The Random Trip Model, Part II: Stationary Regime and Perfect Simulation[1]

Jean-Yves Le Boudec and Milan Vojnović

December 2005

Technical Report
MSR-TR-2005-165

Microsoft Research
Microsoft Corporation
One Microsoft Way
Redmond, WA 98052
http://www.research.microsoft.com

[1]Author affiliations: Jean-Yves Le Boudec, EPFL, CH-1015, Lausanne, Switzerland, Email: jean-yves.leboudec@epfl.ch; Milan Vojnović, Microsoft Research Ltd, CB3 0FB Cambridge, United Kingdom, Email: milanv@microsoft.com.

Abstract— In part I, we defined the random trip class of mobility models, and showed that a necessary and sufficient condition for existence (and convergence to) a stationary regime is that the mean trip duration is finite. In this part, using Palm calculus, we establish properties of the stationary regime for the examples considered in the first part, when this condition holds. Then we provide an algorithm to sample the simulation state in its stationary regime at time 0 (“perfect simulation"), without computing geometric constants. For random waypoint on the sphere, random walk on torus and billiards, we show that, in the stationary regime, the node location is uniform. Our perfect sampling is implemented to use with ns-2, and it is available to download from http://ica1www.epfl.ch/RandomTrip.

I. INTRODUCTION

IN part I, we defined the random trip class of mobility models and showed that a necessary and sufficient condition for existence (and convergence to) a stationary regime is that the mean trip duration is finite. In this part, we are interested in the stationary regime and its simulation without transients.

A. Perfect Simulation

Like many simulation models, when the condition for stability is satisfied, simulation runs go through a transient period and converge to the stationary regime. It is important to remove the transients for performing meaningful comparisons of, for example, different mobility regimes. A standard method for avoiding such a bias is to (i) make sure the used model has a stationary regime and (ii) remove the beginning of all simulation runs in the hope that long runs converge to stationary regime.

However, as we show now, the length of transients may be prohibitively long for even simple mobility models. Our example is the space graph explained in Figure 1. There are a little less than 5000 possible paths; in Figure 1 we show the distribution of the path used by the mobile at time t, given that initially a path is selected uniformly among all possible paths (i.e. the mobile is initially placed at a random vertex (uniformly) and the trip destination vertex is also drawn uniformly at random on the set of the vertices). This was obtained analytically (details are in Appendix). Figure 1 illustrates that the transient period may be long compared to typical simulation lengths (for example 900 sec in [4]). A major difficulty with transient removal is to know when the transient ends; if it may be long, as we illustrated, considerable care should be used. An alternative, called “perfect simulation", is to sample the initial simulation state from the stationary regime. For most models this is hard to do, but, as we show, this is quite easy (from an implementation viewpoint) for the random trip model. Perfect simulation for the random waypoint was advocated and solved by Navidi and Camp in [13] who also give the stationary distribution (assuming location and speed are independent in the stationary regime, an issue later resolved in [9] using the Palm techniques in this paper).

B. The Palm Calculus Framework

The derivations in [13] involve long and sophisticated computations. We use a different approach, based on Palm calculus, a set of formulae that relate time averages to event averages. Palm calculus is now well established, but not widely used or even known in applied areas. For a quick overview of Palm calculus, see [10]; for a full fledged theory, see [1]. This framework allows us to generalise the results in [13] to the broad class of restricted random waypoint models, and obtain a sampling algorithm that, for complicated, non convex areas, does not require a priori computation of geometric integrals. More fundamentally, the Palm calculus framework allows us derive simple sampling algorithms for the generic random trip model—a task that may be formidable without this tool.

C. Organisation of the Paper

In Section II we give a generic representation of the time stationary distribution of *any* random trip model that satisfies the stability condition in Part I [3, Section IV]. In Section III we derive an efficient sampling algorithm for perfect simulation for the sub-family of models that can be represented as restricted random waypoint. In Section IV we show that, for random waypoint on sphere, random walk on torus and billiards, the time-stationary distribution of node location is uniform, i.e. the distribution bias for location does not exist for these models. In Section V we discuss related work.

II. TIME STATIONARY DISTRIBUTIONS

For a perfect simulation, all we need is to sample from the time stationary distribution of the process state. The state of the process is the phase $I(t)$, the path $P(t)$, the trip duration $S(t)$ and where on trip $U(t)$. In this section we give a simple representation of the time stationary distribution of this process state for *any* random trip model. In the next sections we will apply it to the various examples introduced earlier.

Our representation relates this distribution to the stationary distribution π^0 of the Markov chain $Y_n = (I_n, P_n)$ of phase and path sampled at transition instants, and to the mean trip duration $\bar{\tau}(y) := \int_0^{+\infty} sF(y, ds)$ given that the phase and path is y.

Theorem 1: Assume the condition for existence and uniqueness of a stationary distribution in Section IV of Part I is satisfied. The time stationary distribution of the process state at an arbitrary time t is given by the following.

1) *Phase and Path: Let $Y(t) = (I(t), P(t))$.*

$$d\mathbb{P}(Y(t) = y) = \frac{\bar{\tau}(y)}{\int_{\mathcal{Y}} \bar{\tau}(x)\pi^0(dx)}\pi^0(dy).$$

2) *Trip duration, given phase and path:*

$$d\mathbb{P}(S(t) = s | Y(t) = y) = \frac{s}{\bar{\tau}(y)}F(y, ds).$$

3) *Fraction of time elapsed on the trip: $U(t)$ is independent of $(I(t), P(t), S(t))$ and is uniform on $[0, 1]$.*

Note that the factor $\int_{\mathcal{Y}} \bar{\tau}(y)\pi^0(dy)$ in the denominator of item 1 is the mean trip duration, and the stability condition in Part I, Section IV is precisely that it is finite.

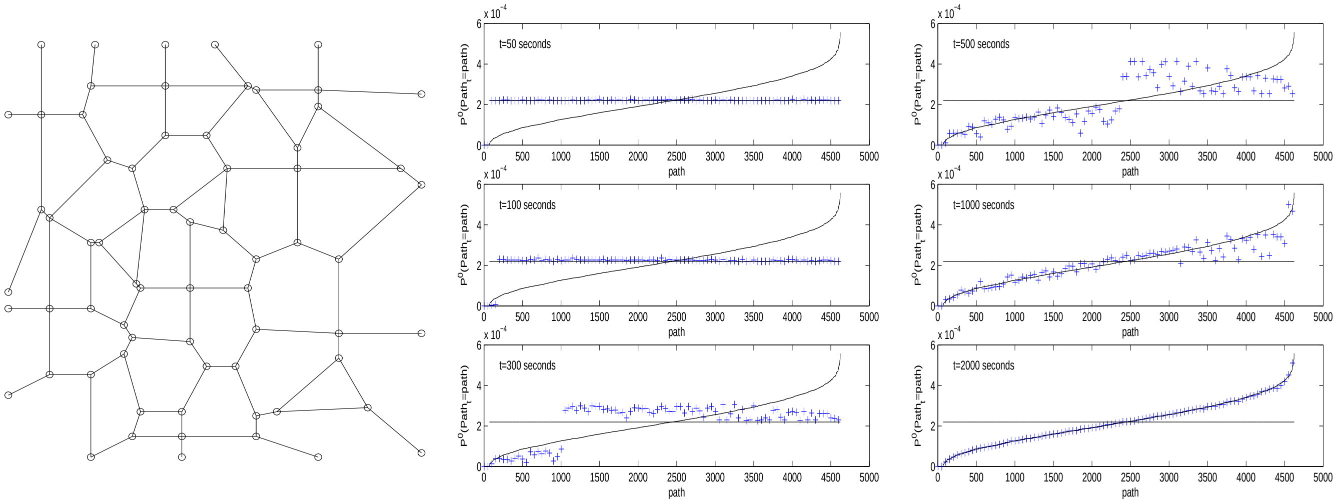

Fig. 1. Left: “Space Graph", a model proposed by Jardosh et al [8]. A mobile starts from a randomly chosen vertex and goes along a shortest path toward another randomly chosen vertex. Numerical speed is constant = 1.25 m/s. Bounding area 1 km ×1 km. Middle and right: Probability distribution of the path used by a mobile at time t. Initially, the path is chosen uniformly among all possible paths. x-axis: path index, sorted by path length; y-axis: probability that this path is used at time t for $t = 50, 100, 300, 500, 1000, 2000$ seconds of simulated time. Horizontal solid line: initial distribution; other solid line: time-stationary distribution. The transient lasts for a long time.

Proof. We use the inversion formula of Palm calculus [1]. Let λ be the intensity of the point process T_n, i.e. the average number of trip origins per time unit. For any bounded, non random, function ϕ of the process state:

$$\begin{aligned}\mathbb{E}(\phi(Y(t),S(t),U(t))) &= \lambda\mathbb{E}^0\left(\int_0^{S_0}\phi(Y_0,S_0,\frac{\tau}{S_0})d\tau\right)\\ &= \lambda\mathbb{E}^0\left(S_0\int_0^1\phi(Y_0,S_0,u)du\right) \quad (1)\end{aligned}$$

where the latter is by the change of variable $\tau = S_0 u$ in the integral. First take $\phi(y,s,u) = 1$ and obtain

$$\lambda = \frac{1}{\int_{\mathcal{Y}}\bar{\tau}(y)\pi^0(dy)}.$$

Second, take $\phi(y,s,u) = \psi(y)$ in Equation (1) and obtain

$$\mathbb{E}(\psi(Y(t))) = \lambda\mathbb{E}^0(S_0\psi(Y_0)) = \lambda\int_{\mathcal{Y}}\bar{\tau}(y)\pi^0(dy)$$

which shows item 1. Now take $\phi(y,s) = \psi(y,s)$ and obtain

$$\begin{aligned}&\mathbb{E}(\psi(Y(t),S(t))) = \lambda\mathbb{E}^0(S_0\psi(Y_0,S_0))\\ &= \lambda\int_{\mathcal{Y}}\int_0^{+\infty}s\psi(y,s)F(y,ds)\pi^0(dy)\end{aligned}$$

which shows item 2. Last, take $\phi(y,s,u) = \psi(y,s)\xi(u)$ and obtain

$$\begin{aligned}&\mathbb{E}(\psi(Y(t),S(t))\xi(U(t)))\\ &= \lambda\mathbb{E}^0\left(S_0\psi(Y_0,S_0)\int_0^1\xi(u)du\right)\\ &= \lambda\left(\int_0^1\xi(u)du\right)\mathbb{E}^0(S_0\psi(Y_0,S_0)).\end{aligned}$$

This factorization shows that $U(t)$ on one hand, $(Y(t),S(t))$ on the other, are mutually independent ([9, Lemma in Appendix]). Further, let $\psi(\cdot) = 1$ and obtain that the distribution of $U(t)$ is uniform on $[0,1]$, which ends the proof of item 3. □

Special Case: Independent Pauses. In many examples with pauses, the set of phases is reduced to $\{pause, move\}$, the model alternates between these two, and $\pi^0(I_0 = i) = 0.5$ for $i = pause$ or $move$. Define $\bar{\tau}_{pause}$ [resp. $\bar{\tau}_{move}$] as the mean pause duration (sampled at trip endpoints) [resp. mean trip duration for a trip that is not a pause]. It follows from Item 1 that

$$P(I(t) = pause) = \frac{\bar{\tau}_{pause}}{\bar{\tau}_{pause} + \bar{\tau}_{move}}$$

and

$$P(I(t) = move) = \frac{\bar{\tau}_{move}}{\bar{\tau}_{pause} + \bar{\tau}_{move}}.$$

III. Application to Examples A to D of Part I

In all of this section, we assume that the condition for stationarity in Part I is satisfied. We focus on restricted random waypoint on general connected area, since examples A to D are special cases of it.

A. Time Stationary Distributions

A direct application of Theorem 1 gives the time stationary distribution of the process. Due to its description complexity, we give it in three pieces, in the following theorems. Special notation local to this section is given below.

The first theorem generalises known statements for the classical random waypoint (Example A) [16], [15]. It relates the time average speed to the distribution of the speed selected at a waypoint, and contains an exact representation of the time stationary distribution of location.

Theorem 2: Under the time stationary distribution, conditional to phase $I(t) = \ell = (i, j, r, move)$:

Notation Used in Section III

• $Q(i,j)$: probability that next subdomain is $\mathcal{A}_j$ given current subdomain is $\mathcal{A}_i$. q^0 is the unique stationary probability of Q given by $q^0Q = q^0$.

• For $r \in \mathbb{N}_+$, $F_i(r)$ is the probability that the number of consecutive trips within subdomain $\mathcal{A}_i$ is smaller or equal r, with $r \geq 0$. $\bar{R}_i = \sum_{r\geq 0} \bar{F}_i(r)$, with $\bar{F}_i(r) = 1 - F_i(r)$, is the expected number of consecutive trips within subdomain $\mathcal{A}_i$.

• $\bar{\Delta}_{i,j}$ is the *average distance* in $\mathcal{A}$ for two points chosen uniformly in $\mathcal{A}_i$ and $\mathcal{A}_j$. $\Delta_{i,j}$ is *an upper bound* on the distance in $\mathcal{A}$ between two points in $\mathcal{A}_i$ and $\mathcal{A}_j$.

• $f^0_{V|\ell}(v)$ is the Palm (= at a transition instant) distribution of speed, given that phase is $\ell = (i,j,r,move)$; $\omega_{i,j} = \mathbb{E}^0\left(\frac{1}{V_0}|I_n = (i,j,r,move)\right)$ is the event average of the inverse of the speed chosen for a trip from subdomain $\mathcal{A}_i$ to $\mathcal{A}_j$. We have $\omega_{i,j} = \int_0^\infty \frac{1}{v} f_{V|i,j,r,move}(v)dv$, assumed to be independent of r.

• $F^0_{S|\ell}(s)$ is the Palm (= at a transition instant) distribution of pause time, given that phase is $\ell = (i,j,r,pause)$; $\tau_{i,j} = \mathbb{E}^0(S_0|I_0 = (i,j,r,pause))$ is the expected pause time of a pause, given that origin and destination subdomains are $\mathcal{A}_i$ to $\mathcal{A}_j$. We have $\tau_{i,j} = \int_0^\infty sF^0_{S|i,j,r,pause}(ds)$, assumed to be independent of r.

1) *The numerical speed is independent of the path and the instantaneous location of the mobile at time t. Its density is*
$$f_\ell(v) = C_\ell \frac{1}{v} f^0_{V|\ell}(v)$$
where $f^0_{V|\ell}(v)$ is the density of the numerical speed sampled at a transition instant and C_ℓ is a normalising constant.
2) *The path endpoints $(P(t)(0), P(t)(1))$ have a joint density over $\mathcal{A}_i \times \mathcal{A}_j$ given by*
$$d\mathbb{P}(P(t)(0) = m_0, P(t)(1) = m_1 | I(t) = \ell) = K_{i,j,r} d(m_0, m_1)$$
where $K_{i,j,r}$ are normalising constants and $d(\cdot)$ is the distance in $\mathcal{A}$.
3) *The distribution of $X(t)$, given $P(t)(0) = p$ and $P(t)(1) = n$, is uniform on the segment $[p,n]$.*

Proof. Apply Theorem 1 to obtain the joint distribution of the path, location and speed $V(t)$, by noting that $V(t) = d(P(t)(0), P(t)(1))/S(t)$. □

Comment 1. As we show later, there is no need to know the value of the constants $K_{i,j,r}$ to use the theorem in a simulation.[1]

Comment 2. The distribution of path endpoints $P(t)(0)$ and $P(t)(1)$ is *not* uniform, and the two endpoints are correlated (they tend to be far apart), contrary to what happens when sampled at transition instants. This was found already for Example A in [13].

Comment 3. One can use Theorem 2 to derive an explicit representation of the density of location $X(t)$ sampled at an arbitrary instant; for example [10] gives a closed form for the density Example A (random waypoint). However, the explicit formula is quite complicated, and is not helpful for perfect simulation. Indeed, we need to sample not only the location, but jointly location and trip, and this is readily done with Theorem 2, as we show next.

Comment 4. The relation between time stationary and event stationary distribution of speed is sometimes interpreted as "speed decay" since it is more likely to produce low speed values than the density $f^0_\ell(v)$. If one desires a uniform speed distribution in time average, then the density of speed at transition instants should be $f^0_\ell(v) = K'_\ell v 1_{\{v_{\min} < v < v_{\max}\}}$. Note that such a speed distribution satisfies the stability condition in Part I [3, Section IV] even if $v_{\min} = 0$.

Theorem 3: Under the time stationary distribution, conditional to phase $I(t) = \ell = (i,j,r,pause)$:

1) *The location $X(t)$ and the time $R(t)$ until end of pause are independent.*
2) *$X(t)$ is uniform in $\mathcal{A}_i$.*
3) *$R(t)$ has density*
$$f_\ell(r) = \frac{1}{\tau_\ell} \bar{F}^0_{S|\ell}(s)$$
where $\bar{F}^0_{S|\ell}(s) = 1 - F^0_{S|\ell}(s)$ is the complementary distribution of pause time, given the phase is ℓ.

Proof. Similar to (but simpler than) Theorem 2. □

We next show the time-stationary distribution for phase, but only for the special case $L = 1$, i.e. one sub-domain. The general case for arbitrary L bears some notational complexity and is for this reason deferred to Appendix.

Theorem 4: The time stationary distribution $\pi(\ell)$ to be in phase ℓ is
$$\pi(pause) = \frac{\tau_{pause}}{\tau_{pause} + \bar{\Delta}\omega}$$
and $\pi(move) = 1 - \pi(pause)$, where τ_{pause} is the average pause time, $\bar{\Delta}$ the average distance in $\mathcal{A}$ between two points in $\mathcal{A}_1$, and
$$\omega = \mathbb{E}^0\left(\frac{1}{V_0} \mid I_0 = move\right)$$
is the event average of the inverse of the speed.

As with Theorem 2, we show later that we do not need to know $\bar{\Delta}$ to use this theorem for sampling.

B. Perfect Simulation Without Computing Geometric Integrals

A straightforward application of the previous section poses the problem of how to sample m_0, m_1 from the density in Theorem 2. Further, in order to sample the phase in Theorem 4 one needs to compute the geometric integrals $\bar{\Delta}_{i,j}$; for simple cases ($L = 1$ and $\mathcal{A}_1$ is a rectangle or disk) there exist closed forms, as mentioned in Comment 1 after Theorem 2. Otherwise, one needs to compute them offline by Monte Carlo simulation. For some cases, this is time consuming (see analysis in Appendix). There is generally more efficient procedure, which avoids computing the geometric integrals when they are not known, as we show now. The solution of these two problems is based on the following lemma.

[1] However, in the special case of convex domains where $d(m,n)$ is the usual Euclidean distance, it is worth noting that there are known formulae: $K^{-1}_{i,j,r} = \text{vol}(\mathcal{A}_i)\text{vol}(\mathcal{A}_j)\bar{\Delta}_{i,j}$, for $r > 0$, and else $K^{-1}_{i,j,r} = \text{vol}(\mathcal{A}_i)^2\bar{\Delta}_{i,i}$, where $\text{vol}(\mathcal{A}_i)$ is the area or volume of $\mathcal{A}_i$ (in square or cubic meters) and $\bar{\Delta}_{i,j}$ is the average distance in $\mathcal{A}$ between two points drawn uniformly in $\mathcal{A}_i$ and $\mathcal{A}_j$. For $r = 0$ and $\mathcal{A}_i$ = a square of a size a, $K^{-1}_{i,j,0} \approx 0.5214a^5$; for a disk of radius a, $K^{-1}_{i,j,0} \approx 0.9054\pi^2 a^5$ [7]. For an arbitrary case, it is generally not possible to obtain either $\text{vol}(\mathcal{A}_i)$ or $\bar{\Delta}_{i,j}$ in a closed form, but $K^{-1}_{i,j,r}$ can be estimated directly by Monte Carlo simulation.

1) Rejection Sampling Lemma: Let (J,Y) be a random vector, where J is in a discrete set $\mathcal{J}$ and $Y \in \mathbb{R}^d$. Assume that $\mathbb{P}(J=j) = \lambda\mu(j)\omega_j$ and the distribution of Y conditional to $J=j$ has a density $\frac{f_j(y)}{\omega_j}$ The problem is to sample from (J,Y) without having to compute the normalising constants of the densities ω_j for all j.

Assume we know factorisations of the form $f_j(y) = k_j(y)g_j(y)$ where $g_j(y)$ is a probability density, i.e. $\int g_j(y)dy = 1$, or in other words there is no normalising constant to compute for $g_j(y)$. Assume also that we know upper bounds κ_j such that $0 \le k_j(y) \le \kappa_j$.

Lemma 1: Let ν be the probability on $\mathcal{J}$ defined by: if ω_j is known $\nu(j) = \alpha\mu(j)\omega_j$ else $\nu(j) = \alpha\mu(j)\kappa_j$, where α is a normalising constant, defined by the condition $\sum_j \nu(j) = 1$. The following algorithm draws a sample from (J,Y):

do forever
 draw j with probability $\nu(j)$
 ***if** ω_j is known*
  *draw y from the density $f_j(y)/\omega_j$;**leave***
 else
  draw y from the density $g_j(y)$
  draw $U \sim \text{Unif}(0,\kappa_j)$
  if** $U \le \frac{k_j(y)}{\kappa_j}$ **leave
end do

Comment. The lemma follows by the structure of the distribution of J and conditional density of Y. The structure is: $\mathbb{P}(J=j)$ is proportional to ω_j, while the conditional density of Y, given $J=j$, is inversely proportional to ω_j. By this structure, twisting the original distribution of J and conditional density of Y, by replacing ω_j with κ_j, indeed results in the original joint density of (J,Y). The lemma is a general result. However, it may be helpful to note that the general form was suggested by particular distributions in Theorem 1. Therein, phase $I(t)$ acts the role of J, while $(P(t),S(t),U(t))$ acts the role of Y.

Proof. (of lemma) Let I_k be the phase drawn at the k iteration of the loop and T be the number of iterations when we exit the loop (if ever). Assume first that ω_i is unknown for all i. We have $\mathbb{P}(T=k) = q_1(1-q_1)^{k-1}$ with

$$q_1 = \sum_i \int_{\mathbb{R}^d} \frac{k_i(y)}{\kappa_i} g_i(y)dy = \sum_i \nu(i)\frac{\omega_i}{\kappa_i} = \alpha\sum_i \mu(i)\omega_i.$$

Note that $0 < q_1 \le 1$ thus the loop terminates with probability 1. I_T is the value of i when we exit the loop and

$$\mathbb{P}(I_T=i) = \sum_{k\ge1}\mathbb{P}(I_T=i \text{ and } T=k|T\ge k)(1-q_1)^{k-1}$$
$$= \sum_{k\ge1}\nu(i)\frac{\omega_i}{\kappa_i}(1-q_1)^{k-1} = \nu(i)\frac{\omega_i}{\kappa_i}\frac{1}{q_1} = \frac{\mu(i)\omega_i}{\sum_j \mu(j)\omega_j}$$

which shows the result in this case. Second, consider some i for which ω_i is known. Let $g_i = f_i/\omega_i$, $k_i(y) = \omega_i$ and $\kappa_i = \omega_i$. When $I=i$ is drawn, it is kept with probability 1. Thus the case ω_i is known, is a special case of the previous one. □

2) The Sampling Method: The following theorem gives the sampling method. The details for the general case have some description complexity, and is for this reason deferred to Appendix. We show all details here for the case $L=1$.

Theorem 5: (Perfect Simulation of Restricted Random Waypoint) The following algorithm draws a sample of the time stationary state of the restricted random waypoint:

1) *Sample a phase $I(t) = \ell = (i,j,r,\phi)$ from the algorithm in Figure 2 (simple case) or in Appendix (general case).*
2) *If $\phi = pause$*
   - *Sample a time τ from the distribution with density $f_\ell(\tau) = \bar{F}^0_{S|\ell}(\tau)/\bar{\tau}_\ell$.*
   - *Sample a point M uniformly in $\mathcal{A}_i$.*
   - *Start the simulation in pause phase at location M and schedule the end of pause at τ.*
3) *If $\phi = move$*
   - *Sample a speed v from the distribution with density proportional to $\frac{1}{v}f^0_{V|\ell}(v)$.*
   - *Set M_0, M_1 to the value returned by the algorithm in Figure 2 (simple case) or in Appendix (general case).*
   - *Sample u uniformly in $(0,1)$.*
   - *Start the simulation in move phase, with initial position $(1-u)M_0 + uM_1$, next trip endpoint $= M_1$, and speed $= v$.*

Note that the algorithm in Figure 2 solves both problems mentioned in the introduction of this section. If $\bar{\Delta}$ is known with little computational cost (i.e. when $\mathcal{A}$ is a rectangle or a disk) it is always preferable to use the former case ("$\bar{\Delta}$ is known"). Else there are two options: (i) compute $\bar{\Delta}$

If $\bar{\Delta}$ is known

$q_0 = \tau_{pause}/(\tau_{pause} + \omega\bar{\Delta})$
Draw $U_1 \sim U(0,1)$
if $U_1 \le q_0$ $I(t) = pause$
else
 $I(t) = move$
 do
  Draw $M_0 \sim \text{Unif}(\mathcal{A}_1), M_1 \sim \text{Unif}(\mathcal{A}_1)$
  Draw $U_2 \sim \text{Unif}(0,\Delta)$
 until $U_2 < d(M_0,M_1)$

else (i.e. $\bar{\Delta}$ is not known)

$q_0 = \tau_{pause}/(\tau_{pause} + \omega\Delta)$
do forever
 Draw $U_1 \sim U(0,1)$
 if $U_1 \le q_0$ $I(t) = pause$; **leave**
 else
  Draw $M_0 \sim \text{Unif}(\mathcal{A}_1), M_1 \sim \text{Unif}(\mathcal{A}_1)$
  Draw $U_2 \sim \text{Unif}(0,\Delta)$
  if $U_2 < d(M_0,M_1)$
  $I(t) = move$; **leave**
end do

Fig. 2. Sampling algorithm for restricted random waypoint with $L=1$, supporting both cases where the average distance between points in $\mathcal{A}_1$ is known or not. The general case $L>1$ is given in Appendix. τ_{pause} is the average pause time, $\bar{\Delta}$ the average distance in $\mathcal{A}$ between two points in $\mathcal{A}_1$, Δ an upper bound on the distance in $\mathcal{A}$ between two points in $\mathcal{A}_1$ and $\omega = \mathbb{E}^0(1/V_0|I_0 = move)$.

offline by Monte-Carlo simulation and use the case $\bar{\Delta}$ is known", or (ii) use the case ("$\bar{\Delta}$ is not known"). Apart from unusually long simulation campaigns with the same model, the optimal choice, in terms of number of operations is to use the latter case (see Appendix). Furthermore, using the latter case simplifies the overall simulation code development. Figure 3 illustrates the sampling method on some examples from Part I.

Proof. First note (Theorem 2) that we need only to consider path and location. Then apply Theorems 2, 3 and 4. When $\bar{\Delta}_{i,j}$ is known, we solve the first problem of sampling m_0, m_1 from the density in Theorem 2 by applying Lemma 1 with $\mathcal{J} = \{1\}$, $y = (m_0, m_1)$, $\omega_1 = \bar{\Delta}_{i,j}$, $f_1(m_0, m_1) = d(m_0, m_1)\text{Unif}_{\mathcal{A}_i}(m_0)\text{Unif}_{\mathcal{A}_j}(m_1)$, $\kappa_1 = \Delta_{i,j}$. The second problem ($\bar{\Delta}_{i,j}$ not known) is solved by setting $\mathcal{J} = I$ and $\omega_\ell = \bar{\tau}_\ell$. □

IV. Application to Examples E to G

These are the examples where the distribution of location at an arbitrary point in time is uniform. In all of this section, we assume that the condition for stationarity in Part I is satisfied.

A. Random Waypoint on Sphere

This model is a special case of restricted random waypoint over a non convex area, with $L = 1$ and $\mathcal{A}_1 = \mathcal{A}$. Thus all findings of Section III apply, in particular, the time stationary speed is independent of location and is given by Theorem 2.

Theorem 6: For the random waypoint on the sphere, the time stationary distribution of the mobile location is uniform.

Proof. Apply Theorem 2. The distribution of $X(t)$ is invariant under any rotation of the sphere around an axis that contains the centre of the sphere, and any distribution that has such an invariance property must be uniform. □

Note that, with the same argument, we can show that, given we are in a move phase, the time stationary distribution of each path endpoint (previous and next) separately is also uniform, but the two endpoints are correlated (it is more likely that they are far apart). This is because, from Theorem 2, a typical path seen in time average is drawn with a probability proportional to its length. This implies that, though the time stationary distribution of points is uniform, *it is not sufficient for perfect simulation to draw an initial position uniformly on the sphere* and start as if it would be a path endpoint (we need in addition to sample a path and where on path according to Theorem 2).

B. Random Walk on Torus

Let $F^0_{pause}(t)$ [resp. $F^0_{move}(t)$] be the distribution of the pause [resp. move] duration, sampled at a transition time. Both distributions are model parameters. Also let $\bar{\tau}_{pause}, \bar{\tau}_{move}$ be the corresponding expected values (thus for example $\bar{\tau}_{pause} = \mathbb{E}^0(S_0|I_0 = pause) = \int_0^\infty t F^0_{pause}(dt)$). Finally, let $f^0_{\vec{V}}(\vec{v})$ be the density of the distribution of the speed vector (sampled at trip endpoints).

Theorem 7: For the random walk on torus, under the time stationary distribution:

1) *The process state at time t is fully described by the phase $I(t)$, the location $X(t)$, the speed vector $\vec{V}(t)$ ($= \vec{0}$ if $I(t) = pause$) and the residual time until end of trip $R(t)$.*
2) *The location $X(t)$ is uniformly distributed.*
3) $\mathbb{P}(I(t) = pause) = 1 - \mathbb{P}(I(t) = move) = \frac{\bar{\tau}_{pause}}{\bar{\tau}_{pause} + \bar{\tau}_{move}}$.
4) *Conditional $I(t) = pause$:*
   - *The residual pause duration $R(t)$ has density $f_{pause}(r) = \bar{F}^0_{pause}(s)/\bar{\tau}_{pause}$;*
   - *$X(t)$ and $R(t)$ are independent.*
5) *Conditional to $I(t) = move$:*
   - *$\vec{V}(t)$ has density $f^0_{\vec{V}}(\vec{v})$;*
   - *The residual trip duration $R(t)$ has density $f_{move}(r) = \bar{F}^0_{move}(s)/\bar{\tau}_{move}$;*
   - *$X(t), \vec{V}(t)$ and $R(t)$ are independent.*

Proof. Item 1 follows from the fact that the speed vector is not altered by wrapping. Item 3 directly follows from Theorem 1 and the discussion after it. We now show item 5. Recall $P(t)(0)$ is the start position of the current path. By Theorem 1, the time stationary joint density of $P(t)(0) = m, \vec{V}(t) = \vec{v}, S(t) = s$, conditional to a move phase is $\frac{s}{\bar{\tau}_{move}} f^0_{\vec{V}}(\vec{v}) f^0_{move}(s)\text{Unif}(m)$, where $\text{Unif}(\cdot)$ is the uniform density on $\mathcal{A}$. Now $X(t) = w(M_n + U(t)S_n\vec{V}_n)$, $T_n \le t < T_{n+1}$, and $R(t) = (1 - U(t))S(t)$, where $w(\cdot)$ is the wrapping function defined in Part I, Section III-F. Take any three bounded functions ϕ, ψ, ξ. Now apply Theorem 1:

$$\begin{aligned} &\mathbb{E}\left(\phi(X(t))\psi(\vec{V}(t))\xi(R(t))|I(t) = move\right) \qquad (2)\\ &= \int_0^1 \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \left(\int_{\mathcal{A}} \phi(w(m + us\vec{v}))\pi^0(dm)\right) \cdot \\ &\quad \cdot \psi(\vec{v})\xi(us)\frac{s}{\bar{\tau}_{move}} f^0_{\vec{V}}(\vec{v}) F^0_{move}(ds)d\vec{v}du \\ &= \int_0^1 \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \left(\int_{\mathcal{A}} \phi(w(m + us\vec{v}))\text{Unif}(dm)\right) \cdot \\ &\quad \cdot \psi(\vec{v})\xi(us)\frac{s}{\bar{\tau}_{move}} f^0_{\vec{V}}(\vec{v}) F^0_{move}(ds)d\vec{v}.du \end{aligned}$$

The last equality is because by Lemma 1 in Part I, the stationary distribution of trip endpoint, sampled at an arbitrary endpoint, is uniform. Now by Lemma 2

$$\int_{\mathcal{A}} \phi(w(m + us\vec{v}))\text{Unif}(dm) = \int_{\mathcal{A}} \phi(m)\text{Unif}(dm)$$

thus

$$\begin{aligned} (2) &= \int_{\mathcal{A}} \phi(m)\text{Unif}(dm) \cdot \\ &\quad \cdot \int_{\mathbb{R}^2} \psi(\vec{v}) f^0_{\vec{V}}(\vec{v})d\vec{v} \int_0^1 \int_0^\infty \frac{s}{\bar{\tau}_{move}} \xi(us) F^0_{move}(ds)du \\ &= \int_{\mathcal{A}} \phi(m)\text{Unif}(dm) \cdot \\ &\quad \cdot \int_{\mathbb{R}^2} \psi(\vec{v}) f^0_{\vec{V}}(\vec{v})d\vec{v} \int_0^\infty \xi(r) F^0_{move}(dr) \end{aligned}$$

where the last equality is by the change of variable (s, u) to (r, s). with $r = us$ This shows item 5. Item 4 is analog.

This also shows that, conditional to the phase $I(t)$ being either move or pause, the location $X(t)$ is uniformly distributed. Item 2 follows immediately. □

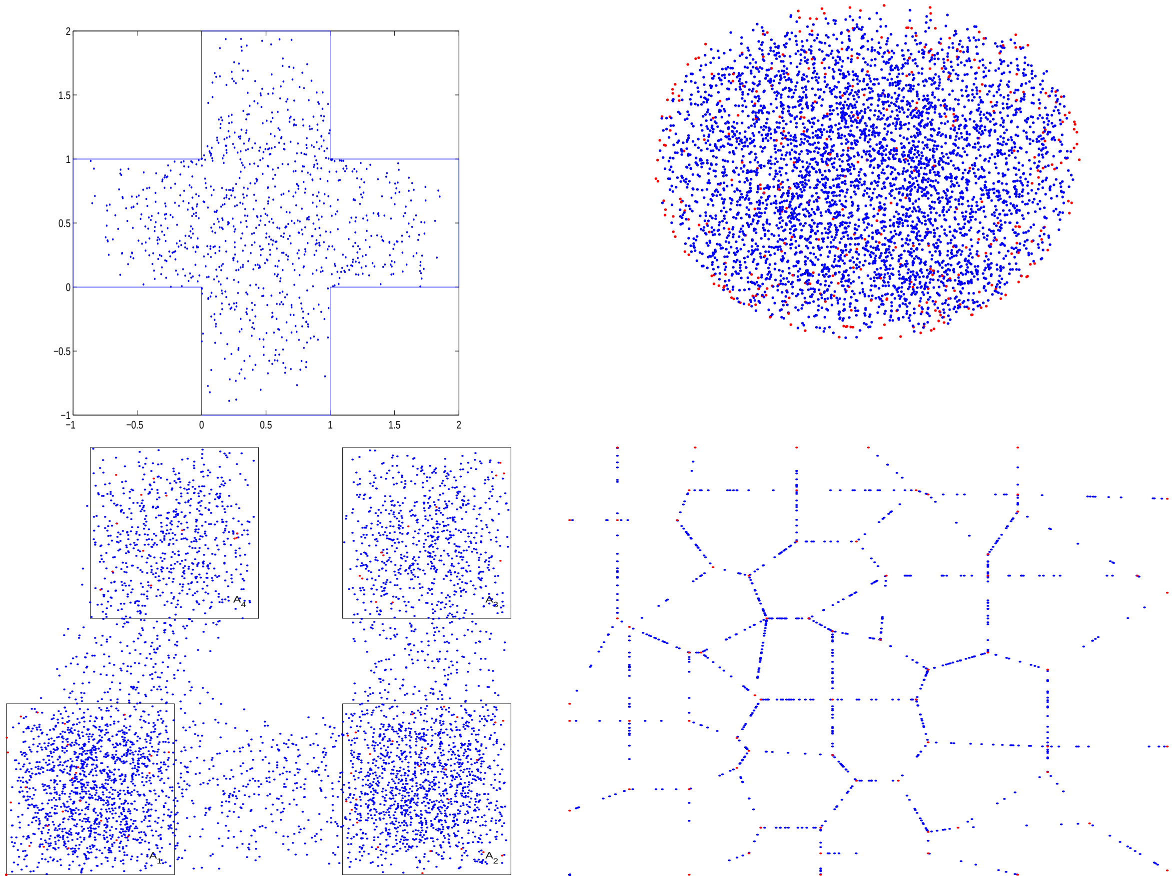

Fig. 3. Perfect sampling of node position from time-stationary distribution for examples introduced in Part I [3]: Swiss Flag (1000 samples), Fish in a Bowl (5000 samples), Four Town restricted random waypoint (5000 samples) and Space Graph (10000 samples). Densities are not uniform, with bias towards central areas and interior corner points.

Lemma 2: Let X be a random point, uniformly distributed in $\mathcal{A} = [0, a_1] \times [0, a_2] \times \ldots [0, a_d]$. For any non random vector $\vec{v} \in \mathbb{R}^d$, the distribution of $w(X + \vec{v})$ is also uniform in $\mathcal{A}$.

Proof. First we show the lemma for $d = 1$. It is also sufficient to show this for $a_1 = 1$. We have

$$X' = X + v \mod 1. \tag{3}$$

Since X' is limited to the interval $[0, 1]$, its distribution is entirely defined by its Fourier coefficients for $n \in \mathbb{Z}$: $c'_n = \mathbb{E}\left(e^{2i\pi n X'}\right)$. By Equation (3) $c'_n = e^{2i\pi n v} c_n$, where c_n is the nth Fourier coefficient of the distribution of X. Now X is uniform over $[0, a]$ thus $c_n = 0$ for $n \neq 0$ and $c_0 = 1$. It follows that $c'_n = c_n$ for all n.

Now back to the general case, we have shown that all coordinates are uniformly distributed. Further, they are independent because X is uniform and $\vec{v}$ is constant. □

Perfect Simulation of the random walk on torus. It follows immediately and, contrary to random waypoint on sphere, it is very simple. Pick a phase in proportion to the average time spent in the phase. Pick a point and, for the *move* phase, a speed vector as if at a transition point, and pick a remaining trip duration according to the general formula for the density of the residual time until next transition, in any stationary system. Also, there is no speed decay [17] as with random waypoint on a sphere.

C. Billiards

There is a similar result for the billiards, but its proof if more elaborate. We assume that the speed vector has a completely symmetric distribution, as defined in Section III-G of Part I (i.e. there is equal probability of going left or right [resp. up or down]). We continue with the same notation, in particular, the state of the simulation at time t is given by the phase $I(t)$, the location $X(t)$, the speed vector $\vec{V}(t)$ ($= \vec{0}$ if $I(t) = pause$) and the residual time until end of trip $R(t)$.

Note that now there is a difference. The instant speed $\vec{V}(t)$ is, in general, not constant during an entire trip and may differ from the unreflected speed $\vec{W}_n$ chosen at the beginning of the trip (as it gets reflected at the boundary of $\mathcal{A}$). Let $f^0_{\vec{W}}(\vec{w})$ be the density of the distribution of the non reflected speed vector (sampled at trip endpoints).

Theorem 8: For the random walk with reflection, the same holds as in Theorem 7 after replacing the first bullet of item 5 by

- *$\vec{V}(t)$ has density $f^0_{\vec{W}}(\vec{v})$.*

Proof. Item 1 follows from Lemma 5 in Part I, which says that, in order to continue a path from an intermediate point m it is not needed to know the unreflected speed vector, the instant speed is enough. The rest follows from Theorem 1 and Lemma 3, in a similar way as for Theorem 7. More precisely, with the same notation as in the proof of Theorem 7, we have (recall that we defined, in Part I, proof of Lemma 4, for any location $m \in \mathbb{R}^2$ the (linear) operator J_m as the one that transforms the unreflected speed vector into the true speed vector, when m is the hypothetical location if there would be no reflection):

$$\begin{aligned} & \mathbb{E}\left(\phi(X(t))\psi(\vec{V}(t))\xi(R(t))|I(t)=move\right) \quad (4)\\ = & \int_0^1 \int_{\mathbb{R}^+} h(s,u)\xi(us)\frac{s}{\tau_{move}} F^0_{move}(ds)du \end{aligned}$$

where

$$\begin{aligned} & h(s,u) := \\ & \int_{\mathbb{R}^2}\int_{\mathcal{A}} \psi\left(J_{m+su\vec{v}}(\vec{v})\right)\phi(b(m+us\vec{v}))\, f^0_{\vec{V}}(\vec{v})\pi^0(dm)d\vec{v} \\ = & \int_{\mathbb{R}^2}\int_{\mathcal{A}} \psi\left(J_{m+su\vec{v}}(\vec{v})\right)\phi(b(m+us\vec{v}))\, f^0_{\vec{V}}(\vec{v})\mathrm{Unif}(dm)d\vec{v} \end{aligned}$$

where the latter equality is by Part I, Lemma 4. Now we apply Lemma 3 to $h(s,u)$ with $\alpha = su$, $M =$ the location of the mobile at beginning of trip (which is uniform under the stationary distribution) and $\vec{W} =$ the unreflected speed vector for this trip. We have, with the notation of the lemma

$$h(s,u) = \mathbb{E}\left(\phi(M')\psi(\vec{W}')\right)$$

and thus

$$\begin{aligned} h(s,u) &= \mathbb{E}\left(\phi(M)\psi(\vec{W})\right) \\ &= \int_{\mathbb{R}^2}\int_{\mathcal{A}} \psi(\vec{v})\phi(m)\, f^0_{\vec{V}}(\vec{v})\mathrm{Unif}(dm)d\vec{v}. \end{aligned}$$

Combine this with Equation (4) and obtain the rest of the theorem. □

The following lemma is used in the proof of Theorem 8; it says that, at the end of a trip that starts from a uniform point M and a completely symmetric initial speed vector $\vec{W}$, the reflected destination point M' and speed vector $\vec{W}'$ are independent and have same distribution as initially.

Lemma 3: Let M be a random point, uniformly distributed in $\mathcal{A}$. Let $\vec{W}$ be a random vector in $\mathbb{R}^2$ independent of M and with completely symmetric distribution. Let $\alpha \in \mathbb{R}$ be a constant. Define $M' = b\left(M + \alpha\vec{W}\right)$ and $\vec{W}' = J_{M+\alpha\vec{W}}\left(\vec{W}\right)$. $(M', \vec{W}')$ has the same joint distribution as $(M, \vec{W})$.

Proof. It is sufficient to consider the case $a_1 = a_2 = 1$. Since J_m is a linear operator, it is also sufficient to consider the case $\alpha = 1$. The mapping that transforms $(M = (x,y), \vec{W} = (u,v))$ into $(M' = (x',y'), \vec{W}' = (u',v'))$ is such that $x = \varepsilon_1(x'-u') + 2n_1$, $y = \varepsilon_2(y'-v') + 2n_2$, $u = \varepsilon_1 u'$ and $v = \varepsilon_2 v'$, where $\varepsilon_1, \varepsilon_2 \in \{-1,1\}$ and $n_1, n_2 \in \mathbb{Z}$. It is differentiable almost everywhere and its Jacobian is 1. Thus, the joint density of $(M', \vec{W}')$ is

$$\begin{aligned} & f_{M',\vec{W}'}(x',y',u',v') \\ = & \sum_{\varepsilon_1,\varepsilon_2\in\{-1,1\},n_1,n_2\in\mathbb{Z}} f_{\vec{W}}(\varepsilon_1 u', \varepsilon_2 v') \cdot 1_{\varepsilon_1(x'-u')+2n_1\in(0,1)} \\ & \cdot 1_{\varepsilon_2(y'-v')+2n_2\in(0,1)}. \end{aligned}$$

Since $f_{\vec{W}}$ is completely symmetric:

$$\begin{aligned} = & f_{\vec{W}}(u',v') \cdot \\ & \cdot \sum_{\varepsilon_1,\varepsilon_2\in\{-1,1\},n_1,n_2\in\mathbb{Z}} 1_{\varepsilon_1(x'-u')+2n_1\in(0,1)} 1_{\varepsilon_2(y'-v')+2n_2\in(0,1)} \\ = & \left(\sum_{\varepsilon_1\in\{-1,1\},n_1\in\mathbb{Z}} 1_{\varepsilon_1(x'-u')+2n_1\in(0,1)}\right) \cdot \\ & \cdot \left(\sum_{\varepsilon_2\in\{-1,1\},n_2\in\mathbb{Z}} 1_{\varepsilon_2(y'-v')+2n_2\in(0,1)}\right). \end{aligned}$$

Now for any $x \in \mathbb{R}\backslash\mathbb{Z}$:

$$\sum_{\varepsilon_1\in\{-1,1\},n_1\in\mathbb{Z}} 1_{\{\varepsilon_1 x+2n_1\in(0,1)\}} = 1.$$

It follows that for all u', v' and $x', y' \in (0,1)$ except on a set of zero mass:

$$f_{M',\vec{W}'}(x',y',u',v') = f_{\vec{W}}(u',v') = f_{M,\vec{W}}(x',y',u',v').$$

□

Remark. It is important to use the instant speed vector $\vec{V}(t)$ and not the unreflected speed vector $\vec{W}(t)$ when describing the simulation state: indeed the description by phase $I(t)$, location $X(t)$, unreflected speed vector $\vec{W}(t)$ ($= \vec{0}$ if $I(t) = pause$) and residual time until end of trip $R(t)$ is not sufficient to continue the simulation (one needs to remember which reflection was applied to the speed vector) and is thus not Markov.

Also note that, in time stationary averages, the location $X(t)$ and the unreflected speed vector $\vec{W}(t)$ are *not* independent. For example, given that the unreflected speed vector is $\vec{W}(t) = (0.5a_1, 0)$ and the trip duration is $S(t) = 1$, it is more likely that $X(t)$ is in the second right half of the rectangle. In contrast, $X(t)$ and the instant speed vector $\vec{V}(t)$ *are* independent, as shown by the theorem.

Perfect simulation of the billiards. It is similar to the random walk on torus.

V. Related Work

For a survey of existing mobility models, see the work by Camp, Boleng, and Davies [5] and the references therein. Bettstetter, Harnstein, and Pérez-Costa [7] studied the time-stationary distribution of a node location for classical random-waypoint model. They observed that the time-stationary node location is non-uniform and it has more mass in the center of a rectangle. A similar problem has been further studied by Bettstetter, Resta, and Santi [2]. A closed-form expression for the time-stationary density of a node location is obtained only for random-waypoint on a one-dimensional interval; for two dimensions only approximations are obtained. Note that in Theorem 2, we do have an exact representation of the distribution of node location as a marginal of a distribution with a known density. Neither [7] nor [2] consider how to run perfect simulations.

It is the original finding of Yoon, Liu, and Noble [16] that the default setting of the classical random-waypoint exhibits speed decay with time. The default random-waypoint assumes the event-stationary distribution of the speed to be uniform on an interval $(0, vmax]$. The authors found that if a node is initialized such that origin is a waypoint, the expected speed decreases with time to 0. This in fact is fully explained by the infinite expected trip duration as sampled at trip transitions, which implies the random process of mobility state is null-recurrent; see Part I [3, Section IV]. In a subsequent work [17], the same authors advocate to run "sound" mobility models by initializing a simulation by drawing a sample of the speed from its time-stationary distribution. We remark that this is only a partial solution as speed is only a component of node mobility state. For this reason, the authors in [17] do not completely solve the problem of perfect simulation. Another related work is that of Lin, Noubir, and Rajaraman [11] that studies a class of mobility models where travel distance and travel speed between transition points can be modeled as a renewal process. The renewal assumption was also made in [16], [17]. We note that this assumption is *not* verified with mobility models such as classical random-waypoint on any non-isotropic domain, such as a rectangle, for example. The renewal assumption has been made to make use of a "cycle" formula from the theory of renewal random processes. From Palm calculus, we know that the "cycle" formula is in fact Palm inversion formula, which we used extensively throughout the paper, and that applies more generally to stationary random processes; this renders the renewal assumption unnecessary.

Perhaps the work closest to ours is that of Navidi, Camp, and Bauer in [15], [13]. As discussed in Section I-B, we provide a systematic framework that allows to formally prove some of the implicit statements in [13] and generalize to a broader class. Further, our perfect sampling algorithm differs in that it works even when geometric constants are not a priori known. In [12], Nain, Towsley, Liu and Liu consider the random walk on torus and billiards models (which they call "random direction"), assuming the speed vectors are isotropic. They find that the stationary regime has uniform distribution, and advocate that this provides an interesting bias-free model.

There are other well established techniques for performing perfect simulation. The method in [14] applies to a large class of Markov chains on which some partial ordering can be defined, and uses coupling from the past (sample trajectories starting in the past at different initial conditions). The technique presented in this paper is much simpler, as, unlike in the case of [14], we can obtain an explicit representation of the stationary distribution.

VI. Conclusion

Our perfect sampling algorithm is implemented to use with ns-2 to produce perfect simulations for a broad set of random trip mobility models. The code is freely available to download from:

- `http://ica1www.epfl.ch/RandomTrip`

This web page contains also further pointers to random trip mobility models.

Acknowledgments

The perfect sampling ns-2 code pointed out in this paper was implemented by Santashil PalChaudhuri. M. V. is grateful to Srdjan Čapkun for referring him to the mobility problem studied in this paper. He also thanks Don Towsley for encouragement and discussions.

References

[1] François Baccelli and Pierre Brémaud. *Palm Probabilities and Stationary Queues*. Springer LNS, 1987.

[2] Christian Bettstetter, Giovanni Resta, and Paolo Santi. The node distribution of the random waypoint mobility model for wireless ad hoc networks. *IEEE Transactions On Mobile Computing*, 2(3), July-September 2003.

[3] Jean-Yves Le Boudec and Milan Vojnović. The Random Trip Model, Part I: Stability. Technical Report MSR-TR-2005-164, Microsoft Research, 2005.

[4] Josh Broch, David A. Maltz, David B. Johnson, Yih-Chun Hu, and Jorjeta Jetcheva. A performance comparison of multi-hop wireless ad hoc network routing protocols. In *Mobile Computing and Networking*, pages 85–97, 1998.

[5] T Camp, J. Boleng, and V. Davies. A survey of mobility models for ad hoc network research. *WCMC: Special issue on Mobile Ad Hoc Networking: Research, Trends and Applications*, 2(5):483–502, 2002.

[6] Erhan Çinlar. *Introduction to Stochastic Processes*. Prentice Hall, 1975.

[7] Hannes Hartenstein Christian Bettstetter and Xavier Pérez-Costa. Stochastic properties of the random waypoint mobility model. *ACM/Kluwer Wireless Networks*, Special Issue on Modeling and Analysis of Mobile Networks 2003.

[8] Amit Jardosh, Elizabeth M. Belding-Royer, Kevin C. Almeroth, and Subhash Suri. Towards realistic mobility models for mobile ad-hoc networking. In *Proc. of ACM Mobicom 2003*, pages 217–229, San Diego, CA, 2003.

[9] J. Y. Le Boudec. On the stationary distribution of speed and location of random waypoint. *IEEE Transactions on mobile computing*, to appear.

[10] Jean-Yves Le Boudec. Understanding the simulation of mobility models with palm calculus. Technical Report EPFL/IC/2004/53, EPFL, 06 2004.

[11] Guolong Lin, Guevara Noubir, and Rajmohan Rajamaran. Mobility models for ad-hoc network simulation. In *Proceedings of Infocom 2004*, April 2004.

[12] Philippe Nain, Don Towsley, Benyuan Liu, and Zhen Liu. Properties of Random Direction Models. In *Proceedings of IEEE Infocom 2005*, Miami, Florida, USA, March 2005.

[13] William Navidi and Tracy Camp. Stationary distributions for the random waypoint model. *IEEE Transactions on Mobile Computing*, 3(1), 2004.

[14] James G. Propp and David B. Wilson. Exact sampling with coupled markov chains and applications to statistical mechanics. *Random Structures and Algorithms*, 9(1 & 2):223–252, 1996.

[15] T. Camp W. Navidi and N. Bauer. Improving the accuracy of random waypoint simulations through steady-state initialization. In *Proceedings of the 15th International Conference on Modeling and Simulation (MS '04)*, pages 319–326, March 2004.

[16] Jungkeun Yoon, Mingyan Liu, and Brian Noble. Random waypoint considered harmful. In *Proc of Infocom*, 2003.

[17] Jungkeun Yoon, Mingyan Liu, and Brian Noble. Sound mobility models. In *Proceedings of Mobicom*, 2003.

Appendix

A. Evaluation of Transient Path Distribution

In this section we describe the computation of the transient path distribution in the example in Figure 1. The model is a restricted random waypoint described as follows. The set of subdomains is finite with a subdomain $\mathcal{A}_i$ defined as the location of a vertex $i \in \mathcal{L}$. The set of paths is finite and a path is specified by the indexes of the origin and destination vertex

i and j. To simplify, we consider a model with no pauses. Our objective is to compute the distribution of the path $P(t)$ at time $t \geq 0$ with $t = 0$ taken as origin of a trip, i.e. $T_0 = 0$. We are thus interested in $\mathbb{P}^0(P(t) = i)$, $t \geq 0$, $i \in \mathcal{L}$. We can compute this *transient* distribution as (P_n, T_n) by using Markov renewal property as follows: for each $i \in \mathcal{L}$,

$$\begin{aligned} & \mathbb{P}^0(P(t) = j|P_0 = i) \\ = \; & \mathbb{P}^0(P(t) = j, T_1 > t|P_0 = i) + \\ & + \sum_{k \in I} \int_0^t Q(i,k) F_i(k, ds) \mathbb{P}^0(P(t-s) = k|P_0 = k) \quad (5) \end{aligned}$$

where $F_i(j,s) := \mathbb{P}^0(P_1 = j, T_1 \leq s|P_0 = i)$. From [6, Proposition 4.9, Chapter 10, Section 4], we have

$$\begin{aligned} & \lim_{t \uparrow \infty} \mathbb{P}^0(P(t) = j|P_0 = i) \\ = \; & \lambda \sum_{k \in \mathcal{L}} q^0(k) \int_0^\infty \mathbb{P}^0(P(s) = j, T_1 > s|P_0 = k) ds \\ = \; & \lambda \sum_{k \in \mathcal{L}} q^0(k) \mathbb{E}^0 \left(\int_0^\infty 1_{P(s)=j} 1_{T_1 > s} ds | P_0 = k \right) \\ = \; & \lambda \sum_{k \in \mathcal{L}} q^0(k) \mathbb{E}^0 \left(\int_0^{T_1} 1_{P(s)=j} ds | P_0 = k \right) \end{aligned}$$

where $\lambda = 1/\sum_{k \in \mathcal{L}} q^0(k) \mathbb{E}^0(T_1|P_0 = k)$. Indeed, the right-hand side in the last equality is precisely what would be given by Palm inversion formula for the time-stationary distribution of path. The above result shows convergence with time to the time-stationary distribution of path, from initial path chosen arbitrarily on the set of paths.

The system of equations (5) is known as Markov renewal equations and can be, in principle, routinely solved numerically. In the example, we assume each path takes a fixed integer number of time units, so for a path $(i,j) \in \mathcal{L}^2$, $F_i(j, ds) = \delta_{\tau_j}(s)$, for some fixed integer $\tau_j > 0$, where $\delta_{\tau_j}(\cdot)$ is a Dirac function. In this case, (5) boils down to the system of difference equations:

$$p(i,j,t) = g(i,j,t) + \sum_{k \in \mathcal{L}} Q(i,k) h(k,t) p(k,j,t-\tau_j), \; t = 1, 2, \ldots$$

where we define $p(i,j,t) := \mathbb{P}^0(P(t) = j|P(0) = i)$, $g(i,j,t) = 1_{\tau_i > t, i=j}$ and $h(k,t) = 1_{t \geq \tau_k}$, $i,j,k \in \mathcal{L}$. The numerical results in Figure 1 are obtained from the last difference equations.

B. Perfect Sampling for Restricted Random Waypoint

The theorem generalises Theorem 4 in Section III, to any number L of sub-domains $\mathcal{A}_i$, $i \in \mathcal{L}$.

Theorem 9: The time stationary distribution π of the phase $I(t) = (i,j,r,\phi)$ is

$$\pi(i,j,r,\phi) = \lambda q^0(i) Q(i,j) \times \\ \times \begin{cases} \bar{F}_i(r-1)(\bar{R}_i + 1)\tau_{i,j} & r \geq 0, \phi = pause \\ \underline{\Delta}_{i,j} \omega_{i,j} & r = 0, \phi = move \\ \bar{F}_i(r-1) \bar{R}_i \bar{\Delta}_{i,i} \omega_{i,i} & r > 0, \phi = move \end{cases}$$

where λ is a normalising constant, defined by the above equation and $\sum_i \pi(i) = 1$, $\bar{F}_i(r) = 1 - F_i(r)$ and $\bar{R}_i = \sum_{r \geq 0} \bar{F}_i(r)$ is the expected number of trips within a subdomain $i \in \mathcal{L}$.

- $\mathcal{L}_0 =$ is the set of (i,j) for which $\bar{\Delta}_{i,j}$ is known. Define $K_{i,j} = \bar{\Delta}_{i,j}$, for $(i,j) \in \mathcal{L}_0$ and otherwise $K_{i,j} = \Delta_{i,j}$.
- The following three distributions are used:

$$\begin{cases} E_p(i,j) = q^0(i) Q(i,j)(\bar{R}_i + 1)\tau_{i,j}/e_p \\ E_{m,0}(i,j) = q^0(i) Q(i,j) K_{i,j} \underline{\omega}_{i,j}/e_{m,0} \\ E_{m,1}(i,j) = q^0(i) Q(i,j) K_{i,i} \bar{R}_i \omega_{i,i}/e_{m,1} \end{cases}$$

where $e_p, e_{m,0}, e_{m,1}$ are normalising constants.

do forever
  Draw $U_1 \sim U(0,1)$
  if $U_1 \leq \frac{e_p}{e_p + e_{m,0} + e_{m,1}}$ // decide $\phi(t) = pause$
    Draw (i,j) from the distribution $E_p(i,j)$
    Draw $r \in \mathbb{Z}_+$ with probability $\frac{\bar{F}_i(r-1)}{\bar{R}_i + 1}$
    $I(t) = (i,j,r,pause)$; **leave**
  else // try $\phi(t) = move$
    // first sample i,j
    if $U_1 \leq \frac{e_{m,0}}{e_{m,0} + e_{m,1}}$
      Draw (i,j) from the distribution $E_{m,0}(i,j)$; $r = 0$
      Set $O := \mathcal{A}_i$, $\mathcal{D} := \mathcal{A}_j$, $\Delta := \Delta_{ij}$
    else
      Draw (i,j) from the distribution $E_{m,1}(i,j)$
      Draw $r \in \mathbb{Z}_+$ with probability $\frac{\bar{F}_i(r-1)}{\bar{R}_i + 1}$
      Set $O := \mathcal{A}_i$, $\mathcal{D} := \mathcal{A}_i$, $\Delta := \Delta_{ii}$
    if $(i,j) \in \mathcal{L}_0$
      $I(t) = (i,j,r,move)$
      do
        Draw $M_0 \sim$ Unif(O), $M_1 \sim$ Unif$(\mathcal{D})$
        Draw $U_2 \sim$ Unif$(0,\Delta)$
      until $U_2 < d(M_0, M_1)$
      leave
    else // $(i,j) \notin \mathcal{L}_0$
      Draw $M_0 \sim$ Unif(O), $M_1 \sim$ Unif$(\mathcal{D})$
      Draw $U_2 \sim$ Unif$(0,\Delta)$
      if $U_2 < d(M_0, M_1)$
        $I(t) = (i,j,r,move)$; **leave**
end do

Fig. 4. Sampling algorithm for restricted random waypoint with an arbitrary value of L, supporting both cases where the average distance between $\mathcal{A}_i$ and $\mathcal{A}_j$ is known or not.

Proof. By substitution in the balance equations, we can verify that the event-stationary distribution of the phase I_n is given by

$$\begin{cases} \pi^0(i,j,r,pause) = \pi^0(i,j,r,move) \\ \pi^0(i,j,r,move) = \alpha q^0(i) Q(i,j) \bar{F}_i(r-1), \; r \geq 0 \end{cases} \quad (6)$$

with α a normalising constant. The rest follows from Theorem 1. □

The perfect sampling algorithm generalising that in Figure 2 to arbitrary number of subdomains is displayed in Figure 4.

C. Details of Perfect Sampling for Restricted Random Waypoint

Complexity. We compare the complexity of the two branches of the algorithm in numbers of calls to the random number generator. Let a be the number of such calls required to simulate one sample (M_0, M_1) uniformly in the $\mathcal{A}_1$ plus one ($a = 5$ for a rectangle or a disk, usually more for non convex domains). By an analysis similar to the proof of Lemma 1, we find, for the former case $C_1 = \frac{\alpha+\bar{\Delta} a}{\alpha+\bar{\Delta}}$ and for the latter $C_2 = \frac{\Delta-\bar{\Delta}}{\alpha+\bar{\Delta}}(1+a) + \frac{\alpha+(1+a)\bar{\Delta}}{\alpha+\bar{\Delta}}$, with $\alpha = \bar{\tau}_{pause}/\omega$.

We always have $C_2 > C_1$; thus if $\bar{\Delta}$ is known with little computational cost, it is always preferable to use the former case ("$\bar{\Delta}$ is known"). In contrast, if $\bar{\Delta}$ is not known, there are two options: (i) compute $\bar{\Delta}$ offline by Monte-Carlo simulation and use the former case ("$\bar{\Delta}$ is known"), or (ii) use the latter case ("$\bar{\Delta}$ is not known"). The optimal choice depends on the number N of mobiles that need to be initialised by the sampling procedure (N includes the number of replications of the simulation). Clearly, since $C_2 > C_1$, as N goes to ∞, and since the cost of the Monte Carlo simulation is incurred only once for all simulation runs, there is a breakpoint N_0 such that for $N \leq N_0$ it is optimal to use the first option, and vice versa. The complexity of Monte Carlo to compute $\bar{\Delta}$ with 99.99% confidence interval and a relative accuracy of $1-\varepsilon$ is of the order of $a(6\frac{\sigma}{\bar{\Delta}\varepsilon})^2$, where σ^2 is the variance of the distance between two points in $\mathcal{A}_1$. σ depends on the regularity of the domain $\mathcal{A}$. For restricted random waypoint or city graph, it is large compared to the mean value. For more regular areas, a crude approximation of σ is $\Delta - \bar{\Delta}$. Comparing C_2/C_1 to this complexity, we find that N_0 is of the order of 10 to 1000 times $\frac{1}{\varepsilon^2}$. In practise, $\varepsilon = 10^{-4}$ and thus N_0 is of the order of 10^9 to 10^{11} for $L = 1$, which is probably larger than the number of simulation runs performed in a campaign by several orders of magnitude. Thus, it should generally be much more efficient to consider the second option.