A General Volume-Parameterized Market Making Framework

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We introduce a framework for automated market making for prediction markets, the *volume parameterized market* (VPM), in which securities are priced based on the market maker's current liabilities as well as the total volume of trade in the market. We provide a set of mathematical tools that can be used to analyze markets in this framework, and show that many existing market makers (including cost-function based markets [Chen and Pennock 2007; Abernethy et al. 2011, 2013], profit-charging markets [Othman and Sandholm 2012], and buy-only markets [Li and Vaughan 2013]) all fall into this framework as special cases. Using the framework, we design a new market maker, the *perspective market*, that satisfies four desirable properties (worst-case loss, no arbitrage, increasing liquidity, and shrinking spread) in the complex market setting, but fails to satisfy information incorporation. However, we show that the sacrifice of information incorporation is unavoidable: we prove an impossibility result showing that *any* market maker that prices securities based only on the trade history cannot satisfy all five properties simultaneously. Instead, we show that perspective markets may satisfy a weaker notion that we call *center-price information incorporation*.

Categories and Subject Descriptors: J.4 [**Social and Behavioral Sciences**]: Economics

Additional Key Words and Phrases: Prediction markets; market making; convex optimization

1. INTRODUCTION

A *prediction market* is a securities market in which traders buy and sell contracts with values that are contingent on the outcome of a future event. Such markets are quite common, ranging from exchanges for stock options and other financial derivatives to bookmakers for sporting events to markets like the Iowa Electronic Markets [Forsythe et al. 1992] that offer betting contracts on political election results. Interest in prediction markets stretches beyond gamblers and investors, as researchers have become quite intrigued at the *information aggregation* properties of market mechanisms. Efficient market theory [Malkiel and Fama 1970] suggests that market prices reflect consensus forecasts that ought not be systematically inaccurate, and indeed these forecasts have been accurate in a variety of empirical settings [Ledyard et al. 2009; Berg et al. 2001; Wolfers and Zitzewitz 2004].

The computer science literature has seen a recent burst in the development of *automated market makers* for facilitating prediction markets [Hanson 2003; Chen and Pennock 2007; Pennock 2010; Abernethy et al. 2011, 2013]. In traditional markets, an agent who arrives with the goal of buying or selling a given security must find a counterparty who is interested in taking the other side of the transaction for this security at a reasonable price. This can be a challenge in thin markets or in markets with a large set of diverse securities. To combat this problem, an automated market

This research was partially supported by the NSF under grant IIS-1054911. Any opinions, findings, conclusions, or recommendations are those of the authors alone.

EC'14, June 8–12, 2014, Stanford, CA, USA. ACM 978-1-4503-2565-3/14/06 ...$15.00.

http://dx.doi.org/10.1145/2600057.2602900

maker acts as a central authority that interacts with traders and facilitates all transactions. The market maker is always willing to both buy and sell the set of securities in question, and can adjust prices based on the history of trades in the market or other factors. While the existence of a market maker can be beneficial for traders, who have available a guaranteed counterparty at all times, the act of market making can be profitable too: a market maker can in principle balance its inventory and profit off the bid-ask spread [Othman and Sandholm 2012; Li and Vaughan 2013]. The market making literature within the computer science community has focused on a mix of algorithmic and economic questions: Can we design a market maker that has a bounded loss (downside) from trading? How should we design the space of contracts? When can the pricing function on these contracts be computed efficiently?

ALGORITHM 1: The cost-function market maker

Market maker announces payoff function $\phi : \Omega \to \mathbb{R}^k$
Market maker initializes share vector $\mathbf{q} \leftarrow \mathbf{0} \in \mathbb{R}^k$
for all traders $t = 1, \ldots, T$ **do**
  Trader t purchases bundle $\mathbf{r}_t \in \mathbb{R}^k$ and pays $C(\mathbf{q} + \mathbf{r}_t) - C(\mathbf{q})$
  Market maker updates the state $\mathbf{q} \leftarrow \mathbf{q} + \mathbf{r}_t$
end for
Outcome ω is revealed and trader t is paid $\boldsymbol{\phi}(\omega) \cdot \mathbf{r}_t$ for every $t = 1, \ldots, T$

Perhaps the most popular automated market making framework is the *cost-function based* market maker [Chen and Pennock 2007; Abernethy et al. 2011, 2013], described explicitly in Algorithm 1. The basic setup is as follows. Let Ω denote a (potentially large or even infinite) set of mutually exclusive and exhaustive states of the world. The market maker selects a set of k possibly-related contracts to offer, e.g., a contract worth $1 if and only if candidate X wins the primary election and another worth $1 if candidate X wins the general election. Formally, these contracts are specified by a payoff function $\boldsymbol{\phi} : \Omega \to \mathbb{R}^k$, where $\phi_i(\omega)$ is the payoff of contract i in the event of outcome ω. The market maker prices these contracts using a convex potential function $C : \mathbb{R}^k \to \mathbb{R}$ called the *cost function*. Formally, the market maker maintains a state vector $\mathbf{q} \in \mathbb{R}^k$, and when a trader wants to purchase a "bundle" of contracts denoted by $\mathbf{r} \in \mathbb{R}^k$, where r_i denotes the quantity of contract i, the trader is charged $C(\mathbf{q}+\mathbf{r})-C(\mathbf{q})$. The state vector is then updated to $\mathbf{q} + \mathbf{r}$, and when the outcome ω is revealed this particular trader is paid $\boldsymbol{\phi}(\omega) \cdot \mathbf{r}$. In this framework, prices depend only on the history of trade, and only through the state vector $\mathbf{q}$.

The cost-function framework is mathematically quite elegant, relating a number of natural concepts in convex analysis to concepts in mechanism design. Furthermore, it is possible to design markets in this framework that satisfy a variety of nice properties such as bounded loss for the market maker and no arbitrage. However, there are several limitations of this framework that limit its value in both theory and practice:

— The bid-ask spread is fixed at 0, and as a result, the market maker cannot take advantage of disagreements between traders in order to obtain profit guarantees, either deterministically or in expectation.
— The liquidity provided is effectively constant. The market maker does not adapt the total number of shares made available at different prices in response to trading volume. This is in contrast to typical financial markets in which an increase in transactions could incentivize more liquidity providers to enter the market.

Prior work has studied the design of market makers that overcome these limitations. Othman and Sandholm [2011, 2012] and Li and Vaughan [2013] proposed various market makers with adaptive liquidity in which the market maker is capable of making a

Class of VPM	WCL	ARB	II	L	SS	Complex
Cost-function market *[Abernethy et al. 2011, 2013]*	✔	✔	✔	✘	✔	✔
Profit-charging market *[Othman and Sandholm 2012]*	✔	✔	✘	✔	✔	✘
Buy-only market *[Li and Vaughan 2013]*	✔	✔	✔	✔	✘	✘
Perspective market	✔	✔	✘	✔	✔	✔

Table I: Desiderata satisfied by various proposed market mechanisms: (WCL) The market satisfies a worst-case loss bound; (ARB) the market prevents arbitrage opportunities; (II) the market possesses the "information incorporation" property; (L) liquidity increases with the volume of trade; (SS) the market has an asymptotically-vanishing bid-ask spread. The final column tracks whether the market can handle combinatorial settings and other complex outcome spaces.

profit. However, these markets obtained these nice features at the expense of others. All are limited to the *complete market setting* in which each outcome is associated with a single contract with a binary payoff (i.e., $\phi_i(\omega) = 1$ if $\omega = i$ and 0 otherwise). Furthermore, the buy-only markets of Li and Vaughan [2013] have the undesirable feature of a growing bid-ask spread, and we show that the profit-charging markets of Othman and Sandholm [2012] do not satisfy information incorporation, the natural property that the price of a contract never decreases as traders purchase more of that contract. Table I summarizes the properties attained by each of these market makers.

In this paper we tell a story with two major subplots. First, we investigate the goal of designing a market with five desirable properties: bounded worst-case loss for the market maker, monotonically increasing liquidity, an asymptotically shrinking bid-ask spread, lack of arbitrage, and information incorporation. We consider a very general framework for market making, in which the cost of a bundle may depend *arbitrarily* on the sequence of bundles purchased so far, and we show that it is *impossible* for such a market to simultaneously possess all five properties. This explains why sacrifices were necessary to achieve adaptive liquidity in previous work.

Second, we introduce a novel market making framework, the *volume-parameterized market* (VPM) framework, which generalizes cost-function markets, allowing the potential function C to depend not only on the market state $\mathbf{q}$ but also on a real-valued measure of the volume of trade in the market. We show that the VPM framework encompasses the other markets shown in Table I; cost-function markets, profit-charging markets [Othman and Sandholm 2012], and buy-only markets [Li and Vaughan 2013] are all special cases. We develop a set of tools that can be used to reason about the properties of markets in this framework. We then go on to introduce a new market, the *perspective market*, a specific VPM that satisfies bounded loss, no arbitrage, increasing liquidity, and shrinking spread. While the impossibility result tells us that we should not expect the perspective market to satisfy information incorporation, we show that in some cases it can satisfy a relaxation of this property, which we call *center-price information incorporation*. Perspective markets are defined not only for complete markets but for arbitrarily complex contract spaces, and are the first markets designed for complex contracts that satisfy these properties.

Tools from convex analysis. Throughout the paper we make use of several tools and definitions from convex analysis which we review here. A set $X \subseteq \mathbb{R}^n$ is *convex* if for all $\mathbf{x}, \mathbf{x}' \in X$ and all $\alpha \in [0, 1]$, $\alpha\mathbf{x} + (1-\alpha)\mathbf{x}' \in X$. The *convex hull* of a set X, denoted $\mathrm{ConvHull}(X)$, is the intersection of all convex sets containing X. The *epigraph* of a function $f : \mathbb{R}^n \to (-\infty, \infty]$ is the set $\{(\mathbf{x}, v) \in \mathbb{R}^n \times \mathbb{R} : v \geq f(\mathbf{x})\}$, and a function

$f : \mathbb{R}^n \to (-\infty, \infty]$ is said to be *convex* if its epigraph is a convex set. A *subgradient* to a function $f : \mathbb{R}^n \to (-\infty, \infty]$ at a point $\mathbf{x}$ is a vector $\mathbf{v} \in \mathbb{R}^n$ such that for all $\mathbf{y} \in \mathbb{R}^n$, $f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{v} \cdot (\mathbf{y} - \mathbf{x})$. We denote the set of subgradients at $\mathbf{x}$ as $\partial f(\mathbf{x})$. Finally, the *convex conjugate* of a function f is a function on $\mathbb{R}^n$ defined by $f^*(\mathbf{v}) = \sup_{\mathbf{x} \in \mathbb{R}^n} [\mathbf{x} \cdot \mathbf{v} - f(\mathbf{x})]$. See Rockafellar [1997] for more details.

2. A GENERAL MODEL AND IMPOSSIBILITY

We begin with a very general framework for market making, where the cost of a bundle may depend arbitrarily on the sequence of bundles purchased so far. This is essentially as general as it is possible to get while allowing the market to depend only on internal information; it does, however, ignore *external* variables such as time, the state of external markets, and the identity of individual traders. Our goal is to determine what fundamental frictions, if any, exist between various desirable market properties. We show that, even in this extremely general framework, such frictions do exist and indeed lead to an impossibility theorem.

2.1. The model

Let Ω denote a set of mutually exclusive and exhaustive states of the world or *outcomes*. Our market will sell "shares" in various securities whose payoffs will be contingent upon the future outcome. We will use the term *contract bundle* to refer to a vector $\mathbf{r} \in \mathbb{R}^k$ that describes the (possibly fractional) number of shares of each of k different securities. We let $\boldsymbol{\phi} : \Omega \to \mathbb{R}^k$ denote the *payoff function* of the k contracts/securities, with $\boldsymbol{\phi}(\omega)$ denoting a vector of payoff amounts when the outcome is $\omega \in \Omega$. If a bundle $\mathbf{r}$ is purchased, and outcome $\omega \in \Omega$ occurs, the trader receives payoff $\boldsymbol{\phi}(\omega) \cdot \mathbf{r}$. Let $\mathcal{S} = (\mathbb{R}^k)^*$ denote the *history space* of the market, consisting of finite (and possibly empty) sequences of bundles. The markets we consider will be defined using a *cost function* $N : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$, where $N(s; s')$ is the cost of purchasing the sequence $s \in \mathcal{S}$ of bundles given the current history $s' \in \mathcal{S}$. The cost function is required to satisfy $N(\mathbf{r} \oplus s; s') = N(\mathbf{r}; s') + N(s; s' \oplus \mathbf{r})$ for all $\mathbf{r} \in \mathbb{R}^k$ and all $s, s' \in \mathcal{S}$, where the $\oplus$ operator denotes concatenation; that is, the cost of a sequence must be the sum of the costs of each element, updating the state in between. The market procedure is analogous to the cost-function-based market maker framework, and is detailed in Algorithm 2.

ALGORITHM 2: The Generic Market Maker

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Market maker initializes state s ← ∅
for all traders t = 1, ..., T do
    Trader t purchases bundle r_t ∈ ℝ^k
    Trader pays N(r_t; s)
    Market maker updates the state s ← s ⊕ r_t
end for
Outcome ω is revealed and trader t is paid φ(ω) · r_t for t = 1, ..., T
```

We will be interested in several quantities and properties of our market maker, which we now introduce. The first is the *worst-case loss*, which is the most money a market maker could lose in any run of the market. In the following, we will use the notation $\Sigma(s)$ to denote the sum of the bundles in s and $|s|$ to denote the length of s, i.e., the number of bundles in s.

Definition 2.1 (*Worst-case loss*). The *worst-case loss* of a market $(\boldsymbol{\phi}, N)$ is

$$\sup_{s \in \mathcal{S}} \sup_{\omega \in \Omega} \boldsymbol{\phi}(\omega) \cdot \Sigma(s) - N(s; \emptyset). \tag{1}$$

Another crucial notion is that of *arbitrage* — a sequence of trades s following market history s' which guarantees positive profit for the trader.

Definition 2.2 (*Arbitrage*). An *arbitrage* is a pair $s, s' \in \mathcal{S}$ such that

$$\inf_{\omega \in \Omega} \boldsymbol{\phi}(\omega) \cdot \Sigma(s) - N(s; s') > 0. \tag{2}$$

A notion which we will use to define several desiderata of our market maker is that of *volume*, which intuitively measures the total amount of activity in the market since trading began. We define this in a very general way for now, but in Section 3 we will hone in on a particular form of this volume measure.

Definition 2.3 (*Volume*). The function $V : \mathcal{S} \to \mathbb{R}_+$ is a *volume function* if for all $s \in \mathcal{S}$ and $\mathbf{r} \in \mathbb{R}^k$ it satisfies

(1) $V(s \oplus \mathbf{r}) \geq V(s)$
(2) $V(s \oplus s')$ is unbounded in $|s'|$ where $s' = \mathbf{r} \oplus \mathbf{r} \oplus \ldots \oplus \mathbf{r}$ for some $\mathbf{r} \neq \mathbf{0}$
(3) For all $s' \in S$, $V(s \oplus (\alpha \mathbf{r}) \oplus s')$ is unbounded in α for $\mathbf{r} \neq \mathbf{0}$.

Using our notion of volume, we wish to say things like "property X holds as the volume in the market approaches infinity." We formalize this "volume limit" now.

Definition 2.4 (*Volume limit*). Given some volume function $V : \mathcal{S} \to \mathbb{R}_+$, we say that the *volume limit* of some function $f : \mathcal{S} \to \mathbb{R}$ is c, denoted $\lim_{V(s)\to\infty} f(s) = c$, if for all $\epsilon > 0$ there exists $\tau \in \mathbb{R}$ such that for all $s \in \mathcal{S}$ with $V(s) > \tau$, we have $|f(s) - c| < \epsilon$.

We can now state the desiderata we will focus on in this paper, relative to some market $(\boldsymbol{\phi}, N)$ and volume function V.

(1) **Bounded worst-case loss (WCL)**. The worst-case loss of the market is finite.
(2) **No arbitrage (ARB)**. For all $s, s' \in \mathcal{S}$, $\exists\, \omega \in \Omega$ such that $\boldsymbol{\phi}(\omega) \cdot \Sigma(s) \leq N(s; s')$.
(3) **Information incorporation (II)**. For all $s \in \mathcal{S}$ and $\mathbf{r} \in \mathbb{R}^k$, $N(\mathbf{r}; s \oplus \mathbf{r}) \geq N(\mathbf{r}; s)$.
(4) **Increasing liquidity (L)**. For all $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^k$, $\lim_{V(s)\to\infty} |N(\mathbf{r}; s) - N(\mathbf{r}; s \oplus \mathbf{r}')| = 0$.
(5) **Shrinking spread (SS)**. For $\mathbf{r} \in \mathbb{R}^k$, $\lim_{V(s)\to\infty} |N(\mathbf{r}; s) + N(-\mathbf{r}; s)| = 0$.

All of these desiderata have appeared multiple times in the literature, though in some cases our definitions differ slightly. Our definition of WCL, ARB, and II exactly correspond to the standard definitions [Abernethy et al. 2011, 2013], and these concepts are central to the theory of automated market making. The terms *liquidity* and *bid-ask spread* do not enjoy such standard definitions, however. We briefly survey the literature on these concepts now.

The term *liquidity* is used in different ways, but generally quantifies a market's ability to execute a trade of a certain size without changing price very much. Liquidity is closely related to the concept of *market depth*.[1] The term *bid-ask spread* refers to the difference in price on either side of "the book" — the difference between the purchase price and the sell price for a given security.

In the complete market setting, Othman and Sandholm [2012] defined *unlimited market depth* as the property that the price of any fixed-size transaction approaches the marginal bid or ask price, and *vanishing bid-ask spread* as the property that the sum of prices of all securities goes to one (for the compete market setting). In Abernethy et al. [2011], liquidity is defined in terms of the bid-ask spread of trading a minimal allowed bundle of size ϵ. Finally, Li and Vaughan [2013] define *liquidity adaptation*

[1]The distinction is generally that liquidity refers to the speed of a sale, whereas market depth refers to the quantity, but speed and quantity are often one and the same in automated market making models.

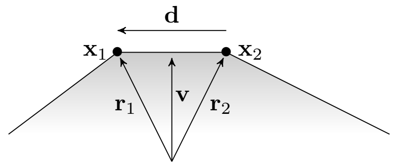

Fig. 1: Construction for the trades in Theorem 2.6.

to mean that the difference in price before and after a certain purchase gets arbitrarily small as volume increases. Our definition of SS is therefore more or less standard, and follows from the intuition above. For L, our definition is very similar to that of Li and Vaughan [2013] except that we use the bundle cost instead of the instantaneous price, as the latter does not always exist in the most general setting (for more about instantaneous prices see Section 3.3).

Note that we have not included profitability as a desideratum, despite it being considered by (and even the central motivation for) other work. The concept of a worst-case loss closely relates to profit, in that a negative WCL translates to guaranteed profit. Of course, one cannot guarantee profit in all situations, but rather when there is sufficient disagreement among the traders (if all traders share the same belief, the market reduces to a zero-sum game between the market maker and the "aggregate trader"). We discuss this disagreement–profit intuition in Section 5.

2.2. Impossibility

Ideally, we would like to design market makers that satisfy all five desiderata. Unfortunately, we show in this section that even given the power to condition arbitrarily on the history of trades, this is impossible for all but the simplest markets. In Section 2.3, we discuss this result at a high level, as well as the implications for our study.

Definition 2.5. We say a market (ϕ, N) is *non-trivial* if ϕ is non-constant; that is, if there exist $\omega_1, \omega_2 \in \Omega$ such that $\phi(\omega_1) \neq \phi(\omega_2)$.

THEOREM 2.6. *No non-trivial market (ϕ, N) with at least two securities ($k \geq 2$) can satisfy all five desiderata (WCL, ARB, II, L, SS), for any choice of volume function V.*

The proof makes use of the following technical lemma that gives us bundles with particular properties. We will use these bundles to construct a sequence of trades which force the market maker to suffer unbounded loss if all other desiderata are satisfied.

LEMMA 2.7. *Let $X := \{\phi(\omega) : \omega \in \Omega\} \subset \mathbb{R}^k$, and define $\sigma(\mathbf{r}) := \max_{\mathbf{x} \in X} \mathbf{r} \cdot \mathbf{x}$ to be the highest payoff possible for bundle $\mathbf{r}$. Then if $|X| > 1$ and $k > 1$, there exist bundles $\mathbf{r}_1, \mathbf{r}_2 \in \mathbb{R}^k$ satisfying the following two properties:*

i. $\sigma(\mathbf{r}_1) + \sigma(\mathbf{r}_2) > \sigma(\mathbf{r}_1 + \mathbf{r}_2)$

ii. *For $i = 1, 2$, we have* $(\text{argmax}_{\mathbf{x} \in X} \mathbf{r}_i \cdot \mathbf{x}) \cap (\text{argmax}_{\mathbf{x} \in X} (\mathbf{r}_1 + \mathbf{r}_2) \cdot \mathbf{x}) \neq \emptyset$.

PROOF. To begin, note that the set $Y = \text{ConvHull}(X) \subset \mathbb{R}^k$ is a convex polytope, $k > 1$, and hence has an edge $[\mathbf{x}_1, \mathbf{x}_2] := \{\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 : \lambda \in [0, 1]\}$ where $\mathbf{x}_1, \mathbf{x}_2 \in X$; here we used the assumption that $|X| > 1$. Moreover, we must have some direction $\mathbf{v} \in \mathbb{R}^k$ exposing this edge, meaning $[\mathbf{x}_1, \mathbf{x}_2] = \text{argmax}_{\mathbf{y} \in Y} \mathbf{v} \cdot \mathbf{y}$.

We first consider the case in which X contains points outside of $[\mathbf{x}_1, \mathbf{x}_2]$, meaning $X \setminus [\mathbf{x}_1, \mathbf{x}_2] \neq \emptyset$. Let $\mathbf{d} := \mathbf{x}_1 - \mathbf{x}_2 \neq 0$, and $c := \sigma(\mathbf{v}) = \mathbf{v} \cdot \mathbf{x}_1 = \mathbf{v} \cdot \mathbf{x}_2$. As $|X|$ is finite, we must have $K_1 := c - \max_{\mathbf{x} \in X \setminus [\mathbf{x}_1, \mathbf{x}_2]} \mathbf{x} \cdot \mathbf{v} > 0$, and $K_2 := \max_{\mathbf{x} \in X} |\mathbf{d} \cdot \mathbf{x}| < \infty$. We also have $K_2 > 0$; to see this note that $\mathbf{d} \cdot \mathbf{x}_1 - \mathbf{d} \cdot \mathbf{x}_2 = \|\mathbf{d}\|^2 > 0$, so at least one of $|\mathbf{d} \cdot \mathbf{x}_1|$ or $|\mathbf{d} \cdot \mathbf{x}_2|$ is strictly positive. Set $\alpha := K_1/(3K_2)$ and define $\mathbf{r}_1 := \mathbf{v} + \alpha \mathbf{d}$, $\mathbf{r}_2 := \mathbf{v} - \alpha \mathbf{d}$, and $\mathbf{r} := \mathbf{r}_1 + \mathbf{r}_2$. See Figure 1 for an illustration.

First some brief calculations. For any $\mathbf{x} \in X$,

$$\mathbf{x} \in [\mathbf{x}_1, \mathbf{x}_2]: \quad \mathbf{r}_i \cdot \mathbf{x} = \mathbf{v} \cdot \mathbf{x} \pm \alpha \mathbf{d} \cdot \mathbf{x} \geq c - \alpha|\mathbf{d} \cdot \mathbf{x}| \geq c - \alpha K_2 = c - K_1/3$$
$$\mathbf{x} \notin [\mathbf{x}_1, \mathbf{x}_2]: \quad \mathbf{r}_i \cdot \mathbf{x} = \mathbf{v} \cdot \mathbf{x} \pm \alpha \mathbf{d} \cdot \mathbf{x} \leq (c - K_1) + \alpha|\mathbf{d} \cdot \mathbf{x}| \leq c - 2K_1/3.$$

Hence, $\mathrm{argmax}_{\mathbf{x} \in X}\, \mathbf{r}_i \cdot \mathbf{x} \subseteq [\mathbf{x}_1, \mathbf{x}_2]$ for $i = 1, 2$. Now as any $\bar{\mathbf{x}} \in [\mathbf{x}_1, \mathbf{x}_2]$ can be written $\bar{\mathbf{x}} = \mathbf{x}_1 - \lambda \mathbf{d}$ for $\lambda \geq 0$, we have $(\mathbf{r}_1 \cdot \mathbf{x}_1 - \mathbf{r}_1 \cdot \bar{\mathbf{x}}) = (\mathbf{v} + \alpha \mathbf{d}) \cdot \lambda \mathbf{d} = \alpha\lambda\|\mathbf{d}\|^2 \geq 0$, so certainly $\mathbf{x}_1 \in \mathrm{argmax}_{\mathbf{x} \in X}\, \mathbf{r}_1 \cdot \mathbf{x}$. Similarly, $\mathbf{x}_2 \in \mathrm{argmax}_{\mathbf{x} \in X}\, \mathbf{r}_2 \cdot \mathbf{x}$, thus establishing property (ii). For property (i), we have $\sigma(\mathbf{r}_1 + \mathbf{r}_2) = \sigma(2\mathbf{v}) = 2c$ and $\sigma(\mathbf{r}_1) + \sigma(\mathbf{r}_2) \geq \mathbf{r}_1 \cdot \mathbf{x}_1 + \mathbf{r}_2 \cdot \mathbf{x}_2 = 2c + \alpha\|\mathbf{d}\|^2 > 2c$.

Finally, we return to the case $[\mathbf{x}_1, \mathbf{x}_2] = Y$. Here we may set α to any positive value and construct the bundles $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}$ in the same way, and the proof still holds. □

We are now ready to prove the impossibility result.

PROOF OF THEOREM 2.6. Let $(\boldsymbol{\phi}, N)$ be a non-trivial market with $k \geq 2$ satisfying ARB, II, L, and SS for volume function V. We will leverage these four properties to construct a sequence of trades with unbounded loss, violating WCL.

Let X and $\sigma(\mathbf{r})$ be defined as in Lemma 2.7. We begin with a claim which bounds cost of a bundle $\mathbf{r}$ by $\sigma(\mathbf{r})$, which corresponds to the highest fixed price one could achieve within the price space $\mathrm{ConvHull}(X)$.

CLAIM 1. If $(\boldsymbol{\phi}, N)$ satisfies (SS, II, ARB), then $N(\mathbf{r}; s) \leq \sigma(\mathbf{r})$ for all $\mathbf{r}, s$.
PROOF. If not, by SS, for $\epsilon = (N(\mathbf{r}; s) - \sigma(\mathbf{r}))/2 > 0$ we have some τ such that $|N(\mathbf{r}; s) + N(-\mathbf{r}; s)| < \epsilon$ for all s with $V(s) > \tau$. Now by definition of a volume function, we know that for some K, the sequence of K copies of $\mathbf{r}$, $s' = \oplus_{i=1}^{K} \mathbf{r}$, satisfies $V(s \oplus s') > \tau$. By repeated applications of II, $N(\mathbf{r}; s \oplus s') \geq N(\mathbf{r}; s)$. Thus, $-N(-\mathbf{r}; s \oplus s') \geq N(\mathbf{r}; s \oplus s') - \epsilon \geq N(\mathbf{r}; s) - \epsilon > \sigma(\mathbf{r})$. Hence, as $\sigma(\mathbf{r}) = -\inf_\omega \boldsymbol{\phi}(\omega) \cdot (-\mathbf{r})$, we have $\inf_\omega \boldsymbol{\phi}(\omega) \cdot (-\mathbf{r}) - N(-\mathbf{r}; s \oplus s') > 0$, violating ARB. □

We can now start building our trades. Non-triviality of the market gives $|X| > 1$, and we have assumed $k > 1$, so the Lemma 2.7 gives us bundles $\mathbf{r}_1$ and $\mathbf{r}_2$ with properties (i) and (ii). Fix $M > 0$, and let $\mathbf{r} := \mathbf{r}_1 + \mathbf{r}_2$ and $\delta := \sigma(\mathbf{r}_1) + \sigma(\mathbf{r}_2) - \sigma(\mathbf{r})$, which is strictly positive by property (i). We now show that if traders buy enough of the combined bundle $\mathbf{r}$, the cost of M copies of either $\mathbf{r}_1$ or $\mathbf{r}_2$ will be bounded away from its maximum price.

CLAIM 2. For sufficiently large K, we have $N(M\mathbf{r}_i; K\mathbf{r}) < M(\sigma(\mathbf{r}_i) - \delta/4)$ for some $i \in \{1, 2\}$.
PROOF. For a contradiction, assume that for all $K_0 > 0$ there is some $K > K_0$ for which $N(M\mathbf{r}_i; K\mathbf{r}) \geq M(\sigma(\mathbf{r}_i) - \delta/4)$ for $i = 1, 2$. Note that by definition of a volume function, $V(K\mathbf{r} \oplus s)$ is unbounded in K for all s. Now let $\epsilon = M\delta/20$ and take K_0 large enough to satisfy the appropriate applications of the volume limit for SS and L to guarantee that for all $K > K_0$, we have

$$N(-M\mathbf{r}_1; K\mathbf{r} \oplus M\mathbf{r}) \leq -N(M\mathbf{r}_1; K\mathbf{r} \oplus M\mathbf{r}) + \epsilon \leq -N(M\mathbf{r}_1; K\mathbf{r}) + 2\epsilon$$
$$N(-M\mathbf{r}_2; K\mathbf{r} \oplus M\mathbf{r} \oplus -M\mathbf{r}_1) \leq \ldots \leq \ldots \leq -N(M\mathbf{r}_2; K\mathbf{r}) + 3\epsilon.$$

Hence, after $K\mathbf{r}$ has already been sold by the market maker, the cost of purchasing $M\mathbf{r}$, $-M\mathbf{r}_1$, and $-M\mathbf{r}_2$ in order can be bounded as follows:

$$\begin{aligned} N\big(M\mathbf{r} \oplus (-M\mathbf{r}_1) \oplus (-M\mathbf{r}_2); K\mathbf{r}\big) &= N(M\mathbf{r}; K\mathbf{r}) + N(-M\mathbf{r}_1; K\mathbf{r} \oplus M\mathbf{r}) + N(-M\mathbf{r}_2; K\mathbf{r} \oplus M\mathbf{r} \oplus -M\mathbf{r}_1) \\ &\leq N(M\mathbf{r}; K\mathbf{r}) - N(M\mathbf{r}_1; K\mathbf{r}) - N(M\mathbf{r}_2; K\mathbf{r}) + 5\epsilon \\ &\leq M\sigma(\mathbf{r}) - M(\sigma(\mathbf{r}_1) - \delta/4) - M(\sigma(\mathbf{r}_2) - \delta/4) + 5\epsilon \\ &= M(\sigma(\mathbf{r}) - \sigma(\mathbf{r}_1) - \sigma(\mathbf{r}_2)) + M\delta/2 + M\delta/4 \;=\; -M\delta/4 < 0, \end{aligned}$$

which violates no arbitrage (ARB). (We applied Claim 1 and the assumption in the second inequality.) Hence, we have shown that for all $K > K_0(M)$, we have $N(M\mathbf{r}_i; K\mathbf{r}) < M(\sigma(\mathbf{r}_i) - \delta/4)$ for some $i \in \{1, 2\}$. □

We now leverage Claim 2 to build our trade: buy $K\mathbf{r}$ and then sell $M\mathbf{r}_i$. We will then use Lemma 2.7 to pick the outcome which gives the trader profit unbounded in M, meaning that we can choose M so that the market maker suffers unbounded loss.

For the i and K guaranteed by Claim 2, and applying Claim 1, we have $N(K\mathbf{r} \oplus M\mathbf{r}_i; \emptyset) = N(K\mathbf{r}; \emptyset) + N(M\mathbf{r}_i; K\mathbf{r}) < K\sigma(\mathbf{r}) + M(\sigma(\mathbf{r}_i) - \delta/4)$. But by Lemma 2.7(ii), we have some ω^* such that $\sigma(\mathbf{r}) = \mathbf{r} \cdot \boldsymbol{\phi}(\omega^*)$ and $\sigma(\mathbf{r}_i) = \mathbf{r}_i \cdot \boldsymbol{\phi}(\omega^*)$. Hence, the worst-case loss of the market maker is

$$\sup_{s \in \mathcal{S}} \sup_{\omega \in \Omega} \boldsymbol{\phi}(\omega) \cdot \Sigma(s) - N(s; \emptyset) \geq \boldsymbol{\phi}(\omega^*) \cdot (K\mathbf{r} + M\mathbf{r}_i) - N(K\mathbf{r} \oplus M\mathbf{r}_i; \emptyset)$$
$$> K\sigma(\mathbf{r}) + M\sigma(\mathbf{r}_i) - K\sigma(\mathbf{r}) - M(\sigma(\mathbf{r}_i) - \delta/4) = M\delta/4.$$

As M was arbitrary and $\delta > 0$ was fixed, the loss is unbounded, violating WCL. □

2.3. Intuition and implications

To understand Theorem 2.6, consider the following two seemingly equivalent markets, A and B, both with outcome space $\Omega = \{\omega_1, \omega_2\}$. In A, a security is offered for each outcome, which pays out \$1 if the outcome occurs and \$0 otherwise, but in B only the security for ω_1 is offered. Formally, $\boldsymbol{\phi}^A(\omega_1) = (1, 0)$ and $\boldsymbol{\phi}^A(\omega_2) = (0, 1)$, while $\phi^B(\omega_1) = 1$ and $\phi^B(\omega_2) = 0$. Note that, as exactly one of ω_1 and ω_2 must occur, a bet for ω_1 is a bet against ω_2. Hence, markets A and B are equivalent in the sense that any bundle $\mathbf{r}^A \in \mathbb{R}^2$ a trader purchases for a cost c^A can be expressed as a bundle $r^B = r_1^A - r_2^A$ with cost $c^B = c^A - r_2^A$ such that the payoffs are equivalent, meaning $\mathbf{r}^A \cdot \boldsymbol{\phi}^A(\omega) - c^A = r^B \phi^B(\omega) - c^B$ for all $\omega \in \Omega$. However, note that Theorem 2.6 applies to A and not B.[2] What is going on here?

In the single security setting B, traders are faced with essentially one choice: buy or sell the security. Market A, though, places distinctions between bundles which get mapped to the same bundle r^B as above. Consider specifically bundles $\mathbf{r}^A$ with $r_1^A = r_2^A$, which get mapped to $r^B = 0$. Such a bundle translates to "simultaneously buy and sell 1 share of the security $\mathbf{1}_{\omega_1}$," so while B interprets this as the null trade $r^B = 0$ and merely leaves the market state unchanged, market A demands that this trade be priced just as any other and that all five desiderata hold while the trade is executed. In particular, from a high level II states that the price should increase, but SS states that it should decrease, since the cost of the trade itself is roughly the bid-ask spread. Thus, we observe friction between the desiderata, whereas in the single security case, such "buy and sell the same security" bundles are simply ignored.

Indeed, one can interpret the edge $[\mathbf{x}_1, \mathbf{x}_2]$ in the proof above as a generalization of this phenomenon. By constructing bundles $\mathbf{r}_1$ and $\mathbf{r}_2$ which add the normal $\mathbf{v}$ to the edge, we are effectively simulating this buy and sell behavior, and hence the price of the combined bundle $\mathbf{v}$ is bound in opposing directions by II and SS. Of course, the other three desiderata are needed to ensure even more bizarre situations do not arise.

To conclude, we note that Theorem 2.6, while relying only on standard desiderata in the literature, may hint that certain desiderata as stated are too strong. The central role of II in the proof above is in the claim: the cost of a bundle cannot leave the price space, since otherwise one could continue purchasing it, and by II and SS, eventually the sell price will also leave the price space, creating an arbitrage. Note however that if

[2]Of course, we have not shown that a market for B *can* satisfy all five desiderata; this scalar case is in general still open, though we show a similar positive result in Section 5 and Proposition 5.4.

the purchase price is outside of the price space, no rational trader would purchase it, as such a purchase is a *guaranteed loss* (by definition of σ). Moreover, it is not clear what "information" should be incorporated by such a trade. Thus, it may be more natural to enforce II solely for potentially rational trades, i.e., those for which the trader would profit for at least one outcome. It is not clear that it is technically tractable to do so however, and even if so, it may be that Theorem 2.6 would still hold.

Rather than restricting II to potentially rational trades, in Section 5.3 we consider a different relaxed notion of II, dubbed *center-price information incorporation (CII)*. CII requires that rather than the cost of a bundle itself, only the center of the bid-ask spread need increase as the bundle is purchased. We give a proof that this can hold with the other 4 desiderata for the single security case (Proposition 5.4) and conjecture that it does so for multiple securities as well. However, even though we recognize CII as a potentially viable alternative to II, we note that relaxing II to CII implies that in some situations, the price of a bundle could *decrease* as it is purchased, which is a wholly unintuitive and arguably problematic property of a market maker.

3. THE VPM FRAMEWORK

The original class of cost-function market makers has the downside that they do not increase the liquidity or guarantee profit to the market maker. Towards achieving such a goal, we considered in Section 2 a highly general model of a market making agent that adjusts the pricing function in response to the entire sequence of prior trades. Unfortunately we showed that five desirable properties are impossible to achieve in tandem even under this generic framework.

We now introduce a market making framework, the *volume-parameterized market (VPM)*, which lies between the cost-function framework (Algorithm 1) and the generic one considered in the previous section (Algorithm 2). The VPM framework is in essence a potential-based market, where the prices are set according to a potential function which tracks a real-valued measure of the market volume v in addition to the total outstanding trade vector $\mathbf{q}$. Despite this generalization, we show in Section 4 that the VPM encompasses many other frameworks considered in the field. In Section 5 we go on to introduce the *perspective market*, a specific VPM market satisfying all the desiderata save information incorporation (II) (see Table I); in light of our impossibility theorem the inclusion of II would have to come at the expense of another property.

3.1. Setting

One of the downsides of the cost-function market making framework is that it does not allow for any kind of "progress" to be achieved by the market maker. It fails in particular on the increasing liquidity goal, and it does not allow the market maker to achieve guaranteed profit. It also lacks incorporation of a bid-ask spread, which is good for price discovery but also precludes the opportunity to financially benefit from the market making activity.

ALGORITHM 3: The Volume-Parameter Market

Market maker initializes share vector $\mathbf{q} \leftarrow \mathbf{0}$ and volume $v \leftarrow 0$
for all traders $t = 1, \ldots, T$ **do**
  Trader t purchases bundle $\mathbf{r}_t \in \mathbb{R}^k$
  Trader pays $N(\mathbf{r}_t; \mathbf{q}, v) := C(\mathbf{q} + \mathbf{r}_t, v + f(\mathbf{r}_t)) - C(\mathbf{q}, v)$
  Market maker updates the state $\mathbf{q} \leftarrow \mathbf{q} + \mathbf{r}_t$
  Market maker updates the volume $v \leftarrow v + f(\mathbf{r}_t)$
end for
Outcome ω is revealed and trader t is paid $\boldsymbol{\phi}(\omega) \cdot \mathbf{r}_t$ for all $t = 1, \ldots, T$

To achieve more flexibility in the hopes of overcoming these drawbacks, we now introduce a market making framework, the *volume-parameterized market (VPM)*, which is described in full detail in Algorithm 3 and Definition 3.2. Much like the market maker of Othman and Sandholm [2012], the VPM builds off of the cost-function framework by introducing an additional parameter, the *volume* $v \in \mathbb{R}_+$ of the market activity thus far. The cost function used for pricing will depend on both the cumulative shares $\mathbf{q}$ and the current volume v and is written as $C(\mathbf{q}, v)$.

To measure the increase in the volume parameter following a trade, we will need the concept of an asymmetric norm. Readers may simply think of f as a norm throughout; the full generality is needed only in Proposition 4.1.

Definition 3.1. A function f is an *asymmetric norm* if it satisfies for all x, y:

— Non-negativity: $f(x) \geq 0$
— Definiteness: $f(x) = f(-x) = 0$ if and only if $x = 0$
— Positive homogeneity: $f(\alpha x) = \alpha f(x)$ for all $\alpha > 0$
— Triangle inequality: $f(x + y) \leq f(x) + f(y)$.

Definition 3.2. A *volume-parameterized market (VPM)* is a tuple $(\boldsymbol{\phi}, C, f)$ with payoff function $\boldsymbol{\phi} : \Omega \to \mathbb{R}^k$, differentiable cost function $C : \mathbb{R}^k \times \mathbb{R}_+ \to \mathbb{R}$, and volume update function $f : \mathbb{R}^k \to \mathbb{R}_+$ which is an asymmetric norm.[3]

Given the generality of our setting in Section 2, it comes as no surprise that we may express a VPM $(\boldsymbol{\phi}, C, f)$ as a $(\boldsymbol{\phi}, N)$ market: for a sequence of trades $s = (\mathbf{r}_1, \ldots, \mathbf{r}_T) \in \mathcal{S}$ define $V(s) = \sum_{t=1}^{T} f(\mathbf{r}_t)$ and $N(\mathbf{r}; s) = C(\Sigma(s) + \mathbf{r}, V(s) + f(\mathbf{r})) - C(\Sigma(s), V(s))$. (Recall that $\Sigma(s) = \sum_{t=1}^{T} \mathbf{r}_t$.) One immediately sees that the cost function N depends on s only through the functions Σ and V, and thus we may overload our $N(\cdots)$ notation by writing $N(\mathbf{r}; \mathbf{q}, v) := C(\mathbf{q}+\mathbf{r}, v+f(\mathbf{r})) - C(\mathbf{q}, v)$, as this expression is valid for all s such that $\Sigma(s) = \mathbf{q}$ and $V(s) = v$. In particular, we have

$$N(\mathbf{r}_1, \ldots, \mathbf{r}_T; \mathbf{q}, v) = C\left(\mathbf{q} + \sum_{t=1}^{T} \mathbf{r}_t,\ v + \sum_{t=1}^{T} f(\mathbf{r}_t)\right) - C(\mathbf{q}, v). \tag{3}$$

Of course, it is easy to see that if the market starts at $(\mathbf{q}, v) = (\mathbf{0}, 0)$, not all states $(\mathbf{q}, v)$ can be achieved. By our requirement that f be an asymmetric norm, in fact, the set of *valid states* which are reachable by the market is precisely the set $\{(\mathbf{q}, v) : f(\mathbf{q}) \leq v\}$.

It is also clear at this stage that the cost-function framework is a special case of the VPM framework: we simply take $C(\mathbf{q}, v) = U(\mathbf{q})$ for some U. We will discuss this special case as well as several others in Section 4.

3.2. Desiderata of the VPM Market

We will again study the five desiderata presented in Section 2.1. For clarity, we now restate them using our more specialized notation.

(1) **Bounded worst-case loss (WCL)**. There exists a constant $L \geq 0$ such that for all sequences $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_T$ and any outcome ω we have $N(\mathbf{r}_1, \ldots, \mathbf{r}_T; \mathbf{0}, 0) - \boldsymbol{\phi}(\omega) \cdot \sum_{t=1}^{T} \mathbf{r}_t \leq L$.

(2) **No arbitrage (ARB)**. For all valid states $(\mathbf{q}, v)$ and for all sequences $\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_T$ there exists $\omega \in \Omega$ such that $\boldsymbol{\phi}(\omega) \cdot \sum_{t=1}^{T} \mathbf{r}_t \leq N(\mathbf{r}_1, \ldots \mathbf{r}_T; \mathbf{q}, v)$.

(3) **Information incorporation (II)**. For all valid states $(\mathbf{q}, v)$ and for all $\mathbf{r}$, $N(\mathbf{r}; \mathbf{q}, v) \leq N(\mathbf{r}; \mathbf{q} + \mathbf{r}, v + f(\mathbf{r}))$.

[3]Note that we sometimes refer to C itself as the VPM when $\boldsymbol{\phi}$ and f are irrelevant or assumed.

(4) **Increasing liquidity (L)**. For all $\mathbf{r}, \mathbf{r}'$ and all $\epsilon > 0$, there exists some τ such that if $v > \tau$ and $(\mathbf{q}, v)$ is valid state, then $|N(\mathbf{r}; \mathbf{q}, v) - N(\mathbf{r}; \mathbf{q} + \mathbf{r}', v + f(\mathbf{r}'))| < \epsilon$.

(5) **Shrinking spread (SS)**. For all $\mathbf{r}$ and all $\epsilon > 0$, there exists some τ such that if $v > \tau$ and $(\mathbf{q}, v)$ is valid state, then $|N(\mathbf{r}; \mathbf{q}, v) + N(-\mathbf{r}; \mathbf{q}, v)| < \epsilon$.

3.3. Useful Tools

The goal is to design a class of VPM market makers that satisfy as many desiderata as possible. To achieve that, we need more insight into the VPM framework and to develop several tools. All proofs in this subsection may be found in Appendix A.[4]

Here and throughout this document, we will use the notation $\nabla_i g(x_1, x_2, \ldots, x_n)$ to be the derivative of g with respect to its ith argument (which may be a vector), evaluated at the point $(x_1, \ldots, x_n)$. For example, $\nabla_1 C(\mathbf{q}, v)$ is the derivative of C with respect to the quantity vector, evaluated at the point $(\mathbf{q}, v)$.

Instantaneous Price. Since the market is smooth and f is directionally differentiable, *instantaneous prices* exist for any valid state $(\mathbf{q}, v)$. The instantaneous price of bundle $\mathbf{r}$ at state $(\mathbf{q}, v)$ is the unit price of purchasing infinitesimal portion of $\mathbf{r}$, denoted as $\delta_{\mathbf{r}} N(\mathbf{q}, v) := \lim_{\epsilon \to +0} N(\epsilon \mathbf{r}; \mathbf{q}, v)/\epsilon$. It can be written in terms of C and f:

$$\delta_r N(\mathbf{q}, v) = \lim_{\epsilon \to +0} \frac{C(\mathbf{q} + \epsilon \mathbf{r}, v + f(\epsilon \mathbf{r})) - C(\mathbf{q}, v)}{\epsilon} = \nabla_1 C(\mathbf{q}, v) \cdot \mathbf{r} + \nabla_2 C(\mathbf{q}, v) \delta_{\mathbf{r}} f(\mathbf{0}), \quad (4)$$

where we use $\delta_{\mathbf{r}} f(\mathbf{x})$ to represent the $\mathbf{r}$-directional derivative of f at $\mathbf{x}$.

No-Trade Belief Set. One could imagine a trader having a belief vector $\mathbf{b}$, where each component b_i is the trader's expectation for the payoff of ith security. Let B denote the set of all valid belief vectors $B := \mathrm{ConvHull}(\boldsymbol{\phi}(\Omega))$, then for any $\mathbf{b} \in B$, there must exist some distribution $\mathbf{p}$ over the outcome space Ω such that $\mathbf{b} = E_{\omega \sim \mathbf{p}}[\boldsymbol{\phi}(\omega)]$.

The no-trade belief set (NTBS) at valid state $(\mathbf{q}, v)$ is the set of all belief vectors $\mathbf{b} \in B$ such that a *risk neutral, myopic* trader with beliefs $\mathbf{b}$ has no incentive to trade. Concretely, we define the NTBS as the set of all belief vectors according to which the expected payoff of any bundle is no more than its instantaneous price:

$$\mathrm{NTBS}(\mathbf{q}, v) = \{\mathbf{b} \in B \mid \forall \mathbf{r}, \delta_{\mathbf{r}} N(\mathbf{q}, v) \geq \mathbf{b} \cdot \mathbf{r}\}.$$

Perhaps surprisingly, we can characterize the NTBS as the subgradient of f, after scaling and shifting.

LEMMA 3.3. *For any valid state* $(\mathbf{q}, v)$*,*

$$\mathrm{NTBS}(\mathbf{q}, v) = B \cap \Big(\nabla_1 C(\mathbf{q}, v) + \nabla_2 C(\mathbf{q}, v) \partial f(0)\Big). \quad (5)$$

We shall observe that the NTBS is a generalization of price vector of cost-function market. In fact, if C is a constant with respect to v, then the term $\nabla_1 C(\mathbf{q}, v)$ becomes the price vector of a cost-function market and the term $\nabla_2 C(\mathbf{q}, v)\partial f(0)$ becomes zero.

Purchase Triangle Inequality. It is common in market mechanisms that a trader wishing to buy a bundle $\mathbf{r}$ is better off buying it all at once rather than splitting the purchase into smaller pieces. We dub this property the *purchase triangle inequality*.

Definition 3.4. A VPM $(\boldsymbol{\phi}, C, f)$ satisfies the *purchase triangle inequality* if for all bundles $\mathbf{r}, \mathbf{r}'$ and all valid states $(\mathbf{q}, v)$, $N(\mathbf{r}, \mathbf{r}'; \mathbf{q}, v) \geq N(\mathbf{r} + \mathbf{r}'; \mathbf{q}, v)$.

The purchase triangle inequality leads to a much simpler analysis of properties such as worst-case loss or no arbitrage, as we need only consider single trades rather than

[4]The appendix can be found in the full version of this paper, available on the authors' websites.

arbitrary sequences. Under minor conditions, a VPM inherits this useful property from the fact that f satisfies the triangle inequality as an asymmetric norm.

LEMMA 3.5. *A VPM (ϕ, C, f) satisfies the purchase triangle inequality whenever C is increasing in v.*

Sufficient Conditions for Desiderata. If we fix v as a constant, then a VPM (ϕ, C, f) yields a cost-function market $U(\mathbf{q}) : \mathbb{R}^k \mapsto \mathbb{R}$ defined by $U(\mathbf{q}) = C(\mathbf{q}, v)$. In general, a cost-function market $U(\mathbf{q})$ satisfies no arbitrage if for all $\mathbf{q}$ and $\mathbf{r}$, $\min_{\omega\in\Omega} \mathbf{r} \cdot \boldsymbol{\phi}(\omega) \leq U(\mathbf{q}+\mathbf{r}) - U(\mathbf{q})$. A useful fact is that if $C(\mathbf{q}, v)$ is increasing in v, the VPM satisfies no arbitrage if the cost-function markets derived by fixing v satisfy no arbitrage.

LEMMA 3.6. *Let VPM (ϕ, C, f) be given. If $C_v : q \mapsto C(q, v)$ satisfies no arbitrage for all v, and if C is nondecreasing in v, then the VPM satisfies no arbitrage.*

Finally, we relate liquidity and shrinking spread to the first-order behavior of C.

LEMMA 3.7. *Let VPM (ϕ, C, f) be given. L and SS are satisfied under the following two conditions:*

(*1*) $\lim_{v\to\infty} \nabla_2 C(\mathbf{q}, v) \to 0$*, uniformly for all* $\mathbf{q}$ *s.t.* $(\mathbf{q}, v)$ *is valid state.*
(*2*) *For any fixed* $\mathbf{r}$*,* $\lim_{v\to\infty} \|\nabla_1 C(\mathbf{q}, v) - \nabla_1 C(\mathbf{q} + \theta\mathbf{r}, v + f(\theta\mathbf{r}))\| \to 0$*, uniformly for all* $\mathbf{q}$ *and* θ *s.t.* $(\mathbf{q}, v)$ *is valid state and* $0 \leq \theta \leq 1$.

4. EXISTING MARKET MODELS AS VPMS

The VPM framework generalizes many previously proposed market makers. In the introduction we presented Table I which compares the three classes of market discussed below, including the subset of desiderata they satisfy, and compares these to the *perspective market* that we introduce in Section 5.

The simplest example of VPM is the cost-function market maker, which is defined by a convex function $U(\mathbf{q})$. This market can be easily viewed as a VPM with $C(\mathbf{q}, v) = U(\mathbf{q})$, ignoring the volume information. The real power of a VPM, however, is the ability to add market properties of a progressive nature, such as increasing liquidity and shrinking spread. It is not surprising that several previous attempts to add such properties can be viewed as special cases of the VPM framework as well.

Both of the frameworks discussed in this section were defined only for *complete markets* in which $\phi_i(\omega) = 1$ if $\omega = i$ and 0 otherwise.

4.1. Profit-Charging Market Maker

The profit-charging market maker proposed by Othman and Sandholm [2012], which builds on the constant-utility market maker [Chen and Pennock 2007], can be viewed as a special case of the VPM framework. It consists of a utility function u, a liquidity function α, a profit function g, a discrete distribution $\mathbf{p}$ over outcomes and a fixed initial liquidity x^0, where u, α and g are all twice differentiable, u is strictly increasing, and α and g are non-decreasing.[5]

In addition, there is an internal scalar s that acts the same way as the volume parameter v in VPM framework. In particular, $s = \sum_{t=1}^{T} f(\mathbf{r}_t)$ where $(\mathbf{r}_1, \ldots, \mathbf{r}_T)$ is the previous trading history and f is a norm.[6] An s-parameterized cost function $\overline{C}$ is solved implicitly using the constant-utility framework: [7] $\sum_{i=1}^{n} p_i \cdot u(\overline{C}(\mathbf{q}, s) - q_i) =$

[5]In Othman and Sandholm [2012], the liquidity function is denoted by f; we use α to avoid confusion.
[6]They consider a slightly more general distance function $d(\mathbf{q}, \mathbf{q}{+}\mathbf{r})$, which becomes a volume update function if d only depends on the difference between its arguments.
[7]In their paper, $\overline{C}$ is only implicitly parameterized by s; we make it explicit for clarity.

$u(x^0 + \alpha(s))$. Finally, the profit function is added and the final cost function becomes $C(\mathbf{q}, s) := \overline{C}(\mathbf{q}, s) + g(s)$. It is straightforward to verify that C satisfies Definition 3.2 with s playing the role of v.

Othman and Sandholm [2012] show that the market satisfies WCL, L, and SS. Using the fact that constant-utility market makers have no arbitrage [Chen and Pennock 2007], we can easily apply Lemma 3.6 to establish ARB for this market as well. Finally, the impossibility result implies that the market must fail to satisfy II.

4.2. Buy-Only Market Maker

Li and Vaughan [2013] proposed a class of market makers for complete markets that have *adaptive liquidity* (similar to increasing liquidity in our discussion) and can guarantee a profit. Their market makers are potential-based, meaning the cost of bundle $\mathbf{r}$ at market state $\mathbf{q}$ is $U(\mathbf{q} + \mathbf{r}) - U(\mathbf{q})$, but with the added restriction that only buying is allowed, i.e., bundles are restricted to $\mathbf{r} \in \mathbb{R}^k_+$. A trader can only "sell" securities by buying the complement ones. For example, if there are 3 securities corresponding to 3 mutually exclusive events in the outcome space, then a trader who would like to sell the bundle $(1, 0, 0)$ (i.e., purchase the bundle $(-1, 0, 0)$), must instead make the equivalent purchase of the complement bundle $(0, 1, 1)$.

As a consequence of the buy-only restriction, each component of the cumulative share vector $\mathbf{q}$ is monotonically increasing. The volume of the market is thus tracked by $\mathbf{q}$ itself, even if no volume parameter is explicitly recorded as is done in VPM framework. This implicit idea of volume is the reason why buy-only market makers can have properties such as increasing liquidity. As we shall show below, if we explicitly write out the volume parameter, then the buy-only market makers falls into VPM framework. We go into further detail in Appendix B.

We wish to show that the buy-only market maker is equivalent to a VPM market in a particular strong sense. Intuitively, we would like to say that T traders who sequentially purchase a sequence of bundles $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_T$ in the VPM market would receive the same net payoffs respectively as T traders who sequentially purchased the same sequence of bundles in the buy-only market. We cannot make this statement, however, as any bundle $\mathbf{r}_t$ with negative entries is invalid in the buy-only market. Instead, we will need to map negative bundles to positive ones via some map ρ, which ensures that $\rho(\mathbf{r})$ has only positive entries. Of course, this transformation ρ is in some sense implicit in the buy-only market itself — if a trader wants to sell a security, she must instead rephrase it as a purchase of other securities.

We now define a particular VPM in terms of a given a buy-only market U. Let

$$C(\mathbf{q}, v) := U\big(\mathbf{q} + w(\mathbf{q}, v)\mathbf{1}\big) - w(\mathbf{q}, v)\ , \qquad f(\mathbf{r}) = \sum_{i=1}^{n} r_i + 2n \operatorname{maxneg}(\mathbf{r})\ , \tag{6}$$

where $w(\mathbf{q}, v) = (v - \sum_{i=1}^n q_i)/(2n)$, and $\operatorname{maxneg}(\mathbf{r}) := \max_i(-r_i)_+$ is the magnitude of the most negative entry of a bundle $\mathbf{r}$ (and 0 if all entries are nonnegative). One can easily verify that f is an asymmetric norm. Finally, we set our bundle map $\rho(\mathbf{r}) := \mathbf{r} + \operatorname{maxneg}(\mathbf{r})\mathbf{1}$, which adds the positive smallest amount of the $\mathbf{1}$ bundle needed to make $\mathbf{r}$ have nonnegative entries.

Writing $C(\mathbf{q}, v)$ in terms of U as above hints at a natural correspondence between states $(\mathbf{q}, v)$ in the VPM market and states $\sigma(\mathbf{q}, v) := \mathbf{q} + \frac{1}{2n}\left(v - \sum_{i=1}^n q_i\right)\mathbf{1}$ in the buy-only market. Noting that $\sigma(\mathbf{0}, 0) = \mathbf{0}$, the equivalence between markets follows immediately from the following proposition, which is proved in Appendix B.

PROPOSITION 4.1. *Consider any buy-only market U, and the VPM market defined as in eq.* (6). *For any valid pair $(\mathbf{q}, v)$ and any outcome $\omega \in \Omega$, purchasing* $\mathbf{r}$ *in the VPM* *market when the current state is* $(\mathbf{q}, v)$ *yields the same net payoff as purchasing* $\rho(\mathbf{r})$ *in the buy-only market when the current state is* $\sigma(\mathbf{q}, v)$*. The buy-only market state after this purchase is made is* $\sigma(\mathbf{q}+\mathbf{r}, v+f(\mathbf{r}))$.

5. A NEW MARKET MAKER

In this section, we describe a new class of VPM called the *perspective market*, which we show satisfies 4 out of 5 desiderata (WCL, ARB, L and SS). From Theorem 2.6, we know that such a market must give up II. In light of this, in Section 5.3 we introduce a relaxed notion of II called *center-price information incorporation (CII)*, and conjecture that CII is also satisfied by the perspective market.

Note that it is possible from Table I that these same four desiderata are satisfied by the profit-charging market [Othman and Sandholm 2012], and moreover a different set of four (WCL, ARB, II, L) are satisfied by the buy-only market [Li and Vaughan 2013]. However, as mentioned above, both of these market makers are defined solely for the complete market setting. In contrast, the perspective market inherits the VPM flexibility of allowing for general functions $\boldsymbol{\phi}$, called the *complex market* setting.[8] The perspective market is the first such model to have all of these properties.

5.1. Definition of the Perspective Market

Our goal is to modify a cost-function market to have increasing liquidity. We start with a cost-function market with cost function U defined via its convex conjugate R [Abernethy et al. 2013], that is, $U(\mathbf{q}) = R^*(\mathbf{q}) = \sup_{\mathbf{p}\in\Pi} \mathbf{p}\cdot\mathbf{q} - R(\mathbf{p})$, where $\Pi := \mathrm{ConvHull}(\boldsymbol{\phi}(\Omega)) \subseteq \mathbb{R}^k$. It is known that U has constant liquidity. We inject liquidity by scaling the conjugate function R. Let $\overline{C}(\mathbf{q}, v) = \sup_{\mathbf{p}\in\Pi} \mathbf{p}\cdot\mathbf{q} - \alpha(v)R(\mathbf{p})$, where $\alpha(v)$ is an increasing function called *liquidity function*. As volume v grows R becomes curvier, $\overline{C}$ becomes flatter, and the market enjoys higher liquidity. By simple calculation, $\overline{C}(\mathbf{q}, v) = \alpha(v)U(\mathbf{q}/\alpha(v))$. This transformation from $U(\mathbf{q})$ to $\overline{C}(\mathbf{q}, v)$ is what is commonly known as a *perspective transformation* in convex analysis.

The liquidity function $\alpha(v)$ needs to be chosen with care for two reasons. First, we would like the properties of the standard path independent market, such as no arbitrage and bounded worst-case loss, to be preserved. Second, the introduction of v brings in a bid-ask spread, or NTBS (see Section 3.3), that we want to shrink.

The perspective market uses a cost function based on $\overline{C}$ with an additional additive term $g(v)$. As in Othman and Sandholm [2012], we use this additive term to ensure no arbitrage and potentially generate profit. We now define our new market maker.

Definition 5.1. A *perspective market* is a VPM $(\boldsymbol{\phi}, C, f)$ with cost function defined by

$$C(\mathbf{q}, v) = \alpha(v)U(\mathbf{q}/\alpha(v)) + g(v) \tag{7}$$

for some liquidity function α and profit function g, and cost potential U.

Some regularity assumptions are needed on R, U, α and g, which we give in the following subsection. We now state the main result of this section.

THEOREM 5.2. *A perspective market satisfying regularity conditions (1–5) below for some particular constants* M, S, *and* B, *satisfies ARB, WCL, L and SS when the following 6 conditions hold:*

(1) $\lim_{v\to\infty} \alpha'(v) = 0$*; (2)* $\lim_{v\to\infty} g'(v) = 0$*; (3)* $\lim_{v\to\infty} \alpha(v) = \infty$*; (4)* $\lim_{v\to\infty} v\alpha'(v)/\alpha(v)^2 = 0$*; (5)* $\forall v \geq 0,\ g(v) \geq M\alpha(v)$*; (6)* $\forall v \geq 0,\ g'(v) \geq M\alpha'(v)$.

[8] See Appendix B for thoughts about how the buy-only market might extend to the complex market setting as well.

It is easy to verify that if we choose $\alpha(v) = v^c$ $(0 < c < 1)$ or $\log(v+1)$, and $g(v) = M\alpha(v)$, then the conditions of the theorem all hold. For a concrete choice of $\boldsymbol{\phi}$ and U that satisfy the regularity conditions, one can take $\Omega = \{0,1\}^2$, $\boldsymbol{\phi}(\omega)_i = \omega_i$, and $U(\mathbf{q}) = \log(1+e^{q_1}) + \log(1+e^{q_2})$. We prove Theorem 5.2 in the following subsection, combining equations (8) through (12).

5.2. Proof of Theorem 5.2

We begin by stating the regularity conditions we will need.

(1) R is a pseudo-barrier,[9] bounded by a constant M on Π.
(2) R is $1/S$ strongly convex for some constant S, and is twice-differentiable. Then U is also twice-differentiable and $\nabla^2 U$ is bounded above by S.[10]
(3) R is closed, and therefore $R = R^{**} = U^*$.[11]
(4) Π is bounded by a constant B, i.e., $\pi \in \Pi \implies \|\pi\| \le B$.
(5) $\alpha(v)$ is a positive increasing function on $[0,\infty)$, $g(0) = 0$, and $\alpha(v)$ and $g(v)$ are continuously differentiable.

Also notice that since f is a norm, by the equivalence of norms on finite dimensional Banach space, there is some largest real number $K > 0$, such that $f(\mathbf{q}) \ge K\|\mathbf{q}\|$ for all $\mathbf{q} \in \mathbb{R}^k$. In the following, we establish sufficient conditions on $\alpha(v)$ and $g(v)$ that guarantee all 4 desiderata.

Increasing Liquidity (L) and Shrinking Spread (SS)

To prove increasing liquidity and shrinking spread, we want establish the two conditions in Lemma 3.7. First,

$$\begin{aligned}
\nabla_2 C(\mathbf{q}, v) &= \alpha'(v) U\left(\tfrac{\mathbf{q}}{\alpha(v)}\right) + \alpha(v)\left\langle \nabla U\left(\tfrac{\mathbf{q}}{\alpha(v)}\right), \tfrac{-\alpha'(v)\mathbf{q}}{\alpha(v)^2}\right\rangle + g'(v) \\
&= \alpha'(v)\left\{\left\langle \nabla U\left(\tfrac{\mathbf{q}}{\alpha(v)}\right), \tfrac{\mathbf{q}}{\alpha(v)}\right\rangle - R\left(\nabla U\left(\tfrac{\mathbf{q}}{\alpha(v)}\right)\right)\right\} - \alpha'(v)\left\langle \nabla U\left(\tfrac{\mathbf{q}}{\alpha(v)}\right), \tfrac{\mathbf{q}}{\alpha(v)}\right\rangle + g'(v) \\
&= -\alpha'(v) R\left(\nabla U\left(\tfrac{\mathbf{q}}{\alpha(v)}\right)\right) + g'(v),
\end{aligned}$$

therefore $|\nabla_2 C(\mathbf{q}, v)| \le \alpha'(v) M + |g'(v)|$. The second equality follows from the fact that $R = U^*$ and Theorem 23.5 of Rockafellar [1997]. Therefore, condition (1) in Lemma 3.7 is satisfied if

$$\lim_{v\to\infty} \alpha'(v) = \lim_{v\to\infty} g'(v) = 0. \tag{8}$$

On the other hand, for fixed $\mathbf{r}$ and $0 \le \theta \le 1$,

$$\begin{aligned}
&\left\|\nabla_1 C(\mathbf{q}, v) - \nabla_1 C(\mathbf{q}+\theta\mathbf{r}, v + f(\theta\mathbf{r}))\right\| = \left\|\nabla U\left(\tfrac{\mathbf{q}}{\alpha(v)}\right) - \nabla U\left(\tfrac{\mathbf{q}+\theta\mathbf{r}}{\alpha(v+\theta f(\mathbf{r}))}\right)\right\| \\
&\le S\left\|\tfrac{\mathbf{q}}{\alpha(v)} - \tfrac{\mathbf{q}+\theta\mathbf{r}}{\alpha(v+\theta f(\mathbf{r}))}\right\| = S\left\|\tfrac{\mathbf{q}}{\alpha(v)} - \tfrac{\mathbf{q}}{\alpha(v+\theta f(\mathbf{r}))} + \tfrac{\mathbf{q}}{\alpha(v+\theta f(\mathbf{r}))} - \tfrac{\mathbf{q}+\theta\mathbf{r}}{\alpha(v+\theta f(\mathbf{r}))}\right\| \\
&\le S\|\mathbf{q}\|\left(\tfrac{1}{\alpha(v)} - \tfrac{1}{\alpha(v+\theta f(\mathbf{r}))}\right) + S\tfrac{\|\mathbf{r}\|}{\alpha(v+\theta f(\mathbf{r}))}.
\end{aligned}$$

The first inequality follows from the Mean Value Theorem and the fact that $\nabla^2 U$ is bounded by S.

[9]We borrow the term pseudo-barrier from Abernethy et al. [2013] to denote a bounded function whose derivative (in terms of magnitude) is a normal barrier function.
[10]This is not restrictive, as any strictly convex function is strongly convex on compact set.
[11]This is not restrictive either as we can always take the closure of R, which has no effect on U.

The second term of the final expression goes to zero as long as

$$\lim_{v\to\infty} \alpha(v) = \infty. \tag{9}$$

For the first term, applying the Mean Value Theorem again and $v \geq f(\mathbf{q}) \geq K\|\mathbf{q}\|$ gives

$$S\|\mathbf{q}\| \left(\tfrac{1}{\alpha(v)} - \tfrac{1}{\alpha(v+\theta f(\mathbf{r}))}\right) = S\theta f(\mathbf{r})\|\mathbf{q}\| \frac{\alpha'(\eta)}{\alpha(\eta)^2} \leq \frac{Sf(\mathbf{r})}{K} \cdot \frac{v\alpha'(\eta)}{\alpha(\eta)^2} \leq \frac{Sf(\mathbf{r})}{K} \cdot \frac{\eta\alpha'(\eta)}{\alpha(\eta)^2},$$

where $v \leq \eta \leq v + \theta f(\mathbf{r})$. Therefore, condition (2) of Lemma 3.7 is satisfied if both Equation (9) and the following equation hold:

$$\lim_{v\to\infty} \frac{v\alpha'(v)}{\alpha(v)^2} = 0. \tag{10}$$

No Arbitrage (ARB)

Since the cost-function market U has no arbitrage and scaling its conjugate preserves this property [Abernethy et al. 2013], the market defined by $C(\cdot, v)$ has no arbitrage for any v. From the previous derivation, we have

$$\nabla_2 C(\mathbf{q}, v) = -\alpha'(v) R\left(\nabla U\left(\tfrac{\mathbf{q}}{\alpha(v)}\right)\right) + g'(v) \geq -\alpha'(v)M + g'(v),$$

hence by Lemma 3.6, the following condition is sufficient for no arbitrage:

$$\forall v \geq 0, \quad g'(v) \geq M\alpha'(v). \tag{11}$$

Bounded Worst-case Loss (WCL)

Let $L(v)$ be the worst-case loss when the market ends up with volume v. Then

$$\begin{aligned}
L(v) &= \max_{\omega\in\Omega} \sup_{\mathbf{q}:(\mathbf{q},v)\text{ valid}} \mathbf{q}\cdot\boldsymbol{\phi}(\omega) - C(\mathbf{q}, v) + C(\mathbf{0}, 0) \leq \max_{\omega\in\Omega} \sup_{\mathbf{q}} \mathbf{q}\cdot\boldsymbol{\phi}(\omega) - C(\mathbf{q}, v) + C(\mathbf{0}, 0) \\
&= \max_{\omega\in\Omega} \sup_{\mathbf{q}} \mathbf{q}\cdot\boldsymbol{\phi}(\omega) - \alpha(v) U\left(\alpha(v)^{-1}\mathbf{q}\right) + \alpha(0)U(\mathbf{0}) - g(v) \\
&= \max_{\omega\in\Omega} \sup_{\mathbf{q}} \alpha(v)\left(\alpha(v)^{-1}\mathbf{q}\cdot\boldsymbol{\phi}(\omega) - U\left(\alpha(v)^{-1}\mathbf{q}\right) + U(\mathbf{0})\right) + (\alpha(0) - \alpha(v))\, U(\mathbf{0}) - g(v) \\
&= \max_{\omega\in\Omega} \sup_{\mathbf{q}} \alpha(v)\left(\mathbf{q}\cdot\boldsymbol{\phi}(\omega) - U(\mathbf{q}) + U(\mathbf{0})\right) + (\alpha(0) - \alpha(v))\, U(\mathbf{0}) - g(v).
\end{aligned}$$

Since R is bounded on Π, Theorem 4.2 of Abernethy et al. [2013] implies that the path independent market defined by U has worst-case loss bounded by $\sup_{\mathbf{p}\in\Pi} R(\mathbf{p}) - \inf_{\mathbf{p}\in\Pi} R(\mathbf{p}) = \sup_{\mathbf{p}\in\Pi} R(\mathbf{p}) + U(\mathbf{0})$. Therefore

$$\max_{\omega\in\Omega} \sup_{\mathbf{q}} \mathbf{q}\cdot\boldsymbol{\phi}(\omega) - U(\mathbf{q}) + U(\mathbf{0}) \leq \sup_{\mathbf{p}\in\Pi} R(\mathbf{p}) + U(\mathbf{0}),$$

$$L(v) \leq \alpha(v) \sup_{\mathbf{p}\in\Pi} R(\mathbf{p}) + \alpha(0)U(\mathbf{0}) - g(v) \leq \alpha(v)M + \alpha(0)U(\mathbf{0}) - g(v).$$

The market is guaranteed to have bounded loss if

$$\forall v \geq 0, \quad g(v) \geq \alpha(v)M. \tag{12}$$

In fact, we can see now that profit can be generated by setting $g(v)$ even larger, such as $\alpha(v)(M + 1)$. (This idea of generating profit via an additive function was also pointed out in Othman and Sandholm [2012].) Note however that care must be taken when setting g, as for information elicitation we do not want zero loss or guaranteed profit in *every* situation, or perhaps even for every v. As we discussed in Section 2.1, traders will simply refuse to participate if they have a guaranteed loss.

It may be possible, however, to guarantee profit when there is sufficient disagreement in the market, a traditional case where market making is profitable in finance.

In terms of our market state $(\mathbf{q}, v)$, we can express "disagreement" as $f(\mathbf{q}) < v$, as agreement would mean all trades were in the same direction, giving us $f(\mathbf{q}) = v$. Intuitively, this says that the traders disagree when the history of trades has "ups and downs." In principle, one could set g in such a way as to guarantee profit which is increasing in $v - f(\mathbf{q})$, thereby profiting when traders disagree, but still gathering information (at a loss) when they agree.

5.3. Center-price Information Incorporation

In light of Theorem 2.6, we cannot hope to achieve information incorporation (II) along with the other 4 desiderata. We now motivate an alternative to II, *center-price information incorporation (CII)*, which we conjecture holds for the perspective market.

In a cost-function market U, the instantaneous price of a bundle $\mathbf{r}$ at market state $\mathbf{q}$ can be written as a dot product $\nabla U(\mathbf{q}) \cdot \mathbf{r}$, where $\nabla U(\mathbf{q})$ is the *instantaneous price vector* with components corresponding to the instantaneous prices of each security. In contrast, the instantaneous price for a VPM is computed in eq. (4) as $\nabla_1 C(\mathbf{q}, v) \cdot \mathbf{r} + \nabla_2 C(\mathbf{q}, v) \delta_{\mathbf{r}} f(\mathbf{0})$. The first term is simply the instantaneous price of $\mathbf{r}$ in the cost-function market $C_v := C(\cdot, v)$, a construct we used in Lemma 3.6. The key difference here is the addition of the vanishing second term which controls the bid-ask spread.

Furthermore, since f is a norm, from eq. (5) we see that the NTBS is simply an f-norm ball centered at this *center price (vector)* $\nabla_1 C(\mathbf{q}, v)$. If traders are believed to be (approximately) myopic and risk neutral, the NTBS can be viewed as capturing the "information" in the market state. It is therefore natural to define information incorporation with respect this center price rather than the instantaneous price.

Definition 5.3. A VPM (ϕ, C, f) satisfies Center-price Information Incorporation (CII) if for all $\mathbf{r}$, $\mathbf{q}$, v s.t. $(\mathbf{q}, v)$ is valid state, $\nabla_1 C(\mathbf{q} + \mathbf{r}, v + f(\mathbf{r})) \cdot \mathbf{r} \geq \nabla_1 C(\mathbf{q}, v) \cdot \mathbf{r}$.

We conjecture that CII holds in the perspective market, and now state two results (proved in Appendix C) which provide strong evidence for this conjecture. The first shows that the perspective market satisfies CII in the single security case under some conditions on α. The second reduces the general problem of proving CII for all $\mathbf{r}$ to verifying it for $\mathbf{r}$ near $\mathbf{0}$.

PROPOSITION 5.4. *The perspective market with $f(r) = |r|$ satisfies CII in the single security case if α is concave and $v\alpha'(v)/\alpha(v) \leq 1$ for all $v \geq 0$.*

LEMMA 5.5. *Fix an instantiation (ϕ, C, f) of the perspective market as in Definition 5.1. Suppose for all $\mathbf{r}$ there exists $\tau > 0$ such that for all $\epsilon \leq \tau$, $\nabla_1 C(\mathbf{q} + \epsilon\mathbf{r}, v + f(\epsilon\mathbf{r})) \cdot \epsilon\mathbf{r} \geq \nabla_1 C(\mathbf{q}, v) \cdot \epsilon\mathbf{r}$ for all valid states $(\mathbf{q}, v)$. Then the market satisfies CII.*

CONJECTURE 5.6. *The perspective market satisfies CII in the general case if α is concave and $v\alpha'(v)/\alpha(v) \leq 1$ for all v.*

6. OTHER CONSIDERATIONS

We conclude by discussing two extra properties of interest when designing VPMs, *increasing market depth* and *expressiveness*. The formal statements and proofs from this section may be found in Appendix D and E.

Increasing Market Depth. Liquidity measures the speed of price changes during transactions. When liquidity is high, traders may be able to make money by moving the market prices to reflect the expected payoffs of securities. To quantify this, we define *market depth* as the most a trader could make starting at some market state $(\mathbf{q}, v)$ if he had the power to choose the outcome ω. We say a market has *increasing market depth* if for any $M > 0$, there exists $\tau > 0$ such that whenever $v > \tau$, $\mathrm{MarketDepth}(\mathbf{q}, v) > M$.

This is implies that the market becomes more attractive as volume increases. It can be shown that the sufficient conditions from Lemma 3.7 are sufficient for this as well.

Expressiveness. Abernethy et al. [2013] defined a notion of *expressiveness* for cost-function markets that requires that the market price can be pushed arbitrarily close to any beliefs $\mathbf{b} \in B$. This property is important when interpreting market prices as collective beliefs. In the VPM setting, the market price generalizes to the NTBS, as discussed in Section 3.3. Accordingly, we define expressiveness via two properties:

(1) **Price mobility:** For any valid state $(\mathbf{q}, v)$, any $\mathbf{b} \in B$, and any $\epsilon > 0$, there exists a bundle r such that for some $\mathbf{b}' \in \mathrm{NTBS}(\mathbf{q}+\mathbf{r}, v+f(\mathbf{r}))$, $||\mathbf{b}-\mathbf{b}'|| \leq \epsilon$.

(2) **Bounded information loss:** There exists $\gamma \geq 0$ such that for any valid state $(\mathbf{q}, v)$ and all $\mathbf{b}, \mathbf{b}' \in \mathrm{NTBS}(\mathbf{q}, v)$, $||\mathbf{b}-\mathbf{b}'|| \leq \gamma$.

We say that a market satisfying these properties is γ-*expressive*. The notion of expressiveness in Abernethy et al. [2013] corresponds to this definition with $\gamma = 0$.

Bounded information loss is not hard to satisfy in practice as Lemma 3.3 tells us that we only need to upper bound $\nabla_2 C(\mathbf{q}, v)$. Interestingly, price mobility is implied by bounded worst-case loss.

ACKNOWLEDGMENTS

We thank Abe Othman for helpful discussions.

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APPENDIX

A. PROOF OF RESULTS IN SECTION 3.3

A.1. Proof of Lemma 3.3

We first state and prove the following useful result.

PROPOSITION A.1. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be convex. Then for any $\mathbf{x} \in \mathbb{R}^n$,*

$$\forall \mathbf{r}\, \delta_{\mathbf{r}} h(\mathbf{x}) \geq \mathbf{b} \cdot \mathbf{r} \iff \mathbf{b} \in \partial h(\mathbf{x}).$$

PROOF. We will make use of the fact that $\delta_{\mathbf{r}} h(\mathbf{x}) = \sup_{\mathbf{g} \in \partial h(\mathbf{x})} \mathbf{g} \cdot \mathbf{r}$. Also, by definition, $\mathbf{g} \in \partial h(\mathbf{x})$ is equivalent to $h(\mathbf{x}+\mathbf{r}) - h(\mathbf{x}) \geq \mathbf{g} \cdot \mathbf{r}$ for all $\mathbf{r}$. For the forward direction, for all $\mathbf{r}$ we have $\mathbf{b} \cdot \mathbf{r} \leq \delta_{\mathbf{r}} h(\mathbf{x}) = \sup_{\mathbf{g} \in \partial h(\mathbf{x})} \mathbf{g} \cdot \mathbf{r} \leq h(\mathbf{x}+\mathbf{r}) - h(\mathbf{x})$, so $\mathbf{b} \in \partial h(\mathbf{x})$. Conversely, if $\mathbf{b} \in \partial h(\mathbf{x})$, then clearly $\mathbf{b} \cdot \mathbf{r} \leq \sup_{\mathbf{g} \in \partial h(\mathbf{x})} \mathbf{g} \cdot \mathbf{r} = \delta_{\mathbf{r}} h(\mathbf{x})$. □

PROOF OF LEMMA 3.3. Consider the function $g(\mathbf{r}) = \nabla_1 C(\mathbf{q}, v) \cdot \mathbf{r} + \nabla_2 C(\mathbf{q}, v) f(\mathbf{r})$, which is convex as f is convex. Then by Equation (4) we see that $\delta_{\mathbf{r}} N(\mathbf{q}, v) = \delta_{\mathbf{r}} g(\mathbf{0})$. Now by definition of the NTBS and Proposition A.1 we have

$$\mathrm{NTBS}(\mathbf{q}, v) = \{\mathbf{b} \in B \mid \forall \mathbf{r}, \delta_{\mathbf{r}} g(\mathbf{0}) \geq \mathbf{b} \cdot \mathbf{r}\} = \partial g(\mathbf{0}) \cap B.$$

Expanding out ∂g gives the result. □

A.2. Proof of Lemma 3.5

PROOF. For all valid states $(\mathbf{q}, v)$, all bundles $\mathbf{r}$ and $\mathbf{r}'$,

$$\begin{aligned}
&N(\mathbf{r}; \mathbf{q}, v) + N(\mathbf{r}'; \mathbf{q}+\mathbf{r}, v + f(\mathbf{r})) - N(\mathbf{r}+\mathbf{r}'; \mathbf{q}, v) \\
&= C(\mathbf{q}+\mathbf{r}, v+f(\mathbf{r})) - C(\mathbf{q}, v) + C(\mathbf{q}+\mathbf{r}+\mathbf{r}', v+f(\mathbf{r})+f(\mathbf{r}')) - C(\mathbf{q}+\mathbf{r}, v+f(\mathbf{r})) \\
&\quad - C(\mathbf{q}+\mathbf{r}+\mathbf{r}', v+f(\mathbf{r}+\mathbf{r}')) + C(\mathbf{q}, v) \\
&= C(\mathbf{q}+\mathbf{r}+\mathbf{r}', v+f(\mathbf{r})+f(\mathbf{r}')) - C(\mathbf{q}+\mathbf{r}+\mathbf{r}', v+f(\mathbf{r}+\mathbf{r}')).
\end{aligned}$$

As C is increasing in v, this expression is nonnegative as $f(\mathbf{r}) + f(\mathbf{r}') \geq f(\mathbf{r}+\mathbf{r}')$ by the assumption that f is an asymmetric norm. □

A.3. Proof of Lemma 3.6

PROOF. Fix any valid state $(\mathbf{q}, v)$ and any sequence of purchases $\mathbf{r}_1, \cdots, \mathbf{r}_T$. Let $\mathbf{r} = \sum_{t=1}^T \mathbf{r}_t$, $u = \sum_{t=1}^T f(\mathbf{r}_t) \geq 0$. We now have

$$\begin{aligned}
\min_{\omega \in \Omega} \mathbf{r} \cdot \boldsymbol{\phi}(\omega) &\leq C(\mathbf{q}+\mathbf{r}, v+u) - C(\mathbf{q}, v+u) && (13) \\
&\leq C(\mathbf{q}+\mathbf{r}, v+u) - C(\mathbf{q}, v+u) + C(\mathbf{q}, v+u) - C(\mathbf{q}, v) && (14) \\
&= C(\mathbf{q}+\mathbf{r}, v+u) - C(\mathbf{q}, v) = N(\mathbf{r}_1, \ldots, \mathbf{r}_T; \mathbf{q}, v)
\end{aligned}$$

where (13) follows from no arbitrage of $C(\cdot, v+u)$ and (14) from monotonicity of $C(\mathbf{q}, \cdot)$. □

A.4. Proof of Lemma 3.7

We make use of the following result.

PROPOSITION A.2. *Given conditions of Lemma 3.7, and for any fixed bundle* $\mathbf{r}$,

$$\lim_{v \to \infty} |\delta_{\mathbf{r}} N(\mathbf{q}, v) - N(\mathbf{r}; \mathbf{q}, v)| = 0, \tag{15}$$

uniformly for all $\mathbf{q}$ *s.t.* $(\mathbf{q}, v)$ *is valid state.*

PROOF. By Equation (4),

$$\begin{aligned}
&|\delta_{\mathbf{r}} N(\mathbf{q},v) - \delta_{\mathbf{r}} N(\mathbf{q}+\theta\mathbf{r}, v+f(\theta\mathbf{r}))| \\
=\ & \Big|\nabla_1 C(\mathbf{q},v)\cdot\mathbf{r} + \nabla_2 C(\mathbf{q},v)\delta_{\mathbf{r}} f(0) - \nabla_1 C(\mathbf{q}+\theta\mathbf{r}, v+f(\theta\mathbf{r}))\cdot\mathbf{r} \\
& - \nabla_2 C(\mathbf{q}+\theta\mathbf{r}, v+f(\theta\mathbf{r}))\delta_{\mathbf{r}} f(0)\Big| \\
\le\ & \Big\|\nabla_1 C(\mathbf{q},v) - \nabla_1 C(\mathbf{q}+\theta\mathbf{r}, v+f(\theta\mathbf{r}))\Big\|\cdot\|\mathbf{r}\| \\
& + \Big|\nabla_2 C(\mathbf{q},v) - \nabla_2 C(\mathbf{q}+\theta\mathbf{r}, v+f(\theta\mathbf{r}))\Big|\cdot|\delta_{\mathbf{r}} f(0)|
\end{aligned}$$

Since $\|\mathbf{r}\|$ and $|\delta_{\mathbf{r}} f(0)|$ are constants, by the two conditions of Lemma 3.7, for any $\epsilon > 0$, there is some τ s.t. when $v > \tau$,

$$|\delta_{\mathbf{r}} N(\mathbf{q},v) - \delta_{\mathbf{r}} N(\mathbf{q}+\theta\mathbf{r}, v+f(\theta\mathbf{r}))| < \epsilon. \tag{16}$$

Therefore when $v > \tau$,

$$\begin{aligned}
&|\delta_{\mathbf{r}} N(\mathbf{q},v) - N(\mathbf{r};\mathbf{q},v)| \\
=\ & \left|\delta_{\mathbf{r}} N(\mathbf{q},v) - \int_0^1 \delta_{\mathbf{r}} N(\mathbf{q}+\theta\mathbf{r}, v+f(\theta\mathbf{r}))\mathrm{d}\theta\right| \\
=\ & \left|\int_0^1 \Big(\delta_{\mathbf{r}} N(\mathbf{q},v) - \delta_{\mathbf{r}} N(\mathbf{q}+\theta\mathbf{r}, v+f(\theta\mathbf{r}))\Big)\mathrm{d}\theta\right| \\
\le\ & \int_0^1 \left|\Big(\delta_{\mathbf{r}} N(\mathbf{q},v) - \delta_{\mathbf{r}} N(\mathbf{q}+\theta\mathbf{r}, v+f(\theta\mathbf{r}))\Big)\right|\mathrm{d}\theta \\
\le\ & \int_0^1 \epsilon\,\mathrm{d}\theta = \epsilon
\end{aligned}$$

□

PROOF OF LEMMA 3.7. Similar to how we get Equation (16), we can derive that for all fixed $\mathbf{r}$, $\mathbf{r}'$ and $\epsilon > 0$, there is some τ s.t. when $v > \tau$,

$$|\delta_{\mathbf{r}} N(\mathbf{q},v) - \delta_{\mathbf{r}} N(\mathbf{q}+\mathbf{r}', v+f(\mathbf{r}'))| < \epsilon.$$

Using Proposition A.2, for any $\epsilon > 0$, there is some τ s.t. when $v > \tau$,

$$\begin{aligned}
&|N(\mathbf{r};\mathbf{q},v) - N(\mathbf{r};\mathbf{q}+\mathbf{r}', v+f(\mathbf{r}'))| \\
\le\ & |N(\mathbf{r};\mathbf{q},v) - \delta_{\mathbf{r}} N(\mathbf{q},v)| + |N(\mathbf{r};\mathbf{q}+\mathbf{r}', v+f(\mathbf{r}')) - \delta_{\mathbf{r}} N(\mathbf{q}+\mathbf{r}', v+f(\mathbf{r}'))| \\
& + |\delta_{\mathbf{r}} N(\mathbf{q},v) - \delta_{\mathbf{r}} N(\mathbf{q}+\mathbf{r}', v+f(\mathbf{r}'))| \le 3\epsilon.
\end{aligned}$$

Increasing liquidity is proved.

By Equation (4),

$$\begin{aligned}
&|\delta_{\mathbf{r}} N(\mathbf{q},v) + \delta_{-\mathbf{r}} N(\mathbf{q},v)| \\
=\ & \Big|\nabla_1 C(\mathbf{q},v)\cdot\mathbf{r} + \nabla_2 C(\mathbf{q},v)\delta_{\mathbf{r}} f(0) + \nabla_1 C(\mathbf{q},v)\cdot(-\mathbf{r}) + \nabla_2 C(\mathbf{q},v)\delta_{-\mathbf{r}} f(0)\Big| \\
=\ & \Big|\nabla_2 C(\mathbf{q},v)\delta_{\mathbf{r}} f(0) + \nabla_2 C(\mathbf{q},v)\delta_{-\mathbf{r}} f(0)\Big| = \Big|\nabla_2 C(\mathbf{q},v)\Big|\cdot\Big|\delta_{\mathbf{r}} f(0) + \delta_{-\mathbf{r}} f(0)\Big|
\end{aligned}$$

By condition 1 of Lemma 3.7, for any $\epsilon > 0$, there is some τ s.t. when $v > \tau$,

$$|\delta_{\mathbf{r}} N(\mathbf{q}, v) + \delta_{-\mathbf{r}} N(\mathbf{q}, v)| < \epsilon.$$

Using Proposition A.2 again, for any $\epsilon > 0$, there is some τ s.t. when $v > \tau$,

$$\begin{aligned} & |N(\mathbf{r}; \mathbf{q}, v) - N(-\mathbf{r}; \mathbf{q}, v)| \\ \leq\ & |N(\mathbf{r}; \mathbf{q}, v) - \delta_{\mathbf{r}} N(\mathbf{q}, v)| + |N(-\mathbf{r}; \mathbf{q}, v) - \delta_{-\mathbf{r}} N(\mathbf{q}, v)| + |\delta_{\mathbf{r}} N(\mathbf{q}, v) + \delta_{-\mathbf{r}} N(\mathbf{q}, v)| \leq 3\epsilon. \end{aligned}$$

Shrinking spread is proved. □

B. THE BUY-ONLY MARKET MAKER

We now give the proof of Proposition 4.1. Afterwards, we show how the VPM framework gives a different perspective on the buy-only market, and conjecture that this yields a method for achieving (WCL, ARB, II, L) in the complex market setting.

PROOF OF PROPOSITION 4.1. We start by showing that the new state of the buy-only market after $\rho(\mathbf{r})$ is purchased is $\sigma(\mathbf{q} + \mathbf{r}, v + f(\mathbf{r}))$. First notice that

$$\rho(\mathbf{r}) = \mathbf{r} + \text{maxneg}(\mathbf{r}) = \mathbf{r} + \frac{1}{2n}\left(\sum_{i=1}^{n} r_i + 2n\text{maxneg}(\mathbf{r}) - \sum_{i=1}^{n} r_i\right)\mathbf{1} = \sigma(\mathbf{r}, f(\mathbf{r})).$$

The new market state after the purchase is simply the sum of the old market state and the bundle $\rho(\mathbf{r})$, i.e.,

$$\sigma(\mathbf{q}, v) + \rho(\mathbf{r}) = \sigma(\mathbf{q}, v) + \sigma(\mathbf{r}, f(\mathbf{r})) = \sigma(\mathbf{q} + \mathbf{r}, v + f(\mathbf{r})), \tag{17}$$

where the last equality follows from the linearity of σ.

We now show that the net payoffs are equivalent. Recall that we are in the complete market setting, so $\phi(\omega) \cdot \mathbf{r} = r_i$, where i is the index of the security corresponding to outcome ω. The net payoff of the trader in the VPM market is

$$\begin{aligned} & r_i - C(\mathbf{q} + \mathbf{r}, v + f(\mathbf{r})) + C(\mathbf{q}, v) \\ &= r_i - U\big(\sigma(\mathbf{q} + \mathbf{r}, v + f(\mathbf{r}))\big) + \frac{1}{2n}\left(v + f(\mathbf{r}) - \sum_{i=1}^{n}(q_i + r_i)\right) \\ &\quad + U\left(\sigma(\mathbf{q}, v)\right) - \frac{1}{2n}\left(v - \sum_{i=1}^{n} q_i\right) \\ &= r_i - U\big(\sigma(\mathbf{q} + \mathbf{r}, v + f(\mathbf{r}))\big) + \frac{1}{2n}\left(f(\mathbf{r}) - \sum_{i=1}^{n} r_i\right) + U\left(\sigma(\mathbf{q}, v)\right) \\ &= r_i + \text{maxneg}(\mathbf{r}) - U\big(\sigma(\mathbf{q} + \mathbf{r}, v + f(\mathbf{r}))\big) + U\left(\sigma(\mathbf{q}, v)\right) \\ &= \rho(\mathbf{r})_i - U(\sigma(\mathbf{q} + \mathbf{r}, v + f(\mathbf{r}))) + U(\sigma(\mathbf{q}, v)), \end{aligned}$$

which is the net payoff of the trader in the buy-only market. □

Viewing the buy-only market as a VPM yields new insights, as we detail now. We can immediately see that the cost function $C(\cdot, \cdot)$ from (6) is convex in both arguments if the original function U is convex, since the first term of C is U applied to a linear function and the second term is linear. C is also smooth if U is.

Since we know that $C(\mathbf{q}, v)$ is convex in both $\mathbf{q}$ and v, one could ask what happens when we view it as a potential-based market. Following the usual duality-based formulation, we can write

$$C(\mathbf{q}, v) = \sup_{\boldsymbol{\pi} \in \Pi} \begin{bmatrix} \mathbf{q} \\ v \end{bmatrix} \cdot \boldsymbol{\pi} - R(\boldsymbol{\pi}), \tag{18}$$

where Π is now a $n+1$-dimensional price space, n being the dimension of the original market. Letting U^* denote the buy-only dual, we can actually work out what the dual R and price space Π have to be:

$$\begin{aligned}
C(\mathbf{q}, v) &= U(\sigma(\mathbf{q}, v)) - \frac{1}{2n}\left(v - \sum_{i=1}^{n} q_i\right) \\
&= \sup_{\mathbf{p}\in\mathbb{Y}} \left(\mathbf{p}\cdot\mathbf{q} + \frac{1}{2n}\left(v - \sum_{i=1}^{n} q_i\right)\sum_{i=1}^{n} p_i - U^*(\mathbf{p})\right) - \frac{1}{2n}\left(v - \sum_{i=1}^{n} q_i\right) \\
&= \sup_{\mathbf{p}\in\mathbb{Y}} \mathbf{p}\cdot\mathbf{q} + \frac{1}{2n}\left(v - \sum_{i=1}^{n} q_i\right)\left(\sum_{i=1}^{n} p_i - 1\right) - U^*(\mathbf{p}).
\end{aligned}$$

We can define a mapping from any buy-only price $\mathbf{p}$ to a buy-sell price $\boldsymbol{\pi} \in \mathbb{R}^{n+1}$ as follows:

$$\boldsymbol{\pi}(\mathbf{p}) := \begin{bmatrix} \mathbf{p} - \frac{1}{2n}(\sum p_i - 1)\mathbf{1} \\ \frac{1}{2n}(\sum p_i - 1) \end{bmatrix}. \tag{19}$$

Let $\mathbb{Y}$ denote the price space for the buy-only market. Now we let $\Pi := \{\boldsymbol{\pi}(\mathbf{p}) : \mathbf{p} \in \mathbb{Y}\}$ and define $R(\boldsymbol{\pi}) := U^*(\boldsymbol{\pi}_{1..n} + \boldsymbol{\pi}_{n+1}\mathbf{1})$, so we have

$$\begin{aligned}
C(\mathbf{q}, v) &= \sup_{\mathbf{p}\in\mathbb{Y}} \mathbf{p}\cdot\mathbf{q} + \frac{1}{2n}\left(v - \sum_{i=1}^{n} q_i\right)\left(\sum_{i=1}^{n} p_i - 1\right) - U^*(\mathbf{p}) \\
&= \sup_{\mathbf{p}\in\mathbb{Y}} \begin{bmatrix}\mathbf{q}\\ v\end{bmatrix}\cdot\boldsymbol{\pi}(\mathbf{p}) - R(\boldsymbol{\pi}(\mathbf{p})) = \sup_{\boldsymbol{\pi}\in\Pi} \begin{bmatrix}\mathbf{q}\\ v\end{bmatrix}\cdot\boldsymbol{\pi} - R(\boldsymbol{\pi}).
\end{aligned}$$

Note that Π is convex as a linear transformation of $\mathbb{Y}$, and R is also convex since U^* is.

This gives us an interesting way to think about the buy-only market: one can interpret a buy-only market with current prices $\mathbf{p} \in \mathbb{Y}$ as letting traders buy *or sell* securities at prices $\mathbf{p} - \frac{1}{2n}(\sum p_i - 1)\mathbf{1}$, but forcing them to additionally buy some quantity (depending on their purchase) of a "bogus" security at a current price of $\frac{1}{2n}(\sum p_i - 1)$.

Finally, we note that the above approach hints at a general way of achieving (WCL, ARB, II, L) in the *complex market* setting. Given a general price space $\Pi^{\phi} := \mathrm{ConvHull}(\boldsymbol{\phi}(\Omega)) \subseteq \mathbb{R}^k$, we can simply add another dimension for the "volume security," and treat this as if it were simply another security. Specifically, if we let $g : \Pi^{\phi} \to \mathbb{R}$ concave, and define $\Pi := \{(\pi, g(\pi)) : \pi \in \Pi^{\phi}\}$, we may be able to recover the useful properties of the buy-only price space $\mathbb{Y}$. In principle, the techniques of [Li and Vaughan 2013] regarding the analysis of the curvature of the edge of $\mathbb{Y}$ (here captured by the g function) are not restricted to the complete market setting, and thus adaptive liquidity (L) could be shown here as well.

C. PROOFS FOR CII IN THE PERSPECTIVE MARKET

PROOF OF PROPOSITION 5.4. Since we are considering a single security, q and r are scalars. Recall that $\nabla C_1(q, v) = U'(q/\alpha(v))$. Then the definition of CII can be written as

$$\left(U'\left(\frac{q+r}{\alpha(v+f(r))}\right) - U'\left(\frac{q}{\alpha(v)}\right)\right)\cdot r \geq 0.$$

Since U is convex, U' is increasing, and we only need to show that

$$\mathrm{sign}\left(\frac{q+r}{\alpha(v+f(r))} - \frac{q}{\alpha(v)}\right) = \mathrm{sign}(r).$$

And since

$$\frac{q+r}{\alpha(v+f(r))} - \frac{q}{\alpha(v)} = \frac{r}{\alpha(v+f(r))} - q\left(\frac{1}{\alpha(v)} - \frac{1}{\alpha(v+f(r))}\right),$$

it suffices to show that

$$\frac{\alpha(v+f(r)) - \alpha(v)}{\alpha(v)}|q| \leq |r|.$$

As $f(r) = |r|$, α concave implies $\alpha(v+f(r)) - \alpha(v) = \alpha(v+|r|) - \alpha(v) \leq |r|\alpha'(v)$. Finally, $|q| \leq v$ for any valid state (q, v). Therefore if $v\alpha'(v)/\alpha(v) \leq 1$ as assumed,

$$\frac{\alpha(v+f(r)) - \alpha(v)}{\alpha(v)}|q| \leq \frac{v\alpha'(v)}{\alpha(v)}|r| \leq |r|$$

as desired. □

PROOF OF LEMMA 5.5. Let $F(\mathbf{r}; \mathbf{q}, v) = (\nabla_1 C(\mathbf{q}+\mathbf{r}, v+f(\mathbf{r})) - \nabla_1 C(\mathbf{q}, v)) \cdot \mathbf{r}$, the difference of the two terms in Definition 5.3; CII holds if and only if $F(\mathbf{r}; \mathbf{q}, v) \geq 0$ for all valid states $(\mathbf{q}, v)$ and all $\mathbf{r}$. By definition of the perspective market, we have

$$F(\mathbf{r}; \mathbf{q}, v) = \left(\nabla U\left(\frac{\mathbf{q}+\mathbf{r}}{\alpha(v+f(\mathbf{r}))}\right) - \nabla U\left(\frac{\mathbf{q}}{\alpha(v)}\right)\right) \cdot \mathbf{r}. \tag{20}$$

We note the following identity for any $\lambda > 0$, using the fact that f is an asymmetric norm in the second equality:

$$\begin{aligned}
&F(\mathbf{r}; \mathbf{q}, v) + \frac{1}{\lambda}F(\lambda\mathbf{r}; \mathbf{q}+\mathbf{r}, v+f(\mathbf{r})) \\
&= \left(\nabla U\left(\tfrac{\mathbf{q}+\mathbf{r}}{\alpha(v+f(\mathbf{r}))}\right) - \nabla U\left(\tfrac{\mathbf{q}}{\alpha(v)}\right)\right) \cdot \mathbf{r} + \frac{1}{\lambda}\left(\nabla U\left(\tfrac{\mathbf{q}+\mathbf{r}+\lambda\mathbf{r}}{\alpha(v+f(\mathbf{r})+f(\lambda\mathbf{r}))}\right) - \nabla U\left(\tfrac{\mathbf{q}+\mathbf{r}}{\alpha(v+f(\mathbf{r}))}\right)\right) \cdot (\lambda\mathbf{r}) \\
&= \left(\nabla U\left(\tfrac{\mathbf{q}+(1+\lambda)\mathbf{r}}{\alpha(v+(1+\lambda)f(\mathbf{r}))}\right) - \nabla U\left(\tfrac{\mathbf{q}}{\alpha(v)}\right)\right) \cdot \mathbf{r} \\
&= (1+\lambda)^{-1}F((1+\lambda)\mathbf{r}; \mathbf{q}, v).
\end{aligned} \tag{21}$$

We now prove the lemma by induction. Take any $\mathbf{r}$ and assume $F(\lambda\mathbf{r}; \mathbf{q}, v) \geq 0$ for all (valid) $\mathbf{q}, v$ and all $\lambda \in [0, n]$ for some integer n. Now let $(\mathbf{q}, v)$ be any valid state; then by eq. (21), we have

$$F((1+\lambda)\mathbf{r}; \mathbf{q}, v) = (1+\lambda)F(\mathbf{r}; \mathbf{q}, v) + \tfrac{1+\lambda}{\lambda}F(\lambda\mathbf{r}; \mathbf{q}+\mathbf{r}, v+f(\mathbf{r})) \geq 0$$

for $\lambda \in [0, n]$. Thus, we have $F(\lambda\mathbf{r}; \mathbf{q}, v) \geq 0$ for all valid $(\mathbf{q}, v)$ and $\lambda \in [0, n+1]$. Finally, the supposition of the lemma provides the base case ($n = 1$) for bundle $\tau\mathbf{r}$. □

D. MARKET DEPTH

Here we show conditions under which a VPM has increasing market depth. We formalize our notion of depth as follows:

$$\mathrm{MarketDepth}(\mathbf{q}, v) = \max_{\omega\in\Omega} \sup_{T} \sup_{\mathbf{r}_1,\ldots,\mathbf{r}_T} \sum_{t=1}^{T} \mathbf{r}_t \cdot \boldsymbol{\phi}(\omega) - N(\mathbf{r}_1, \ldots, \mathbf{r}_T; \mathbf{q}, v). \tag{22}$$

Note that increasing market depth does not imply unbounded loss for the market maker, as the payoff to one trader may be offset by previous trades made by others.

To satisfy increasing market depth, the market needs to have increasing liquidity. The market must also have a shrinking spread; otherwise it could be the case that either selling or buying a security is not profitable. Hence, it is natural that the sufficient conditions from Lemma 3.7 are relevant to market depth as well.

LEMMA D.1. *Consider a non-trivial VPM (ϕ, C, f). Increasing market depth is satisfied if the two conditions from Lemma 3.7 hold.*

PROOF. Since the market is non-trivial, there exist $\omega_1, \omega_2 \in \Omega$, such that $\phi(\omega_1) \neq \phi(\omega_2)$. Let $d = \|\phi(\omega_1) - \phi(\omega_2)\|/4 > 0$. Recall that k is the number of securities offered in the market, and let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k, -\mathbf{e}_1, -\mathbf{e}_2, \ldots, -\mathbf{e}_k\}$ be the set of standard basis and their opposites. Pick any $M > 0$ and consider $2k$ bundles $\mathcal{R} = M\mathcal{E} = \{M\mathbf{e}_1, M\mathbf{e}_2, ..., M\mathbf{e}_k, -M\mathbf{e}_1, -M\mathbf{e}_2, \ldots, -M\mathbf{e}_k\}$. Let $D = \max_{\mathbf{e}\in\mathcal{E}} |\delta_{\mathbf{e}} f(0)|$. Then $\max_{\mathbf{r}\in\mathcal{R}} |\delta_{\mathbf{r}} f(0)| = MD$.

By Proposition A.2, there is some τ_1 such that $|\delta_{\mathbf{r}} N(\mathbf{q}, v) - N(\mathbf{r}; \mathbf{q}, v)| < 1$ for all $\mathbf{r} \in \mathcal{R}$ and all $\mathbf{q}$ if $v > \tau_1$ and $(\mathbf{q}, v)$ is valid. For any $\mathbf{q}$, let $\mathbf{p} = \nabla_1 C(\mathbf{q}, v)$. By the triangle inequality, $\|\mathbf{p} - \phi(\omega_i)\| \geq 2d$ for some $i \in \{1, 2\}$; henceforth, we fix $\omega := \omega_i$ for this i. We now have $4d^2 \leq \|\mathbf{p} - \phi(\omega)\|^2 = \sum_{i=1}^{k} ((\mathbf{p} - \phi(\omega)) \cdot \mathbf{e}_i)^2$, and thus there exists some i such that $|(\mathbf{p} - \phi(\omega)) \cdot \mathbf{e}_i| \geq 2d/\sqrt{k}$. By the definition of $\mathcal{R}$, there is some $\mathbf{r} \in \mathcal{R}$ such that $(\phi(\omega) - \mathbf{p}) \cdot \mathbf{r} \geq 2dM/\sqrt{k}$.

By condition (1) of Lemma 3.7, there is some τ_2, such that $|\nabla_2 C(\mathbf{q}, v)| < d/2D\sqrt{k}$ if $v \geq \tau_2$. Recall that $\max_{\mathbf{r}\in\mathcal{R}} |\delta_{\mathbf{r}} f(0)| = MD$, thus $|\nabla_2 C(\mathbf{q}, v)\delta_{\mathbf{r}} f(0)| < dM/2\sqrt{k}$ for all $\mathbf{r} \in \mathcal{R}$. Therefore, by Equation (4),

$$\phi(\omega) \cdot \mathbf{r} - \delta_{\mathbf{r}} N(\mathbf{q}, v) = (\phi(\omega) - \mathbf{p}) \cdot \mathbf{r} - \nabla_2 C(\mathbf{q}, v)\delta_{\mathbf{r}} f(0) \geq 3dM/2\sqrt{k}.$$

Suppose a trader purchases the bundle $\mathbf{r}$ at $v > \max\{\tau_1, \tau_2\}$ and suppose ω happens. Then the payoff is

$$\phi(\omega) \cdot \mathbf{r} - N(\mathbf{r}; \mathbf{q}, v) \geq \phi(\omega) \cdot \mathbf{r} - \delta_{\mathbf{r}} N(\mathbf{q}, v) - |\delta_{\mathbf{r}} N(\mathbf{q}, v) - N(\mathbf{r}; \mathbf{q}, v)| \geq 3dM/2\sqrt{k} - 1.$$

Since M is arbitrary, increasing market depth is proved. □

E. EXPRESSIVENESS

Recall the following two properties:

(1) **Price mobility:** For any valid state $(\mathbf{q}, v)$, any $\mathbf{b} \in B$, and any $\epsilon > 0$, there exists a bundle $\mathbf{r}$ such that for some $\mathbf{b}' \in \text{NTBS}(\mathbf{q} + \mathbf{r}, v + f(\mathbf{r}))$, $||\mathbf{b} - \mathbf{b}'|| \leq \epsilon$.
(2) **Bounded information loss:** There exists $\gamma \geq 0$ such that for any valid state $(\mathbf{q}, v)$ and all $\mathbf{b}, \mathbf{b}' \in \text{NTBS}(\mathbf{q}, v)$, $||\mathbf{b} - \mathbf{b}'|| \leq \gamma$.

We say that a market satisfying these properties is γ-*expressive*. The notion of expressiveness in Abernethy et al. [2013] corresponds to this definition with $\gamma = 0$.

Bounded information loss is not hard to satisfy in practice as Lemma 3.3 tells us that we only need to upper bound $\nabla_2 C(\mathbf{q}, v)$. Interestingly, price mobility is implied by bounded worst-case loss.

LEMMA E.1. *Consider any VPM (ϕ, C, f) such that f is a norm, C is continuously differentiable, and the purchase triangle inequality is satisfied. If at some state $(\mathbf{q}_0, v_0)$, a trader with belief $\mathbf{b}$ cannot move the NTBS within ϵ distance from $\mathbf{b}$ by purchasing any single bundle, then for any $M > 0$ there is some bundle $\mathbf{r}$ such that the expected payoff of purchasing $\mathbf{r}$ at state $(\mathbf{q}_0, v_0)$ is at least M.*

The intuition behind the proof is that if a trader's belief being always at least ϵ away from the NTBS, then there is always some small bundle the trader can purchase to make a small expected payoff. Since this process can repeat any number of times, the purchase triangle inequality gives an arbitrarily large expected payoff for the trader, which implies unbounded loss to the market maker.

PROOF. The intuition behind the proof is simple, but some technicalities must be addressed. First, we want to argue that the price movement is not arbitrarily large (i.e., the liquidity is not arbitrarily small). Otherwise, even if the belief $\mathbf{b}$ is at least ϵ away from the NTBS, the price might change so fast (i.e., the liquidity might be too small) that we can not guarantee any positive expected payoff.

Since we assumed $f(\cdot)$ to be a norm, we can write $f(\mathbf{r})$ as $\|\mathbf{r}\|$. We will use $\|\cdot\|_2$ when talking about 2-norm. We will sometimes interchange $\|\cdot\|$ and $\|\cdot\|_2$ when showing something is bounded as any two norms are equivalent in finite-dimensional Banach space.

Fix some ϵ and M and suppose the market is at state $(\mathbf{q}, v)$. If the trader purchases some bundle $\mathbf{r}$ with $\|\mathbf{r}\| \leq 2M/\epsilon$, then the market state becomes $(\mathbf{q}', v') = (\mathbf{q}+\mathbf{r}, v+\|\mathbf{r}\|)$. The 2-norm of the new state is bounded by $\|(\mathbf{q}+\mathbf{r}, v+\|\mathbf{r}\|)\|_2 \leq \|\mathbf{q}\|_2 + v + 2\|\mathbf{r}\|_2 = \|\mathbf{q}\|_2 + v + 4M/\epsilon = M'$. Since $C(\mathbf{q}, v)$ is continuously differentiable, both $\nabla_1 C(\mathbf{q}, v)$ and $\nabla_2 C(\mathbf{q}, v)$ are continuous functions. Since any continuous function is uniformly continuous on compact set, if we take the closed ball $\mathcal{B}$ of market states centered at origin with radius M', $\nabla_1 C(\mathbf{q}, v)$ and $\nabla_2 C(\mathbf{q}, v)$ are uniformly continuous on $\mathcal{B}$. In fact, for all $\mathbf{r}$ such that $\|\mathbf{r}\|_2 < 1$, by Equation (4), the bundle price $\delta_{\mathbf{r}} N(\mathbf{q}, v) = \nabla_1 C(\mathbf{q}, v) \cdot \mathbf{r} + \nabla_2 C(\mathbf{q}, v)\|\mathbf{r}\|$ [12] is a uniformly equicontinuous family on $\mathcal{B}$. This ensures that there exists some $\theta > 0$ such that if $\|\mathbf{q}_1 - \mathbf{q}_2\| \leq \theta$, $|v_1 - v_2| \leq \theta$, $\|\mathbf{r}\| \leq 1$, both $(\mathbf{q}_1, v_1)$ and $(\mathbf{q}_2, v_2)$ are in $\mathcal{B}$, then $|\delta_{\mathbf{r}} N(\mathbf{q}_1, v_1) - \delta_{\mathbf{r}} N(\mathbf{q}_2, v_2)| \leq \epsilon/2$.

We now turn to an induction proof of the claim. We will show that for all $0 \leq i \leq 2M/\epsilon\theta$, there is some bundle $\mathbf{r}_i$ such that $\|r_i\| \leq i\theta$ and the expected payoff of purchasing $\mathbf{r}_i$ is at least $i\epsilon\theta/2$.

The base case, $i = 0$, is trivial.

Now consider $i \leq 2M/\epsilon\theta - 1$ and assume such an $\mathbf{r}_i$ exists. We try to construct $\mathbf{r}_{i+1}$ by adding a small bundle $\mathbf{u}$. Let $(\mathbf{q}, v)$ be the market state after the purchase of $\mathbf{r}_i$. Then $\mathbf{b}$ must be at least ϵ away from the NTBS at $(\mathbf{q}, v)$ by assumption in the lemma. By convexity of the NTBS, there must be some direction $\mathbf{u}$, $\|\mathbf{u}\| = 1$, such that

$$\mathbf{u} \cdot \mathbf{b} \geq \epsilon + \mathbf{u} \cdot \mathbf{x} \quad \forall \mathbf{x} \in \text{NTBS}.$$

By the characterization of the NTBS in Lemma 3.3, we have

$$\mathbf{u} \cdot \mathbf{b} \geq \epsilon + \nabla_1 C(\mathbf{q}, v) \cdot \mathbf{u} + \nabla_2 C(\mathbf{q}, v)\, \mathbf{x} \cdot \mathbf{u} \quad \forall \mathbf{x} \in \partial f(0).$$

Equivalently, $\mathbf{u} \cdot \mathbf{b} \geq \epsilon + \nabla_1 C(\mathbf{q}, v) \cdot \mathbf{u} + \nabla_2 C(\mathbf{q}, v) \sup_{\mathbf{x} \in \partial f(0)} \mathbf{x} \cdot \mathbf{u}$. Using $\|\cdot\|_*$ to denote the dual norm of $\|\cdot\|$, we have $\sup_{\mathbf{x} \in \partial f(0)} \mathbf{x} \cdot \mathbf{u} = \sup_{\|\mathbf{x}\|_* \leq 1} \mathbf{x} \cdot \mathbf{u} = \|\mathbf{u}\|$, thus

$$\mathbf{u} \cdot \mathbf{b} \geq \epsilon + \nabla_1 C(\mathbf{q}, v) \cdot \mathbf{u} + \nabla_2 C(\mathbf{q}, v)\|\mathbf{u}\| = \epsilon + \delta_{\mathbf{u}} N(\mathbf{q}, v).$$

For any $0 \leq t \leq \theta$, $\|(\mathbf{q}+t\mathbf{u}) - \mathbf{q}\| \leq \theta$, $|(v+t) - v| \leq \theta$, and both $(\mathbf{q}, v)$ and $(\mathbf{q}+t\mathbf{u}, v+t)$ are in $\mathcal{B}$, hence we have by our uniform equicontinuity that $|\delta_{\mathbf{u}} N(\mathbf{q}, v) - \delta_{\mathbf{u}} N(\mathbf{q}+t\mathbf{u}, v+t)| < \epsilon/2$. Combining the two inequalities, we have for all $0 \leq t \leq \theta$

$$\mathbf{u} \cdot \mathbf{b} \geq \epsilon/2 + \delta_{\mathbf{u}} N(\mathbf{q}+t\mathbf{u}, v+t).$$

Integrate with respect to t from 0 to θ. Then

$$\theta \mathbf{u} \cdot \mathbf{b} \geq \theta\epsilon/2 + C(\mathbf{q}+\theta\mathbf{u}, v+\theta) - C(\mathbf{q}, v),$$

which tells us that the purchase of bundle $\theta\mathbf{u}$ at $(\mathbf{q}, v)$ yields an expected payoff of at least $\theta\epsilon/2$. Let $\mathbf{r}_{i+1} = \mathbf{r}_i + \theta\mathbf{u}$. Then by purchase triangle inequality and induction hypothesis, the purchase of $\mathbf{r}_{i+1}$ yields an expected payoff of at least $\theta\epsilon/2 + i\epsilon\theta/2 = (i+1)\epsilon\theta/2$. Also, $\|\mathbf{r}_{i+1}\| \leq \|\mathbf{r}_i\| + \theta \leq (i+1)\theta$.

Finally, set $i = 2M/\epsilon\theta$, then the expected payoff of purchasing $\mathbf{r}_i$ is at least $2M/\epsilon\theta \cdot \epsilon\theta/2 = M$, which finishes the proof. □

[12]We used the fact that $\delta_{\mathbf{r}} f(0) = \|\mathbf{r}\|$ if f is norm.