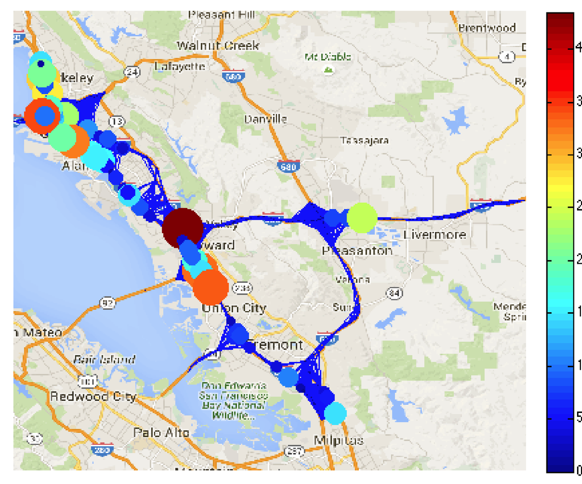
Modeling Signals Embedded in a Euclidean Domain

Philip A. Chou[1] and Ricardo L. De Queiroz[2]

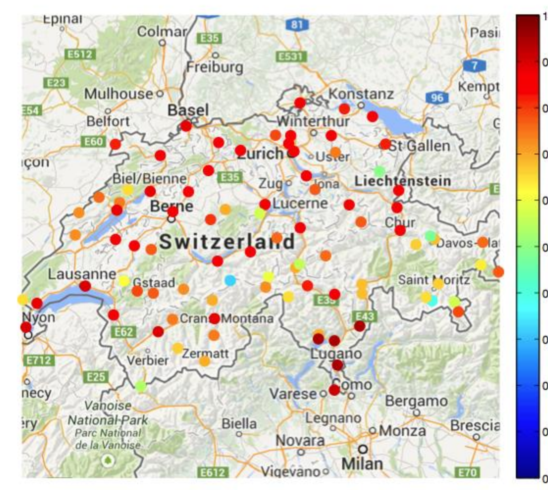
[1]Microsoft Research and [2]Universidade de Brasilia

Introduction

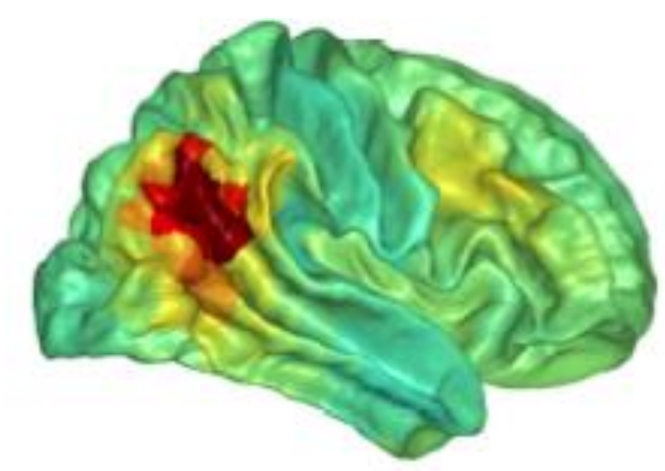
Many signals are defined on irregular domains embedded in Euclidean space. Often, they are modeled with GSP. But should they be?



Transportation networks



Weather data



Biological networks

3D point clouds

Modeling signals on a sparse graph

Graph Signal Processing Framework

Signal $x(v)$ defined on discrete domain $\mathcal{V}$

Undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $\mathcal{E} = \{(v_i, v_j)\}$

Weights $W = [w_{ij}]$, $w_{ij} > 0$ if $(v_i, v_j) \in \mathcal{E}$ else 0

Diagonal $D = [d_{ij}]$, $d_{ii} = w_{ii} + \sum_j w_{ij}$

Laplacian $L = D - W = \chi \Lambda \chi^T$

Graph Fourier Transform (GFT) of x: $\hat{\boldsymbol{x}} = \chi^T \boldsymbol{x}$

Modeling signal as GMRF(L)

Vector $\boldsymbol{x} = \left(x(v_1), \ldots, x(v_n)\right)^T$ is a GMRF wrt $\mathcal{G}$ with mean $\boldsymbol{\mu}$ and precision $Q = [q_{ij}]$ iff $q_{ij} \neq 0 \Leftrightarrow (v_i, v_j) \in \mathcal{E}\ \forall i \neq j$ and

$$p(\boldsymbol{x}) = (2\pi)^{-\frac{n}{2}} |R|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T R^{-1} (\boldsymbol{x}-\boldsymbol{\mu})\right)$$

where covariance matrix $R = Q^{-1}$.

If $w_{ij} = -q_{ij}\ \forall i \neq j$ and $w_{ii} = \sum_j q_{ij}\ \forall i$, then $L = Q = R^{-1}$. Thus $R = L^{-1} = \chi \Lambda^{-1} \chi^T$. KLT is GFT

Modeling signals as samples of a stationary Gaussian Process

Gaussian Process Definition

Stationary Gaussian Process (GP) $x(\boldsymbol{v})$, $\boldsymbol{v} \in \mathbb{R}^N$, with mean μ and covariance function $R_{xx}(\boldsymbol{d}) = E(x(\boldsymbol{v}) - \mu)(x(\boldsymbol{v}+\boldsymbol{d}) - \mu)$ is a random process s.t. any sample vector $\boldsymbol{x} = \left(x(v_1), \ldots, x(v_n)\right)^T$ has density

$$p(\boldsymbol{x}) = (2\pi)^{-\frac{n}{2}} |R|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T R^{-1} (\boldsymbol{x}-\boldsymbol{\mu})\right)$$

where $\boldsymbol{\mu} = (\mu, \ldots, \mu)^T$ and $R = [r_{ij}]$, $r_{ij} = R_{xx}(\boldsymbol{v}_i - \boldsymbol{v}_j)$.

Modeling signal as GP(R)

Model mean-removed signal $x(\boldsymbol{v}_1), \ldots, x(\boldsymbol{v}_n)$ as samples of a GP with mean 0 and covariance function $R_{xx}(\boldsymbol{d})$ either

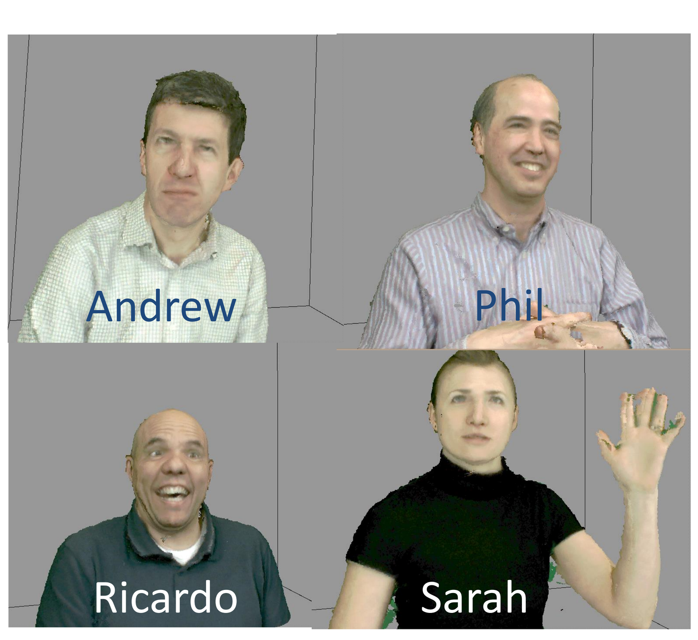
- Modeled, e.g., as $R_{xx}(\boldsymbol{d}) = \sigma_x^2 \rho^{-|\boldsymbol{d}|}$ or
- Estimated, e.g., as $R_{xx}(\boldsymbol{d}) = \frac{1}{|\Phi_d|} \sum_{(\boldsymbol{u},\boldsymbol{v}) \in \Phi_d} x(\boldsymbol{u}) x(\boldsymbol{v})$

"GPT" is KLT

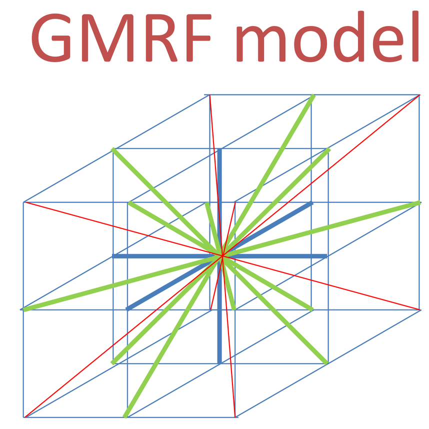
Comparison

Data sets

Man



GMRF model

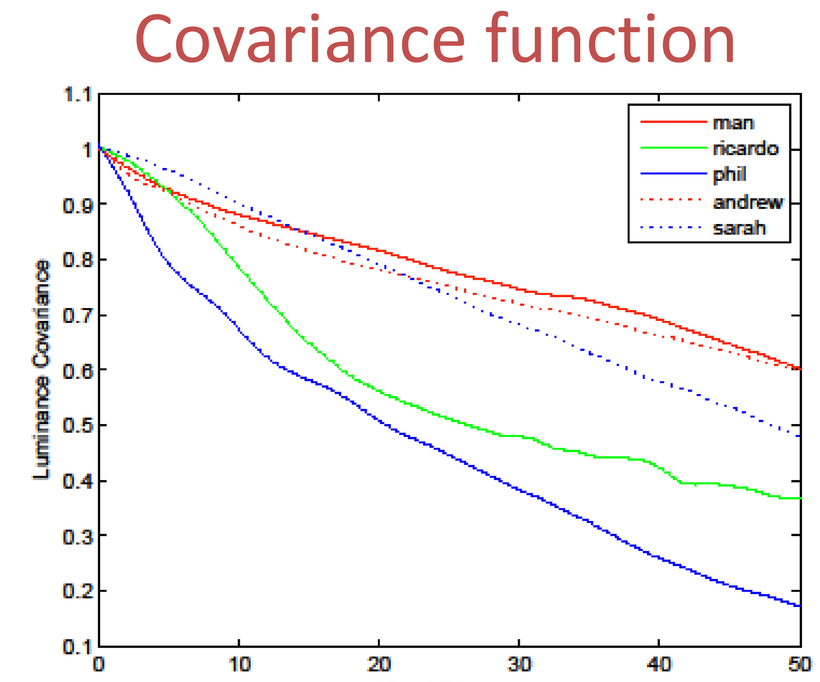


Blue: $w_{ij} = 1$

Green: $w_{ij} = 1/\sqrt{2}$

Red: $w_{ij} = 1/\sqrt{3}$

Covariance function

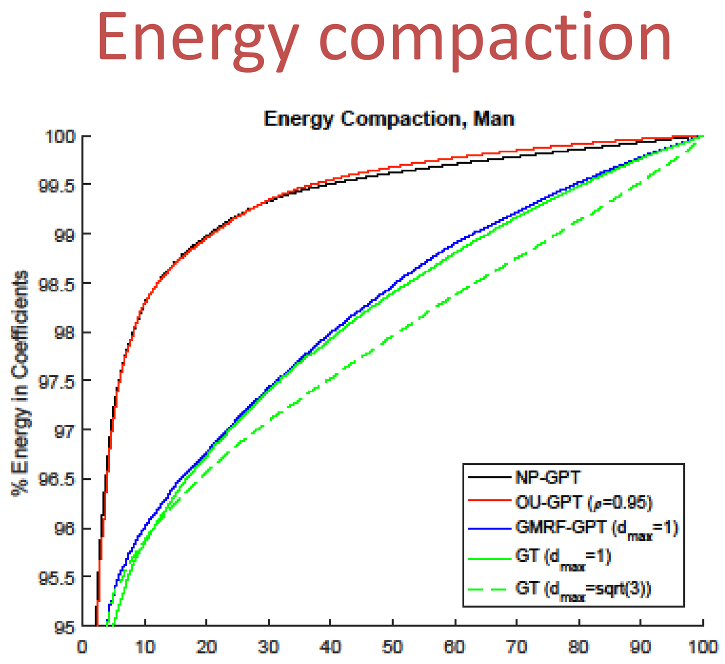


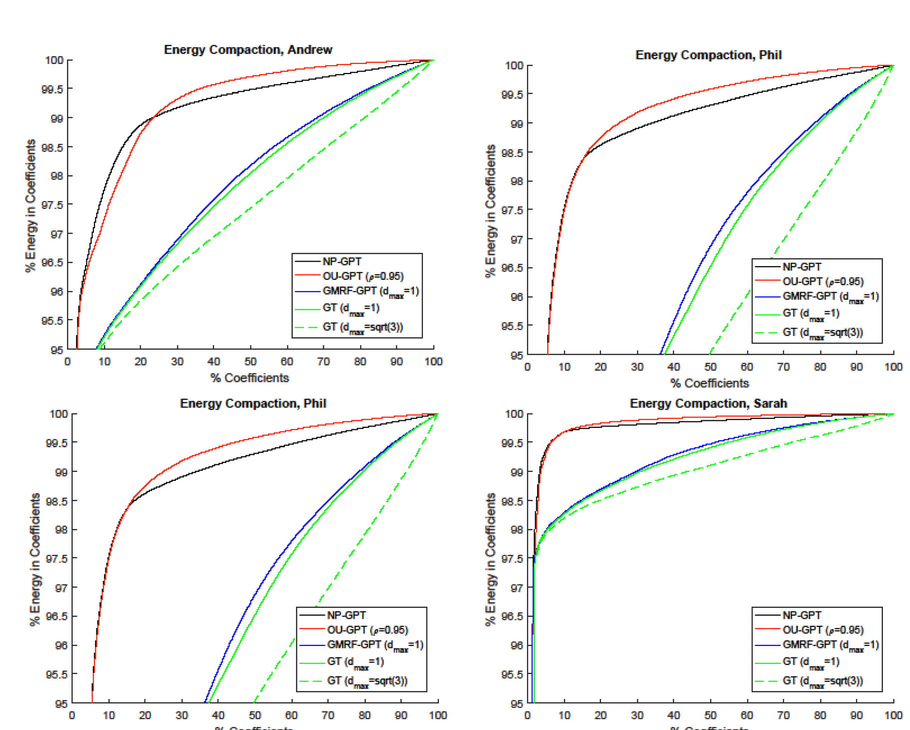
Transform coding gain (dB)

Data set	GFT	GPT
Man	14.9	**18.9**
Andrew	13.2	**17.6**
Ricardo	9.8	**16.4**
Phil	17.9	**24.8**
Sarah	20.0	**28.3**
Average	*15.1*	***21.2***

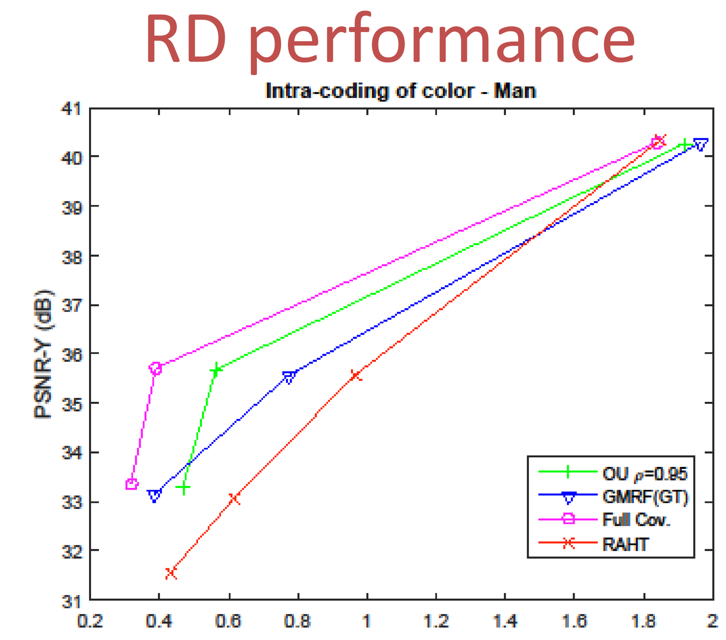
6 dB difference

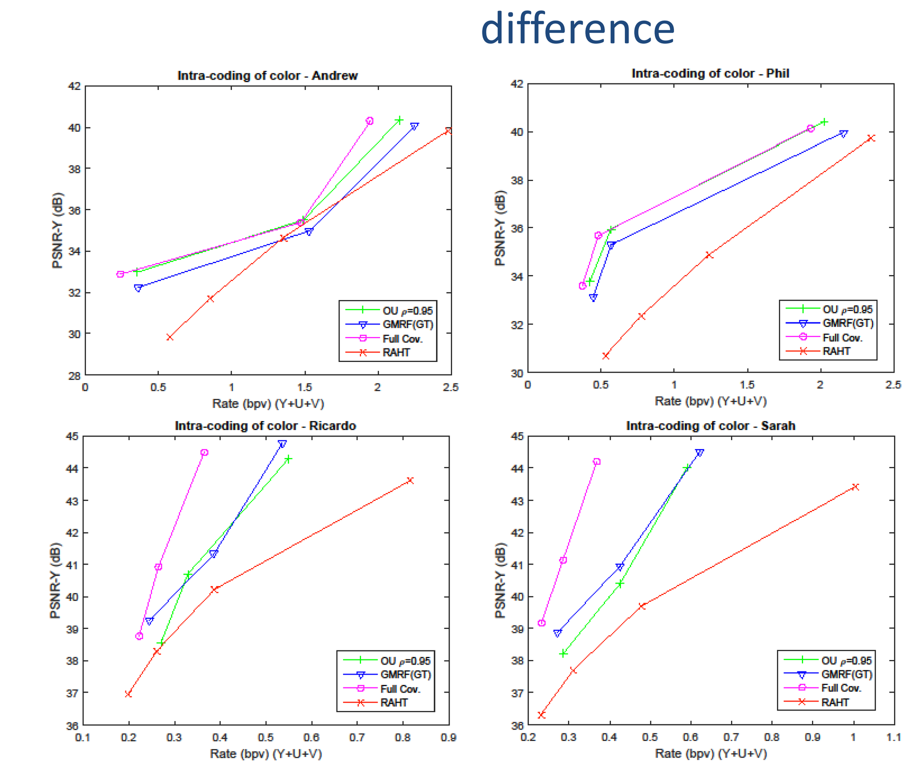
Energy compaction





RD performance





Conclusion

For signals defined on an irregular domain embedded in Euclidean space, consider modeling them as Gaussian Processes.

References

P.A. Chou and R.L. de Queiroz, *Gaussian Process Transforms,* ICIP 2016, to appear.

R.L. de Queiroz and P.A. Chou, *Transform Coding for Point Clouds using a Gaussian Process Model,* TIP, submitted.

Microsoft Research