Finite work

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Introduction

In the appendix of [1], Gurevich defines *finite exploration* for small-step algorithms which do not interact with the environment. Although satisfying finite exploration is a seemingly weaker property than satisfying bounded exploration, he proves the two notions are equivalent for this class of algorithms. We investigate what happens in the case of ordinary small-step algorithms – in particular, these algorithms *do* interact with the environment – as described by Blass and Gurevich in [2]. Our conclusion is that every algorithm satisfying the appropriate version of finite exploration is equivalent to an algorithm satisfying bounded exploration. We provide a counterexample to the stronger statement that every algorithm satisfying finite exploration satisfies bounded exploration. This statement becomes true if the definition of bounded exploration is modified slightly. The proposed modification is natural for algorithms operating in isolation, but not for algorithms belonging to a larger systems of computation. We believe the results generalize to general interactive small-step algorithms.

Definitions

We use the vocabulary in [2], with the following additional definitions.

Definition 0.1. The *universe* $U(\alpha)$ of an answer function α is the set of elements of states occurring in the queries and replies of its domain and range.

Definition 0.2. The *size* of an answer function α, or $Size(\alpha)$, is the number of elements occurring in the queries and replies of its domain and range, *including repetition.* Note that $Size(\alpha)$ can be larger than $||U(\alpha)||$, for example, the answer function $\{((a, a), b)\}$ has size 3 although its universe has size 2.

Definition 0.3. An answer function α is *tame* for the state X if $\alpha \subset \alpha'$ and α' is well founded with respect to $\vdash_X$. Say α is tame if it is tame for some state X.

Definition 0.4. An *ordered answer function* is a pair $(\alpha, \bar{a})$ where α is an answer function and the tuple $\bar{a}$ is a particular ordering of the universe $U(\alpha)$. Ordered answer functions $(\alpha, a_1 \dots a_n)$ and $(\beta, b_1 \dots b_n)$ are *isomorphic* if the map sending a_i to b_i is an isomorphism from α to β.

We now introduce the modified version of the bounded exploration postulate. The only real difference with the bounded work postulate presented in [2] is the restriction of arbitrary answer functions to tame answer functions.

Modified bounded exploration postulate

- There is a bound, depending only on the algorithm A, on the size of contexts.
- There exists a finite set of terms T, depending only on A, with the following properties. If α is tame for states X and X' and each term in T has the same value in X and in X' when the variables are given the same values in $Range(\alpha)$, then
  - $\alpha \vdash_X q$ if and only if $\alpha \vdash_{X'} q$, and
  - if additionally α is a context for X, then either the algorithm fails for both (X, α) and (X, α'), or it fails for neither and $\Delta^+(X, \alpha) = \Delta^+(X', \alpha)$.

Finite exploration postulate

- There is a bound, depending only on the algorithm A, on the size of contexts.
- There exists a map

  $$t : \Sigma\text{-structures} \rightarrow \text{ finite sets of } \Sigma\text{-terms}$$

  depending only on A, with the following properties.

- $t(X)$ is closed under subterms and contains at least one variable.
- $t(X) = t(X')$ if $X \cong X'$.
- If α is tame for states X and X' and each term in $t(X)$ has the same value in X and in X' when the variables are given the same values in *Range*(α), then
  * $\alpha \vdash_X q$ if and only if $\alpha \vdash_{X'} q$, and
  * if additionally α is a context for X, then either the algorithm fails for both (X, α) and (X, α'), or it fails for neither and $\Delta^+(X, \alpha) = \Delta^+(X', \alpha)$.
- Given Z and α, if for every state X for which α is tame there is a term in $t(X)$ with different values in X and Z given some values for the variables in *Range*(α), then reciprocally for every state X for which α is tame, there is a term in $t(Z)$ (as opposed to $t(X)$) with different values in X and Z given some values in *Range*(α).

Definition 0.5. The pair (Z, α) described in the final clause of finite exploration is called an *error pair*.

Definition 0.6. Two Σ-structures X and X' *coincide* over an answer function α and a set of terms T if each term in T has the same value in X and in X' when the variables are given the same values in *Range*(α) (and additionally α is an answer function for both X and X') . The above postulates can be written more concisely with this terminology.

The first clause of the finite exploration postulate implies that there is a finite bound, depending only on the algorithm, on the size of a tame answer function (see corollary 2.31 of [2]). For a fixed n, a finite tuple $\bar{a}$ belongs finitely many answer functions α where $(\alpha, \bar{a})$ is an ordered answer function of size n. Therefore we have the following.

Fact 0.7. There are finitely many isomorphism classes of ordered tame answer functions.

Facts about finite and modified bounded exploration

The main goal of this section is to show that every algorithm satisfying modified bounded exploration is equivalent to an algorithm satisfying bounded exploration. In the preliminary lemma we analyze the behavior of well foundedness in structures satisfying finite exploration (and therefore modified bounded exploration as well). This lemma is also used in the proof that finite exploration implies modified bounded exploration.

Lemma 0.8. *Assume algorithm A satisfies finite exploration, α is an answer function for X, X', and X, X' coincide over α and $t(X)$. Then*

- *α is well founded for X if and only if α is well founded for X', and*
- *α is a context for X if and only if α is a context for X'.*

Proof. Assume X, X' and α are given as above. Define the monotone operators

$$\Gamma_X(Z) = \{q \ : \ (\exists \xi \subset \alpha \restriction Z) \ \xi \vdash_X q\}$$

$$\Gamma_{X'}(Z) = \{q \ : \ (\exists \xi \subset \alpha \restriction Z) \ \xi \vdash_{X'} q\}$$

The answer function α is $\vdash_X$ well founded if $Dom(\alpha) \subset \Gamma_X^\infty$. We prove inductively that $\Gamma_X^n = \Gamma_{X'}^n$ for all $n \in \mathbb{N}$, and this is sufficient to complete the proof. The base case is trivial: $\Gamma_X^0 = \Gamma_{X'}^0$ is empty. Inductively assume $\Gamma_X^{n-1} = \Gamma_{X'}^{n-1}$. Now $\Gamma_X^n = \{q \ : \ (\exists \xi \subset \alpha \restriction \Gamma_X^{n-1}) \ \xi \vdash_X q\}$. Since Γ_X^n is well founded, every $\xi \subset \alpha \restriction \Gamma_X^{n-1}$ is tame. Therefore by finite exploration $\xi \vdash_X q$ if and only if $\xi \vdash_{X'} q$ for such ξ and $\Gamma_X^n = \Gamma_{X'}^n$.

∎

Lemma 0.9. *If algorithm A satisfies modified bounded exploration then the normalization A' of A satisfies bounded exploration.*

Proof. Assume that X, X' are states for A, that α is an answer function for both, and that X, X' coincide over α. Since contexts are not changed by normalization, the final clause of the bounded exploration postulate is satisfied by A' and it is enough to show the following.
Claim: If $\alpha \vdash'_X q$ then $\alpha \vdash'_{X'} q$.

Recall the causality relation $\alpha \vdash'_X q$ for A' means that α is $\vdash_X$ well-founded and q is $\vdash_X$ reachable under α, that is, there exist $\beta_1 \dots \beta_n$ and $q_1 \dots q_n = q$ where each $\beta_i \subset \alpha \restriction \{q_j \ : \ j < i\}$ and$\beta_i \vdash_X q_i$. By the previous lemma, α is well founded for $\vdash_X$ implies α is well founded for $\vdash_{X'}$. Therefore each β_i is tame for X' and by modified bounded exploration $\beta_i \vdash_{X'} q_i$. Thus $\alpha \vdash'_{X'} q$.

∎

Main result

Theorem 0.10. *Every algorithm satisfying the finite exploration postulate satisfies the modified bounded exploration postulate.*

Proof. Let A be an algorithm satisfying finite exploration, with map t. We will obtain a modified bounded exploration witness W. As in the original proof, the main tool is König's lemma in the guise of the compactness theorem of first order logic and the compactness of Cantor space. The most significant difference in the structure of this proof is the consideration of more than one tree with Cantor space structure.

Recall that Cantor space consists of infinite sequences $\xi = \langle \xi_i : i \in \mathbb{N} \rangle$ where $\mathbb{N}$ is the set of natural numbers and each $\xi_i \in \{0, 1\}$. Every function f from a finite set of natural numbers to $\{0, 1\}$ gives a basic open set

$$O(f) \equiv \{\xi \ : \ \xi_i = f(i) \text{ for all } i \in \text{Domain}(f)\}$$

of the Cantor space. An arbitrary open set of the Cantor space is a union of basic open sets $O(f)$.

Say there are n isomorphism classes of tame ordered answer functions, and choose representatives $(\alpha_1, \bar{a}_1) \dots (\alpha_n, \bar{a}_n)$ for these isomorphism classes. Call a term *equational* if it has the form $term_1 = term_2$. For each isomorphism class $[\alpha_i, \bar{a}_i]$ list in some order all the equational terms in the signature $\Sigma \cup \{\bar{a}_i\}$:

$$e_{i1}, e_{i2}, e_{i3}, \dots$$

Let Tr_i be the corresponding Cantor space for $[\alpha_i, \bar{a}_i]$, that is, the set of infinite sequences representing characteristic functions for these equational terms. Additionally, equip Tr_i with a special subset $\mathfrak{O}_i$ of sequences representing $(Y, (\beta, \bar{b}))$ where $(\beta, \bar{b}) \cong (\alpha_i, \bar{a}_i)$, Y is a Σ-structure, and β is an answer function for Y.

Definition 0.11. Every (structure, ordered answer function) pair $(Y, (\beta, \bar{b}))$ where $||\bar{b}|| = ||\bar{a}_i||$ gives rise to a sequence $\xi \in Tr_i$, where $\xi_j = 1$ if e_{ij} holds in Y under the substitition $\bar{a}_i \to \bar{b}$. If additionally $(\beta, \bar{b}) \cong (\alpha_i, \bar{a}_i)$ we say that $\xi \in T_i$ *represents* $(Y, (\beta, \bar{b}))$.

Lemma 0.12. *For each isomorphism class $[\alpha_i, \bar{a}_i]$ the subset $\mathfrak{O}_i \subset Tr_i$ is closed.*

Proof. Given $\xi \notin \mathfrak{O}_i$ we find an open set U with $\xi \subset U$ and U disjoint from $\mathfrak{O}_i$. Let $\Phi = \{e_{ij} \ : \ \xi_j = 1\} \cup \{\neg e_{ij} \ : \ \xi_j = 0\}$. Since ξ does not represent any $\Sigma \cup \{\bar{a}_i\}$-structure, the set of sentences Φ is not satisfiable. By the compactness theorem for first order logic, there is a finite subset $\Phi' \subset \Phi$ which is not satisfiable and a corresponding basic open set U containing ξ and not containing any sequences representing $\Sigma \cup \{\bar{a}_i\}$-structures. ∎

Since Cantor space is compact, $\mathfrak{O}_i \subset Tr_i$ is compact for each i. We now define open covers for each $\mathfrak{O}_i$, so that the finite subcovers of the finitely many $\mathfrak{O}_1, \ldots \mathfrak{O}_n$ will give us a candidate for the modified bounded exploration witness.

Definition 0.13. Let T be a set of Σ-terms. Two branches $\xi, \psi \in Tr_i$ *agree* on T if $\xi_j = \psi_j$ for every e_{ij} which is an equality between terms in T with elements of $\bar{a}_i$ substituted for the variables.

We say $(X, (\alpha, \bar{a}))$ and $(Y, (\beta, \bar{b}))$ *agree* on a set of Σ-terms T if 1) $(\alpha, \bar{a})$ and $(\beta, \bar{b})$ belong to the same isomorphism class $[\alpha_i, \bar{a}_i]$ and 2) the sequences $\xi_1, \xi_2 \in T_i$ representing $(X, (\alpha, \bar{a}))$ and $(Y, (\beta, \bar{b}))$ agree on T.

Definition 0.14. Say $(Y, (\beta, \bar{b}))$ is represented in Tr_i by ξ. Define

$$O((Y, (\beta, \bar{b}))) = \{\psi \in Tr_i \ : \ \psi \text{ agrees with } \xi \text{ on t(Y)}\}$$

Note that $O((Y, (\beta, \bar{b})))$ is a basic open set.

Lemma 0.15. *For each isomorphism class $[\alpha_i, \bar{a}_i]$ define $Legal_i$ to be the set of pairs $(Y, (\beta, \bar{b}))$ where Y is a legal state, β is tame for Y and $(\alpha_i, \bar{a}_i) \cong (\beta, \bar{b})$. Also, define $Error_i$ to be the set of pairs $(Y, (\beta, \bar{b}))$ where (Y, β) is an error pair and $(\beta, \bar{b}) \cong (\alpha_i, \bar{a}_i)$. Then we claim*

$$\mathfrak{O}_i \subset \bigcup_{(Y,(\beta,\bar{b})) \subset Legal_i \cup Error_i} O(Y, (\beta, \bar{b}))$$

Proof. Consider the sequence ξ representing $(Z, (\gamma, \bar{c}))$ where Z is a Σ-structure, γ is an answer function for Z and $(\gamma, \bar{c}) \cong (\alpha_i, \bar{a}_i)$. If (Z, γ) is an error pair then since ξ agrees with itself clearly $\xi \in O((Z, (\gamma, \bar{c})))$. If not, then there exists $(Y, (\gamma, \bar{c}))$ where Y is legal, γ is tame for Y, and the sequence ξ' representing $(Y, (\gamma, \bar{c}))$ agrees with ξ on $t(Y)$. In this case, $\xi \in O((Y, (\gamma, \bar{c})))$ and $(Y, (\gamma, \bar{c})) \in Legal_i$. ∎

By the compactness of $\mathfrak{O}_i$, there exist finite subsets $Legal'_i \subset Legal_i$ and $Error'_i \subset Error_i$ where

$$\mathfrak{O}_i \subset \bigcup_{(Y,(\beta,\bar{b}))\subset Legal'_i \cup Error'_i} O(Y,(\beta,\bar{b}))$$

Definition 0.16.

$$W = \bigcup_{i\leq n} \bigcup_{(Y,(\beta,\bar{b}))\subset Legal'_i} t(Y)$$

Recall that n is the number of isomorphism classes of ordered answer functions. Since each $t(Y)$ is closed under subterms and contains at least one variable, the same is true for W. We need the following lemmas to prove that W is indeed a modified bounded exploration witness.

Lemma 0.17. *If $(X,(\alpha,\bar{a}))$ and $(Y,(\beta,\bar{b}))$ agree on a set of Σ-terms T (closed under subterms and containing at least one variable) then there exists Z where*

- $(Z,(\alpha,\bar{a})) \cong (Y,(\beta,\bar{b}))$ *and*
- X, Z *coincide over* α *and* T*, that is, each term in* T *has the same value in* X *and* Z *when the variables are given the same values in* $Range(\alpha)$.

Proof. First make X and Y disjoint. The Σ-structure Z is obtained from Y by removing each element equal to some $t(\bar{b}')$ where $t \in T$ and $\bar{b}' \subset \bar{b}$ and replacing it with the corresponding element in X equal to $t(\bar{a}')$, where $\bar{a}'$ is the isomorphic image of $\bar{b}'$ under the unique isomorphism from $(\beta,\bar{b})$ to $(\alpha,\bar{a})$. By the choice of Z we clearly have X, Z coinciding over α and T. By the definition of agreement the notion of equality remains undisturbed and indeed since T contains a variable we have $(Z,(\alpha,\bar{a})) \cong (Y,(\beta,\bar{b}))$. Note in particular that if β is tame for Y then α is tame for Z. ∎

Lemma 0.18. *Say $\xi \in Tr_i$ represents $(X,(\alpha,\bar{a}))$ where X is legal and α is tame for X. Then $\xi \in \cup_{(Y,(\beta,\bar{b}))\in Legal'} O(Y,(\beta,\bar{b}))$.*

Proof. Assume $\xi \notin \cup_{(Y,(\beta,\bar{b}))\in Legal'} O(Y,(\beta,\bar{b}))$ for contradiction. Then $\xi \in O(Z,(\beta,\bar{b}))$ for some error pair (Z,β). By the definition of $O(Z,(\beta,\bar{b}))$ we know that $(X,(\alpha,\bar{a}))$ and $(Z,(\beta,\bar{b}))$ agree on $t(Z)$. By the previous lemma we can find Z' where $(Z',(\alpha,\bar{a})) \cong (Z,(\beta,\bar{b}))$ and where Z', X coincide over α and $t(Z)$. By the isomorphism postulate (Z',α) is an error pair and by the definition of t we have $t(Z) = t(Z')$. But by the definition of error pairs and by the final clause of the finite exploration postulate, Z' must fail to coincide with X over α and $t(Z')$. Contradiction ∎

We now show that W is a modified bounded exploration witness for our algorithm. Say states X, X' coincide over β and W, and β is tame for both. Let $\bar{b}$ order the universe of β and say $(\beta, \bar{b}) \cong (\alpha_i, \bar{a}_i)$. Take $(Y, (\alpha_i, \bar{a}_i)) \in Legal'_i$ such that $(X, (\beta, \bar{b}))$ and $(Y, (\alpha_i, \bar{a}_i))$ agree on $t(Y)$. Since $t(Y) \subset W$ it follows that $(X', (\beta, \bar{b}))$ and $(Y, (\alpha_i, \bar{a}_i))$ agree on $t(Y)$ as well. By the second to last lemma there exists a state Z with $(Z, (\beta, \bar{b})) \cong (Y, (\alpha_i, \bar{a}_i))$ such that X, X' and Z all coincide over $t(Z) = t(Y)$ and β. Since α_i is tame for the legal state Y, it follows that β is tame for the legal state Z.

- Suppose $\alpha \vdash_X q$. By the finite exploration postulate $\alpha \vdash_Z q$ and again $\alpha \vdash_{X'} q$.

- Suppose α is a context for X. By the previous section, α is a context for Z as well. By two applications of finite exploration, either the algorithm fails for both (X, α) and (X', α) or it fails for neither and $\Delta^+(X, \alpha) = \Delta^+(X', \alpha)$.

This ends the proof of the main theorem. ∎

Corollary 0.19. *If A satisfies finite exploration then A is equivalent to an algorithm satisfying bounded exploration.*

Proof. Since A satisfies modified bounded exploration, it follows by the results of the previous section that the normalization of A satisfies bounded exploration. ∎

Counterexample

We give an example of an algorithm satisfying finite exploration but not bounded exploration. The construction in fact refutes the weaker statement that *strong* finite exploration implies bounded exploration, where strong finite exploration is the natural analog to bounded exploration obtained by removing the tameness condition from the definition of finite exploration.

Lemma 0.20. *There exists an algorithm A satisfying strong finite exploration but not bounded exploration.*

Proof. The signature for the algorithm is $\Sigma \equiv \{R_n \mid n \in \mathbb{N}\}$ where R_n is a n-ary relation. A *legal state* for A is any state isomorphic to X_k for some $k \geq 1$, where the universe of X_k is $\{a_n : n \leq \omega\}$ and the single positive relation $R_k(a_1 \ldots a_k)$ holds in X_k. For $Y \cong X_k$ define $t(Y) = \{R_i \mid i \leq k\}$. For Σ-structure Z which is not a state define $t(Z) = \{R_1\}$. Let s be a label, let $\bar{s}_n$ be the query of s repeated n times, and α_n be the answer function $((s, a_1), \ldots, (s, a_n))$. Note that $Range(\alpha_n) = a_1 \ldots a_n$. Given $1 \leq j \leq k$, $Y \cong X_k$, and β_j the isomorphic image of α_j, define $\beta_j \vdash_Y \bar{s}_j$. There are no other causality relations. In particular the only well founded answer function is the empty function. Finally the algorithm executes no updates.

Claim 1: If (Z, β) is an error pair then for every state X where β is an answer function for X the structures Z, X fail to coincide over β and $t(Z)$.

Proof. Let (Z, β) be an error pair and assume the range of β is $\bar{b} = b_1 \ldots b_n$. We assert that there are at least two elements of $\bar{b}$ satisfying R_1. First, say there is only one element b_i satisfying R_1. Without loss of generality assume $b_i = b_1$ and let Y be the state isomorphic to X_1 with $\bar{b}$ substituted for $a_1 \ldots a_n$: the structures Z and Y coincide over β and $t(Y) = R_1$. Second, if there are no elements of $\bar{b}$ satisfying R_1 then let Y be the state isomorphic to X_1 with $\bar{b}$ substituted for $a_2 \ldots a_{n+1}$. Again, the structures Z and Y coincide over β and $t(Y) = R_1$. Now if X is a legal state it will never contain two elements satisfying R_1. Therefore Z will fail to coincide with any legal state X over β and $t(Z) = R_1$.

$\square$

Claim 2: The algorithm satisfies strong finite exploration.

Proof. The size of contexts is bounded by zero, and the witness map t satisfies the first and second clauses. By the previous lemma the fourth clause is satisfied. Since there are no updates we are left to prove the following: if α is an answer function for states X and X' which coincide over α and $t(X)$ then $\alpha \vdash_X q$ if and only if $\alpha \vdash_{X'} q$. Without loss of generality we can assume $X = X_i$ and and $X' = X_j$ with $1 \leq i < j$. We can also assume that $\alpha = \alpha_k$ where $k \leq j$. First assume $k \leq i$. In both X_i and X_j the answer function α_k causes a single query, namely $\bar{s}_k$. Thus $\alpha_k \vdash_X q$ if and only if $\alpha \vdash_{X'} q$. Finally assume $i < k \leq j$. Note that $R_i \in t(X_i) \cap t(X_j)$ and additionally $R_i(\alpha_i)$ holds in X_i but not in X_j. SInce $\alpha_i \subset \alpha_k$ it follows that X_i, X_j fail to coincide over α_k and $t(X_i) \cap t(X_j)$. $\square$

We finish the proof by showing that the algorithm fails to satisfy bounded exploration. Assume for contradiction that W is a bounded exploration witness. Since W is finite there exists a maximum j with $R_j \in W$. Thus X_{j+1}, X_{j+2} coincide over α_{j+2} and W, but α_{j+2} causes $\bar{s}_{j+2}$ in X_{j+2} and fails to cause $\bar{s}_{j+2}$ in X_{j+1}. Contradiction.

$\square$

Remark 0.21. The above counterexample can be translated into an algorithm whose signature consists if a binary predicate $\rightarrow$ and a unary predicate U as follows. Let $G_1, G_2, \ldots$ be a sequence of $\rightarrow$ graphs where no G_i is a subgraph of G_j. The n-ary predicate $R_n(\bar{x})$ is replaced by the quantifier free formula $\theta(\bar{x}, \bar{y})$, where

$$\theta(\bar{x}, \bar{y}) \equiv x_1 \rightarrow x_2 \ldots \rightarrow x_n \wedge \bigwedge_i U(x_i) \wedge \bigwedge_j \neg U(y_j) \wedge G_n(\bar{y}) \wedge \bigwedge_{i,j} x_i \rightarrow y_j$$

This replacement induces natural replacements of X_i, t, α_i and $\vdash_X$. We leave the verification that the new algorithm satisfies strong finite exploration but not modified exploration as an exercise for the reader.

Conclusion

As mentioned in the introduction, we believe the above arguments extend to general interactive small-step algorithms, with attainable histories taking the place of tame answer functions.

References

[1] Yuri Gurevich, "Sequential Abstract State Machines Capture Sequential Algorithms," *A.C.M. Trans. Computational Logic* 1 (2000) 77-111.

[2] Andreas Blass and Yuri Gurevich, "Ordinary Interactive Small-Step Algorithms, I," *to appear in ACM Trans. Computational Logic*