FOURIER TRANSFORM FOR THE DIRECTED QUINCUNX LATTICE

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ABSTRACT

We introduce a new signal transform for computing the spectrum of a signal given on a two-dimensional directional quincunx lattice. The transform is non-separable, but closely related to a two-dimensional (separable) discrete Fourier transform. We derive the transform using recently discovered connections between signal transforms and polynomial algebras. These connections also yield several important properties of the new transform.

1. INTRODUCTION

It is well-known that an application of the two-dimensional discrete Fourier transform (DFT) assumes the data reside on a two-dimensional directed square lattice (shown in Figure 1) with periodic boundary conditions (not shown in Figure 1), which effectively turns the lattice into a torus. Two shift operators x and y operate on this lattice. Their action is diagonalized by the DFT, i.e., shifting on the lattice corresponds to scaling in the frequency domain.

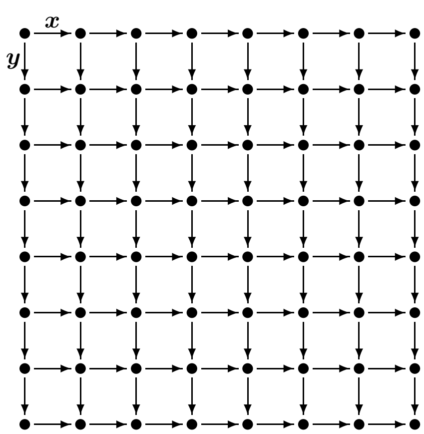

Fig. 1. Lattice associated with the two-dimensional discrete Fourier transform (the periodic boundary conditions are omitted) including the two shift operators x, y. The associated polynomial algebra (explained in the text) is $\mathcal{A} = \mathbb{C}[x, y]/\langle x^n - 1, y^n - 1\rangle = \mathbb{C}[x]/\langle x^n - 1\rangle \otimes \mathbb{C}[y]/\langle y^n - 1\rangle$.

In this paper, we derive a new signal transform called *discrete directed quincunx transform (DDQT)* for a signal given on the directed quincunx lattice, which arises from the lattice in Figure 1 by omitting every other vertex (see Figure 2). We call the associated two shift operators for this lattice u and v; as required, shifting with u and v on the lattice corresponds to a scaling in the quincunx lattice's frequency domain. We will see that the DDQT is non-separable, but closely related to a separable two-dimensional Fourier transform.

Our approach exploits the connection between discrete signal transforms and *polynomial algebras*, which is well-known for the DFT. We exhibited and studied this connection for the 16 discrete cosine and sine transforms in [1, 2], and used this connection to derive a new non-separable two-dimensional signal transform for a triangular lattice [3]. In this paper, we use the same underlying theory, but use a different derivation strategy for the DDQT.

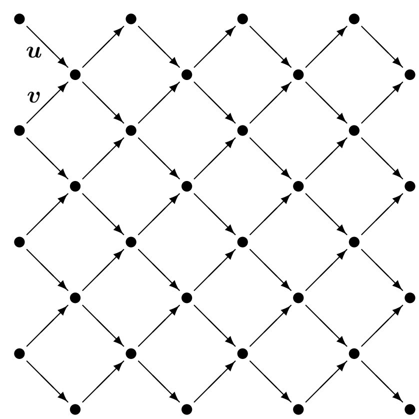

Fig. 2. The quincunx lattice including the two shift operators u and v (again the boundary conditions are omitted). The associated polynomial algebra is $\mathcal{A} = \mathbb{C}[u, v]/\langle u^n - 1, u^{n/2} - v^{n/2}\rangle$.

Background literature. The quincunx lattice has been studied in signal processing already. Applications include subband coding with respect to arbitrary sampling lattices where the quincunx lattice along with other non-separable sampling lattices has been used [4, 5]. In [6] the authors describe how to convert a signal given on the regular lattice to a signal on the quincunx lattice by performing a fractional Fourier transform.

Organization. In Section 2 we provide the necessary background on polynomial algebras and explain the connection between these algebras and signal transforms. In Section 3, we derive the Fourier transform for the quincunx lattice, called DDQT, and its diagonalization properties. We offer conclusions in Section 4.

2. POLYNOMIAL ALGEBRAS AND TRANSFORMS

In this section, we introduce some background on polynomial algebras and their associated polynomial transforms. First, we focus on polynomial algebras in one variable and explain that the DFT

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and the discrete cosine transforms (DCTs) are polynomial transforms for suitably chosen polynomial algebras. Then we extend the theory to polynomial algebras in two variables, first explaining the general and then the separable case. We will use this theory in Section 3 to construct a polynomial algebra in two variables for the quincunx lattice; the associated polynomial transform will be called the DDQT. The knowledge of the underlying polynomial algebra then immediately provides several properties of the DDQT.

2.1. One Variable

Polynomial algebras. An algebra is a vector space $\mathcal{A}$ which is also a ring, i.e., equipped with a multiplication operation for its elements such that the distributive law holds. Examples include the complex numbers $\mathcal{A} = \mathbb{C}$ or the algebra of polynomials in one variable $\mathcal{A} = \mathbb{C}[x]$.

The key players in this paper are *polynomial algebras*. Fixing a polynomial $p(x) \in \mathbb{C}[x]$, the polynomial algebra $\mathbb{C}[x]/p(x)$ is the set of all polynomial of degree less than the degree of p,

$$\mathbb{C}[x]/p(x) = \{q(x) \,|\, \deg(q) < \deg(p)\},$$

with addition and multiplication modulo p. As a vector space, the dimension of $\mathcal{A}$ is $\deg(p)$. As an algebra, $\mathbb{C}[x]/p(x)$ is generated (by consecutively forming sums, products, and scalar multiples) by x.

Polynomial transforms. Let $\mathcal{A} = \mathbb{C}[x]/p(x)$ be a polynomial algebra and assume that p has pairwise distinct zeros $\alpha = (\alpha_0, \dots, \alpha_{n-1})$, where $n = \deg(p)$. Then the Chinese remainder theorem (CRT) [7] decomposes $\mathcal{A}$ into a direct sum (or Cartesian product with elementwise operation) of one-dimensional polynomial algebras:

$$\begin{aligned} \Phi :\ \mathbb{C}[x]/p(x) &\rightarrow \mathbb{C}[x]/(x-\alpha_0) \oplus \ldots \oplus \mathbb{C}[x]/(x-\alpha_{n-1}), \\ q(x) &\mapsto (q(\alpha_0), \dots, q(\alpha_{n-1})). \end{aligned}$$

The CRT states that Φ is one-to-one (bijective) and an isomorphism of algebras. Φ maps a polynomial $q \in \mathcal{A}$ to the list of its projections $q(x) \equiv q(\alpha_k) \bmod x - \alpha_k$, $0 \le k < n$.

If we choose a basis $b = (p_0, \dots, p_{n-1})$ in $\mathbb{C}[x]/p(x)$, then Φ (which in particular is a linear mapping) is represented by the matrix

$$\mathcal{P}_{b,\alpha} = [p_\ell(\alpha_k)]_{0 \le k, \ell < n}, \qquad (1)$$

which contains in its columns the images of p_ℓ under Φ. We call $\mathcal{P}_{b,\alpha}$ the *polynomial transform* for $\mathcal{A}$ with basis b. From a signal processing point of view, $\mathcal{P}_{b,\alpha}$ is a Fourier transform for the polynomial algebra $\mathcal{A}$.

Diagonalization properties. Let $h(x) \in \mathcal{A} = \mathbb{C}[x]/p(x)$. Then multiplication by $h(x)$ is a linear mapping in $\mathcal{A}$ because of the distributivity law, i.e., for $q(x), q_1(x), q_2(x) \in \mathcal{A}$, $c \in \mathbb{C}$,

$$h(q_1 + q_2) = hq_1 + hq_2, \quad h(cq) = c(hq).$$

Thus, with respect to the chosen basis b of $\mathcal{A}$, multiplication by $h(x)$ is represented by a matrix $M_{h(x)}$. The CRT implies that every such matrix is diagonalized by $\mathcal{P}_{b,\alpha}$. More precisely

$$\mathcal{P}_{b,\alpha} M_{h(x)} \mathcal{P}_{b,\alpha}^{-1} = \operatorname{diag}(h(\alpha_0), \dots, h(\alpha_{n-1})). \qquad (2)$$

In particular,

$$\mathcal{P}_{b,\alpha} M_x \mathcal{P}_{b,\alpha}^{-1} = \operatorname{diag}(\alpha_0, \dots, \alpha_{n-1}). \qquad (3)$$

Associated graph. We can associate a (possibly weighted) graph with $\mathbb{C}[x]/p(x)$ with basis $b = (p_0, \dots, p_{n-1})$ by choosing the basis polynomials p_ℓ as vertices and expressing the operation of x as edges. The adjacency matrix of this graph is M_x.

Examples: DFT and DCTs. As an example, it is well-known that the DFT_n is a polynomial transform for $\mathcal{A} = \mathbb{C}[x]/(x^n - 1)$ with basis $b = (1, x, \dots, x^{n-1})$. Namely, the zeros of $p(x) = x^n - 1$ are given by $\alpha_k = \omega_n^k$, where $\omega_n = \exp(-2\pi i/n)$, which shows that $\mathrm{DFT}_n = \mathcal{P}_{b,\alpha}$. The matrices $M_{h(x)}$ in (2) are the circulant matrices in this case (and are thus diagonalized by the DFT); in particular

$$M_x = Z_n = \begin{bmatrix} 0 & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}$$

is the cyclic shift. The graph associated with $\mathbb{C}[x]/(x^n - 1)$ and thus the DFT is a directed cycle, which has Z_n as adjacency matrix. Note that the equation $x^n - 1 = 0$ in $\mathcal{A}$, or equivalently the equation $x^n = 1$, captures the periodic boundary condition.

We have also shown that all 16 DCTs and DSTs are polynomial transforms [1, 2]. For example, the DCT, type III, of size n, is a polynomial transform for the polynomial algebra $\mathbb{C}[x]/T_n(x)$ with basis $b = (T_0(x), \dots, T_{n-1}(x))$, where $T_k(x)$ denotes the kth Chebyshev polynomial.

2.2. Two variables

The above theory extends to polynomial algebras in *two* variables. We first develop the theory in the general case and then consider the important special case of separable algebras corresponding to separable transforms.

Polynomial algebras. A polynomial algebra in two variables is given by[1]

$$\mathcal{A} = \mathbb{C}[x,y]/\langle p(x,y), q(x,y)\rangle, \qquad (4)$$

with addition and multiplication modulo p *and* q, which are two polynomials in two variables. If p and q are properly chosen, then $\mathbb{C}[x,y]/\langle p(x,y), q(x,y)\rangle$, viewed as a vector space, is of finite dimension.

Polynomial transforms. To define a polynomial transform for the algebra (4) we assume that the total degrees[2] of p and q are n and m, respectively, and that the equations $p(x,y) = q(x,y) = 0$ have precisely mn solutions $\alpha = (\alpha_k, \beta_k)_{0 \le k < mn}$. The CRT now becomes the one-to-one mapping (isomorphism of algebras)

$$\Phi :\ \mathcal{A} \rightarrow \bigoplus_{0 \le k < mn} \mathbb{C}[x,y]/\langle x - \alpha_k, y - \beta_k\rangle,$$

which also implies that $\dim(\mathcal{A}) = mn$. With respect to a basis $b = (p_\ell(x,y) | 0 \le \ell < mn)$ of $\mathcal{A}$, the polynomial transform for $\mathcal{A}$ is the matrix $\mathcal{P}_{b,\alpha} = [p_\ell(\alpha_k, \beta_k)]_{0 \le k, \ell < mn}$. As in the case of one variable, $\mathcal{P}_{b,\alpha}$ can be considered as a Fourier transform for $\mathcal{A}$.

Diagonalization properties. If $h(x,y) \in \mathcal{A}$, then the multiplication by $h(x,y)$ is a linear mapping in $\mathcal{A}$ and w.r.t. the basis b

[1] We restrict ourselves to computing modulo *two* polynomials p and q. In the general case of two variables—contrary to the univariate case—there are situations where we need more than two polynomials.

[2] The total degree of $p(x,y)$ is defined as the maximum value of $i + j$, as i and j range over all summands cx^iy^j of $p(x,y)$.

represented by a matrix $M_{h(x,y)}$, which is diagonalized as

$$\mathcal{P}_{b,\alpha} M_{h(x,y)} \mathcal{P}_{b,\alpha}^{-1} = \operatorname{diag}_{0 \le k < mn}(h(\alpha_k, \beta_k)).$$

Associated graph. Again we associate a graph with $\mathcal{A}$ by choosing the basis polynomials in b as vertices and represent the action, now of both x and y, through edges. The adjacency matrix of the graph is $M_x + M_y$.

Separable algebras. A particularly simple situation arises, if in (4), $p = p(x)$ and $q = q(y)$ depend only on x and y, respectively. In this case, we can choose as basis in $\mathcal{A}$ the product[3] of the chosen bases b_x and b_y of $\mathbb{C}[x]/p(x)$ and $\mathbb{C}[y]/q(y)$, respectively. Further, the solutions of $p(x) = q(y) = 0$ are given by the Cartesian product $\alpha \times \beta$ of the zeros α of p and β of q. Mathematically, this shows that

$$\mathbb{C}[x,y]/\langle p(x), q(y)\rangle \cong \mathbb{C}[x]/p(x) \otimes \mathbb{C}[y]/q(y) \qquad (5)$$

is the *tensor product* of one-dimensional algebras, or a *separable* polynomial algebra. The polynomial transform in this case is then also *separable* in the signal processing sense, i.e., given by

$$\mathcal{P}_{b_x,\alpha} \otimes \mathcal{P}_{b_y,\beta},$$

where $\otimes$ is here the Kronecker product of *matrices*, defined as $A \otimes B = [a_{k,\ell} B]_{k,\ell}$, for $A = [a_{k,\ell}]$.

Example. As an example, which we use later, we consider

$$\begin{aligned}\mathbb{C}[x,y]/\langle x^m - 1, y^n - 1\rangle \\ \cong \mathbb{C}[x]/(x^m-1) \otimes \mathbb{C}[y]/(y^n-1),\end{aligned} \qquad (6)$$

with basis

$$\begin{aligned} b \;=\; & (x^0y^0, x^1y^0, \dots, x^{m-1}y^0, \\ & \dots\dots \\ & x^0y^{n-1}, x^1y^{n-1} \dots, x^{m-1}y^{n-1}). \end{aligned} \qquad (7)$$

The corresponding polynomial transform is separable, namely the two-dimensional $\mathrm{DFT}_n \otimes \mathrm{DFT}_m$.

The graph corresponding to the algebra is shown in Figure 1 (without the edges for the periodic boundary conditions). The vertices are considered to be indexed by the basis polynomials in (7) from top to bottom in row major order. The horizontal arrows correspond to the operation of x on the basis; likewise, the vertical arrows depict the operation of y. The corresponding matrices are

$$M_x = \mathrm{I}_n \otimes Z_m \quad \text{and} \quad M_y = Z_n \otimes \mathrm{I}_m$$

(with I_n the identity matrix), which both are clearly diagonalized by $\mathrm{DFT}_n \otimes \mathrm{DFT}_m$ as required.

As a non-separable example, we derived a signal transform for an undirected, i.e., spatial triangular lattice in [3].

3. THE QUINCUNX LATTICE AND TRANSFORM

With the theory in place, the derivation of the Fourier transform for the quincunx lattice in Figure 2 becomes surprisingly easy. Obviously, we have two shifts operating on the lattice, called u and v in Figure 2. Thus, the task is to find polynomials $p(u,v)$ and $q(u,v)$, and a basis for $\mathbb{C}[u,v]/\langle p(u,v), q(u,v)\rangle$, such that the corresponding graph is the quincunx lattice. The theory in Section 2.2 then allows us to read off the transform and its diagonalization properties.

Construction as a subalgebra. We start by recognizing that the quincunx lattice arises from the standard square lattice by omitting every other vertex (observed from Figures 1 and 2). Thus, the idea is to identify the algebra for the quincunx lattice with a subalgebra of (6).

The vertices in Figure 1 are indexed by the elements x^iy^j of the basis (7). Thus, the vertices left over in Figure 2 are precisely those in the set

$$b = \{x^iy^j \mid i + j \equiv 0 \bmod 2\}. \qquad (8)$$

Traversing the vertices in Figure 2 by rows from top to bottom, the order in b is

$$\begin{aligned} b \;=\; & (x^0y^0, x^2y^0, \dots, x^{n-2}y^0, \\ & x^1y^1, x^3y^1, \dots, x^{n-1}y^1, \\ & x^0y^2, x^2y^2, \dots, x^{n-2}y^2, \\ & \dots\dots). \end{aligned} \qquad (9)$$

The set b in (9) spans a subvector space of (6), but this space is also a subalgebra (closed under multiplication) only if n is even. (Otherwise, $x^{n-1}, x^2 \in b$, but their product $x^{n+1} \equiv x^1$ is not.) This is equivalent to requiring that the rows in Figure 2 have equally many elements, namely $n/2$ each. The dimension of this subalgebra is $n^2/2$. Further, we read off that

$$u = xy \quad \text{and} \quad v = xy^{-1},$$

i.e., the desired algebra for the quincunx lattice is the subalgebra of (6) generated by u and v with basis the b in (9).

Constructing the algebra. Next, we have to find the polynomials $p(u,v)$ and $q(u,v)$. Clearly, $u^n = x^ny^n = 1$ and $u^{n/2} = x^{n/2}y^{n/2} = x^{n/2}y^{-n/2}y^n = x^{n/2}y^{-n/2} = v^{n/2}$. Thus, we get the polynomials $p(u,v) = u^n - 1$ and $q(u,v) = u^{n/2} - v^{n/2}$, which yields the polynomial algebra

$$\mathcal{A} = \mathbb{C}[u,v]/\langle u^n - 1, u^{n/2} - v^{n/2}\rangle. \qquad (10)$$

The question is whether the polynomials p and q are sufficient to describe the quincunx lattice (i.e., the conditions $p = q = 0$ could be too “loose”). Applying Hilbert’s method [8, ch. 9] to compute the dimension of $\mathcal{A}$ yields $\dim(\mathcal{A}) = n^2/2$, as desired, which confirms that p and q are properly chosen. Obviously, $\mathcal{A}$ in (10) is not separable.

At this point we have the polynomial algebra and its basis b (which we would need to express in u and v) and could thus apply our theory to compute the desired transform. However, there is a simpler way of obtaining the transform.

Simplification through variable change. We can change the variables in (10), keeping u, but setting $w = uv = x^2$, which implies $w^{n/2} = 1$. In the variables u, w, (10) now takes the *separable* form

$$\mathbb{C}[u,w]/\langle u^n - 1, w^{n/2} - 1\rangle.$$

For this algebra, we know the polynomial transform, which is $\mathrm{DFT}_n \otimes \mathrm{DFT}_{n/2}$, *provided* the chosen basis has the form (7) of a product of two bases. In our case, the basis is already chosen, namely b in (9). Thus, we first express b in the new variables u, w; then, we identify the permutation P that permutes b to take the product form in (7). The desired transform for the quincunx lattice will then be $(\mathrm{DFT}_n \otimes \mathrm{DFT}_{n/2})P$.

[3] Recall that the product of two lists $(c_1, \dots, c_n)$ and $(d_1, \dots, d_m)$ is the list of all products $(c_1d_1, c_2d_1, \dots)$ of length mn.

Expressing b in (9) in the variables w and u is straightforward and yields

$$\begin{aligned} b \quad = \quad & (w^0u^0, w^1u^0, \dots w^{n/2-1}u^0, \\ & w^0u^1, w^1u^1, \dots, w^{n/2-1}u^1, \\ & w^{n/2-1}u^2, w^0u^2, \dots, w^{n/2-2}u^2, \\ & w^{n/2-1}u^3, w^0u^3, \dots, w^{n/2-2}u^3, \\ & \dots\dots). \end{aligned} \tag{11}$$

To explain the structure of b, we assume that the rows in (11) are numbered from 0 to $n-1$. A row with an odd number $2k+1$ arises from row $2k$ by multiplying with u. A row with even number $2k$ arises from row $2k-2$ by multiplying with u^2 *and* applying a cyclic shift $Z_{n/2}$. Thus, to permute b into the form (7) (with w as x), we have to undo these cyclic shifts for each row. Doing so yields the permutation

$$P_n = (Z^0_{n/2} \oplus Z^0_{n/2}) \oplus (Z^{-1}_{n/2} \oplus Z^{-1}_{n/2}) \oplus \dots \oplus (Z^{-(n/2-1)}_{n/2} \oplus Z^{-(n/2-1)}_{n/2}),$$

where

$$A \oplus B = \begin{bmatrix} A & \\ & B \end{bmatrix}$$

is the *direct sum* of matrices. In a more compressed form, P_n can be written as

$$P_n = \bigoplus_{0 \le i < n/2} (\mathrm{I}_2 \otimes Z^{-i}_{n/2}). \tag{12}$$

DDQT. The *discrete directed quincunx transform (DDQT)* for a signal given on a quincunx lattice (Figure 2) with periodic boundary conditions is given by

$$\mathrm{DDQT}_{n\times n/2} = (\mathrm{DFT}_n \otimes \mathrm{DFT}_{n/2}) P_n,$$

where P_n is given in (12). The signal is assumed to be scanned in row-major order from top to bottom in Figure 2, i.e., in n rows of length $n/2$. The $\mathrm{DDQT}_{n\times n/2}$ is a polynomial transform for the polynomial algebra (10) with basis b (expressed in u and v). By construction, the corresponding graph is exactly the quincunx lattice. Note that this lattice inherits the periodic boundary conditions from the square lattice for the DFT.

Diagonalization properties. Being a polynomial transform the general diagonalization properties of the $\mathrm{DDQT}_{n\times n/2}$ follow directly from the theory presented in Section 2.2. We start with deriving the explicit form of the two shift matrices M_u and M_v.

Letting u operate on b in (11) yields the matrix

$$\begin{aligned} M_u \quad &= \quad \begin{bmatrix} 0_{n/2} & & & & & Z_{n/2} \\ \mathrm{I}_{n/2} & & & & & \\ & Z_{n/2} & & & & \\ & & \mathrm{I}_{n/2} & & & \\ & & & \ddots & & \\ & & & & \mathrm{I}_{n/2} & 0_{n/2} \end{bmatrix} \\ &= \quad (Z_n \otimes \mathrm{I}_{n/2})(\mathrm{I}_{n/2} \otimes (\mathrm{I}_{n/2} \oplus Z_{n/2})). \end{aligned}$$

Similarly, letting w operate on b yields

$$M_w = \mathrm{I}_n \otimes Z_{n/2}.$$

Since $v = u^{-1}w$, we get, using properties of $\otimes$ and $\oplus$,

$$M_v = M_u^{-1} M_w = (\mathrm{I}_{n/2} \otimes (Z_{n/2} \oplus \mathrm{I}_{n/2}))(Z_n^{-1} \otimes \mathrm{I}_{n/2}).$$

As a consequence of our general theory, each of the matrices M_u, M_v, M_w is diagonalized by the $\mathrm{DDQT}_{n\times n/2}$; this can also be confirmed via a direct computation. If $h = h(x, y)$ is any polynomial in two variables, then $h(M_u, M_v)$ is diagonalized by the $\mathrm{DDQT}_{n\times n/2}$.

Remark. The reader familiar with group theory may have observed that the quincunx lattice can be viewed as a subgroup of index 2 of the direct product $C_n \times C_n$ of two cyclic groups of order n. As such, this subgroup must be isomorphic to $C_n \times C_{n/2}$, and hence it is not surprising that the DDQT is "almost" a $\mathrm{DFT}_n \otimes \mathrm{DFT}_{n/2}$. The actual problem is to derive the *exact* form of this transform.

4. CONCLUSIONS

We have derived a new signal transform, which is a Fourier transform for a two-dimensional signal on the quincunx lattice with periodic boundary conditions. The derivation was made comparatively easy using the theory of polynomial algebras in one and two variables. This paper is an extension of the algebraic approach towards signal transforms already used in [1, 3] and part of an ongoing effort to work towards an algebraic theory of signal processing [9, 10].

5. ACKNOWLEDGMENT

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