Approximability of subspace approximation

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Credits

Based on joint work with Madhur Tulsiani and Nisheeth Vishnoi,

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and disjoint (?) work of

- Varadarajan, Venkatesh, Ye, and Zhang (SICOMP, 2007)
- Kindler, Naor, and Schechtman (Math of OR, 2010)
- Guruswami, Raghavendra, Saket, and Wu (preprint)
- ...
- the anonymous heroes who discovered eigenvalues, eigenvectors, gaussians etc.

Subspace approximation

Find a low-dimensional representation of high-dimensional data up to a small error.

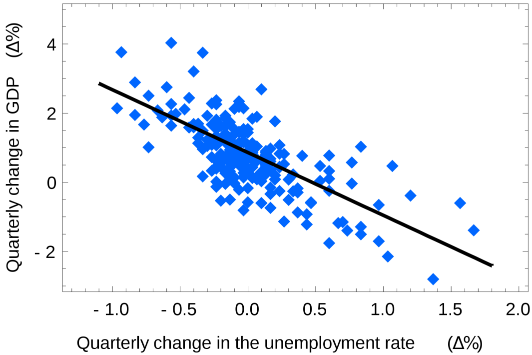
or for simplicity,

Given n points $a_1, a_2, \ldots, a_n \in \mathbb{R}^d$, find a k-dimensional linear subspace V that minimizes

$$\left(\sum_{i=1}^{n} d(a_i, V)^p \right)^{1/p}.$$

Special cases

$p = 2$ (ordinary least squares) and $p = \infty$ (radii of point sets).



http://en.wikipedia.org/wiki/File:Okuns_law_quarterly_differences.svg

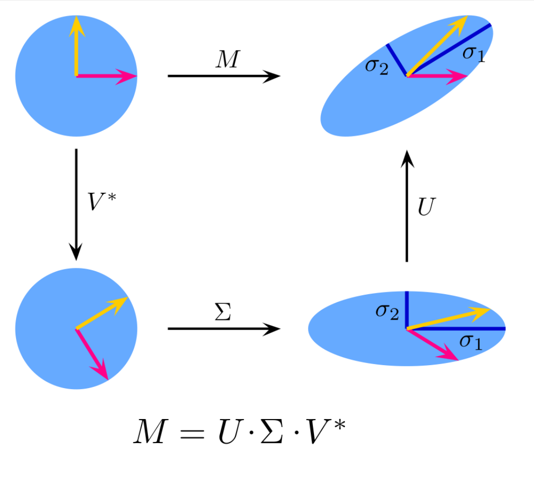
Simplest case: $\dim(V) = d - 1$ and $p = 2$

Subspace V is uniquely identified by its unit normal x.

$$\min_{\dim(V)=d-1} \left(\sum_{i=1}^{n} d(a_i, V)^2 \right)^{1/2} = \min_{\|x\|_2=1} \left(\sum_{i=1}^{n} \langle a_i, x \rangle^2 \right)^{1/2}$$
$$= \min_{\|x\|_2=1} \|Ax\|_2 \,.$$

So the optimal x is the smallest singular vector of $A \in \mathbb{R}^{n \times d}$, which has $a_1, \ldots, a_n$ as its rows.

Singular Value Decomposition (SVD)



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- Minimizing $\|Ax\|_p$ over $\|x\|_2 \geq 1$ is not a convex program.
- Magic of SVD: But we can do this *efficiently* for $p = 2$ (Ref. Matrix Computations, Golub and Van Loan).

Convex relaxation and randomized rounding

Using $\langle a_i, x\rangle^p = (a_i^T x x^T a_i)^{p/2}$,

$$\min_{\|x\|_2=1} \left(\sum_{i=1}^{n} \langle a_i, x\rangle^p \right)^{1/p} \xrightarrow{\text{relax}} \min_{\substack{X\,:\,X\succcurlyeq 0 \\ X \text{ symmetric} \\ \text{trace}(X)=1}} \left(\sum_{i=1}^{n} \left(a_i^T X a_i \right)^{p/2} \right)^{1/p}.$$

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- X may not have rank one (and thus no expression as xx^T); SVD gives $X = V\Sigma V^T = \sum_{i=1}^{n} \sigma_i v_i v_i^T$.
- Compute SVD of X to get its singular values $\sigma_1, \ldots, \sigma_n$ and singular vectors $v_1, \ldots, v_n$. Output as vector x the (normalized) random linear combination

$$\sum_{i=1}^{n} r_i \sqrt{\sigma_i} v_i, \quad \text{where } r_i\text{'s are i.i.d. } N(0,1).$$

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More generally, the convex relaxation and rounding (with tiny modifications) give an approximation factor of $\sqrt{2}\gamma_p$ for k-dimensional subspace approximation, for any k and $p \geq 2$. For $p = \infty$, the approximation factor becomes $O(\sqrt{\log n})$.

Continuous analog and integrality/rank gap

Continuous analog of subspace approximation

$$\min_{\|x\|_2=1} \sum_{i=1}^{n} \langle a_i, x \rangle^p \quad \xrightarrow{\text{relax}} \quad \min_{\substack{X \,:\, X \succcurlyeq 0 \\ X \text{ symmetric} \\ \text{trace}(X)=1}} \sum_{i=1}^{n} \left(a_i^T X a_i\right)^{p/2}$$

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$$\Big\downarrow \text{continuous} \qquad\qquad\qquad\qquad \Big\downarrow \text{continuous}$$

$$\min_{\|x\|_2=1} \int_{\mathbb{R}^d} \langle g, x\rangle^p \, \mu(g) dg \xrightarrow{\text{relax}} \min_{\substack{X \,:\, X \succcurlyeq 0 \\ X \text{ symmetric} \\ \text{trace}(X)=1}} \int_{\mathbb{R}^d} \left(g^T X g\right)^{p/2} \mu(g) dg$$

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$$\text{LHS} = \int_{\mathbb{R}} g_1^p e^{-g_1^2/2} dg_1 \quad \text{vs.} \quad \text{RHS} \leq \frac{1}{d^{p/2}} \int_{\mathbb{R}^d} \|g\|^p \, e^{-\|g\|^2/2} dg$$

Dictatorship test

$$\text{IsDictator(x)} : x \mapsto \mathrm{E}\left[\langle a, x\rangle^{p}\right] = \frac{1}{2^d} \sum_{a \in \{-1,1\}^d} \langle a, x\rangle^{p} .$$

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- Dictator: If $x = (1, 0, \ldots, 0)$ then $\mathrm{E}\left[\langle a, x\rangle^p\right] = 1$, for p even.
- Far-from-dictator: If all the coordinates of x are small, then

$$\begin{aligned}
\mathrm{E}\left[\langle a, x\rangle^p\right] &\approx \mathrm{E}\left[\langle g, x\rangle^p\right] && \text{by invariance principle} \\
&= \mathrm{E}\left[\langle g, (1, 0, \ldots, 0)\rangle^p\right] && \text{by spherical symmetry} \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g_1^p e^{-g_1^2/2} dg_1 = \gamma_p^p .
\end{aligned}$$

Thank you. Any questions?