Introduction to LP and SDP Hierarchies

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Local Constraints in Approximation Algorithms

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- How does one reason about increasingly larger local constraints?
- Does approximation get better as constraints get larger?

LP/SDP Hierarchies

- Various hierarchies give increasingly powerful programs at different levels (rounds).
  - Lovász-Schrijver (LS, LS_+)
  - Sherali-Adams
  - Lasserre
  - “Mixed”

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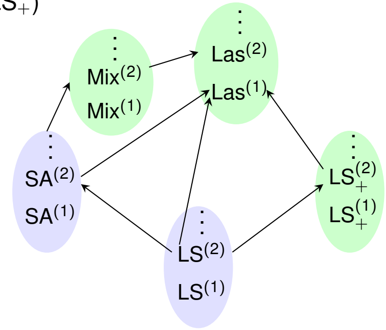

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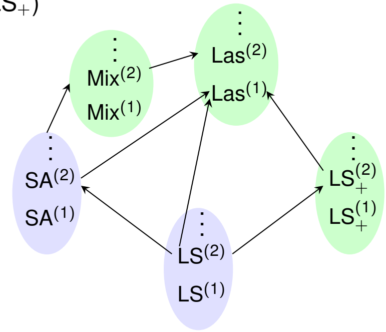

- Can optimize over r^{th} level in time $n^{O(r)}$. n^{th} level is tight.

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- Lower bounds rule out large and natural class of algorithms.
- Performance measured by considering integrality gap at various levels.

$$\text{Integrality Gap} = \frac{\text{Optimum of Relaxation}}{\text{Integer Optimum}} \qquad \text{(for maximization)}$$

Why bother?

- Conditional
- All polytime algorithms

NP-Hardness

UG-Hardness

- Unconditional
- Restricted class of algorithms

LP/SDP Hierarchies

What Hierarchies want

Example: Maximum Independent Set for graph $G = (V, E)$

$$\begin{aligned} \text{minimize} \quad & \sum_u x_u \\ \text{subject to} \quad & x_u + x_v \leq 1 \qquad \forall (u, v) \in E \\ & x_u \in [0, 1] \end{aligned}$$

- Hope: $x_1, \ldots, x_n$ is convex combination of 0/1 solutions.

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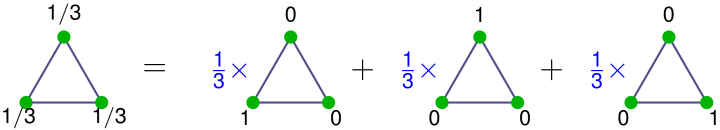

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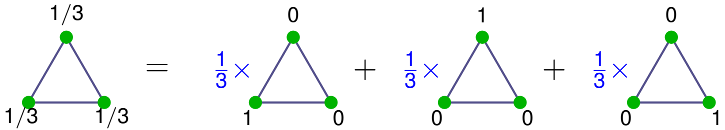

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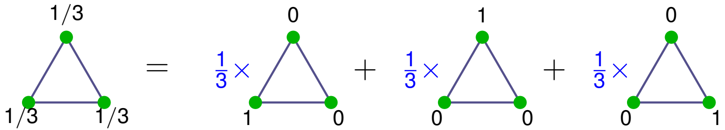

- Hierarchies add variables for conditional/joint probabilities.

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- Start with a 0/1 integer linear program.
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$$\sum_i a_i z_i \leq b$$

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\sum_i a_i \cdot (X_{\{i,5,7\}} - X_{\{i,5,7,9\}}) &\leq b \cdot (X_{\{5,7\}} - X_{\{5,7,9\}})
\end{aligned}$$

- LP on n^r variables.

Sherali-Adams $\approx$ Locally Consistent Distributions

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- Using $0 \leq z_1 \leq 1$, $0 \leq z_2 \leq 1$

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- $SA^{(r)} \implies LCD^{(r)}$. If each constraint has at most k vars, $LCD^{(r+k)} \implies SA^{(r)}$

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- $(Y \succeq 0)$ + original constraints + consistency constraints.

The Lasserre hierarchy (constraints)

- Y is psd. (i.e. find vectors $\mathbf{U}_S$ satisfying $Y_{S_1,S_2} = \langle \mathbf{U}_{S_1}, \mathbf{U}_{S_2} \rangle$)

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- Original quadratic constraints as inner products.

SDP for Independent Set

$$
\begin{array}{lll}
\text{maximize} & \sum_{i \in V} \left|\mathbf{U}_{\{i\}}\right|^2 & \\
\text{subject to} & \langle \mathbf{U}_{\{i\}}, \mathbf{U}_{\{j\}} \rangle = 0 & \forall\, (i,j) \in E \\
& \langle \mathbf{U}_{S_1}, \mathbf{U}_{S_2} \rangle = \langle \mathbf{U}_{S_3}, \mathbf{U}_{S_4} \rangle & \forall\, S_1 \cup S_2 = S_3 \cup S_4 \\
& \langle \mathbf{U}_{S_1}, \mathbf{U}_{S_2} \rangle \in [0, 1] & \forall S_1, S_2
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The “Mixed” hierarchy

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- Captures what we actually know how to use about Lasserre solutions.
- Level r has
  - Variables X_S for $|S| \leq r$ and all Sherali-Adams constraints.
  - Vectors $\mathbf{U}_0, \mathbf{U}_1, \ldots, \mathbf{U}_n$ satisfying

$$\langle \mathbf{U}_i, \mathbf{U}_j \rangle = X_{\{i,j\}}, \langle \mathbf{U}_0, \mathbf{U}_i \rangle = X_{\{i\}} \text{ and } |\mathbf{U}_0| = 1.$$

Hands-on: Deriving some constraints

The triangle inequality

- $|\mathbf{U}_i - \mathbf{U}_j|^2 + |\mathbf{U}_j - \mathbf{U}_k|^2 \;\geq\; |\mathbf{U}_i - \mathbf{U}_k|^2$ is equivalent to

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$$\langle \mathbf{U}_i - \mathbf{U}_j, \mathbf{U}_k - \mathbf{U}_j \rangle \geq 0$$
- $\mathrm{Mix}^{(3)} \implies \exists$ distribution on z_i, z_j, z_k such that $\mathbb{E}[z_i \cdot z_j] = \langle \mathbf{U}_i, \mathbf{U}_j \rangle$ (and so on).

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- For all integer solutions $(z_i - z_j) \cdot (z_k - z_j) \geq 0$.
$$\therefore \langle \mathbf{U}_i - \mathbf{U}_j, \mathbf{U}_k - \mathbf{U}_j \rangle = \mathbb{E}\left[(z_i - z_j) \cdot (z_k - z_j)\right] \geq 0$$

“Clique constraints” for Independent Set

- For every clique B in a graph, adding the constraint

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- For $i, j \in B$, $\langle \mathbf{U}_i, \mathbf{U}_j \rangle = 0$. By Pythagoras,

$$\sum_{i \in B} \left\langle \mathbf{U}_0, \frac{\mathbf{U}_i}{|\mathbf{U}_i|} \right\rangle^2 \leq |\mathbf{U}_0|^2 = 1 \implies \sum_{i \in B} \frac{x_i^2}{x_i} \leq 1.$$

- Derived by Lovász using the ϑ-function.

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- Restriction: $\mathbf{x} = (x_1, \ldots, x_n) \in LS(P)$ if $\exists Y$ satisfying
  (think $Y_{ij} = \mathbb{E}[z_i z_j] = \mathbb{P}[z_i \wedge z_j]$)
  - $Y = Y^T$
  - $Y_{ii} = x_i \qquad \forall i$
  - $\frac{Y_i}{x_i} \in P, \ \frac{\mathbf{x} - Y_i}{1 - x_i} \in P \qquad \forall i$
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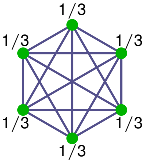
- Above is an LP (SDP) in $n^2 + n$ variables.

Lovász-Schrijver in action

- r^{th} level optimizes over distributions conditioned on r variables.



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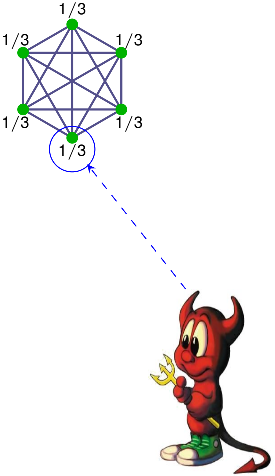

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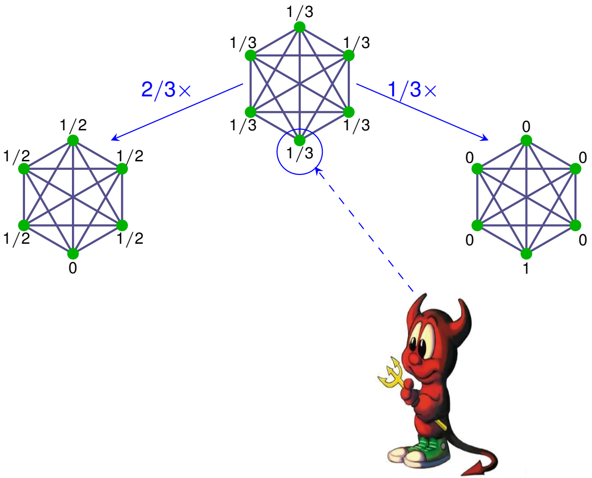

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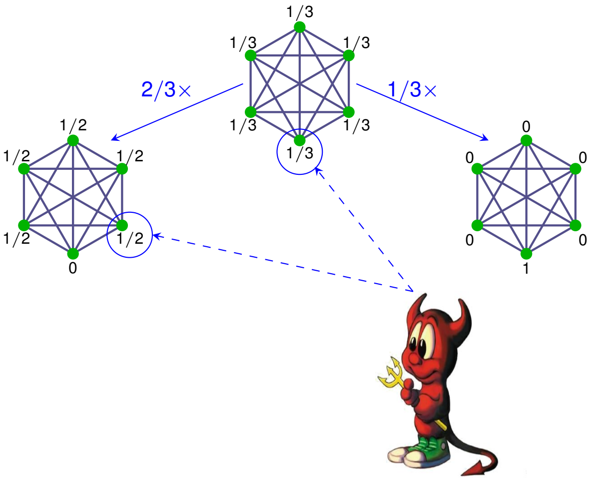

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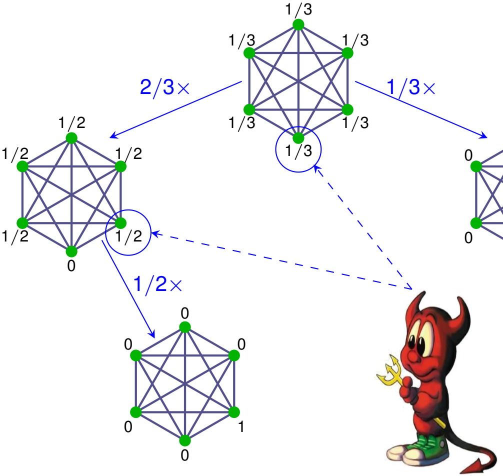

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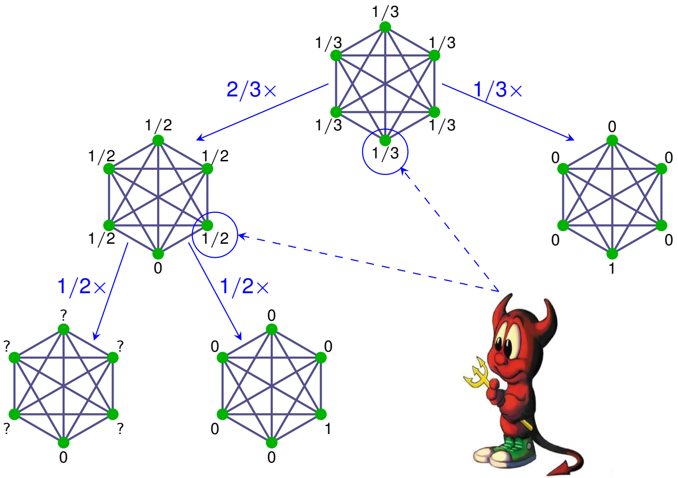

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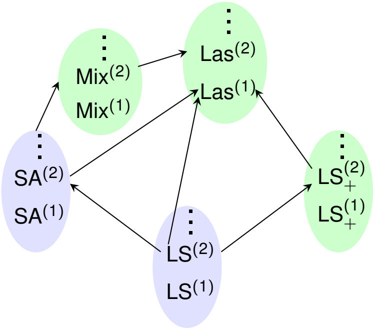

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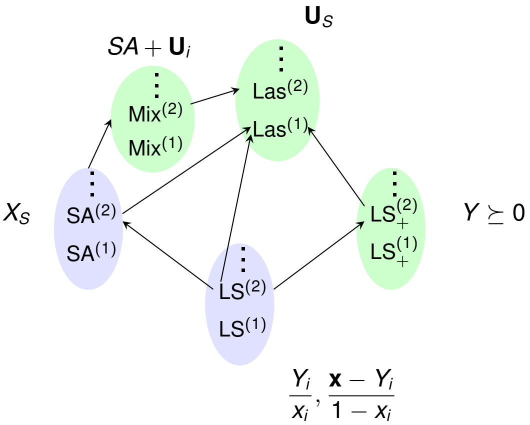

Integrality Gaps for Expanding CSPs

CSP Expansion

- MAX k-CSP: m constraints on k-tuples of (n) boolean variables. Satisfy maximum. e.g. MAX 3-XOR (linear equations mod 2)

$$z_1 + z_2 + z_3 = 0 \qquad z_3 + z_4 + z_5 = 1 \qquad \cdots$$

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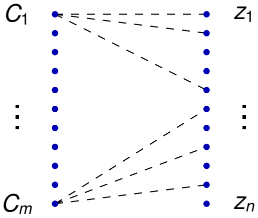

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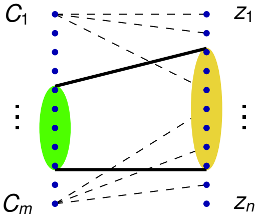

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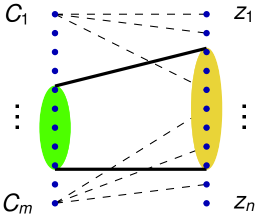

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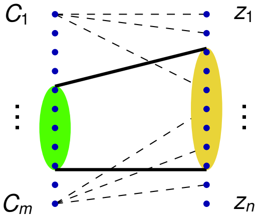

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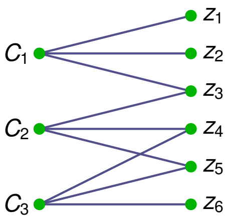
- Used extensively in proof complexity e.g. [BW01], [BGHMP03]. For LS_+ by [AAT04].

Local Satisfiability



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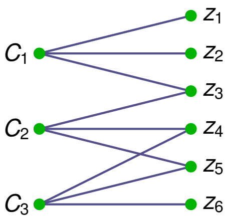

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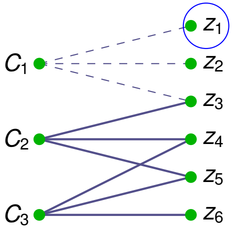

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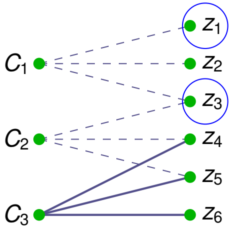

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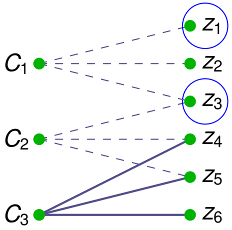

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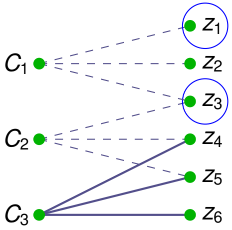

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- Can show any three 3-XOR constraints are simultaneously satisfiable.
- Can take $\gamma \approx (k-2)$ and any αn constraints.
- Just require $\mathbb{E}[C(z_1, \ldots, z_k)]$ over any $k-2$ vars to be constant.

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Sherali-Adams LP for CSPs

Variables: $X_{(S,\alpha)}$ for $|S| \leq t$, partial assignments $\alpha \in \{0,1\}^S$

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\text{maximize} & \displaystyle\sum_{i=1}^{m} \sum_{\alpha \in \{0,1\}^{T_i}} C_i(\alpha) \cdot X_{(T_i,\alpha)} & & \\
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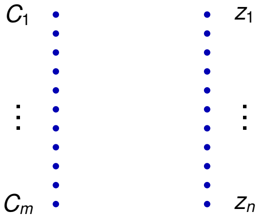
- Distributions should “locally look like" supported on satisfying assignments.

Obtaining integrality gaps for CSPs [BGMT 09]



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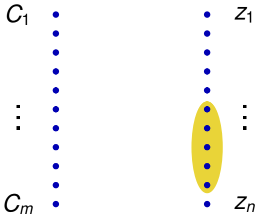

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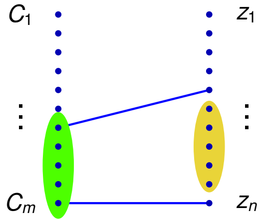

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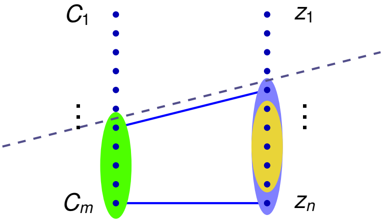

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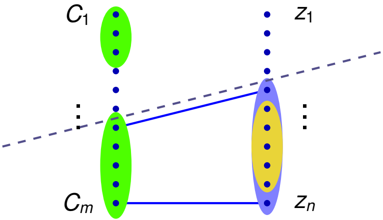

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- Remaining constraints “independent" of this assignment.
- Gives optimal integrality gaps for $\Omega(n)$ levels in the mixed hierarchy.

Vectors for Linear CSPs

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- Start with a $\{-1, 1\}$ quadratic integer program.
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- Write program for inner products of vectors $\mathbf{W}_S$ s.t.
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Gaps for 3-XOR

SDP for MAX 3-XOR

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- Used by [FO 06], [STT 07] for LS_+ hierarchy.

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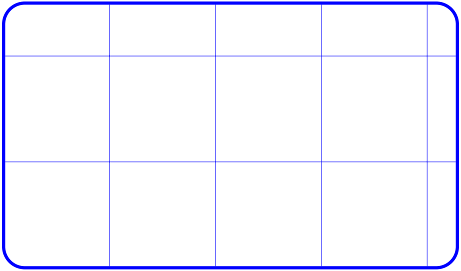

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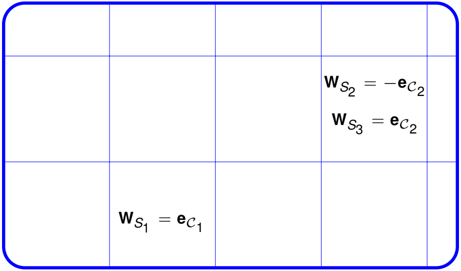

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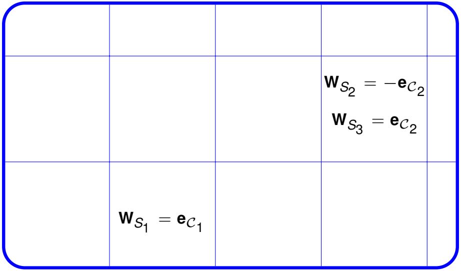
- Relies heavily on constraints being linear equations.



Reductions

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- Question posed in [AAT 04]. First done by [KV 05] from Unique Games to Sparsest Cut.

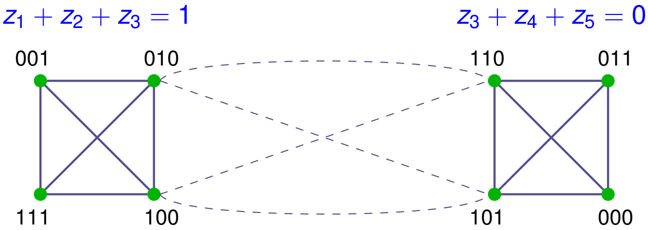
What can be proved

	NP-hard	UG-hard	Gap	Levels
MAX k-CSP	$\frac{2^k}{2^{\sqrt{2k}}}$	$\frac{2^k}{k+o(k)}$	$\frac{2^k}{2k}$	$\Omega(n)$
Independent Set	$\frac{n}{2^{(\log n)^{3/4+\epsilon}}}$		$\frac{n}{2^{c_1\sqrt{\log n \log\log n}}}$	$2^{c_2\sqrt{\log n \log\log n}}$
Approximate Graph Coloring	l vs. $2^{\frac{1}{25}\log^2 l}$		l vs. $\frac{2^{l/2}}{4l^2}$	$\Omega(n)$
Chromatic Number	$\frac{n}{2^{(\log n)^{3/4+\epsilon}}}$		$\frac{n}{2^{c_1\sqrt{\log n \log\log n}}}$	$2^{c_2\sqrt{\log n \log\log n}}$
Vertex Cover	1.36	2 - ϵ	1.36	$\Omega(n^\delta)$

- All the above results are for the Lasserre hierarchy.

The FGLSS Construction

- Reduces MAX k-CSP to Independent Set in graph G_Φ.



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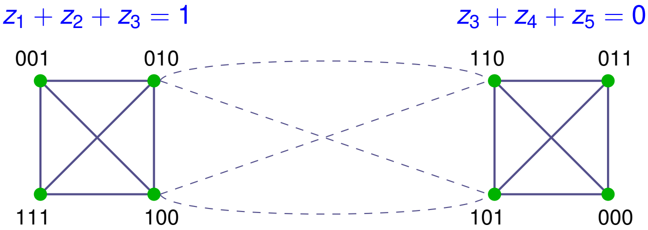
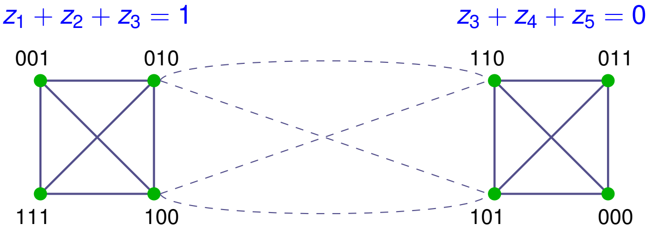


- Need vectors for subsets of vertices in the G_Φ.

The FGLSS Construction

- Reduces MAX k-CSP to Independent Set in graph G_Φ.



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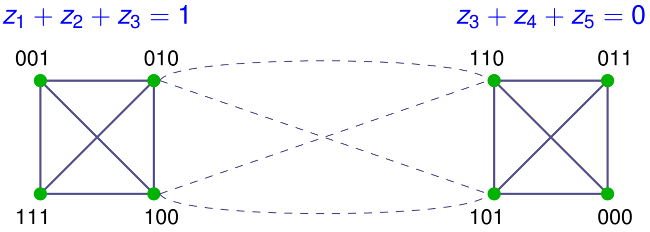

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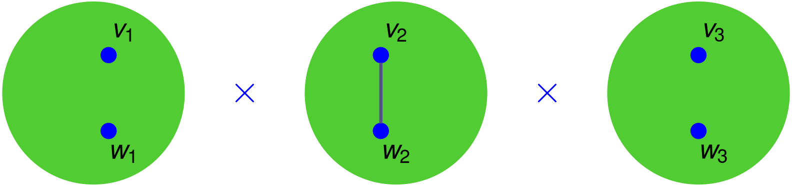
$\mathbf{U}_{\{(z_1,z_2,z_3)=(0,0,1)\}} = \frac{1}{8}(\mathbf{W}_\emptyset + \mathbf{W}_{\{1\}} + \mathbf{W}_{\{2\}} - \mathbf{W}_{\{3\}} + \mathbf{W}_{\{1,2\}} - \mathbf{W}_{\{2,3\}} - \mathbf{W}_{\{1,3\}} - \mathbf{W}_{\{1,2,3\}})$

Graph Products



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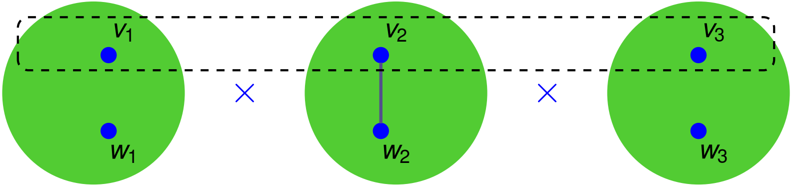

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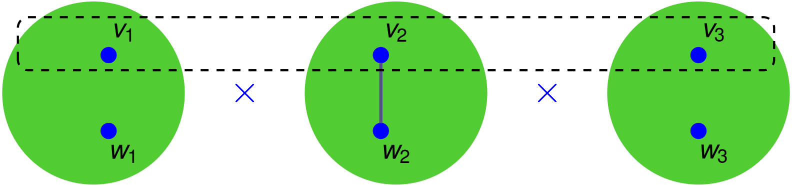

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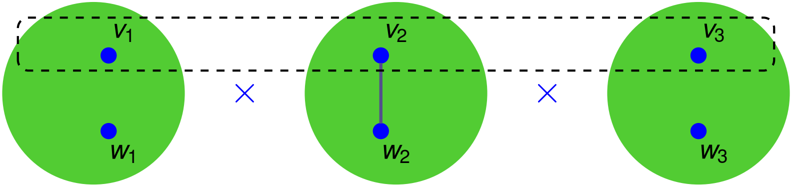

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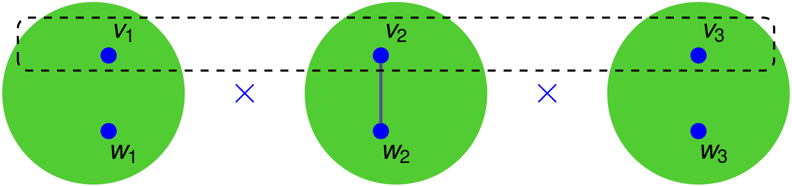

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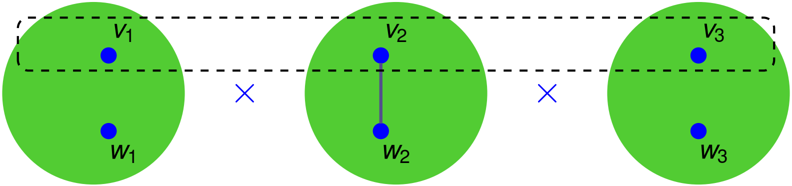

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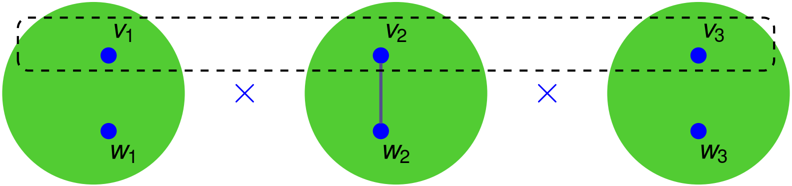

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- Together give a gap of $\frac{n}{2^{O(\sqrt{\log n \log\log n})}}$.

A few problems

Problem 1: Lasserre Gaps

- Show an integrality gap of $2 - \epsilon$ for Vertex Cover, even for $O(1)$ levels of the Lasserre hierarchy.
- Obtain integrality gaps Unique Games (and Small-Set Expansion)
  - Gaps for $O((\log \log n)^{1/4})$ levels of mixed hierarchy were obtained by [RS 09] and [KS 09].
  - Extension to Lasserre?

Problem 2: Generalize Schoenebeck's technique

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- What extra constraints do vectors capture?

Thank You

Questions?