}^2 : Q(i, j) > 0\}$. Here $\mathcal{R}_i = \mathbb{Z}_+$, if there exists no finite r_i such that $F_i(r_i) = 1$, else $\mathcal{R}_i = \{0, 1, \ldots, r_i\}$. Conditional on all observed past for transitions $k \le n$, including phase I_n and path $P_n = (M_n, M_{n+1})$, the distribution of (I_{n+1}, P_{n+1}), depends only on (I_n, P_n), thus $Y_n = (I_n, P_n)$ is a Markov chain. This shows that H1 holds. Condition H4 indeed

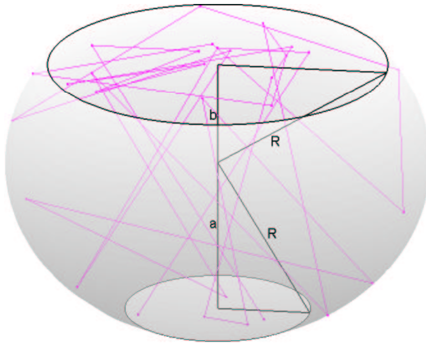

Fig. 4. Fish in a bowl is a restricted random waypoint. $\mathcal{A}$ is the volume of the sphere comprised between two horizontal planes. Waypoints are on the boundary $\mathcal{A}_1$ of $\mathcal{A}$.

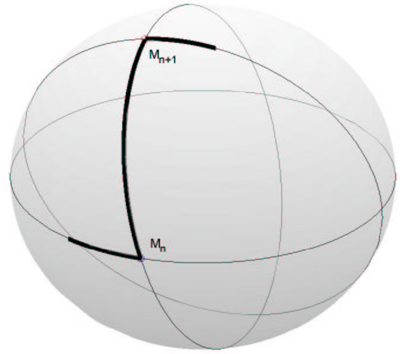

Fig. 5. Random waypoint on a sphere.

holds by the model definition. We next show that H2 holds. Note that the prevailing model permits us to verify H2 by essentially considering only the Markov chain I_n, which takes values on a countable state space, and thus standard stability results can be employed.

Condition H2.i holds for a recurrent set $R = (i, j, r) \times \mathcal{A}^2$, where (i, j, r) is any fixed element of I. Indeed, first note that I_n is an irreducible Markov chain, which follows from the assumed irreducibility of the transition matrix Q that specifies the Markov walk on the subdomains. Define the function $V : I \to \mathbb{R}$ as $V((i, j, r)) = r$, for $(i, j, r) \in I$ and let H be a set given by $H := \mathcal{L}_2 \times \{0\}$. It holds that for any $(i, j, 0) \in H$

$$\mathbb{E}(V(I_1)|I_0 = (i, j, 0)) = \mathbb{E}(R_1|I_0 = j) < +\infty,$$

and, for any $(i, j, r) \notin H$,

$$\mathbb{E}(V(I_1)|I_0 = (i, j, r)) - V(i, j, r) = -1 < 0.$$

The former relation is by hypothesis of the theorem and latter by the fact that $R_{n+1} = R_n - 1$, whenever $R_n > 0$. By Foster's theorem [7, Theorem 1.1–Chapter 5], this implies I_n is a positive recurrent chain and thus H2.i is true by taking any subset of I as a recurrent set.

We next show that H2.ii holds as well. First, we assume that $\mathcal{R}_i$ is finite for all $i \in \mathcal{L}$ and then later remove this assumption. Note that for any $n \geq 2$ and any $y \in I \times \mathcal{P}$

$$\begin{aligned} & \mathbb{P}_y(I_n = (i, j, r), P_n \in A_1 \times A_2) \\ = & \begin{cases} \mathrm{Unif}_i(A_1)\mathrm{Unif}_j(A_2)\mathbb{P}_y(I_n = (i, j, 0)) & r = 0, \\ \mathrm{Unif}_i(A_1)\mathrm{Unif}_i(A_2)\mathbb{P}_y(I_n = (i, j, r)) & r > 0. \end{cases} \end{aligned}$$

where $\mathrm{Unif}_i(\cdot)$ is uniform distribution on a subdomain $i \in \mathcal{L}$. It follows that it is sufficient to verify H2.ii for I_n. We already noted that I_n is positive recurrent and thus has a unique invariant probability distribution π_I^0. Assume first that I_n is aperiodic. The Markov chain I_n is ergodic and thus the following result holds [7, Theorem 2.1–Chapter 4]:

$$\lim_{n \to +\infty} \sum_{v \in I} |p_{uv}(n) - \pi_I^0(v)| = 0, \quad \text{all } u \in I,$$

where $p_{uv}(n)$ is the probability of the transition from a state u to a state v in n transitions. This implies that for any $\varepsilon > 0$ there exists $n_0 \geq 1$ such that for all $n \geq n_0$,

$$p_{uv}(n) \geq \pi_I^0(v) - \varepsilon.$$

It thus follows that in H2.ii we can define the probability measure $\varphi(B) = (\pi_I^0(B) - \varepsilon)/(1 - \varepsilon)$, $B \in I$, and $\beta = 1 - \varepsilon$ with any fixed $\varepsilon \in (\min_{e \in I} \pi_I^0(e), 1)$. Finally, assume I_n is periodic. As all the states in I communicate, I_n is periodic with a period $d > 1$ common to all the states. Define the regeneration set to be a cycle class R of I. Then, H2.ii follows similarly as in the aperiodic case, but using instead this convergence result [7, Theorem 2.3–Chapter 4]:

$$\lim_{n \to +\infty} \sum_{i \in R} |p_{uv}(nd) - d\pi_I^0(v)| = 0, \quad \text{all } u \in R.$$

This allows us to identify a probability measure $\varphi(\cdot)$ in H2.ii that puts all mass on R, i.e. $\varphi(R) = 1$. As an aside remark, note that we already chosen a regeneration set as a cycle class for a special periodic Markov chain of phases of classical random waypoint with pauses in the proof of Theorem 2.

To complete the proof that H2.ii is verified, it is left to consider the case: $\mathcal{R}_i = \mathbb{Z}_+$, for some $i \in \mathcal{L}$. This case is considered separately as in the case when I is countable and infinite, $\min_{e \in I} \pi_I^0(e) = 0$, the above proof cannot be taken verbatim, but with only a few slight modifications. The little technical difficulty is resolved next for I_n aperiodic; it follows similarly for the periodic case. We let $\mathcal{J} = \{(i, j, r) \in I : r \leq r_0\}$, for a finite integer $r_0 \geq 0$ such that $\min_{e \in \mathcal{J}} \pi_I^0(e) > 0$. Then, define $\varphi(B) = (\pi_I^0(B) - \varepsilon)/(\pi_I^0(\mathcal{J}) - \varepsilon)$ for $B \in \mathcal{J}$ and $\varphi(B) = 0$, for $B \in I/\mathcal{J}$, and $\beta = \pi_I^0(\mathcal{J}) - \varepsilon$. We can now indeed choose $\varepsilon \in (\min_{e \in \mathcal{J}} \pi_I^0(e), \pi_I^0(\mathcal{J}))$.

Condition H3 indeed holds for a recurrent set $R = \{e\} \times \mathcal{A}^2$, $e \in I$, by positive recurrence of the chain I_n. $\square$

In addition to the model in Figure 3, we give two particular examples of the restricted random waypoint model.

1) Fish in a Bowl: The model is restricted random waypoint on the domain defined by the volume of the bowl, as in Figure 4. The waypoints are restricted to the subset $\mathcal{A}_1$ of the domain $\mathcal{A}$, where $\mathcal{A}_1$ is the set of the points on the bowl's surface (see Figure 4). The set of phases is $I = \{pause, move\}$.

2) Space Graph: We defined this model in Section I. It is a special case of restricted random waypoint with $\mathcal{A} =$ the space graph and $\mathcal{A}_1 =$ the set of vertices. Note that it differs from the City Section in that the waypoints are restricted to be vertices. The set of phases is $I = \{pause, move\}$.

Note that all models III-A to III-D.2 and III-E are special cases of the restricted random waypoint, with $L = 1$, $r = 0$, and $\mathcal{A}_1 = \mathcal{A}$ for examples III-A to III-C.2, $\mathcal{A}$ a strict subset of $\mathcal{A}$ for examples III-D.1 and III-D.2. Note that the subdomains $\mathcal{A}_\ell$ may be convex as in Figure 3 or not as in Figure 4.

E. Random Waypoint on Sphere

Here $\mathcal{A}$ is the unit sphere of $\mathbb{R}^3$. $\mathcal{P}$ is the set of shortest paths plus pauses. The shortest path between two points is the shortest of the arcs on the great circle that contains the two points. If the two points are on the same great circle diameter, the two arcs have same length (this occurs with probability 0). The trip transition rule picks a path endpoint uniformly on the sphere, and the path is the shortest path to it (if there are two, one is chosen with probability 0.5). The set of phases is $I = \{pause, move\}$. The numerical speed is chosen independently. Initially, a point is chosen uniformly.

This model is in fact a special case of the random waypoint on a connected, non convex domain. However, we mention it separately as it enjoys special properties (the stationary location is uniform, unlike for the random waypoint models described earlier).

F. Random Walk on Torus

This model is called a random waypoint on a torus in [15]. It is used primarily because of its simplicity: unlike for the random waypoint, the distribution of location and speed at a random instant are the same as at a transition instant, as we show later.

The domain $\mathcal{A}$ is the rectangle $[0,a_1]\times[0,a_2]$. Paths are *wrapped* segments, defined as follows. The trip selection rule chooses a speed vector $\vec{V}_n$ and a trip duration S_n independently, according to some fixed distributions. Choosing a speed vector $\vec{V}_n$ is the same as choosing a direction of movement and a numerical speed. The mobile moves from the endpoint M_n in the direction and at the numeric speed given by the speed vector. When it hits the boundary of $\mathcal{A}$, say for example at a location (x_0,a_2), it is wrapped to the other side, to location $(x_0,0)$, from where it continues the trip (Figure 6). Let $w: \mathbb{R}^2 \to \mathcal{A}$ be the wrapping function:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto w\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & \bmod a_1 \\ y & \bmod a_2 \end{pmatrix}. \quad (3)$$

The path P_n (if not a pause) is defined by $(M_n, \vec{V}_n, S_n)$, such that $P_n(u) = w\left(M_n + uS_n\vec{V}_n\right)$. Note that wrapping does not modify the speed vector (Figure 6). After a trip, a pause time is drawn independent of all past from some fixed distribution. Initially, the first endpoint is chosen in $\mathcal{A}$ according to some arbitrary distribution. As we see later, the distribution of endpoint tends to uniform distribution (when sampled at transition instants).

For $a_1 = a_2 = 1$, and if there are no pauses, the sequence $M_0,\ldots,M_n,\ldots$ is a random walk on the torus, in the sense that $M_n = M_0 \oplus \vec{U}_0 \oplus \vec{U}_1 \oplus \cdots \oplus \vec{U}_{n-1}$ where $\oplus$ is addition modulo 1 (componentwise) and $\vec{U}_n \equiv S_n\vec{V}_n$. This is why this mobility model is itself called random walk.

Assumptions H1 and H4 are obviously satisfied by the random walk, with set of phases $I = \{pause, move\}$. The other assumptions of the random trip model are satisfied modulo some mild assumption on distributions:

Theorem 4: Assume that the distribution of the speed vector $\vec{V}_n$ chosen by the trip selection rule has a density (with respect to the Lebesgue measure in $\mathbb{R}^2$). Further assume that either the distribution of trip durations or distribution of pause times have a density. The random walk on torus satisfies the random trip assumptions.

Note that we do not assume any form of symmetry for the direction of the speed vector, contrary to [18].

Proof. H1 and H4 are obviously satisfied (the non lattice condition H4.iii follows from the assumption that either the distribution of the trip durations or distribution of pause times have a density). To show H2, by Theorem 2 we can restrict to the random walk without pauses.

In this case we have $Y_n = (M_n, \vec{V}_n, S_n)$. We can use Lemma 1. Let R_0 be a recurrent set for the chain M_n, and let $R = R_0 \times \mathbb{R}_+^3$. H2.i holds because

$$\mathbb{P}_y(Y_n \in R) = \mathbb{P}_y(M_n \in R_0) = \mathbb{P}(M_n \in R_0 | M_0 = m) = 1 \quad (4)$$

where $y = (m, \vec{v}, s)$.

Since M_n is Harris recurrent there exists some positive integer n_0, $\beta_0 \in (0,1)$ and a probability ϕ_0 on $\mathcal{A}$ such that for any measurable subset of $\mathcal{A}$ and any initial position $m \in \mathcal{A}$:

$$\mathbb{P}(M_{n_0} \in B | M_0 = m) \geq \beta_0 \phi_0(B).$$

Let $f^0_{\vec{V}}(v)$ be the density of the speed vector and $F^0_S(s)$ distribution of the trip duration. Since S_n and $\vec{V}_n$ are drawn independently of the past and M_n, we have, for any $y \in R$:

$$\begin{aligned} & \mathbb{P}_y((M_{n_0} \in B, \vec{V}_{n_0} \in B_1, S_{n_0} \in B_2) \\ = \; & \mathbb{P}(M_{n_0} \in B | M_0 = m)\psi_1(B_1)\psi_2(B_2) \\ \geq \; & \beta_0 \phi(B) \psi_1(B_1)\psi_2(B_2) \end{aligned}$$

where $\psi(B_1) = \int_{B_1} f^0_{\vec{V}}(v)dv$ and $\psi_2(B_2) = \int_{B_2} F^0_S(ds)$. This shows that H2.ii holds for any measurable set of the form $B \times B_1 \times B_2$. It follows that H2.ii is also true for any union of disjoint sets of this form, and thus for any measurable set. □

Lemma 1: In the random walk without pause, the sequence $M_0,\ldots,M_n,\ldots$ is a Harris recurrent Markov chain, with stationary distribution uniform on $\mathcal{A}$.

Proof. (of lemma) We give the proof for $a_1 = a_2 = 1$. The sequence M_n is a random walk on the torus, which can be viewed as the set $[0,1)\times[0,1)$ endowed with componentwise addition modulo 1. This is a compact group, and, in general, a random walk on a compact group converges to a uniform distribution. More specifically, the assumption that the speed vector has a density implies that $b(\vec{V}_n)$ also has one with respect to the uniform measure on the torus, and by [12, Section 5.2], this implies that the distribution of M_n converges weakly to uniform distribution on the torus.

We now show Harris recurrence, i.e. that conditions H2.i and H2.ii hold for the chain M_n. We take as recurrence set a small neighbourhood of the origin $R_\alpha := [0,\alpha]\times[0,\alpha]$, where α will be fixed later.

We first show that H2.i holds for any choice of $\alpha \in (0,1)$. By the uniform convergence (Lemma 2), we have that for any $\varepsilon > 0$, there exists $d \geq 1$ such that

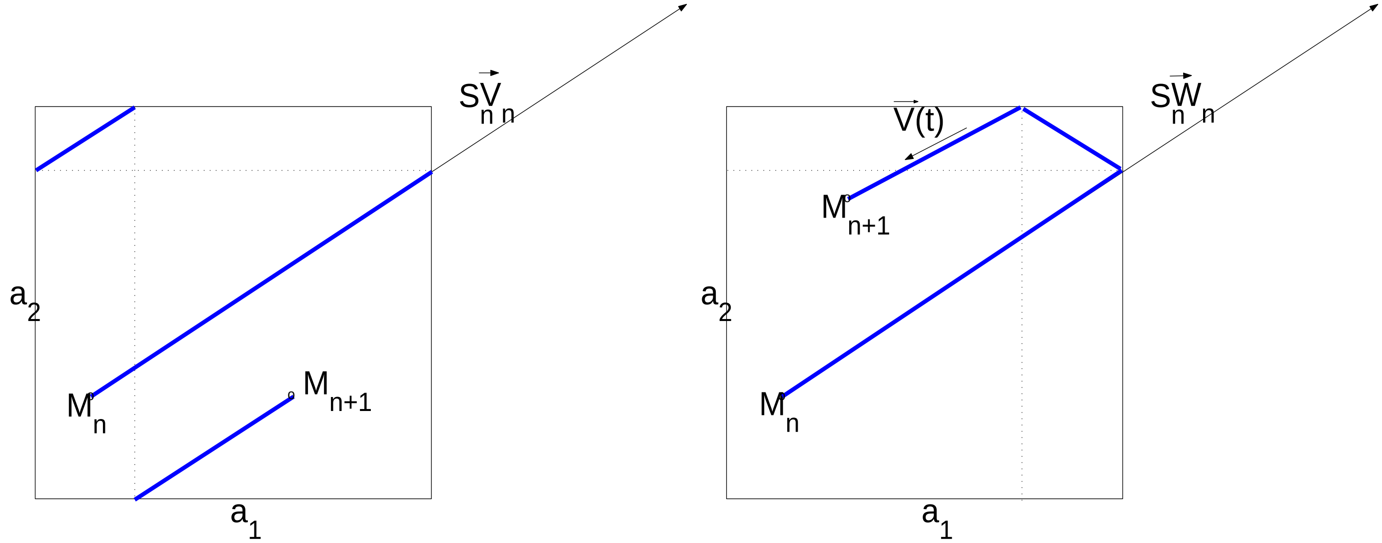

Fig. 6. Definition of random walk on torus (left) and billiards (right).

$$\mathbb{P}_m(M_n \in B) \leq \mathrm{Unif}(B) + \varepsilon, \text{ all } n \geq d, \tag{5}$$

for all product of intervals B in $\mathcal{A}$ and all $m \in \mathcal{A}$. Now, fix $\varepsilon \in (0, \alpha^2)$ and d such that (5) holds. Consider the sampled chain M_{nd}, $n = 0, 1, \ldots$. Note

$$\begin{aligned}
& \mathbb{P}_m(M_{2d} \notin R_\alpha, M_d \notin R_\alpha) \\
= & \int_{\mathcal{A} \setminus R_\alpha} \mathbb{P}_y(M_d \notin R_\alpha) \mathbb{P}_m(M_d \in dy) \\
\leq & (\mathrm{Unif}(\mathcal{A} \setminus R_\alpha) + \varepsilon) \mathbb{P}_m(M_d \notin R_\alpha) \\
\leq & (\mathrm{Unif}(\mathcal{A} \setminus R_\alpha) + \varepsilon)^2 \\
\leq & (1 - \alpha^2 + \varepsilon)^2.
\end{aligned}$$

Following the same argument, we have

$$\mathbb{P}_m\left(\cap_{n=1}^k (M_{nd} \notin R_\alpha)\right) \leq (1 - \alpha^2 + \varepsilon)^k. \tag{6}$$

Now, the left-hand side of H2.i reads as

$$\begin{aligned}
\mathbb{P}_m\left(\cup_{n=1}^\infty M_n \in R\right) &= 1 - \lim_{k \to +\infty} \mathbb{P}\left(\cap_{n=1}^k (M_n \notin R_\alpha)\right) \\
&\geq 1 - \lim_{k \to +\infty} \mathbb{P}\left(\cap_{n=1}^k (M_{nd} \notin R_\alpha)\right) \\
&\geq 1 - \lim_{k \to +\infty} (1 - \alpha^2 + \varepsilon)^k \\
&= 1
\end{aligned}$$

where we used (6) and the fact that we chosen $\varepsilon > 0$ such that $1 - \alpha^2 + \varepsilon < 1$. As we arbitrarily fixed $\alpha \in (0,1)$, this shows that H2.i holds for any $\alpha \in (0,1)$.

We next show that H2.ii holds for $n_0 = 1$ and for some appropriate choice of $\alpha \in (0,1)$. It is sufficient to prove H2.ii for B equal to a box of the form $[x_1, y_1] \times [x_2, y_2]$, (with $x_1 < y_1$ and $x_2 < y_2$) since any measurable set can be approximated, up to a 0 measure set, by a disjoint union of such boxes.

Define B^α as the set derived from B by removal of an upper and right band of width α:

$$B^\alpha = B \cap ([0, y_1 - \alpha] \times [0, y_2 - \alpha]).$$

Note that B^α is non empty if $\alpha < |y_1 - x_1|$ or $\alpha < |y_2 - x_2|$.

Also define, for $i, j \in \mathbb{Z}$: $B_{i,j} = B + (i,j)$ and $B^\alpha_{i,j} = B^\alpha + (i,j)$ (Figure 7). We have

$$\begin{aligned}
\mathbb{P}_m(M_1 \in B) &= \mathbb{P}_m(w(m + \vec{V}_1) \in B) \\
&= \mathbb{P}_m(m + \vec{V}_1 \in \cup_{i,j} B_{i,j}) \\
&= \sum_{i,j} \mathbb{P}_m(m + \vec{V}_1 \in B_{ij}) \\
&\geq \sum_{i,j} \mathbb{P}_0(\vec{V}_1 \in B^\alpha_{ij}) \\
&= \mathbb{P}_0(\vec{V}_1 \in \cup_{i,j} B^\alpha_{ij}) := g_\alpha(B).
\end{aligned}$$

Thus, in H2.ii, we can set $\beta = g_\alpha([0,1) \times [0,1))$ and $\varphi_\alpha(B) = g_\alpha(B)/\beta$, provided that $g_\alpha([0,1) \times [0,1)) > 0$. But this indeed holds for some $\alpha \in (0,1)$ as follows from $\lim_{n \to +\infty} g_{\alpha_n}([0,1) \times [0,1)) = 1$, where α_n is any sequence decreasing to 0. □

Lemma 2: For any $m \in \mathcal{A}$,

$$\lim_{n \to +\infty} \sup_B |\mu_m^{*n}(B) - \mathrm{Unif}(B)| = 0 \tag{7}$$

where the supremum is over all product intervals in $\mathcal{A}$ and μ_m^{*n} is the conditional distribution of M_n given $M_0 = m$.

Proof. We use Erdös-Turán-Koksma inequality [17, Theorem 1.21], which says that for any positive integer H, any probability distribution μ on the torus, and any product of intervals B we have

$$\begin{aligned}
&|\mu(B) - \mathrm{Unif}(B)| \\
&\leq \left(\tfrac{3}{2}\right)^2 \left(\tfrac{2}{H+1} + \textstyle\sum_{0 < \max(h_1,h_2) \leq H} \tfrac{|\hat{\mu}(h_1,h_2)|}{R(h_1,h_2)}\right)
\end{aligned} \tag{8}$$

where h_1, h_2 are integers, $R(h_1,h_2) = \max(1,h_1)\max(1,h_2)$ and $\hat{\mu}(h_1,h_2)$ are the Fourier coefficients of μ:

$$\hat{\mu}(h_1,h_2) := \int_{[0,1)\times[0,1)} e^{-2i\pi(h_1 x_1 + h_2 x_2)} d\mu(x_1,x_2). \tag{9}$$

We apply Equation (8) to $\mu = \mu_m^{*n}$ and obtain

$$\begin{aligned}
e^{-2i\pi\langle (h_1,h_2), M_n\rangle} &= e^{-2i\pi\left\langle (h_1,h_2), b(M_0 + \sum_{j=0}^{n-1} \vec{V}_j)\right\rangle} \\
&= e^{-2i\pi\left\langle (h_1,h_2), M_0 + \sum_{j=0}^{n-1} \vec{V}_j\right\rangle}
\end{aligned}$$

where the last equality is because h_1, h_2 are integers ($\langle \cdot, \cdot \rangle$ is the scalar product). Thus

$$\begin{aligned} \hat{\mu}_m^{*n}(h_1,h_2) &:= \mathbb{E}_m(e^{-2i\pi\langle (h_1,h_2),M_n\rangle}) \\ &= e^{-2i\pi\langle (h_1,h_2),m\rangle}\mathbb{E}_m\left(\Pi_{j=0}^{n-1} e^{-2i\pi\left\langle (h_1,h_2),\vec{V}_j\right\rangle}\right) \\ &= e^{-2i\pi\langle (h_1,h_2),m\rangle}\Pi_{j=0}^{n-1}\mathbb{E}_m\left(e^{-2i\pi\left\langle (h_1,h_2),\vec{V}_j\right\rangle}\right) \end{aligned}$$

where $\mathbb{E}_m$ denotes the conditional expectation given that $M_0 = m$. Thus

$$|\hat{\mu}_m^{*n}(h_1,h_2)| \leq |\hat{f}_{\vec{V}}(h_1,h_2)|^n \tag{10}$$

where $\hat{f}_{\vec{V}}(h_1,h_2)$ are the Fourier coefficients of the distribution of the speed vector.

Now $\vec{V}_n$ has a density thus if $(h_1,h_2) \neq (0,0)$, the scalar product $\left\langle (h_1,h_2),\vec{V}_n \right\rangle$ also has a density, and is thus non-lattice. By Lemma 3:

$$|\hat{f}_{\vec{V}}(h_1,h_2)| < 1 \text{ for } (h_1,h_2) \neq (0,0). \tag{11}$$

We can now use Equations (8), (10) and (11) to Equation (7). Indeed, fix some arbitrary $\varepsilon > 0$, and fix some H such that $\left(\frac{3}{2}\right)^2 \frac{2}{H+1} < \varepsilon/2$. The summation in Equation (8) is finite, thus by Equation (11) it goes to 0 as n grows to infinity. Thus, for n large enough, it is smaller than $\varepsilon/2$. This shows that there is some n_1 (independent of m and I) such that for $n \geq n_1$, the right-hand side in Equation (8) is less than ε. □

Lemma 3: Let X be a real random variable that is non-lattice. For $h \in \mathbb{Z}$, $h \neq 0$:

$$\left|\mathbb{E}\left(e^{2i\pi hX}\right)\right| < 1 \tag{12}$$

Proof. We apply the Cauchy Schwartz [10, Section 6.5–p.132] inequality to the complex valued random variables $e^{2i\pi hX}$ and 1. We have

$$\left|\mathbb{E}\left(e^{2i\pi hX}\right)\right|^2 \leq \mathbb{E}\left(|e^{2i\pi hX}|^2\right) = 1 \tag{13}$$

and equality implies that $e^{2i\pi hX} = c$ a.s. for some constant $c \in \mathbb{C}$, and X has to be lattice. □

We are left only to verify the condition H3. But this follows from (6) that says the number of transitions between successive visits to the recurrent set R_α is stochastically smaller than (finite integer) d times a geometric random variable with a fixed parameter in $(0,1)$.

G. Billiards

This is similar to example III-F, but with billiards-like reflections instead of wrapping (Figure 6). The definition is identical to example III-F, with the wrapping function replaced by the *billiards reflection* function $b : \mathbb{R}^2 \to \mathcal{A}$, defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto b\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 b_1\left(\frac{x}{a_1}\right) \\ a_2 b_1\left(\frac{y}{a_2}\right) \end{pmatrix}$$

where $b_1 : \mathbb{R} \to [0,1]$ is the 2-periodic function defined by

$$b_1(x) = |x|, \text{ for } -1 \leq x \leq 1.$$

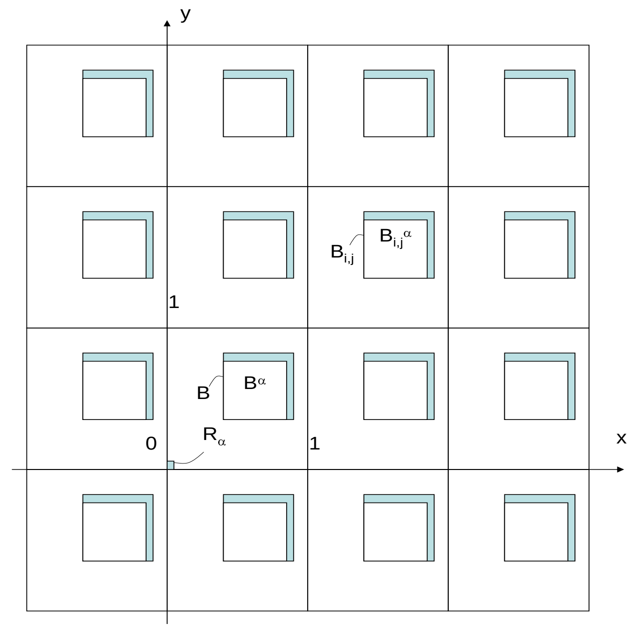

Fig. 7. The tessellation used for random walk on torus. $R_\alpha = [0,\alpha] \times [0,\alpha]$ is the chosen recurrent set. The bounding rectangles are the sets B_{ij}, the interior ones (i.e. minus the shaded areas) are B_{ij}^α.

Unlike the wrapping function, the billiards reflection may alter the speed vector (Figure 6). Therefore we differentiate the *unreflected speed vector* $\vec{W}_n$ from the instant speed vector $\vec{V}(t)$ at time t. In the model without pause, the sequence of node locations $M_0, M_1, \ldots$ is a Markov chain, defined by

$$M_{n+1} = b(M_n + \vec{U}_n), \quad n \geq 0,$$

where $\vec{U}_n := S_n \vec{W}_n$ is the driving sequence of i.i.d. random variables. The path P_n (if not a pause) is defined by $(M_n, \vec{W}_n, S_n)$, such that $P_n(u) = b\left(M_n + uS_n\vec{W}_n\right)$.

The billiards is similar to the random walk on torus, but is not quite as simple (M_n is not a random walk). We need to impose that the speed vector has equal probability of going up or down [resp. left or right].

Definition 1: We say that a random vector (X,Y) has a *completely symmetric distribution* iff $(-X,Y)$ and $(X,-Y)$ have the same distribution as (X,Y).
This is true for example if the direction of $\vec{W}$ is uniformly chosen on the unit circle, or if the two coordinates of $\vec{W}$ are independent and have even distributions. With this assumption, we have a similar result as for the random walk:

Theorem 5: Assume that the distribution of the speed vector $\vec{W}_n$ chosen by the trip selection rule has a density (with respect to the Lebesgue measure in $\mathbb{R}^2$) and is completely symmetric. Further assume that either the distribution of the trip durations or distribution of pause times have a density. The billiards satisfies the random trip assumptions.

Proof. The proof is similar to that of Theorem 4, with Lemma 1 replaced by Lemma 4. □

Lemma 4: In the billiards without pause (with the assumptions of Theorem 5), the sequence $M_0, \ldots, M_n, \ldots$ is a Harris

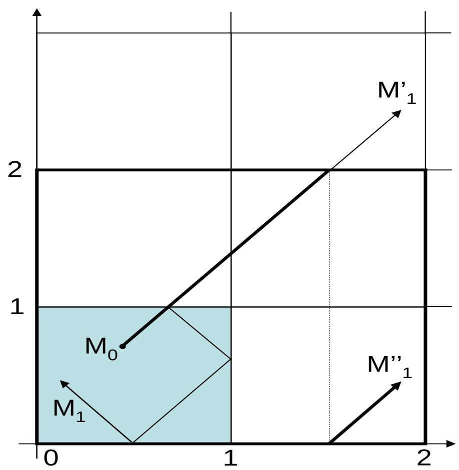

Fig. 8. Random walk M''_n on $[0,2)\times[0,2)$ associated with billiards M_n on $[0,1)\times[0,1)$.

recurrent Markov chain, with stationary distribution uniform on $\mathcal{A}$.

It suffices to consider $a_1 = a_2 = 1$. The lemma relies on a reduction to a random walk, with wrapping. First define, for any location $(x_0, y_0) \in \mathbb{R}^2$ the (linear) operator $J_{(x_0,y_0)}$ by

$$J_{(x_0,y_0)}(x,y) = \left((-1)^{\lfloor x_0 \rfloor} x, (-1)^{\lfloor y_0 \rfloor} y\right). \tag{14}$$

The linear operator J_m expresses exactly the transformation on the unwrapped speed vector (Figure 6), i.e.

$$\vec{V}_{n+1} = J_{M_n + S_n \vec{W}_n}\left(\vec{W}_n\right).$$

We now associate to the sequence M_n two sequences M'_n (non reflected) defined by

$$M'_0 = M_0 \quad M'_{n+1} = M'_n + J_{M'_n}\left(\vec{U}_n\right) \tag{15}$$

and M''_n (wrapped modulo 2) defined by

$$M''_n = w_2(M'_n) \tag{16}$$

where $w_2(\cdot)$ is the wrapping mapping defined by (3) with $a_1 = a_2 = 2$ (Figure 8). We now show by induction on $n \in \mathbb{N}$ that $b(M'_n) = M_n$ for all n, i.e. M'_n is the unreflected version of M_n. This is true by definition for $n = 0$. Assume it holds for n. By application of Lemma 5 to $m = M'_n$ and $\vec{v} = J_{M'_n}\left(\vec{U}_n\right)$ we obtain

$$\begin{aligned} b(M'_{n+1}) &= b\left(M_n + J_{M'_n}\left(J_{M'_n}\left(\vec{U}_n\right)\right)\right) \\ &= b\left(M_n + \vec{U}_n\right) = M_{n+1} \end{aligned} \tag{17}$$

where we have used the fact that J_m is its own inverse.

It follows now from and by Lemma 6 that

$$b(M''_n) = M_n \quad \text{all } n \in \mathbb{Z}_+, \tag{18}$$

i.e. the billiards M_n is derived from the wrapped sequence M''_n by reflection.

Now we show that M''_n is a random walk on the torus with $a_1 = a_2 = 2$, as defined in Section III-F. It follows from Equations (15) and (16) that

$$M''_{n+1} = M_0 \oplus \vec{U}''_0 \oplus \cdots \oplus \vec{U}''_n$$

where here $\oplus$ is addition modulo 2 and $\vec{U}''_n = J_{M'_n}(\vec{U}_n)$. In general, we do not have a random walk (with independent increments) due to the dependence of $\vec{U}''_n$ on M'_n (M'_n is a Markov chain). However, in our particular setting we do, because we assume the distribution of the speed vector (thus of $\vec{U}_n$) is completely symmetric. Indeed, for any non random point m, $J_m(\vec{U}_n)$ has the same distribution as $\vec{U}_n$. It follows that for any point M, $J_M(\vec{U}_n)$ is independent of M and in particular, $J_{M'_n}(\vec{U}_n)$ is independent of M'_n and thus $\vec{U}''_n$ is independent of $(M''_k)_{k=0}^n$.

By Lemma 2 we can now conclude that M''_n uniformly converges to the uniform distribution on $[0,2)\times[0,2)$, where "uniformly" is in the sense of Lemma 2. Now for any measurable part $B \subset [0,1)\times[0,1)$ and for any initial value $m \in [0,1)\times[0,1)$:

$$\mathbb{P}_m(M_n \in B) = \sum_{i,j \in \{0,1\}} \mathbb{P}_m(M''_n \in B''_{i,j}) \tag{19}$$

where

$$B''_{i,j} = J_{(i,j)}(B) + (i,j). \tag{20}$$

(J is defined in Equation (14), see Figure 9.) The uniform convergence of M''_n to the uniform distribution, in the sense of Lemma 2, follows immediately. So do the proofs of H2.i and H2.ii, using a similar reasoning as in Lemma 1.

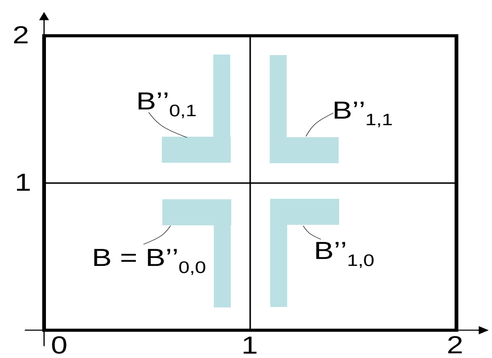

Fig. 9. Mapping from random walk to billiards.

Lemma 5: For any non random point $m \in \mathcal{A}$ and vector $\vec{v} \in \mathbb{R}^2$: $b(m + \vec{v}) = b\left(b(m) + J_m(\vec{v})\right)$.

Proof. It is enough to show the lemma in dimension 1. In this case, the result to prove is

$$b_1(x+v) = b_1\left(b_1(x) + (-1)^{\lfloor x \rfloor} v\right)$$

for any $x, v \in \mathbb{R}$. Both sides of the equation are 2-periodic in x, thus we can restrict to the cases $-1 \le x < 0$ and $0 \le x < 1$. In the former case, the equation is trivial. In the latter, it becomes $b_1(x+v) = b_1(-x-v)$, which is true because $b_1(\cdot)$ is even. □

Lemma 6: For any $m \in \mathbb{R}^2$:

$$b(m) = b(w_2(m))$$

Proof. Each coordinate of $b(\cdot)$ is 2-periodic and the the wrapping $w_2(\cdot)$ is a translation by integer multiples of 2. □

Remark. Note that we need the complete symmetry of the speed vector for Lemma 4 to hold. Consider as counter-example a speed vector with density supported by the set $[0,0.1a_1]\times\mathbb{R}$, i.e. it always goes to the right, by a little amount. After a few iterations, the sequence M_n is always in the set $[0.9a_1,a_1]\times[0,a_2]$, i.e. in a band on the right of the domain. So it cannot converge to a uniform distribution.

IV. Time Stationarity and Convergence

The state of a mobile node at time t is described by the continuous-time Markov process

$$\Phi(t) := (Y(t), S(t), S^-(t)),$$

which takes values on the state space $I\times\mathcal{P}\times\mathbb{R}_+^2$. Here $S(t)$ is the duration of the trip at time t and $S^-(t)$ is the elapsed time on the trip at time t. The random trip model definitions introduced in Section II imply Φ is a Markov renewal process. The following is the main stability result:

Theorem 6: For the random trip model specified by H1–H4 in Section II:

(i) *There exists a time-stationary distribution π for Φ if and only if $\mathbb{E}^0(S_0)$ is finite. Whenever π exists, it is unique and given by:*

$$\pi(B) = \frac{\mathbb{E}^0\left(\int_0^{T_1} 1_{\Phi(s)\in B}ds\right)}{\mathbb{E}^0(S_0)},\ B\in I\times\mathcal{P}\times\mathbb{R}_+^2.$$

(ii-a) *If $\mathbb{E}^0(S_0)$ is finite, then from π^0 almost any trip initiated at time 0, $\Phi(t)$ converges in distribution to π, as $t\to+\infty$.*

(ii-b) *Else, if $\mathbb{E}^0(S_0)=+\infty$, then*

$$\lim_{t\to+\infty}\mathbb{P}_y(\Phi(t)\in A)\to 0,$$

for any set A in $I\times\mathcal{P}\times\mathbb{R}_+^2$ such that

$$\mathbb{E}^0\left(\int_0^{T_1} 1_{\Phi(s)\in A}ds\right) < +\infty.$$

Comment 1. The convergence result ii follows from the Markov renewal theorem [1]. We note that the result holds under assumption that the driving chain Y is only Harris recurrent, not necessarily positive Harris recurrent. If the driving chain Y is null-recurrent, i.e. the mean number of transitions between successive visits to regeneration sets is infinite, then it still may be that $\mathbb{E}^0(S_0)$ is finite and that the asserted limit hold. Similar convergence results are known for a positive Harris recurrent Markov process in continuous time, under a condition on the distribution of the regeneration epochs. See for instance [2, Proposition 3.8] for a convergence in total variation.

Comment 2. The item ii-b formalises the reported "harmfulness" of the random waypoint. It says that for a random trip model, if the mean trip duration $\mathbb{E}^0(S_0)$ is infinite, then the process Φ is in fact *null-recurrent*. The asserted convergence to 0 was originally found for node numeric speed [20].

Comment 3. The conditions introduced in [6] are sufficient conditions for H1-H3 to hold. Condition H4.iii is new and is needed for the asserted convergence in item ii, *not* for item i.

Corollary 1: For examples III-A to III-E, there is a stationary regime if and only if the pause time and inverse speed (sampled at a transition) have a finite expectation. For examples III-F and III-G the condition is that the pause time and trip duration (sampled at a transition) have a finite expectation.

Proof. (of Theorem) Item i. Conditions H1-H3 define the driving chain of phase and path Y to be positive Harris recurrent so that there exists a unique stationary distribution π^0 for the driving chain that is a solution of (1). Combined with H4.i, we have that $(S_n)_{n=0}^\infty$ can be defined as a Palm stationary sequence by letting π^0 be the distribution of Y_0. We now appeal to the conditions of the Slivnyak's inverse construction [3]: (S.i) $0<\mathbb{E}^0(T_1)<+\infty$, (S.ii) $\mathbb{P}^0(T_1>0)=1$, and (S.iii) $\mathbb{E}^0(N(0,t])<+\infty$, for all $t\le t_0$ and some $t_0>0$. Here $N(0,t]$ is the number of trip transitions that fall in the interval $(0,t]$. S.ii is true by the model (H4.ii), which also implies $\mathbb{E}^0(T_1)>0$ in S.i. The rest of S.i is hypothesis of the result. Condition S.iii follows from H4.iii. Indeed,

$$\begin{aligned}\mathbb{E}^0(N(0,t]) &= \sum_{n=1}^\infty \mathbb{P}^0(T_n\le t)\\ &\le e^{ut}\sum_{n=1}^\infty \mathbb{E}^0(e^{-uT_n}),\ \text{any } u>0,\end{aligned}\quad (21)$$

where the last inequality is Chernoff inequality. Define the function $m(u,x)=\mathbb{E}^0(e^{-uT_1}|Y_0=x)$, $u>0$, $x\in I\times\mathcal{P}$. By definition of trip durations (H4.i):

$$\begin{aligned}\mathbb{E}^0(e^{-uT_n}) &= \int_{I\times\mathcal{P}}\cdots\int_{I\times\mathcal{P}} m(u,x)m(u,y_1)\cdots m(u,y_{n-1})\\ &\quad\cdot\pi^0(x)P(x,dy_1)P(y_1,dy_2)\cdots P(y_{n-1},dy_n).\end{aligned}$$

Recall that $P(\cdot,\cdot)$ is the transition semigroup of the chain Y as introduced in Equation (1).

Now, from H4.iii, we have that $m(u,y)<1$ for any $u>0$ and π^0 almost any y. It thus follows that there exists $\varepsilon>0$ such that $\mathbb{E}^0(e^{-uT_n})<(1-\varepsilon)^n$, and in view of (21), this implies S.iii holds.

The existence of a time-stationary distribution π follows by the Palm inversion formula [3]:

$$\mathbb{E}(f(\Phi(0))) = \frac{1}{\mathbb{E}^0(S_0)}\mathbb{E}^0\left(\int_0^{T_1} f(\Phi(s))ds\right), \quad (22)$$

by taking $f(x)=1_{x\in B}$, $B\in I\times\mathcal{P}\times\mathbb{R}_+^2$. Recall that here we admit the convention under which 0 is an arbitrary time. The Palm inversion also implies uniqueness of π and the asserted expression.

Item ii-a. In view of a convergence result in [1], it suffices to show that for π^0 almost any $y\in I\times\mathcal{P}$ and any bounded function $g: I\times\mathcal{P}\times[0,+\infty)^2\to\mathbb{R}^+$, the following limit holds:

$$\begin{aligned}&\lim_{t\to+\infty}\mathbb{E}_y\left(g(Y(t),S^-(t),S^+(t))\right) \quad (23)\\ &= \lambda\int_{y\in I\times\mathcal{P}}\int_{[0,\infty)}\int_{[0,s)} g(y,u,s-u)du F(y,ds)\pi^0(dy),\end{aligned}$$

where $1/\lambda := \int_{y\in I\times\mathcal{P}}\int_{[0,+\infty)}F(y,ds)\pi^0(y)$. Recall that $S(t)=S_n$, $T_n\le t<T_{n+1}$ is the trip duration of the trip on-going at time t and $S^-(t)=t-S(t)$ is the time elapsed on the trip at

time t. The notation $S^+(t) := S(t) - S^-(t)$ denotes the time until the next trip transition instant as seen at time t.

In order to show that the distribution of Φ converges to the asserted limit, it suffices to show that (23) holds for functions $g(\cdot)$ of the form $g(y,u,v) = 1_{y\in A} 1_{u>u_0} 1_{v>v_0}$, for $u_0, v_0 \geq 0$ and A a product of intervals in $I \times \mathcal{P}$.

It follows from [1, Corollary 1] that the limit (23) holds if both of the two following conditions hold:

(C.i) $f(y,\cdot)$ is continuous almost everywhere, for any $y \in I \times \mathcal{P}$ that does not lie in a set of zero π^0 measure,

(C.ii) $\int_{I\times\mathcal{P}} \sum_{n\in\mathbb{Z}} \sup_{nd\leq s<(n+1)d} |f(y,s)|\pi^0(y) < +\infty$, for some $d > 0$,

where $f(y,s) := \mathbb{E}_y(g(y,s,T_1 - s)1_{T_1>s})$.

First, we check C.i for $g(y,u,v) = 1_{y\in A} 1_{u>u_0} 1_{v>v_0}$. We have

$$f(y,s) = 1_{y\in A} 1_{s>u_0} \mathbb{P}_y(T_1 > s + v_0),$$

which for any fixed $y \in I \times \mathcal{P}$ is almost everywhere continuous with s.

Second we check C.ii for $g(\cdot)$ a bounded function. We have that there exists $K < +\infty$ such that $|f(y,s)| \leq K\mathbb{P}_y(T_1 > s)$. It is readily seen that $K \sum_{n=0}^{\infty} \mathbb{P}_y(T_1 > nd)$ upper bounds the left-hand side in the inequality C.ii. Further,

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{P}_y(T_1 > nd) &\leq \sum_{n=0}^{\infty} \int_{[nd,(n+1)d)} \mathbb{P}_y(T_1 > s)ds \\ &= \int_{[0,+\infty)} \mathbb{P}_y(T_1 > s)ds = 1/\lambda. \end{aligned}$$

Hence, C.ii is implied by the boundedness of $g(\cdot)$ and finiteness of $1/\lambda$.

Item ii-b follows from (23). □

V. Conclusion

The random trip model provides a framework to analyse and simulate stable mobility models that are guaranteed to have a unique time-stationary distribution. The model definition comprises conditions that guarantee convergence in distribution of a node mobility to steady-state, by starting a simulation from origin of an arbitrary initial trip. It is showed that many known random mobility models are random trip models.

In the case when the initial state is not sampled from the time-stationary distribution, there is an initial transience due to the convergence of the mobility state distribution to the time-stationary distribution. The rate of this convergence depends on the geometry of the mobility domain and specifics of the trips selection. In Part II, we provide a perfect sampling algorithm that alleviates the initial transience altogether, by drawing initial mobility state from a time-stationary distribution, so that movement is a time-stationary realisation.

The web page "random trip model":

- `http://ica1wwww.epfl.ch/RandomTrip`

provides a repository of random trip models and a free to download perfect sampling software to use in simulations.

References

[1] Gerold Alsmeyer. The markov renewal theorem and related results. *Markov Proc. Rel. Fields*, 3:103–127, 1997.

[2] Soren Asmussen. *Applied Probability and Queues*. Springer, 2 edition, 2003.

[3] Francois Baccelli and Pierre Brémaud. *Elements of Queueing Theory: Palm Martingale Calculus and Stochastic Recurrences*, volume 26. Applications of Mathematics, Springer-Verlag, 2 edition, 2003.

[4] Ljubica Blažević, Jean-Yves Le Boudec, and Silvia Giordano. A location based routing method for mobile ad hoc networks. *IEEE Transactions on Mobile Computing*, 3(4), December 2004.

[5] Jean-Yves Le Boudec and Milan Vojnović. The Random Trip Model, Part II: Stationary Regime and Perfect Simulation. (MSR-TR-2005-165), 2005.

[6] Jean-Yves Le Boudec and Milan Vojnović. Perfect Simulation and Stationarity of a Class of Mobility Models. In *Proceedings of IEEE Infocom 2005*, Miami, Florida, USA, March 2005.

[7] Pierre Brémaud. *Markov Chains: Gibbs Fields, Monte Carlo Simulation and Queues*. Springer, 1999.

[8] T Camp, J. Boleng, and V. Davies. A survey of mobility models for ad hoc network research. *WCMC: Special issue on Mobile Ad Hoc Networking: Research, Trends and Applications*, 2(5):483–502, 2002.

[9] Hannes Hartenstein Christian Bettstetter and Xavier Pérez-Costa. Stochastic properties of the random waypoint mobility model. *ACM/Kluwer Wireless Networks*, Special Issue on Modeling and Analysis of Mobile Networks 2003.

[10] G. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge University Press, 2 edition, 1952.

[11] Amit Jardosh, Elizabeth M. Belding-Royer, Kevin C. Almeroth, and Subhash Suri. Towards realistic mobility models for mobile ad-hoc networking. In *Proc. of ACM Mobicom 2003*, pages 217–229, San Diego, CA, 2003.

[12] B. M. Kloss. Probability Distributions on Bicompact Topological Groups. *Theory of Probability and its Application*, IV:237–270, 1959.

[13] J. Y. Le Boudec. On the stationary distribution of speed and location of random waypoint. *IEEE Transactions on mobile computing*, to appear.

[14] Jean-Yves Le Boudec. Understanding the simulation of mobility models with palm calculus. Technical Report EPFL/IC/2004/53, EPFL, 06 2004.

[15] Guolong Lin, Guevara Noubir, and Rajmohan Rajamaran. Mobility models for ad-hoc network simulation. In *Proceedings of Infocom 2004*, April 2004.

[16] Torgny Lindvall. *Lectures on the Coupling Method*. Dover (originally published by John Wiley & Sons in 1992), 2002.

[17] M.Drmota and R. F. Tichy. *Sequences, Discrepancies and Applications*. Lecture Notes in Math. 1651, Springer, 1997.

[18] Philippe Nain, Don Towsley, Benyuan Liu, and Zhen Liu. Properties of Random Direction Models. In *Proceedings of IEEE Infocom 2005*, Miami, Florida, USA, March 2005.

[19] T. Camp W. Navidi and N. Bauer. Improving the accuracy of random waypoint simulations through steady-state initialization. In *Proceedings of the 15th International Conference on Modeling and Simulation (MS '04)*, pages 319–326, March 2004.

[20] Jungkeun Yoon, Mingyan Liu, and Brian Noble. Random waypoint considered harmful. In *Proc. of Infocom 2003*, 2003.

[21] Jungkeun Yoon, Mingyan Liu, and Brian Noble. Sound mobility models. In *Proceedings of Mobicom*, 2003.